Admissible Generalized Bayes Estimators and Exterior Boundary Value Problems

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Abstract. The problem of admissibility of generalized Bayes estimators under quadratic loss is investigated by relating it to a boundary value problem in partial differential equations. The main theorem generalizes L. Brown's result. Towards the end some applications are considered.

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Admissible Generalized Bayes Estimators and Exterior Boundary Value Problems

1. Introduction. In recent years the problem of determining admissible estimators of the mean of a multivariate normal distribution has drawn the attention of many authors. In this paper, we revist the problem and develop more general admissibility results for generalized Bayes estimators of the mean under the squared error loss function.

L. Brown [3] related the problem of admissibility of an estimator to a calculus of variation problem and obtained a necessary and sufficient condition for the admissibility of bounded risk generalized Bayes estimators. Brown's result includes the admissibility and inadmissibility results of the best invariant estimator due to Stein [12].

Even though Brown's theorem covers a lrage class of estimators, there are many unbounded risk estimators some of which are known to be admissible. We have been able to obtain a generalization which covers, besides bounded risk estimators, a large class of unbounded risk estimators.

Our approach to the problem, though similar to that of Brown, uses the exterior boundary value problem in partial differential equations. Moreover, our proof is shorter and does not require such technical results as mean value property of minimizing functions of the calculus of variation problem. We give a brief outline of the paper below.

In Sections 2 and 3 we give the notations, assumptions and relation of the admissibility problem to the calculus of variation problem. Section 4 deals with the calculus of variation problem and the associated exterior boundary problem. The main theorem of Section 4 relates the calculus of variation problem to the exterior boundary problem, thus enabling us to construct suitable minimizing sequence of functions for the calculus of variation problem. The proofs of results in Section 4, since they do not fit in to the mainstream of this article, are presented in the appendix at the end of the paper. The discussion of our assumptions are given in Section 5. In particular, we show that Brown's assumptions imply ours. Some technical results, which are needed in the proof of our admissibility theorem, are presented in Section 6. The proof of the main thereom is contained in Section 7. Examples and applications are given in Section 8.

2. Preliminaries.

Let X be an m-dimensional normal random variable with unknown mean vector θ and the identity matrix as the dispersion matrix. Let $p_\theta(x)$ denote the m-dimensional normal density $(2\pi)^{-m/2}\exp-\frac{1}{2}\sum_1^m(n_1-u_i)^2$ by E^m and its norm by $|\cdot|$. We consider the problem of estimating θ with respect to the squared error loss function $L(\theta,t) = |\theta-t|^2$ where t is an m-dimensional vector. For any estimator $\delta(x) = (\delta_1(x),\ldots,\delta_m(x))$ its risk function is denoted by $R(\theta,\delta)$.

Let G be an non-negative σ-finite Borel measure on E^m such that $g(x) = \int p_\theta(x)G(d\theta) < \infty$ for almost all x in E^m. Then the generalized Bayes estimator of G exists and is denoted by δ_G. The estimator $\delta_G(x)$ is given by

$$\delta_G(x) = \frac{\int \theta\ p_\theta(x)G(d\theta)}{\int p_\theta(x)G(d\theta)}.$$

It is easy to see, by differentiating under integral sign, that $\delta_G(x) = \frac{\nabla g(x)}{g(x)} + x$ where "∇" stands for the gradient of a function.

An estimator $\delta(\cdot)$ is said to be admissible if, for any other estimator δ', $R(\theta,\delta') \leq R(\theta,\delta)$ for all θ implies $R(\theta,\delta') \equiv R(\theta,\delta)$. By a generalized prior F we mean a non-negative σ-finite measure such that $f(\ ) = \int p_\theta(x)F(d\theta) < \infty$ almost everywhere with respect to the m-dimensional Lebsque measure. For any generalized prior F, let K_F denote the closed convex hull of its support. If x is any point in E^m define

$$d_F(x) = \inf\{|x-y|: \ y \in K_F\}$$

to be the distance of x from K_F. Let $\pi(x)$ denote the projection of x onto K_F. Clearly, $d_F(x) = |\pi(x)-x|$.

Finally, if u: $E^m \to E^1$, we shall say that u is piecewise differentiable if there exists a collection (countable) of disjoint open sets $\{0_i\}$, such that $E^m = \bigcup_{i=1}^{\infty} \bar{0}_i$ ($\bar{0}_i$ is the closure of 0_i) and u is continuously differentiable in each 0_i.

3. The Problem and Assumptions.

The basic problem we are concerned with is to obtain necessary and sufficient conditions for the admissibility of an estimator $\delta(x)$ of θ. Brown [3] has shown that admissible estimators of θ are generalized Bayes (see also Farrell [4], Sacks [10]. Therfore, our study of admissibility of estimators can be confined to generalized Bayes estimators. The main aim of this paper is to obtain sufficient conditions on f(x) for δ_F to be admissible. Through out the rest of this paper we assume that F is a fixed nonnegative σ-finite Borel measure with unbounded support. (If the support of F is bounded then F is a finite measure and therefore δ_F is proper Bayes and hence admissible.

The basic tool for our study is the following necessary and sufficient condition for admissibility due to Farrell [5] (see also Stein [11])). An estimator $\delta(x)$ is admissible if and only if there exists a sequence of finite measures $\{G_n\}$ satisfying (i) G_n has compact support (ii) for some compact set C and a constant $\beta > 0$ such that $G_n(C)$ _ for all n and

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))G_n(d\theta) \to 0 \qquad \text{as } n \to \infty \tag{3.1}$$

Using the fact that $\delta_G(x) = \frac{\nabla g(x)}{g(x)} + x$ and interchanging the order of integration in (2.1) we have, as in Brown [3],

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))G_n(d\theta) = \int |h_n^{1/2}(x)|^2 f(x)dx \tag{3.2}$$

where $h_n = g_n(x)/f(x)$ and $g_n(x) = \int p_\theta(x)G_n(d\theta)$. This identity (3.2) plays a crucial role in the rest of the paper.

