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Monotonicity of Power Functions of Tests Based on Traces of Multivariate Complex Beta and Canonical Correlation Matrices

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Abstract

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It is proved that the power functions of tests based on Pillai's trace for MANOVA and canonical correlation in the complex case are monotonically increasing in each noncentrality parameter provided that the cutoff point is not too large, extending the work of Perlman in the real case. An illustrative table is also given of the smallest error degrees of freedom less the number of variables in MANOVA guaranteeing the above monotonicity property given other sample arguments and significance level.

Monotonicity of Power Functions of Tests Based on Traces of Multivariate Complex Beta and Canonical Correlation Matrices

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1. Introduction

For the MANOVA problem in the real case, Perlman [2] has shown that the power function of the test based on Pillai's trace of a multivariate beta matrix is monotonically increasing in each noncentrality parameter provided that the cutoff point is not too large. This result has also been proved true for the problem of testing independence of two sets of real variates. In this paper, both of these results are extended to the complex case. An illustrative table is also given, of the smallest error degrees of freedom less the number of variables in MANOVA guaranteeing the above monotonicity property given other sample arguments and significance level.

2. Invariant tests for the MANOVA problem

Let $\underset{\sim}{Z}_1(p\times r)$ and $\underset{\sim}{Z}_2(p\times n)$ be independent complex matrix variates. The columns of $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$ are mutually independent and complex normally distributed with common nonsingular covariance matrix $\underset{\sim}{\Sigma}$,

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and $E\underset{\sim}{Z}_1 = \underset{\sim}{\theta}$, $E\underset{\sim}{Z}_2 = \underset{\sim}{0}$. The joint density function of $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$ is given by

(2.1) $\pi^{-p(r+n)}|\underset{\sim}{\Sigma}|^{-(r+n)}\exp[-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\{(\underset{\sim}{Z}_1-\underset{\sim}{\theta})(\bar{\underset{\sim}{Z}}_1-\bar{\underset{\sim}{\theta}})'+\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2'\}]$.

The problem is to test

$$\underset{\sim}{\theta} = \underset{\sim}{0} \quad \text{against} \quad \underset{\sim}{\theta} \neq \underset{\sim}{0}.$$

This problem is invariant under all transformations of the form:

(2.2) $(\underset{\sim}{Z}_1,\underset{\sim}{Z}_2) \longrightarrow (\underset{\sim}{B}\underset{\sim}{Z}_1\underset{\sim}{F}_1,\underset{\sim}{B}\underset{\sim}{Z}_2\underset{\sim}{F}_2)$,

where $\underset{\sim}{B}(p\times p)$ is nonsingular and $\underset{\sim}{F}_1(r\times r)$ and $\underset{\sim}{F}_2(n\times n)$ are unitary. Here, we assume that $p < n + r$. Let $t = \min\{p,r\}$. A maximal invariant statistic is $(\ell_1,\ldots,\ell_t)$, where $1 \geq \ell_1 \geq\cdots\geq \ell_t \geq 0$ are the ordered t largest characteristic roots of the multivariate complex beta matrix $\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'+\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}$. An invariant parameter is $(\omega_1,\ldots,\omega_t)$, where $\omega_1 \geq\cdots\geq \omega_t \geq 0$ are the ordered t largest characteristic roots of $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}'\underset{\sim}{\Sigma}^{-1}$.

