# ON THE EXACT NON-NULL DISTRIBUTION OF WILKS' Lvc CRITERION FOR THE CLASSICAL AND COMPLEX NORMAL POPULATIONS AND RELATED PROBLEMS

by

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# CHAPTER I

# ON THE EXACT NON-NULL DISTRIBUTION OF WILKS' $L_{vc}$ CRITERION

## 1. INTRODUCTION AND SUMMARY

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N$ be a random sample of size N from a p-variate normal population with unknown mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$, i.e., $\underset{\sim}{x}_i \sim N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma}$ is symmetric and positive definite. Let

$$\bar{\underset{\sim}{x}} = N^{-1} \sum_{i=1}^{N} \underset{\sim}{x}_i \quad \text{and} \quad \underset{\sim}{S} = \sum_{i=1}^{N} (\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})' \tag{1.1}$$

Then $\bar{\underset{\sim}{x}}$ has a normal distribution $N(\underset{\sim}{\mu}, N^{-1/2}\underset{\sim}{\Sigma})$ and $\underset{\sim}{S}$ has an independent Wishart distribution $W(\underset{\sim}{\Sigma}, p, n)$ with $n = N-1$. Wilks [25] likelihood ratio criterion $L_{vc}$ for testing $H: \underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho \underset{\sim}{e}\underset{\sim}{e}']$, $\rho, \sigma$ unknown, against the alternative $A \neq H$ can be expressed as

$$L_{vc} = |\underset{\sim}{S}| \; [tr(\underset{\sim}{E}\underset{\sim}{S})/p]^{-1} / [tr(p\underset{\sim}{I} - \underset{\sim}{E})\underset{\sim}{S}/p^{p-1}]^{p-1} \tag{1.2}$$

where $\underset{\sim}{E} = \underset{\sim}{e}\underset{\sim}{e}'$ and $\underset{\sim}{e}' = (1, 1, \ldots, 1)$ and $\underset{\sim}{\mu}$ is unknown. Wilks [25] derived the exact null distribution of $L_{vc}$ for the special cases $p=2$ and $p=3$. Varma [24] obtained the exact null distribution in a series form using Mellin integral transform (see [19]) and factorial series expansion [16], and computed some percentage points for specific values of p. Nagarsenker [17] derived the null distribution employing Box's chisquare series approximation and tabulated percentage points for p=4(1)10. Khatri and Srivastava [13] obtained the exact non-null distribution of $L_{vc}$ in a series form involving Meijer's G-function [14] and certain $a_\delta(j)$ coefficients which are not easy to compute. In this paper, we derive the distribution of $L_{vc}$ in three series forms and compute powers for p=2 and 3 for 5% critical points for various values of N and the parameters. In Section 2, we present some definitions and lemmas which are needed in the sequel. We derive in Section 3, the non-null density of $L_{vc}$ as a series involving Meijer's G-functions using Mellin integral transform. Some special cases have also been discussed which are used to compute powers for the case p=2. In Section 4, we obtain the non-null density in an alternative series form through the method of contour integration (as in [18]) and in Section 5, the non-null moments of the criterion are used to obtain the distribution as a chisquare series employing methods similar to those of Box [2]. Section 6 is devoted to power computations. The densities derived in Sections 4 and 5 have been used for power computation for various alternatives for the case $p=3$ and various values of $N$.

## 2. SOME DEFINITIONS AND RESULTS

In this section we give a few definitions and some lemmas which will be used in the sequel.

Definitions. Let $k$ be a non-negative integer and let $\kappa=(k_1,k_2,\ldots,k_p)$ be a portion of $k$ such that $k_1\geq k_2\geq\ldots\geq k_p\geq 0$, $\sum_{i=1}^{p} k_i=k$ and let

$$(a)_\kappa = \prod_{i=1}^{p} (a-(i-1)/2)_{k_i} = \Gamma_p(a,\kappa)/\Gamma_p(a), \tag{2.1}$$

where

$$(a)_k = (a)(a+1)\ldots(a+k-1) \tag{2.2}$$

and

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(a-(i-1)/2) \tag{2.3}$$

Now Meijer's G-function [14] may be defined by

$$G_{p\ q}^{m\ n}\left[x\,\middle|\,\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\right] = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s\, ds \tag{2.4}$$