We can take, without loss of generality, the compact Set C to be the unit sphere and $\beta = 1$. This implies, as shown by Brown [3], that $h_n(x) \geq 1$ for $|x| \leq 1$ for all n (if necessary normalize F on the unit sphere). The condition (i) that G_n's here compact supports implies the following. For any $\alpha \geq 0$

$$\lim_{r\to\infty} \sup_{\{x:\ x\in K_F^\alpha,\ |x| \geq r\}} h_n(x) = 0 \qquad \text{for all n.} \tag{3.3}$$

See Brown [3] for proof. Now, let J be the class of all non-negative piecewise differentiable real valued functions j defined in E^m satisfying

(i) $j(x) \geq 1$ for $|x| \leq 1$

$$\text{(ii)} \quad \lim_{r\to\infty} \sup_{\{n:\ x\in K_F^{\alpha},\ |x|\geq r\}} j(x) = 0 \quad \text{for all} \quad 0.$$

Then it is easy to see using (2.1) that

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_n}))G_n(d\theta) \geq \inf_{j\in J} \int |j(x)|^2 f(x)dx \tag{3.4}$$

for all n. Consequently, if δ_F is admissible then (3.1) goes to zero as $n \to \infty$ and we have

$$\inf_{j\in J} \int |\nabla j(x)|^2 f(x)dx \tag{3.5}$$

equal to zero. In particular, if (3.5) is positive then δ_F is inadmissible. The converse, that (3.5) is zero implies δ_F is admissible, was proved by Brown [3] under the assumption $\left|\frac{\nabla f(x)}{f(x)}\right|$ is bounded in K_F. This assumption, as shown by Brown [3], is equivalent to the fact that the risk of δ_F is bounded in K_F. We generalize the result of Brown in this paper. We make the following assumptions on F through the rest of the paper.

I. $\Delta \log f(x) < B \qquad$ for all x

II. $\frac{x}{|x|} \frac{\nabla f(x)}{f(x)} + \left|\frac{\nabla f(x)}{f(x)}\right| < K$ for all $x \in K_F$.

where Δ is the Laplacian operator. We discuss these assumptions in Section 5.

4. The Calculus of Variation Problem

We observed in the previous section, that the following calculus of variation problem

$$\inf_{j\in J} \int |\nabla j(x)|^2 f(x)dx$$

is crucial to our study of admissibility of δ_Γ. We are, in particular, interested in finding under what conditions this infimum is zero or not. We introduce below an exterior boundary value problem which describes when this infimum is zero.

Let L_f denote the elliptic differential operator given by

$$L_f u = \Delta u + \nabla u \cdot \frac{\nabla f}{f} \tag{4.1}$$

where u is a twice differentiable function. We say that Exterior Boundary Problem for L_f(BP for L_f) is solvable if there exists a unique bounded solution u_0 for the equation $L_f u(x) = 0$ in the exterior domain $S_1^C = \{x: |x| > 1\}$ satisfying the condition $u_0(x) = 1$ on $|x| = 1$. i.e. u(x) take the value 1 continuously on the boundary ∂S_1 of the unit sphere. We are now in a position to state our result. Note that the unique bounded solution u_0 is identically equal to 1.

Theorem 4.1. A necessary and sufficient condition for $\inf_{j \in J} \int |\nabla j(x)|^2 f(x)dx = 0$ is BP for L_f is solvable.

Proof. See appendix.

The above result enables us to obtain a smooth minimizing sequence of functions for our calculus of variation problems. Indeed, if $k_n(x)$ is a sequence of functions satisfying

$$L_f k_n(n) = 0 \qquad \text{if } 1 < |x| < n \tag{4.2}$$

$$k_n(x) = 1 \qquad \text{if } |x| = 1 \tag{4.3}$$

$$K_n(n) = \varphi_n \qquad \text{if } |x| = n \tag{4.4}$$

where $1 > \varphi_n(x) \geq 0$, then $k_n \to 1$ uniformly on compacta and $\lim_{n\to\infty} \int |\nabla k_n(x)|^2 f(x)dx = 0$. We would need suitably chosen such K_n's to prove the admissiblity result.

It is also a known fact that, by Maximum modulas principle, $0 < k_n(x) < 1$ for $1 < |x| < n$. For a proof see Miranda [9]. The Exterior boundary problem has been studied by Meyers and Serrin [8] and he has given sufficient conditions for its Sdvability. Also see Brown [3] for some sufficient conditions. We list a few towards the end.

5. Discussion of the Assumptions

We show in this section that our assumptions I and II are weaker than the assumption $|\frac{\nabla f}{f}| < B$ on K_F made by Brown [3].

The assumption (I) $\Delta \log f(x) < B$ is equivalent to the boundedness of the posteriori risk. Indeed,

$$\Delta \log f(x) = \int |x-\theta|^2 \frac{p_\theta(x) F(d\theta)}{f(x)} - |\frac{\nabla f}{f}|^2 - m$$

$$= \int |\delta_F(x)-\theta|^2 \frac{p_\theta(x) F(d\theta)}{f(x)} - m \tag{5.1}$$

Therefore, $\Delta \log f(x)$ is bounded if and only if the posterior risk is bounded.

<u>Theorem 5.1</u>. Suppose $|\frac{\nabla f(x)}{f(x)}| < B$ for $x \in K_F$. Then there exists B_1 (depending only on B and m) such that $\Delta \log f(x) < B_1$.

We need the following technical result (see Brown [3]) to prove Theorem 5.1.

<u>Lemma 5.2</u>. Let $K > 0$ be a constant. Then there exists a constant $K_1 > 0$ (depending only on K and m) such that

$$e^{K|x-\theta|} p_\theta(x) \le K_1 \int_{|\xi|<K+1} p_\theta(x+\xi)^2 \, d\xi .$$

Proof. See Brown [3].

We now prove Theorem 5.1.

Proof of Theorem 5.1.

Assume $|\frac{\nabla f(x)}{f(x)}| < B$. Let $x \in K_F$. Then, plainly,

$$\Delta \log f(x) \leq \int |x-\theta|^2 \frac{p_\theta(x) F(d\theta)}{f(x)}$$

$$\leq e^{|x-\theta|} \frac{p_\theta(x) F(d\theta)}{f(x)}$$

$$\leq K_1 \int \int_{|\xi|<K_1+1} p_\theta(x+\xi) \frac{F(d\theta)}{f(x)} \tag{5.2}$$

for some constant $K_1 > 0$ by lemma 5.2. Now,

$$(5.2) \leq K_1 \int_{|\xi|<K_1+1} f(x+\xi)d\xi/f(x) \leq K_1 e^{K_2 |\frac{\nabla f(x)}{f(x)}|} \int_{|\xi|<K_1+1} d\xi \tag{5.3}$$

for some constant K_2 depending on K_1. Therefore, for $x \in K_F$, $\Delta \log f(x) < B_1$ for some $B_1 > 0$ (depending on B, K_1, K_2 and m) since $|\frac{\nabla f(x)}{f(x)}| < B$ for $x \in K_F$.