Start with equation (2.1) and put $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}' = \begin{pmatrix}\underset{\sim}{\mu}_1\\ \underset{\sim}{\mu}_2\end{pmatrix}\underset{\sim}{D}_\omega(t\times t)(\bar{\underset{\sim}{\mu}}_1',\bar{\underset{\sim}{\mu}}_2')$ and $\underset{\sim}{\Sigma}_1 = \begin{pmatrix}\underset{\sim}{\mu}_1 & \underset{\sim}{\mu}_3\\ \underset{\sim}{\mu}_2 & \underset{\sim}{\mu}_4\end{pmatrix}\begin{pmatrix}\bar{\underset{\sim}{\mu}}_1' & \bar{\underset{\sim}{\mu}}_2'\\ \bar{\underset{\sim}{\mu}}_3' & \bar{\underset{\sim}{\mu}}_4'\end{pmatrix} = \underset{\sim}{\mu}\bar{\underset{\sim}{\mu}}'$ where $\underset{\sim}{\mu}_1((p-t)\times t)$, $\underset{\sim}{\mu}_2(t\times t)$, $\underset{\sim}{\mu}_3((p-t)\times(p-t))$ and $\underset{\sim}{\mu}_4(t\times(p-t))$; and $\underset{\sim}{\mu}_2$ and $\underset{\sim}{\mu}_3$ are nonsingular; and $\underset{\sim}{D}_\omega$ denotes the diagonal matrix with characteristic roots $\omega_1 \geq\cdots\geq \omega_t$ of $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}'\underset{\sim}{\Sigma}^{-1}$ as its diagonal elements. Put $\underset{\sim}{\theta} = \begin{pmatrix}\underset{\sim}{\theta}_1\\ \underset{\sim}{\theta}_2\end{pmatrix} = \begin{pmatrix}\underset{\sim}{\mu}_1\\ \underset{\sim}{\mu}_2\end{pmatrix}\underset{\sim}{D}_{\sqrt{\omega}}\underset{\sim}{\phi}(t\times r)$ where $\underset{\sim}{\phi}$ is determined by $\underset{\sim}{\phi} = \underset{\sim}{D}_{\omega^{-\frac{1}{2}}}\underset{\sim}{\mu}_2^{-1}\underset{\sim}{\theta}_2$ and

$\underset{\sim}{\phi}\bar{\underset{\sim}{\phi}}' = \underset{\sim}{I}$ and complete $\bar{\underset{\sim}{\phi}}'(r\text{x}t)$ into a unitary matrix $\bar{\underset{\sim}{\psi}}'(r\text{x}r)$. Finally transform

$$\underset{\sim}{V} = \underset{\sim}{\mu}^{-1}\underset{\sim}{Z}_1\bar{\underset{\sim}{\psi}}', \quad \underset{\sim}{W} = \underset{\sim}{\mu}^{-1}\underset{\sim}{Z}_2.$$

Then the joint density of $\underset{\sim}{V}$ and $\underset{\sim}{W}$ is

$$\pi^{-p(r+n)}\exp[-\text{tr}(\underset{\sim}{W}\bar{\underset{\sim}{W}}'+\underset{\sim}{V}\bar{\underset{\sim}{V}}'-2\text{Re}\underset{\sim}{V}\underset{\sim}{D}^{**}_{\sqrt{\omega}}+\underset{\sim}{D}^{*}_{\omega})],$$

where $\underset{\sim}{D}^{*}_{\omega}(p\text{x}p) = \begin{pmatrix} \underset{\sim}{D}_{\omega} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}$ and $\underset{\sim}{D}^{**}_{\sqrt{\omega}}(r\text{x}p) = \begin{pmatrix} \underset{\sim}{D}_{\sqrt{\omega}} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0}_1 \end{pmatrix}$ and $\underset{\sim}{0}_1$ is $(r-t)\text{x}(p-t)$ zero matrix. Note that the characteristic roots of $\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1' + \underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}$ are the same as those of $\underset{\sim}{V}\bar{\underset{\sim}{V}}'(\underset{\sim}{V}\bar{\underset{\sim}{V}}'+\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}$.

Now, for any region $Q \subseteq C^{p(r+n)}$ invariant under all transformations (2.2), define

$$\varphi_Q(\omega_1,\ldots,\omega_t) = P_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\{(\underset{\sim}{Z}_1,\underset{\sim}{Z}_2) \notin Q\} = P_{\underset{\sim}{D}^{**'}_{\sqrt{\omega}},\underset{\sim}{I}}\{(\underset{\sim}{V},\underset{\sim}{W}) \notin Q\},$$

so φ_Q is the power function of the test with acceptance region Q. For each $i = 1,\ldots,r$, denote the v_i-section of Q for fixed v_j, $j \neq i$ and fixed $\underset{\sim}{W}$ by

$$Q^{(i)}(\tilde{\underset{\sim}{V}}_i,\underset{\sim}{W}) = \{v_i | (\underset{\sim}{V},\underset{\sim}{W}) \in Q\} \subseteq C^p,$$

where $\tilde{\underset{\sim}{V}}_i(p\text{x}(r-1)) \equiv (v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_r)$.