where an empty product is interpreted as unity and $C$ is a curve separating the singularities of $\prod_{j=1}^{m}(b_j-s)$ from those of $\prod_{j=1}^{n}(1-a_j+s)$, $q\geq 1$, $0\leq n\leq p\leq q$, $0\leq m\leq q$; $x\neq 0$ and $|x|\leq 1$ if $q=p$; $x\neq 0$ if $q>p$. The definition above is an application of lemma 2.4 below. Also we need the following special case

$$G_{2\ 2}^{2\ 0}\left[x\,\middle|\,\begin{matrix}a_1 & a_2\\ b_1 & b_2\end{matrix}\right] = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}\, {}_2F_1(a_2-b_2, a_1-b_2, a_1+a_2-b_1-b_2; 1-x) \tag{2.5}$$

where

$${}_2F_1(a,b,c;x) = \sum_{k=0}^{\infty} (a)_k\, (b)_k\, x^k/(c)_k\, k! \tag{2.6}$$

Further, the hypergeometric function of a matrix variate (see James [8])

$${}_pF_q\,(a_1,a_2,\ldots,a_p;\, b_1,b_2,\ldots,b_q;\, \underset{\sim}{S}) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\, \frac{C_\kappa(\underset{\sim}{S})}{k!}\,. \tag{2.7}$$

where $C_\kappa(\underset{\sim}{A})$ denotes to zonal polynomial of the symmetric matrix $\underset{\sim}{A}$ of degree $k$ corresponding to the partition $\kappa$. In particular we have

$${}_0F_0(\underset{\sim}{S}) = \exp(\mathrm{tr}\underset{\sim}{S}) \quad\text{and}\quad {}_0F_1(a;\, \underset{\sim}{S}) = |\underset{\sim}{I}-\underset{\sim}{S}|^{-a} \tag{2.8}$$

Lemmas: We now state a few lemmas without proof which will be used in the following sections.

Lemma 2.1. Let $\Sigma$ be the matrix having the form $\Sigma=\sigma I+\rho ee'$ where $e'=(1,1,\ldots,1)$. $\Sigma$ can be represented in the form $\Sigma=H'DH$ where $H$ is any pxp orthogonal matrix having first row $p^{-1/2}e'$ and $D=\text{diag}((\sigma+\rho p), \sigma, \sigma,\ldots\sigma)$.

Thus using lemma 2.1, we note that the test of hypothesis $H$: $\Sigma=\sigma^2[(1-\rho)I+\rho ee']$ is equivalent to that of $\Sigma=\text{diag}(\sigma_1, \sigma_2,\ldots,\sigma_2)$, $\sigma_1,\sigma_2>0$ and unknown (see [7]).

Lemma 2.2. If $R$ is a positive definite mxm matrix then

$$\int_0^I (\det S)^{t-(m+1)/2}\,(\det(I-S))^{u-(m+1)/2}\,C_\kappa(RS)\,dS = \frac{\Gamma_m(t,\kappa)\Gamma_m(u)}{\Gamma_m(t+u,\kappa)}\,C_\kappa(R)$$

Proof. See Constantine [4].

Lemma 2.3. Let $R$ be a complex symmetric matrix whose real part is positive definite and let $T$ be an arbitrary complex symmetric matrix. Then

$$\int_{S>0}\exp(-\text{tr}RS)(\det S)^{t-(m+1)/2}C_\kappa(ST)\,dS=\Gamma_m(t,\kappa)(\det R)^{-t}\,C_\kappa(TR^{-1})$$

the integration being over the space of positive definite real mxm matrices, and valid for all complex numbers $t$ satisfying $R(t)>(m-1)/2$.

Proof. See Constantine [4].

Finally, we give a lemma defining the Mellin integral transform (see [19]).

Lemma 2.4. If $s$ is any complex variate and $f(x)$ is a function of a real variate $x$, such that

$$F(s) = \int_0^\infty x^{s-1} f(x)dx$$

exists, then under certain regularity conditions

$$f(x) = (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} x^{-s} F(s)ds$$

F(s) is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$.