Now let us consider the case $x \notin K_F$. Let $\pi(x)$ be the projection of x on to K_F. It suffices to show that $|\Delta \log \frac{f(x)}{f(\pi(x))}|$ is uniformly bounded in view of the fact $|\Delta \log f(\pi(x))| < B_1$. Assume without loss of generality that $x = (-d(x), 0, \ldots, 0$, $\pi(x) = 0$ and $K_F \subseteq \{\theta: \theta_1 \geq 0\}$ (To see this, consider the hyperplane tangential to the boundary of K_F at $\pi(x)$. Now rotate and translate the space so that the normal coincides with the axis $(-1, 0, \ldots, 0)$ and $\pi(x) = 0$.). Then

$$\frac{f(x)}{f(0)} = \int e^{-1/2|x_1-\theta_1|^2} e^{-1/2 \sum_{i=0}^{m} \theta_i^2} F(d\theta) \tag{5.4}$$

where $-x_1 = d(x)$ and absorbing $f(0) = e^{-1/2|\theta|^2} F(d\theta)$ in F we have

$$\Delta \log \frac{f(x)}{f(0)} \leq \int \frac{\theta_1^2 e^{x_1\theta_1} e^{-1/2|\theta|^2} F(d\theta)}{e^{x_1-\theta_1} e^{-1/2|\theta|^2} F(d\theta)}$$

$$- \left[\int \theta_1 \frac{e^{x_1 \theta_1} e^{-1/2|\theta|^2} F(d\theta)}{e^{x_1 \theta_1} e^{-1/2|\theta|^2} F(d\theta)} \right]^2 \tag{5.5}$$

Now conditioning with respect to θ_1 and integrating with respect to the other variables we have

$$\Delta \log \frac{f(x)}{f(0)} \leq \frac{\int_0^\infty \theta_1^2 e^{x_1 \theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)}{\int_0^\infty e^{x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)} - \left[\frac{\int_0^\infty \theta_1 e^{x_1\theta_1} e^{-\frac{1}{2}\theta_1^2} F_1(d\theta_1)}{\int_0^\infty e^{x_1\theta_1} e^{-\frac{1}{2}\theta_I^2} F_1(\theta_1)} \right]^2 \tag{5.6}$$

where $F_1(\theta_1) = e^{-1/2 \sum_2^m \theta_i^2} F(d\theta_2,\ldots,d\theta_m/\theta.)$. (Note that fixing θ_1 amounts to fixing a hyperplane).

Therefore, integrating (5.5) by parts, we have

$$\Delta \log \frac{f(x)}{f(0)} \leq \frac{\int \theta_1^2 e^{\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}{\int e^{\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1} \tag{5.6}$$

In obtaining (5.6) we have used the fact that $\theta_1 x_1 < 0$. Observe that $\tilde{F}(\theta_1)$ is non-decreasing and lies between 0 and 1 because f(0) has been so absorbed in F as to normalize $\tilde{F}(\theta_1)$. Moreover, since $\left|\frac{\nabla f(y)}{f(y)}\right| < B$ for $y = \pi(x)$ it follows, by Chebyshev's inequality, that for some constant $K > 0$, $\int_{|y-\theta|<K} \frac{e^{-1/2|y-\theta|^2} F(d\theta)}{f(y)} > 1/4$ for $y = \pi(x)$. Since $\pi(x) = (0,\ldots,0)$ therefore we have $\tilde{F}(\theta_1) > 1/4$ for $\theta_1 > K$. We shall use this fact to get an upper bound for (5.6) as follows.

$$\Delta \log \frac{f(x)}{f(0)} \leq \frac{\int \theta_1^2 e^{\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1}{\int e^{\theta_1 x_1} \tilde{F}(\theta_1) d\theta_1} \leq 4K^2 + \frac{\int_{2K}^{\infty} \theta_1^2 e^{\theta_1 x_1} d\theta_1}{1/4 \int_K^{2K} e^{\theta_1 x_1} d\theta_1} \tag{5.7}$$

Therefore, since $\theta_1 > 0$ and $x_1 < 0$, it follows from (5.7) that

$$\left|\Delta \log \frac{f(x)}{f(0)}\right| < B_1$$

where B_1 depends only on B and m. This completes the proof.

The assumption II that $\frac{x}{|x|} \nabla \log f(x) + |\nabla \log f(x)| < K$ in K_1 is a growth condition on f(x). It implies $f(x) \leq K_1 e^{K_1|x|}$ for $x \in K_F$. It is also easy to see that Brown's assumption $\left|\frac{\nabla f(x)}{f(x)}\right| < B_1$ implies assumption II.

6 Technical Results

The proof of the admissibility theorem requires certain technical results which we present in this section.

Let u be a bounded piecewise differentiable function defined on E^m. The following result is a rather standard one.

Lemma 6.1. There exists a constance $C_2 > 0$ such that

$$\int (u(\theta) - u(x))^2 p_\theta(x) dx \leq C_2 \left[\int |\nabla u(x)|^2 \frac{1}{|x-\theta|^{m-1}} p_\theta(x) dx + \int |\nabla u(x)|^2 p_\theta(x) dx\right]$$

Proof. Write x in polar co-ordinates around θ i.e. $x = (r(x), \phi)$ where $r(x) = ||x-\theta||$. Assume that ϕ has been nonmalized. By Schwartz inequality we have

$$(u(\theta) - u(x))^2 \leq r(x) \int_0^{r(x)} ||\nabla u(x,\phi)||^2 \, ds$$

Therefore, denoting r(x) by r, we have

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \, dx \leq \int r\Big(\int_0^r ||\nabla u(s,\phi)||^2 ds (e^{-1/2r^2} r^{m-1} dr \, ds d\phi$$

$$= \int_0^\infty ||\nabla u(x)\phi||^2 \int_s^\infty r^m e^{-1/2r^2} \, dr \, ds d\phi \tag{6.1}$$

Now integrating by parts we have

$$\int_s^\infty r^m e^{-1/2r^2} \, dr \leq C_2 (s^{m-1} \; e^{-\frac{1}{2}S^2} + e^{-\frac{1}{2}S^2})$$

for some constant $C_2 > 0$.

Therefore,

$$(6.1) \leq C_2 [\int_0^\infty ||\nabla u(x,\phi)||^2 \, s^{m-1} e^{-\frac{1}{2}S^2} \, ds d\phi + \int_0^\infty ||\nabla u(s,\phi)||^2 \, e^{-\frac{1}{2}S^2} \, ds d\phi]$$

$$= C_2 [\int ||\nabla u(x)||^2 p_\theta(x) \, dx + \int ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) dx]$$

Hence the lemma.

Lemma 6.2. Let ρ be a constant such that $0 < \rho < 1/2$. Then there exist constants K_1 and K_2 such that

$$\int_{||x-\theta|| \leq \rho} ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) \, dx \leq K_1 \int_{||x-\theta|| < K_2} ||\nabla u(x)||^2 p_\theta(x) dx.$$

Proof. Fix θ. Define a density function $r(\theta,x)$ by

$$r(\theta,x) = C \, I(\theta,x) \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x)$$

where $I(\theta,x) = 1$ if $||x-\theta|| \leq \rho$ and $= 0$ otherwise and C is the normalizing constant so that $\int r_\theta(x)dx = 1$. Note that C depends only on ρ. Define a new density function $s(\theta,x)$ by setting

$$s(\theta,x) = \int \ldots \int \quad r(\theta,t_1)r(t_1,t_2)\ldots r(t_{2\lambda},x)dt_1\ldots dt_\lambda$$

where $\lambda \geq 1$ is a fixed integer. Plainly $\int s(\theta,x)dx = 1$. Moreover, $s(\theta,x) = 0$ for $||x-\theta|| > \frac{2\lambda}{\rho}$ and $s(\theta,x)$ is bounded.