Pillai and Li [3] have proved the following theorem. They assumed that $p \leq n$, but their proof is also valid if $p < r + n$.

<u>Theorem 1</u>. Let $Q \subseteq C^{p(r+n)}$ be invariant under all transformations (2.2). Suppose that for each $i = 1,..,r$, $Q^{(i)}(\tilde{\underset{\sim}{V}}_i,\underset{\sim}{W})$ is convex, then

$\varphi_Q(\omega_1,\ldots,\omega_t)$ increases monotonically in each ω_j.

Now consider the following acceptance regions:

(1) Roy's largest root test:

$$Q_1 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \ell_1 \leq k_1\}, \qquad 0 < k_1 < 1,$$

(2) Hotelling's trace test:

$$Q_2 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i(1-\ell_i)^{-1} \leq k_2\}, \quad 0 < k_2,$$

(3) Likelihood ratio test:

$$Q_3 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \prod_{i=1}^{t} (1-\ell_i) \geq k_3\}, \qquad 0 < k_3 < 1,$$

(4) Pillai's trace test:

$$Q_4 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i \leq k_4\}, \qquad 0 < k_4 < t.$$

The tests are defined for $p \leq n$; but the last will be defined for $p < n + r$ in the sequel. Pillai and Li [3] have shown that Q_1, Q_2 and Q_3 satisfy the conditions of Theorem 1 and hence their power functions are monotonically increasing in each population root. However, the monotonicity property has not yet been established for test Q_4 based on the trace statistic $\sum_{i=1}^{t} \ell_i$. In this paper, we show that $\varphi_{Q_4}(\omega_1,\ldots,\omega_t)$ is monotonically increasing in each ω_j provided that the cutoff point k_4 is not too large. In order to show this, we need the following lemma:

<u>Lemma 1</u>. Let $\xi' = (z_1,\ldots,z_p)$ and $\eta' = (x_1,y_1,\ldots,x_p,y_p)$ where $z_j = x_j + iy_j$, $j = 1,\ldots,p$, and let T be a one-one transformation between ξ and η such that $T(\xi) = \eta$ with the following properties:

(1) $T(\xi_1 + \xi_2) + T(\xi_1) + T(\xi_2)$ and

(2) $T(a\xi) = aT(\xi)$ where a is a real number. Let Q be a subset of ξ's in p-dimensional complex sample C^p, and Q* be its corresponding subset of η's in the 2p-dimensional real sample space R^{2p}. If Q is convex in C^p and symmetric in ξ, then Q* is convex in R^{2p} and symmetric in η and conversely.

The proof is given by Pillai and Li [3].

For $0 < \alpha < 1$ and $p < r + n$, define $k_4(\alpha,p,r,n)$ to be the size α cutoff point, i.e.

$$P(\sum_{i=1}^{t} \ell_i > k_4(\alpha,p,r,n) | \underset{\sim}{\theta} = \underset{\sim}{0}) = \alpha.$$

<u>Theorem 2</u>. The invariant acceptance region Q_4 satisfies the conditions of Theorem 1 if and only if

$$k_4(\alpha,p,r,n) \leq \max\{1,p-n\}.$$

<u>Proof</u>. $Q_4 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i \leq k_4\}$

$$= \{(\underset{\sim}{V},\underset{\sim}{W}) | tr[\underset{\sim}{V}\underset{\sim}{\bar{V}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}] \leq k_4\},$$

where $0 < k_4 < t$. Since Q_4 is symmetric in the columns $v_1,\ldots,v_r$ of $\underset{\sim}{V}$, it suffices to prove that $Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ is convex for almost all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ if and only if $k_4 \leq \max\{1,p-n\}$. Now