## 3. EXACT NON-NULL DISTRIBUTION OF $L_{vc}$.

In this section, we derive the non-null density of $L_{vc}$ as a series of Meijer's G-functions [14] using Mellin-integral transform (lemma 2.4). Using Lemma 2.1, the test of $H: \underset{\sim}{\Sigma}=\sigma^2[(1-\rho)\underset{\sim}{I}+\rho\underset{\sim}{e}\underset{\sim}{e}']$ reduces to that of $H: \underset{\sim}{\Sigma}=\begin{bmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2 \underset{\sim}{I}_{p_2}\end{bmatrix}$, $\sigma_1,\sigma_2>0$ and unknown, and $p_2=p-1$. The $L_{vc}$ can be expressed as

$$L_{vc} = |\underset{\sim}{S}|/[s_{11}(\mathrm{tr}\underset{\sim}{S}_{22}/p_2)^{p_2}] \tag{3.1}$$

where $\underset{\sim}{S}=\begin{bmatrix} s_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}_{12}' & \underset{\sim}{S}_{22}\end{bmatrix}\begin{matrix}1\\p_2\end{matrix}$, $n=N-1$, $N$ being the size of a sample from $N(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma}>0$. Now, we can make a transformation $\begin{bmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{bmatrix} \to \begin{bmatrix}\underset{\sim}{x}_1/\sigma_1\\ \underset{\sim}{x}_2/\sigma_2\end{bmatrix}\begin{matrix}1\\p_2\end{matrix}$.

Under this transformation the problem reduces to that of testing $H : \underset{\sim}{\Sigma}=\begin{bmatrix}1 & 0\\ 0 & \underset{\sim}{I}_{p_2}\end{bmatrix}$ versus $A_1 \neq H_1$ where $\underset{\sim}{\Sigma}=\begin{bmatrix}1 & \underset{\sim}{\Sigma}_{12}/\sigma_1\sigma_2 \\ \underset{\sim}{\Sigma}_{12}'/\sigma_1\sigma_2 & \underset{\sim}{\Sigma}_{22}/\sigma_2^2\end{bmatrix}\begin{matrix}1\\p_2\end{matrix}$, $\sigma_1$ and $\sigma_2$ unknown. From now on we assume that this has been done and we are testing $H_1$ versus $A_1 \neq H_1$. Let us define

$$T = s_{11}^{-1/2}\ \underset{\sim}{S}_{12}\ \underset{\sim}{S}_{22}^{-1}\ \underset{\sim}{S}_{12}'\ s_{11}^{-1/2} \tag{3.2}$$

Then, the $L_{vc}$ can be written as

$$L_{vc} = |\underset{\sim}{S}_{22}|(1-T)/(\text{tr}\,\underset{\sim}{S}_{22}/p_2)^{p_2} \tag{3.3}$$

We now need the following lemma in order to compute the non-null moments of $L_{vc}$.

Lemma 3.1. The joint p.d.f. of $\underset{\sim}{T}$, $\underset{\sim}{S}_{11}$ and $\underset{\sim}{S}_{22}$ is given by

$$f(\underset{\sim}{T},\underset{\sim}{S}_{11},\underset{\sim}{S}_{22}) = k(p_1,p_2,n,\underset{\sim}{\Sigma})\ |\underset{\sim}{S}_{11}|^{(n-p_1-1)/2}\ |\underset{\sim}{S}_{22}|^{(n-p_2-1)/2} \tag{3.4}$$

$$\exp(-1/2\text{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{11})\ \exp(-1/2\text{tr}\underset{\sim}{\Sigma}_{2.1}^{-1}\underset{\sim}{S}_{22})\ |\underset{\sim}{I}-\underset{\sim}{T}|^{(n-p_1-p_2-1)/2}\ |\underset{\sim}{T}|^{(p_2-p_1-1)/2}\ .$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa\ (\underset{\sim}{S}_{11}^{'1/2}\ \underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{12}\underset{\sim}{\beta}'\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{11}^{1/2}\underset{\sim}{T}/4)/[k!(p_2/2)\kappa],\ \underset{\sim}{0}<\underset{\sim}{T}<\underset{\sim}{I},\ \underset{\sim}{S}_{11},\underset{\sim}{S}_{22}>0$$

where

$$\underset{\sim}{\Sigma}_{1.2} = \underset{\sim}{\Sigma}_{11}-\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{12}'$$

$$\underset{\sim}{\Sigma}_{2.1} = \underset{\sim}{\Sigma}_{22}-\underset{\sim}{\Sigma}_{12}'\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12}$$

$$\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{12}' & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix}$$

and

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}_{12}' & \underset{\sim}{S}_{22} \end{bmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix}$$

and $p_1+p_2=p$, $p_2 \geq p_1 \geq 1$ without loss of generality.

$$k^{-1}(p_1,p_2,n,\underset{\sim}{\Sigma}) = 2^{n(p_1+p_2)/2}\Gamma_{p_1}(p_2/2)\Gamma_{p_1}((n-p_2)/2)\Gamma_{p_2}(n/2)|\underset{\sim}{\Sigma}_{1.2}|^{n/2}|\underset{\sim}{\Sigma}_{22}|^{n/2} \tag{3.5}$$

$\underset{\sim}{T} = \underset{\sim}{S}_{11}^{-1/2}\ \underset{\sim}{S}_{12}\ \underset{\sim}{S}_{22}^{-1}\ \underset{\sim}{S}_{12}'\ (\underset{\sim}{S}_{11}^{-1/2})'$ (See Khatri and Srivastava [13]). Now before finding $E[L_{vc}^{h}]$, we will prove the following theorem.