The bound of $s(\theta,x)$, say K_3, depends only on ρ, m and λ. It is also easy to see

$$\int ||\nabla u(x)||^2 \; r(\theta,x)dx = \int||\nabla u(x)||^2 \; s(\theta,x)dx \tag{6.2}$$

Now,

$$\begin{aligned}\int_{||x-\theta||} ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \rho_\theta(x) \; dx &= \int||\nabla u(x)||^2 \; r_\theta(x)dx \\ &= \int||\nabla u(x)||^2 \; s(\theta,x)dx \\ &= \int_{||x-\theta|| \leq \frac{2\lambda}{\rho}} ||\nabla u(x)|| \; s(\theta,x)dx \\ &\leq K_3 \int_{||x-\theta|| \leq \frac{2\lambda}{\rho}} ||\nabla u(x)||^2 dx \qquad (6.3)\end{aligned}$$

Since,

$$e^{-1/2 \; ||x-\theta||^2} \geq e^{-1/2(\frac{2\lambda}{\rho})^2} \quad \text{for } x \text{ in } \{x: \; ||x-\theta|| \leq \frac{2\lambda}{\rho} \},$$

we have

$$(6.2) \leq K_3 \, e^{1/2(\frac{2\lambda}{\rho})^2} \int_{||x-\theta|| \leq \frac{2\lambda}{\rho}} ||\nabla u(x)||^2 \, p_\theta(x) dx.$$

Letting $K_1 = K_3 \, e^{1/2(\frac{2\lambda}{\rho})^2}$ and $K_2 = \frac{2\lambda}{\rho}$, the lemma follows.

q.e.d.

Corollary 6.3. Let u be a piecewise differentiable function in E^m. Then there exists a constant $K > 0$ such that

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \, dx \leq K \int ||\nabla u(x)||^2 \, e^{-1/2||x-\theta||^2} dx$$

Proof. By lemma (4.3) we have

$$\int (u(\theta)-u(x))^2 \, p_\theta(x) \, dx \leq C_2 \int ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) \, dx$$

$$+ C_2 \int ||\nabla u(x)||^2 \, p_\theta(x) \, dx$$

Now, write,

$$\int ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) \, dx = \int_{||x-\theta|| \leq \rho} ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) dx$$

$$+ \int_{||x-\theta|| > \rho} ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} \, p_\theta(x) dx \qquad (6.4)$$

where $\rho = \rho$ is a fixed positive constant, say 1/4.

The first term in the right side of (6.4) can be bounded using Lemma 6.2 as follows

$$\int_{||x-\theta||\leq 1/4} ||\nabla u(x)||^2 \frac{1}{||x-\theta||^{m-1}} p_\theta(x)dx \leq K_1 \int_{||x-\theta||\leq K_2} ||\nabla u(x)||^2 p_\theta(x)dx$$

$$\leq K_1 \int ||\nabla u(x)||^2 p_\theta(x)dx \qquad (6.5)$$

The second term in right side of (6.4) is bounded by

$$4^m \int p_\theta(x) \quad ||\nabla u(x)||^2 dx \qquad (6.6)$$

Combining (4.16) and (4.17) we have

$$\int (u(\theta)-u(x))^2 p_\theta(x) dx \leq (C_2+K_1+4^m)\int ||\nabla u(x)||^2 p_\theta(x)dx.$$

Hence the corollary.

The next lemma is an extension of Corollary 6.3.

Lemma 6.4. Given a constant $K_1 > 0$ there exists a constant $K_2 > 0$ (depending on K_1 and m) such that

$$\int (u(\theta)-u(x))^2 |x-\theta|^{K_1} p_\theta(x)dx \leq K_2 \int \int_{|\xi|<K_2+1} |\nabla u(x)|^2 p_\theta(x+\xi)d\xi dx.$$

Proof. The result follows from lemma 5.2 and an argument similar to lemmas 6.1, 6.2 and Corollary 6.3.

Let us now prove some consequences of our assumptions I and II. The following result is trivial.

Lemma 6.5. Suppose $\nabla \log f(x) < B$. Then there exists a constant C, depending only on B and m, such that $\left|\frac{\nabla f(x)}{f(x)} - \frac{\nabla f(y)}{f(y)}\right| < C|x-y|$.

Lemma 6.6. Assume I and II. Then there exists a constant K_1, depending only on K, m and B, such that

$$\frac{x}{|x|}\ \frac{\nabla f(x)}{f(x)} + \left|\frac{\nabla f(x)}{f(x)}\right| < K_1 \qquad \text{for all } x \in K_F^2$$

where $K_F^2 = \{x: \ d(x) \le 2\}$. $(d(x) = d_F(x))$.

Proof. Follows from Lemma 6.5.

Next few results are consequences of Assumption II on the calculus of variation problem.

Lemma 6.7. Assume I and II. Suppose $\inf_{j\in J} \int |\nabla j|^2 f(x)dx = 0$. Then for any $K_2 > 0$,

$$\inf_{j\in J} \int_{K_F^2} |\nabla j|^2 \, e^{K_2\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx = 0.$$

Proof. Define, for any constant $K > 0$, x-K to be $\frac{x}{|x|}(|x|-K)$. It is easy to see that $\inf_{j\in J} \int|\nabla j|^2 f(x)dx = 0$ implies $\inf_{j\in J} \int_{|x|>K+1} |\nabla j(x)|^2 f(x-K)dx=0$ and hence $\inf_{j\in J}\int|\nabla j(x)|^2 f(x-K)\,dx = 0$. Now, by assumption II and Lemmas 6.5 and 6.6, we have for x in K_F^2

$$e^{K_2\left|\frac{\nabla f(x)}{f(x)}\right|} f(x) \le e^{K_2\left|\frac{\nabla f(x)}{f(x)}\right|} \frac{f(x)}{f(x-K_2)} \cdot f(x-K_2)$$

$$\le C\, e^{K_2\left|\frac{\nabla f(x)}{f(x)}\right| + K_2\frac{x}{|x|}\ \frac{\nabla f(x)}{f(x)}} : f(x-K_2)$$

where C is a constant depending on K_2, K, B and m. Therefore, by assumption II, for $x \in K_F^2$

$$e^{K_2\left|\frac{\nabla f(x)}{f(x)}\right|} f(n) \le C_1\ f(x-K_2)$$

for some constant $C_1 > 0$. The result now follows easily.