$$tr[\underset{\sim}{V}\underset{\sim}{\bar{V}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}] = p-tr[\underset{\sim}{W}\underset{\sim}{\bar{W}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}]$$

and $(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1} = (v_1\bar{v}_1'+\tilde{\underset{\sim}{V}}_1\bar{\tilde{\underset{\sim}{V}}}_1'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1} = \underset{\sim}{U}^{-1} - \dfrac{\underset{\sim}{U}^{-1}v_1\bar{v}_1'\underset{\sim}{U}^{-1}}{1+\bar{v}_1'\underset{\sim}{U}^{-1}v_1}$, where $\underset{\sim}{U} = \tilde{\underset{\sim}{V}}_1\bar{\tilde{\underset{\sim}{V}}}_1' + \underset{\sim}{W}\bar{\underset{\sim}{W}}'$ is nonsingular for almost all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ since $p \leq n+r-1$.

Hence, excluding a null set of $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ values,

$$Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \{v_1 \left| \frac{\bar{v}_1'\underset{\sim}{U}^{-1}\underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-1}v_1}{1+\bar{v}_1'\underset{\sim}{U}^{-1}v_1} \leq \text{tr}\, \underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-1}-p+k_4\}.\right.$$

Let $\underset{\sim}{\Lambda} = \underset{\sim}{U}^{-\frac{1}{2}}\underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-\frac{1}{2}}$, let $y = \underset{\sim}{U}^{-\frac{1}{2}} v_1$, and define the region $M = M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) \subseteq C^p$ by

$$M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \{y | \frac{\bar{y}'\Lambda y}{1+\bar{y}'y} \leq \text{tr}\Lambda-p+k_4\}.$$

Thus $Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{U}^{\frac{1}{2}}[M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})]$,

where $\underset{\sim}{U}^{\frac{1}{2}}[M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})] = \{\underset{\sim}{U}^{\frac{1}{2}}y | y \in M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})\}$. Choose $\underset{\sim}{\psi}$ to be a pxp unitary matrix such that

$$\underset{\sim}{\Lambda} = \underset{\sim}{\psi}\underset{\sim}{D}_{\lambda_i}(pxp)\bar{\underset{\sim}{\psi}}',$$

where $\lambda_1 \geq \cdots \geq \lambda_p$ are the ordered characteristic roots of $\underset{\sim}{\Lambda}$. Let $z = \bar{\underset{\sim}{\psi}}'y = (z_1,\ldots,z_p)'$ and define the region $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) \subseteq C^p$ by

(2.3) $$H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})=\{z| \sum_{j=1}^{p} z_j\bar{z}_j(\lambda_j-\sum_{m=1}^{p}\lambda_m+p-k_4) \leq \sum_{m=1}^{p}\lambda_m-p+k_4\}.$$

Then $M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{\psi}[H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})]$, so that except for a null set of $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ values,

(2.4) $$Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{U}^{\frac{1}{2}}\underset{\sim}{\psi}[H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})].$$

Now assume that $k_4 \leq \max\{1,p-n\}$. In view of (2.4) and the linearity of the operator $\underset{\sim}{U}^{\frac{1}{2}} \underset{\sim}{\psi}$, to verify that Q_4 satisfies the conditions of Theorem 1, it suffices to show that $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ is convex for all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$. Now define the region $H^* \subseteq R^{2p}$ by

$$(2.5)\quad H^* = \{(x_1,y_1,\ldots,x_p,y_p)' \mid \sum_{j=1}^{p}(x_j^2+y_j^2)(\lambda_j-\sum_{m=1}^{p}\lambda_m+p-k_4) \leq \sum_{m=1}^{p}\lambda_m-p+k_4\},$$

where $x_j+iy_j=z_j$, $j = 1,\ldots,p$.