Theorem 3.1.

$$E[\exp(-t\ \mathrm{tr}\underset{\sim}{S}_{22}/2)|\underset{\sim}{S}_{22}|^{h}\ (1-T)^{h}] = k_3(p_2,n,\underset{\sim}{\Sigma},h).\sum_{j=0}^{\infty}\sum_{J}\sum_{k=0}^{\infty} \tag{3.6}$$

$$\sum_{\kappa}(t+1)^{-(p_2(h+n/2)+k+j)}.(h)_{\kappa}(n/2)_{J}\ C_{\kappa}(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1})C_{J}(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1}+\underset{\sim}{\Sigma}_{1.2}^{-1}\beta'\beta)/k!j!$$

where

$$k_3(p_2,n,\underset{\sim}{\Sigma},h) = 2^{p_2h}\ \Gamma_{p_2}((n-1)/2+h)/[\Gamma_{p_2}((n-1)/2)|\underset{\sim}{\Sigma}_{22}|^{n/2}] \tag{3.7}$$

Proof. Let $w = \exp(-t\ \mathrm{tr}\underset{\sim}{S}_{22}/2)\ |\underset{\sim}{S}_{22}|^{h}\ (1-T)^{h}$ . Using lemma (3.1) with $p_1=1$ we obtain

$$E[w] = k(1,p_2,n,\underset{\sim}{\Sigma})\int_{s_{11}>0}\int_{\underset{\sim}{S}_{22}>\underset{\sim}{0}}\int_{\underset{\sim}{T}:\underset{\sim}{0}}^{\underset{\sim}{I}}(s_{11})^{n/2-1}|\underset{\sim}{S}_{22}|^{n/2+h-(p_2+1)/2} \tag{3.8}$$

$$\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}s_{11}).\exp(-\mathrm{tr}(\underset{\sim}{\Sigma}_{2.1}^{-1}+t\underset{\sim}{I})\underset{\sim}{S}_{22}/2)\ |T|^{(p_2-p_1-1)/2}.$$

$$|\underset{\sim}{I}-\underset{\sim}{T}|^{(n-p_1-p_2-1)/2+h}.\sum_{k=0}^{\infty}\sum_{\kappa}C_{\kappa}(s_{11}^{1/2}\ \underset{\sim}{\Sigma}_{1.2}^{-1}\beta\underset{\sim}{S}_{22}\beta'\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{11}^{1/2}\underset{\sim}{T})/[(p_2/2)_{\kappa}k!]$$

$$.ds_{11}d\underset{\sim}{S}_{22}d\underset{\sim}{T}$$

Now using monotone convergence theorem, the interchange of the integral and summation signs is valid and using lemma (2.2) in order to integrate with respect to T, one obtains

$$E[w] = k_2 \sum_{k=0} \sum_{\kappa} \int_{s_{11}\geq 0} s_{11}^{n/2-1} \exp(-\mathrm{tr}\Sigma_{1.2}^{-1} s_{11}/2) \int_{S_{22}\geq 0} |S_{22}|^{n/2+h-(p_2+1)/2}$$

$$\exp(-\mathrm{tr}(tI+\Sigma_{2.1}^{-1})S_{22}/2)\ C_\kappa(s_{11}\beta'\beta\Sigma_{1.2}^{-2}S_{22}/4)/(k!(\tfrac{n}{2}+h)_\kappa)\, ds_{11}\, dS_{22} \tag{3.9}$$

where

$$k_2 = k(1,p_2,n,\Sigma)\Gamma(p_2/2)\Gamma((n-p_2)/2+h)/\Gamma(n/2+h) \tag{3.10}$$