Theorem 6.8. Assume I and II. Suppose $\inf_{j\in J}\int|\nabla j(x)|^2 f(x)dx = 0$. Then, for any $K > 0$, $\inf_{j\in J}\int|\nabla j(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx = 0$.

Proof. By Lemma 6.7, given any $\varepsilon > 0$ there exists $j_0(x)$ such that

$$\int_{K_F^2} |\nabla j_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx < \varepsilon. \tag{6.7}$$

Representing x in terms of $\pi(x)$ and $d(x)$ it follows from (6.7) that there exists α, $1 < \alpha \leq 2$ such that

$$\int|\nabla j_0(x^{-1}(\pi,\alpha))|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x^{-1}(\pi,\alpha))d\pi < \varepsilon. \tag{6.8}$$

With α as above, consider the set $K_F^\alpha = \{x: \ d(x) \leq \alpha\}$. Plainly,

$$\int_{K_F^\alpha}|\nabla j_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx < \varepsilon . \tag{6.9}$$

Let $\pi_\alpha(x)$ and $d_\alpha(x)$ be the projection and distance of x from K_F. Define now the function u_0 as follows

$$u_0 = j_0(\pi_\alpha(x)) \tag{6.10}$$

We shall now prove that

$$\int|\nabla u_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx < C_1 \cdot \varepsilon \tag{6.11}$$

where C_1 is constant depending only on B, K and m. The left side of (6.11) can be written as

$$\int_{K_F^\alpha} |\nabla u_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx + \int_{E^m-K_F^\alpha} |\nabla u_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx \tag{6.12}$$

Clearly the first term in (6.12) is less than ε by (6.9). To deal with the second term, observe that

$$f(x) \le f(\pi_\alpha(x)) \cdot e^{-\frac{1}{2}d_\alpha^2(x)} \tag{6.13}$$

For a proof of this fact see Brown [3]. Moreover, by lemma 6.5

$$e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} \le e^{KCd_\alpha(x)} \cdot \exp\left(K\left|\frac{\nabla f(\pi_\alpha(x))}{f(\pi_\alpha(x))}\right|\right) \tag{6.14}$$

Therefore representing x in terms of π_α and d_α we have

$$\int_{E^m - K_F^\alpha} |\nabla u_0(x)|^2 e^{K\left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx$$

$$\le \int \int_{d_\alpha > 0} |\nabla u_0(x)|^2 e^{KCd_\alpha} e^{K\left|\frac{\nabla f(\pi_\alpha)}{f(\pi_\alpha)}\right|} f(x^{-1}(\pi_\alpha, d_\alpha))Jd\pi_\alpha dd_\alpha. \tag{6.15}$$

Where J is the Jacobian of the transformation $x \to (\pi_\alpha, d_\alpha)$ and $|J| \ll (d_\alpha+1)^{m-1}$. Hence

$$(6.15) \le \int \int_{d_\alpha > 0} |\nabla j_0(\pi_\alpha)|^2 e^{(KC+1)d_\alpha} e^{-\frac{1}{2}d_\alpha^2} e^{K\left|\frac{\nabla f(\pi_\alpha)}{f(\pi_\alpha)}\right|} f(\pi_\alpha)d\pi_\alpha dd_\alpha \tag{6.16}$$

Integrating with respect to π_α, it follows from (6.8) that

$$(6.16) \le \varepsilon \int_{d_\alpha > 0} e^{(KC+1)d_\alpha} e^{-\frac{1}{2}d_\alpha^2} dd\alpha. \tag{6.17}$$

Setting $C_1 - 1 = \int e^{(KC+1)d\alpha} e^{-\frac{1}{2}d_\alpha^2} dd\alpha$, we have the result.

7. Admissibility Result

We prove the main theorem of this article in this section.

Theorem 7.1. Assume I and II. Then δ_F is admissible if and only if Exterior Boundary problem is solvable for $L_F u = 0$.

Proof. The "only if" part is trivial. We shall prove the sufficiency below. The proof involves constructing a sequence of finite measures $\{G_n\}$ as in Farrell's result and then showing (3.1).

Since the Boundary problem is solvable for $L_f u = 0$ it follows by Theorem 6.8, $\inf_{j \in J} \int |\nabla j(x)|^2 \, e^{K_1 \left|\frac{\nabla f(x)}{f(x)}\right|} f(x)dx = 0$ for a large constant K_1 larger than (B+K+2). Let $j_R(x)$ be a sequence of non-negative functions satisfying

(i) $j_R(x) = 1$ for $|x| \le 1$

(ii) $j_R(x) = e^{-2KR}$ for $R \le |x| \le 2R$

(iii) $j_R(x) = 0$ for $|x| > 2R$

and (iv) $L_{\tilde{f}} j_R(x) = 0$ for $1 \le |x| < R$

here $\tilde{f}(x) = e^{K_1 \left|\frac{\nabla f(x)}{f(x)}\right|} \cdot f(x)$. Such a sequence exists by Theorem 4.1. Let $G_R(d\theta) = j_R^2(\theta)F(d\theta)$ and $g_R(x) = \int p_\theta(x) G_R(d\theta)$. Also, let $\psi_R(x) = \int j_R(\theta) p_\theta(x) F(d\theta)/f(x)$. We shall now prove that

$$\int (R(\theta,\delta_F) - R(\theta,\delta_{G_R}))G_R(d\theta) \to 0 \text{ as } R \to \infty \tag{7.1}$$

We can write 7.1 as

$$\int_{|x|<2R} \left[\int (j_R^2(\theta) - \psi_R^2(x))(\theta - \delta_F(x)) \frac{p_\theta(x)F(d\theta)}{g_R(n)}\right]^2 g_R(x)dx$$

$$+ \int_{|x| \geq 2R} [\int j_R^2(\theta) \frac{(\theta-\delta_F(x))p_\theta(x)}{g_R(x)} F(d\theta)]^2 g_R(x)dx \tag{7.2}$$

We will call the two terms in (7.2) T_1 and T_2 respectively. Consider first the second term T_2. Since $j_R(\theta) = 0$ for $|\theta| > 2R$, using Schwartz inequality and Lemma 6.5 we have

$$T_2 \leq C \int_{|x|<2R} |x|^2 \int_{|\theta|<2R} j_R^2(\theta)p_\theta(x)F(d\theta)dx. \tag{7.3}$$

Now let $D = \{\theta: |\theta| \leq R\}$ and $E = \{x: |x| > 3R\}$. It is easy to see that, by assumption II, $\int e^{-K|\theta|} F(d\theta) < \infty$ and therefore

$$\int_D p_\theta(x)F(d\theta) \leq e^{KR} \int_D p_\theta(x) e^{-K|\theta|}F(d\theta) \tag{7.4}$$