Since $\underset{\sim}{\Lambda}$ and $\underset{\sim}{I}-\underset{\sim}{\Lambda}$ are positive semi-definite and and rank $(\underset{\sim}{\Lambda})$ = rank $(\underset{\sim}{W}) \leq \min\{p,n\} \equiv s$, say, we have that

$$1 \geq \lambda_1 \geq \cdots \geq \lambda_s \geq 0 = \lambda_{s+1} = \cdots = \lambda_p.$$

Hence for each $j = 1,\ldots,p$,

$$\begin{aligned}\lambda_j-\sum_{m=1}^{p}\lambda_m+p-k_4 &\geq -\min\{p-1,s\} + p-k_4\\ &= -\min\{p-1,n\} + p-k_4\\ &= \max\{1,p-n\} - k_4\\ &\geq 0.\end{aligned}$$

Therefore from (2.5), H^* is an ellipsoid (possibly degenerate or empty) in R^{2p} and hence is convex in R^{2p}. By Lemma 1, we have that $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ is convex for all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ in C^p. So Q_4 satisfies the conditions of Theorem 1.

Conversely, suppose that $k_4 > \max\{1,p-n\}$. Since $k_4 < t = \min\{p,r\}$, this requires that $r > 1$. Let $\delta = \delta(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \sum_{m=1}^{p}\lambda_m-p+k_4$ and $\beta_j=\beta_j(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \lambda_j-\delta$, so that

$$H^* = \{(x_1,y_1,\ldots,x_p,y_p)' \mid \sum_{j=1}^{p}(x_j^2+y_j^2)\beta_j \leq \delta\}.$$

We shall show later that there exists $(\tilde{V}_1, W)$ such that

$$(2.6) \qquad \beta_1(\tilde{V}_1, W) > 0 > \beta_p(\tilde{V}_1, W).$$

Since $\beta_j(\tilde{V}_1, W)$ is a continuous function of $(\tilde{V}_1, W)$, there must exist an open set $\Delta \subseteq C^{p(n+r-1)}$ such that (2.6) holds for all $(\tilde{V}_1, W) \in \Delta$. Thus H* fails to be a convex set whenever $(\tilde{V}_1, W) \in \Delta$, which is a non-null set, so by Lemma 1, $H(\tilde{V}_1, W)$ also fails to be a convex set whenever $(\tilde{V}_1, W) \in \Delta$, so Q_4 cannot satisfy the conditions of Theorem 1. Back to the existence of $(\tilde{V}_1, W)$ satisfying (2.6), which can be rewritten as

$$(2.7) \qquad \sum_{j=2}^{p} \lambda_j < p-k_4 < \sum_{j=1}^{p-1} \lambda_j.$$

By assumption, $\max\{1,p-n\} < k_4 < t = \min\{p,r\}$, or

$$(2.8) \qquad \max\{0,p-r\} < p-k_4 < \min\{p-1,n\}$$

Case (i). $p \leq n$, $p < r$.

Choose $(\tilde{V}_1, W)$ such that $W\bar{W}' = I$ and $\tilde{V}_1\bar{\tilde{V}}_1' = D_{d_i}$, where $0 \leq d_1 \leq \cdots \leq d_p$ are defined below. For such $(\tilde{V}_1, W)$ we have

$$(\lambda_1, \ldots, \lambda_p) = ((1+d_1)^{-1}, \ldots, (1+d_p)^{-1}),$$

so (2.7) becomes

$$(2.9) \qquad \sum_{j=2}^{p} (1+d_j)^{-1} < p-k_4 < \sum_{j=1}^{p-1} (1+d_j)^{-1}.$$

Also (2.8) reduces to $0 < p-k_4 < p-1$, so $0 \leq a \leq p-2$ where $a \equiv [p-k_4]$. If $a = 0$ choose the d_j such that $(1+d_1)^{-1} = 1$ and $(1+d_j)^{-1} < (p-k_4)/(p-1)$ for $2 \leq j \leq p$. If $1 \leq a \leq p-2$ select

θ,ε such that $(p-k_4) - a < \theta < 1$ and $0 < \varepsilon < 1-\theta$, and choose the d_j such that $(1+d_1)^{-1} = \ldots = (1+d_a)^{-1} = 1, (1+d_{a+1})^{-1} = \theta$, and $\sum_{j=a+2}^{p} (1+d_j)^{-1} = \varepsilon$. Then it is easy to see that for all $0 \le a \le p-2$, (2.9) is satisfied. Similarly for the other cases (ii) $p \le n$, $p \ge r$, (iii) $p > n$, $p < r$, and (iv) $p > n$, $p \ge r$.