Hence, using maximum modulus principle on $j_R(\theta)$, we have

$$\int_{|x|>2R} |x|^2 \int_D j_R^2(\theta)p_\theta(x)F(d\theta)dx \leq e^{KR} \int_{|x|>2R} |x|^2 p_\theta(|x|-R)dx \tag{7.5}$$

The right side of (7.5) is easily seen to go to zero as $R \to \infty$. By a similar argument it is easy to prove that

$$\int_E |x|^2 \int_{|\theta|\leq 2R} j_R^2(\theta)p_\theta(x) F(d\theta) \to 0 \text{ as } R \to \infty \tag{7.6}$$

Therefore, to prove T_2 goes to zero as $R \to \infty$, it sufficies to show

$$\int_{2R<|x|<3R} |x|^2 \int_{R<|\theta|<2R} j_R^2(\theta)p_\theta(x)F(d\theta) \to 0 \text{ as } R \to \infty \tag{7.7}$$

This follows immediately from the fact that $j_R(\theta) = e^{-2KR}$ for $R<|\theta|<2R$. We will now complete the proof of the theorem by showing T_1 goes to zero as $R \to \infty$.

Expressing $(j_R^2(\theta) - \psi_R^2(x))$ as $(j_R(\theta) - \psi_R(x))(j_R(\theta) + \psi_R(x))$ and using Schwartz inequality we have

$$T_2 \leq 2 \int_{|x|<2R} [\int (j_R(\theta) - \psi_R(x))^2 |\theta-\delta_F(x)|^2 p_\theta(x) F(d\theta)]$$

$$\times [\int j_R^2(\theta) + \psi_R^2(x) \frac{p_\theta(x)}{g_R(x)} F(d\theta)] \, dn \tag{7.8}$$

Since $\psi_R^2(x) \leq \frac{g_R(x)}{f(x)}$ by Schwartz inequality,

$$(7.8) \leq 4 \int_{|x|<2R} \int (j_R(\theta) - \psi_R(x))^2 |\theta-\delta_F(x)|^2 p_\theta(x) F(d\theta) dx \tag{7.9}$$

$$\leq 4 \int_{|x|<2R} \int (j_R(\theta) - j_R(x))^2 |\theta-\delta_F(x)|^2 p_\theta(x) F(d\theta) dx$$

$$+ 4 \int_{|x|<2R} (j_R(x) - \psi_R(x))^2 |\theta-\delta_F(x)|^2 F(d\theta) dx. \tag{7.10}$$

Let the two terms in 7.10 be denoted by T_3 and T_4.

Now, by Lemmas 6.3, 6.4 and 6.5 we have

$$T_3 \leq C \int_{|x|<2R} \int (j_R(\theta) - j_R(x))^2 |\theta-x|^2 p_\theta(x) F(d\theta) dx$$

$$+ \left|\frac{\nabla f(\theta)}{f(\theta)}\right|^2 \int_{|x|<2R} (j_R(\theta) - j_R(x))^2 p_\theta(x) dx F(d\theta) \tag{7.11}$$

$$\leq C_1 \int \int_{|\xi|<2} |\nabla j_R(x)|^2 (1+\left|\frac{\nabla f(x)}{f(x)}\right|^2) f(x+\xi) dx d\xi \tag{7.12}$$

$$\leq C_2 \int |\nabla j_R(x)|^2 e^{K_1 \left|\frac{\nabla f(x)}{f(x)}\right|} f(x) dx \tag{7.13}$$

Where C_1 and C_2 are some constants depending only on B, K and m. The choice of j_R's imply that (7.13) goes to zero and hence T_3 goes to zero as $R \to \infty$. Let us now consider T_4. Using Schwartz inequality and assumption I

$$T_4 \le \int_{|x|<2R}\int [\int (j_R(x) - j_R(\eta))^2 p_\eta(x) F(d\eta)] \frac{|\theta - \delta_F(x)|^2}{f(x)} \quad p_\theta(x) F(d\theta) dx$$

$$\le B \int_{|x|<2R}\int (j_R(x) - j_R(\eta))^2 p_\eta(x) F(d\eta) dx \tag{7.14}$$

Appealing to Lemma 6.3, it is easy to see that

$$T_4 \le B \int |\nabla j_R(x)|^2 f(x) dx \to 0 \text{ as } R \to \infty.$$

Hence T_2 goes to zero as $R \to \infty$ and this completes the proof.

8. Applications and Examples

We present some applications of our admissibility theorem in this section. Observe that, by Theorem 7.1, in order to verify admissibility of a generalized Bayes estimator δ_F we need to verify the solvability of the associated boundary problem. Brown [3] has given various necessary and sufficient conditions for the solvability of the boundary problems. We list a couple of such results which are not found in Brown's article. We require the following definition to state the result.

Definition. A non-negative function $\delta(t)$, $t > 0$ is called a dini function if $\int_{t_0}^{\infty} \delta(t)/t \, dt < \infty$ for some $t_0 > 0$.

Theorem 8.1. If there exists a $L > 0$ such that

$$\frac{x}{|x|}\frac{\nabla f(x)}{f(x)} \le \frac{2-m}{|x|} + \frac{\varepsilon(|x|)}{|x|} \quad \text{for all } |x| > L \tag{8.1}$$

where $\varepsilon(|x|)$ is such that $\Delta(t) = \exp(-\int^{t} \varepsilon(|x|)d|x|)$ is a definition then the boundary problem of δ_F is solvable. Conversely, if

$$\frac{x}{|x|}\frac{\nabla f(x)}{f(x)} \ge \frac{2-m}{|x|} + \frac{\varepsilon(|x|)}{|x|} \quad \text{for all } |x| > L \tag{8.2}$$

where $\varepsilon(|x|)$ is such that $\Delta(t)$ is a non-dini function, then the boundary problem is not solvable and hence δ_F is inadmissible.

Proof. See Meyers and Serrin [8].

It is well known (see Brown [3]) that the solvability of the boundary problem is equivalent to the recurrence of the diffusion process associated with the differential operator L_f. Recently, more general necessary and sufficient conditions (but similar to Theorem 8.1) for the recurrence of a diffusion process have been obtained by R. N. Bhattacharya [*]. These results can be used to verify the solvability of the boundary problem. We do not, however, pause to list those results here.

The problem of verifying whether a given estimator is generalized Bayes or not has been dealt with in great detail by various authors. Strawderman and Cohen [14] have given necessary and sufficient conditions for a spherically symmetric estimator to be generalized Bayes. Using their conditions and Brown's result, they have obtained necessary and sufficient conditions for the

*Personal correspondence.

admissibility of spherically symmetric estimators. Their results can be proved in greater generality using Theorem 7.1. The general ease (non-spherically symmetric case) has been studied by Berger and Srinivasan [2]. They have obtained necessary and sufficient conditions for an estimator to be generalized Bayes. Combining Theorem 7.1 with the results of Berger and Srinivasan one can obtain easily verifiable necessary and sufficient conditions for an estimator to be admissible. We do not pause to list these results here.