Since $k_4(\alpha,p,r,n)$ is decreasing in α and n, while increasing in p,r. The power function of Pillai's trace test for the MANOVA problem has the monotonicity property with respective to each population root provided that α and n are not too small and p and r are not too large.

Approximate values of $k_4(\alpha,p,m+p,s+p)$, where $m = r-p$, $s = n-p$, has been tabulated by Krishnaiah and Schuurmann [1]. In Table 1 and 2, we give the values of $S^*(\alpha,p,m)$ for $\alpha = 0.05$ and $\alpha = 0.01$, where $S^*(\alpha,p,m)$ is the smallest value of $s \ge 0$ such that $k_4(\alpha,p,m+p,s+p) \le 1$. Thus, by Theorem 2, Pillai's trace test has the monotonicity property if $s \ge S^*(\alpha,p,m)$.

TABLE 1

Values of $S^*(0.05,p,m)$

p \ m	2	4	6	8	10
2	6	9	12	14	17
4	22	29	36	42	49
6	45	56	69	79	90
8	77	92	109	126	140
10	119	138	155	177	194

TABLE 2

Values of S*(0.01,p,m)

p \ m	2	4	6	8	10
2	8	11	14	17	20
4	25	33	40	48	55
6	50	62	74	85	97
8	83	99	118	134	147
10	127	145	165	185	203

3. Invariant tests for independence of variates

Let $\underset{\sim}{Z}((p_1+p_2)\text{x}m)$ be a complex random matrix whose columns are independent and complex normally distributed with mean $\underset{\sim}{0}$ and common nonsingular hermitian covariance matrix $\underset{\sim}{\Sigma}$. Let

$$\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \text{ and } \underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix},$$

where $\underset{\sim}{Z}_1(p_1\text{x}m)$, $\underset{\sim}{Z}_2(p_2\text{x}m)$, $\underset{\sim}{\Sigma}_{11}(p_1\text{x}p_1)$, $\underset{\sim}{\Sigma}_{21}(p_2\text{x}p_1)$, $\underset{\sim}{\Sigma}_{22}(p_2\text{x}p_2)$.

The problem of testing independence of two sets of variates is to test

$$\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0} \quad \text{against } \underset{\sim}{\Sigma}_{12} \neq \underset{\sim}{0}.$$

This problem is invariant under all transformations of the form:

$$\begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \underset{\sim}{F},$$

where $\underset{\sim}{B}_1(p_1\text{x}p_1)$ and $\underset{\sim}{B}_2(p_2\text{x}p_2)$ are nonsingular and $\underset{\sim}{F}(m\text{x}m)$ is unitary.

A maximal invariant statistic is $(r_1^2,\ldots,r_t^2)$, where $t = \min\{p_1,p_2\}$ and $1 \geq r_1^2 \geq\ldots\geq r_t^2 \geq 0$ are the ordered t largest characteristic roots of

$$(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1')^{-1}(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_2')(\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}(\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_1') = \underset{\sim}{S}_{11}^{-1}\ \underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21},$$

where $\underset{\sim}{S} = \underset{\sim}{Z}\bar{\underset{\sim}{Z}}'$ is partitioned as $\underset{\sim}{\Sigma}$. Here we assume that $\max\{p_1,p_2\}<m$ to ensure the nonsingularity of $\underset{\sim}{S}_{11}$ and $\underset{\sim}{S}_{22}$. An invariant parameter is the vector of canonical correlations $(\rho_1,\ldots,\rho_t)$, where $\rho_1^2 \geq\ldots\geq \rho_t^2 \geq 0$ are the ordered t largest characteristic roots of $\underset{\sim}{\Sigma}_{11}^{-1}\ \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21}$. The power function of any invariant test is a function of $(\rho_1,\ldots,\rho_t)$.