We consider now the problem of improving inadmissible estimators. Let δ_F be a generalized Bayes estimator such that the associated boundary problems is not solvable. Then, by Theorem 7.1, δ_F is inadmissible and there exists a better estimator. How does one get hold of this better estimator? Brown [3] conjectured that probably one could use a solution of the boundary problem to obtain better estimators. We solve this problem partially below. Let δ_F be a generalized Bayes estimator. Then the risk of δ_F can be written as

$$R(\theta,\delta_F) = m - E_\theta[|\frac{\nabla f(x)}{f(x)}|^2 - 2\frac{\Delta f(x)}{f(x)}] \qquad (8.3)$$

This representation of the risk of a generalized Bayes estimator is due to Stein [13]. We use this to obtain better estimator than δ_F if δ_F is inadmissible. Suppose δ_F is inadmissible. Then by Theorem 7.1 the boundary problem is not solvable and hence the calculus of variation problem has a minimizing solution, say $j_0(x)$, such that $j_0(x) = 1$ for $(x) \le 1$ and $j_0(x) < 1$ for $|x| > 1$. We will use this j_0 to construct a better estimator.

<u>Theorem 8.2</u>. Suppose δ_F is inadmissible and j_0 is the minimizing function. Then the estimator $\delta(x) = \frac{\nabla(j_0(x)f(x))}{j_0(x)f(x)} + x$ is better than δ_F. i.e.

$R(\theta,\delta) \leq R(\theta,\delta_F)$, $\forall\theta$.

Proof. Observe that the risk of $\delta(x)$ has similar representation as (8.3). Using this fact it is easy to see that $R(\theta,\delta) \leq R(\theta,\delta_F)$. $\forall\theta$ if $L_f \left(\frac{j_0(f(x))}{f(x)}\right)^{1/2} \leq 0$ for almost all x. Since j_0 is such that $j_0(x) = 1$ for $|x| < 1$ and $L_f j_0(x) = 0$ for $|x| > 1$, we have $L_f j_0^{1/2}(x) \leq 0$ for almost all x and hence the result.

The above argument yields, in particular, yields whole class of minimax estimators for dimension $m \geq 3$. It is well known that the best invariant estimator is inadmissible minimax estimator for $m \geq 3$ and it corresponds to the function $f(x) = 1$. The corresponding differential operator L_f is the Laplacian operator. It follows, therefore, from Theorem 8.2 that every estimator $\delta(x) = \frac{\nabla j(x)}{j(x)} + x$ is minimax where $j(x)$ is such that $j(x) = 1$ for $|x| \leq 1$ and $j(x)$ is superharmonic for $|x| > 1$. Indeed, it is easy to show that the minimax estimators constructed by Baranchik [1] and Strawderman [15] correspond to such superharmonic functions. We end this discussion with the following result which generalizes Strawderman's theorem. Strawderman [16] showed that there do not exist spherically symmetric proper Bayes minimax estimators for m = 3 and 4. We show below there does not exist any kind of proper Bayes minimax estimators for m = 3 and 4. Our proof depends only in the maximum modulus principle for elliptic equations (see section 3).

Theorem 8.3. There do not exist proper Bayes minimax estimators for m = 3 and 4.

Proof. Suppose there exists a proper Bayes minimax estimator. Let it be denoted by δ_G, where G is the finite prior. Clearly, $\delta_G(x) = \frac{\nabla g(x)}{g(x)} + x$

where $g(x) = \int p_\theta(x)G(d\theta)$. Moreover, using Stein's representation (8.3), we have

$$R(\theta,\delta_G) - m = E_\theta[|\frac{\nabla g(x)}{g(x)}|^2 - \frac{2\Delta g(x)}{g(x)}] \leq 0 \tag{8.4}$$

since δ_G is minimax. It follows using Schwartz inequality and integration by parts, that (8.4) implies

$$2\nabla\psi(\theta) + |\nabla\psi(\theta)|^2 \leq 0 \text{ for all } \theta. \tag{8.5}$$

where $\psi(\theta) = (2\pi)^{-m/2} \int \log g(x) p_\theta(x)dx$. Therefore, we have, from(8.5), that $e^{\frac{1}{2}\psi(\theta)}$ is superharmonic for all θ. Now, an application of maximum modulus principle implies that

$$e^{\frac{1}{2}\psi(\theta)} \geq C_1 \frac{1}{|\theta|^{m-2}} \quad \text{for } |\theta| > 1 \tag{8.6}$$

For some constant $C_1 > 0$, since $u(\theta) = \frac{1}{|\theta|^{m-2}}$ is a harmonic function for $|\theta| > 1$. Hence, by Jensen's inequality,

$$(2\pi)^{-m/2} \int g(x)p_\theta(x)dx \geq C_1^2 \frac{1}{|\theta|^{2(m-2)}} \quad \text{for } |\theta| > 1 \tag{8.7}$$

and therefore

$$\int g(x)dx \geq C_1^2 \int_{|\theta|} \frac{1}{|\theta|^{2(m-2)}} d\theta \tag{8.8}$$

The right side of (8.8) is infinity for m = 3 and 4 which implies $\int g(x)dx = \infty$, contradicting that G is a finite measure. This completes the proof.

Appendix.

In this appendix we present a brief description of the boundary value problem and the proofs of Theorem 4.1 and 8.2.

Let L_f denote the elliptic differential operator given by the equation

$$L_f\, u(x) = \Delta u(x) + \frac{\nabla f(x)}{f(x)} \nabla u(x) \tag{1}$$

where u is a twice continuously differentiable function. Let Ω be a bounded open connected set with boundary $\partial\Omega$. Then we have the following maximum principle for twice continuously differentiable function defined on Ω.

Theorem A1: If $L_f\, u(x) \geq 0$ ($L_f\, u(x) \leq 0$) for x in Ω and u is continuous on $\partial\Omega$ then

$$u(x) \leq \max_{y \in \partial\Omega} u(y) \qquad (u(n) \geq \min_{y \in \partial\Omega} u(y))$$

for every $x \in \Omega$.

Proof. See Miranda [9]

Then next result deals with the existence of solutions of boundary value problems on bounded domains. Let Ω be an annulus i.e. $\Omega = \{n:\ r_1 < |x| < r_2\}$. Let φ_1 and φ_2 be two continuous functions defined on $\{x:\ |x| = r_1\}$ and $\{x:\ |x| = r_2\}$ respectively.

Theorem A2: There exists a unique continuous function u defined on $\bar{\Omega}$ (the closure of Ω) such that $L_f\, u(x) = 0$ for $x \in \Omega$ and $u(x) = \varphi_1(x)$ as $\{x:\ |x| = r_1\}$, $u(x) = \varphi_2(x)$ on $\{x:\ |x| = r_2\}$.

Proof: See Miranda [9].

Now we consider the exterior boundary problem defined in Section 4. Let E be the region $\{x: |x| > 1\}$ and let $\partial E = \{x: |x| = 1\}$. We say the exterior boundary problem is solvable if there exists a unique continuous function u defined on $E \cup \partial E$ such that $L_f u(x) = 0$ if $x \in E$ and $u(x) = 1$ for $|x| = 1$. Note that such a unique solution ought to be identically equal to 1. Let $S_n = \{x: |x| < n\}$ and $\partial S_n = \{x: |x| = n\}$. Let u_n be a continuous function such that $L_f u_n(x) = 0$ for $x \in E \cap S_n$ and $u_n(x) = 1$ for $|x| = 1$ and $u_n(n) = 0$ for $|x| = n$. Such a function u_n exists and is unique by Theorem A2. Moreover, it is a general fact the sequence $\{u_n\}$ has a convergent subsequence.

Theorem A3: The exterior boundary problem is solvable if and only if every convergent subsequence of $\{u_n\}$ converges uniformly on compacta to 1.

Proof: "only if". Suppose there exists a subsequnce of $\{u_n\}$ which converges to a function $u_0 \neq 1$. It is easy to show using maximum principle $L_f u_0(x) = 0$ for $x \in E$ and $u_0(x) = 1$ for $|x| = 1$. This contradicts the fact that the boundary problem is solvable.

"if" Let every convergent subsequence of $\{u_n\}$ converge to 1 uniformly on compacta. Suppose that the boundary problem is not solvable. Then there exists a function $u_0 \neq 1$ such that $L_f u_0(x) = 0$ for $x \in E$ and $u_0(x) = 1$ for $|x| = 1$. Now appealing to maximum principle we have $u_0(x) > 0$ and $u_n(x) \leq u_0(x)\ \forall x \in E$ for every n. Since every convergent subsequence of $\{u_n\}$ converges to 1, it follows $u_0(x) = 1$ and hence a contradiction.

Proof of Theorem 4.1:

"Necessity". Assume $\inf_{j \in J} \int ||\nabla j(x)||^2 f(x)\, dx = 0$. We shall give the proof in three stages.

(a) Let $\{k_n(x)\}$ be a sequence of functions satisfying

$$
\begin{aligned}
L\, k_n(x) &= 0 && \text{for } 1 < ||x|| < n \\
k_n(x) &= 1 && \text{for } ||x|| = 1 \\
&= \phi_n(x) && \text{for } ||x|| = n \\
&= 0 && \text{for } ||x|| > n
\end{aligned}
$$

where $\phi_n(x)$ is a smooth function such that $\phi_n(x) = 1$ for $x \notin K$ and $1 > \phi_n(x) \geq \varepsilon > 0$ for $x \in K$ and $||x|| = n$ with equality holding at some point and $\varepsilon < 1$.

By hypothesis there exists a sequence $\{j_n\} \subset J$ such that $\int ||\nabla j_n(x)||^2 f(x)dx \downarrow 0$. Let $\{u_{m_n}\}$ be a sequence of functions satisfying $Lu_{m_n} = 0$ for $1 < ||x|| < m_n$, $u_{m_n}(x) = 1$ for $||x|| = 1$ and $u_{m_n}(x) = j_n(x)$ for $||x|| \geq m_n$, where m_n is so chosen to satisfy $\sup_{\{x:||x||=m_n, x\varepsilon K\}} j_n(x) \leq \varepsilon$.

The minimizing property of u_{m_n} implies

$$\int ||\nabla j_{m_n}(x)||^2 f(x)dx \leq \int ||\nabla j_n(x)||^2 f(x)dx.$$

Assume without loss of generality that $\{m_n\}$ is an increasing sequence. It follows from Schauder's estimates that u_{m_n} converges to a solution of u of L in E^m satisfying $u(x) = 1$ for $||x|| = 1$.

From a result of Serrin's on lower semicontinuity (See Morrey [17]) it follows that for every compact set

$$\int_C ||\nabla u(x)||^2 f(x)dx \leq \liminf_{m_n \to \infty} \int ||\nabla u_{m_n}||^2 f(x)dx$$

$$\leq \liminf_{n \to \infty} \int ||\nabla j_{n(x)}||^2 \, f(x)dx = 0$$

This implies $u(x) \equiv 1$. On the other hand $K_{m_n}(x) \geq u_{m_n}(x)$ for $1 \leq ||x|| \leq m_n$ by maximum modulus principle. Therefore $K_{m_n}(x)$ converges uniformly on compacta to 1.

(b) Let $\{(\psi_n)(x)\}$ be a sequence of functions satisfying $L(\psi_n)(x) = 0$ for $1 < ||x|| < n$, $(\psi_n)(x) = 1$ for $||x|| = 1$ and $(\psi_n)(x) = 0$ for $||x|| \geq n$. Let $\{V_n(x)\}$ be another sequence such that $Lv_n = 0$ for $1 < ||x|| < n$, $v_n(x) = 1$ for $||x|| = 1$ and $v_n(x) = W_n(x)$ for $||x|| = n$ where $W_n(x)$ is a smooth function satisfying $\varepsilon < W_n(x) \leq 1 - \varepsilon$, $0 < \varepsilon < 1/2$ and $W_n(x) = 1 - \varepsilon$ for $x \notin K$, $||x|| = n$ and $W_n(x) > \epsilon > 0$ for $x \in K$, $||x|| = n$. Suppose $v_n(x)$ converges to 1 uniformly on compacta. Then $v_n(x) = 1/\varepsilon[v_n(x) - (1 - \varepsilon)]$ also converges to 1 uniformly on compacta. Moreover $L\,v_n(x) = 0$ for $1 < ||x|| < n$. By maximum modulus principle it is easy to see that $(\psi_n)(x) \geq V_n(x)$ for $1 \leq ||x|| \leq n$ for all n. Hence $(\psi_n)(x) \to 1$ uniformly on compacta.

(c) Observe that the result in (a) goes through if the boundary functions ϕ_n's of K_n's satisfy $\phi_n = 1 - \varepsilon_n$ for $x \notin K_n$ $||x|| = n$ and $1-\epsilon_n > \phi_n(x) \geq \epsilon_n$ for $x \in K$, $||x|| = n$, where ϵ_n monotonically decreases to zero. Now, using (b) we can get a subsequence of $\{(\psi_n)\}$ which converges to 1 uniformly on compacta. But $\{(\psi\ )\}$is a montonically increasing sequence. Therefore $\psi_n \to 1$ uniformly on compacta.

<u>"Sufficiency"</u>. The proof is similar to the one given by Brown [3].

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