The conditional distribution given $\underset{\sim}{S}_{22}$ of the matrix

$$\underset{\sim}{S}_{11}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21} = (\underset{\sim}{S}_{11.2}+\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21})^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21}$$

is of the same form as the distribution of the matrix $(\underset{\sim}{V}\bar{\underset{\sim}{V}}'+\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}\underset{\sim}{V}\bar{\underset{\sim}{V}}'$ in the MANOVA problem if we take $(p,r,n) = (p_1,p_2,m-p_2)$, and their unconditional null distributions are the same. (See Pillai and Li [3]). Therefore, we have the following:

Theorem 3. If the power function of an invariant test for the MANOVA problem, which accepts the hypothesis $\underset{\sim}{\theta} = \underset{\sim}{0}$ if and only if $(\ell_1,\ldots,\ell_t) \in Q$, increases monotonically in each noncentrality parameter ω_j, then the power function of the invariant test for the independence problem, which accepts the hypothesis $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}$ if and only if $(r_1^2,\ldots,r_t^2) \in Q$, increases monotonically in each canonical correlation ρ_j.

For $p_1+p_2 \leq m$, Pillai and Li [3] have shown that the power function of Roy's largest root test based on r_1^2, Hotelling's trace test based on $\sum_{i=1}^{t} r_i^2/(1-r_i^2)^{-1}$, and the likelihood ratio test based on $\prod_{i=1}^{t}(1-r_i^2)$ all have the monotonicity property.

Now consider Pillai's trace that for testing the independence. This test accepts $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}$ if and only if

$$\sum_{i=1}^{t} r_i^2 \leq k_4(\alpha, p_1, p_2, m-p_2),$$

where k_4 was defined in section 2.

Theorem 1, 2 and 3 imply the following:

Theorem 4. The power function of Pillai's trace test for independence increases in each canonical correlation ρ_j if $k_4(\alpha, p_1, p_2, m-p_2) \leq \max\{1, p_1+p_2-m\}$.

Since $k_4(\alpha, p_1, p_2, m-p_2)$ is decreasing in α and m, while increasing in p_1 and p_2. The power function of Pillai's trace test for the independence problem has the monotonicity property provided that α and m are not too small and p_1 and p_2 are not too large. Table 1, 2 can also be used in this case.

References

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[2] Perlman, M. D. (1974). On the monotonicity of the power functions of tests based on traces of multivariate beta matrices. J. Multivariate Anal., 4, 22-30.

[3] Pillai, K.C.S. and Li, H. C. (1970). Monotonicity of the power functions of some tests of hypotheses concerning multivariate complex normal distributions. Ann. Inst. Statist. Math., 22, 307-318.

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List of Symbols

Greek letters:

Σ	u. c. sigma	(e.g. p. 1, line 22)
θ	l. c. theta	(e.g. p. 2, line 1)
π	l. c. pi	(e.g. p. 2, line 3)
ω	l. c. omega	(e.g. p. 2, line 13)
μ	l. c. mu	(e.g. p. 2, line 15)
ϕ	l. c. phi	(e.g. p. 2, line 20)
ψ	l. c. psi	(e.g. p. 3, line 1)
ξ	l. c. xi	(e.g. p. 4, line 20)
η	l. c. eta	(e.g. p. 4, line 20)
α	l. c. alpha	(e.g. p. 5, line 8)
Λ	u. c. lambda	(e.g. p. 6, line 5)
λ	l. c. lambda	(e.g. p. 6, line 11)
δ	l. c. delta	(e.g. p. 7, line 19)
β	l. c. beta	(e.g. p. 7, line 19)
Δ	u. c. delta	(e.g. p. 8, line 4)
ε	l. c. epsilon	(e.g. p. 9, line 1)
ρ	l. c. rho	(e.g. p. 11, line 7)

Non-italic symbols:

exp	(e.g. p. 2, line 3)
tr	(e.g. p. 2, line 3)
min	(e.g. p. 2, line 9)
Re	(e.g. p. 3, line 5)
max	(e.g. p. 5, line 13)

Short title:

Monotonicity of Powers of Trace Tests for Complex Matrices.