Now using lemma (2.3) to integrate with respect to $S_{22}$ and then in turn using monotone convergence theorem and the relation ${}_0F_0(S) = \exp(\mathrm{tr}S)$, we get

$$E[w]=k_4\ |(tI+\Sigma_{2.1}^{-1})/2|^{-(n/2+h)} \int_{s_{11}\geq 0} s_{11}^{n/2-1} \exp(-(\Sigma_{1.2}^{-1}-\beta\Sigma_{1.2}^{-2}$$

$$(tI+\Sigma_{2.1}^{-1})^{-1}\beta')s_{11}/2)\ ds_{11} \tag{3.11}$$

where $k_4 = k_2\ \Gamma_{p_2}(n/2+h)$. Now integrating with respect to $s_{11}$, we get

$$E[w]=k_4|(tI+\Sigma_{2.1}^{-1})/2|^{-(n/2+h)}\Gamma(n/2)((\Sigma_{1.2}^{-1}-\beta\Sigma_{1.2}^{-2}(tI+\Sigma_{2.1}^{-1})^{-1}\beta')/2)^{-n/2} \tag{3.12}$$

Rewriting (3.12), one obtains

$$E[w] = k_3(p_2,n,\Sigma,h)|tI+\Sigma_{2.1}^{-1}|^{-h}|tI+\Sigma_{2.1}^{-1}-\beta'\beta\Sigma_{1.2}^{-1}|^{-n/2} \tag{3.13}$$

Now adding and subtracting $I$ inside each of the two determinants and using (2.8) we have

$$E[w] = k_3(p_2,n,\Sigma,h)(t+1)^{-p_2(h+n/2)}\ {}_1F_0(h;(t+1)^{-1}(I-\Sigma_{2.1}^{-1}))$$

$${}_1F_0(n/2;(t+1)^{-1}(I-\Sigma_{2.1}^{-1}+\Sigma_{1.2}^{-1}\beta'\beta)) \tag{3.14}$$

which can be expressed as (3.6) after using (2.7).

Theorem 3.2. For any finite $p$, the p.d.f. of $L_{vc}$ is given by

$$p(L_{vc}) = C_1(p_2,n;\underset{\sim}{\Sigma})(L_{vc})^{-(p_2/2+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}p_2^{-(k+j)}A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})$$

$$G_{2p_2\ \ 2p_2}^{2p_2\ \ 0}\left[L_{vc}\left|\begin{matrix}c_1,c_2,\ldots,c_{p_2};\ \ d_1,d_2,\ldots,d_{p_2}\\ a_1,a_2,\ldots,a_{p_2};\ \ b_1,b_2,\ldots,b_{p_2}\end{matrix}\right.\right] \tag{3.15}$$

where

$$C_1(p_2,n,\underset{\sim}{\Sigma}) = (2\pi)^{(p_2-1)/2}\,p_2^{(1-np_2)/2}/\left(\prod_{i=1}^{p_2}\Gamma((n-i)/2)|\underset{\sim}{\Sigma}_{22}|^{n/2}\right) \tag{3.16}$$

$$A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) = (n/2)_J\Gamma(np_2/2+k+j)C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1})C_J(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1}+\Sigma_{1.2}^{-1}\underset{\sim}{\beta}'\underset{\sim}{\beta})/j!k!$$

$$a_i = (p_2-i+1)/2+k_i,\qquad b_i = (n+p_2-i)/2;\qquad i=1,2,\ldots,p_2$$

$$c_i = (p_2-i+1)/2,\qquad d_i = (p_2+n)/2+(k+j+i-1)/p_2;\quad i=1,2,\ldots,p_2 \tag{3.17}$$

Proof. First we evaluate the h-th moment of $L_{vc}$ as the method of derivation of the density of $L_{vc}$ depends on lemma (2.4) concerning the Mellin-transform. Integrating both sides of (3.6) with respect to $t$, $p_2h$ times under the integral sign and putting $t=0$ in the final result we get

$$E[L_{vc}]^h = k_3(p_2,n,\underset{\sim}{\Sigma},h)(p_2/2)^{p_2h}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}(n/2)_J(h)_\kappa C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1})$$

$$C_J(I-\Sigma_{2.1}^{-1}+\Sigma_{1.2}^{-1}\beta'\beta)/(j!k!(np_2/2+k+j)_{hp_2}) \ . \tag{3.18}$$

Now let

$$C(p_2,n,\underset{\sim}{\Sigma}) = 1/\left[\prod_{i=1}^{p_2} \Gamma((n-i)/2)|\underset{\sim}{\Sigma}_{22}|^{n/2}\right] , \tag{3.19}$$

then

$$E[L_{vc}]^h = C(p_2,n,\underset{\sim}{\Sigma})\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})p_2^{p_2 h}$$

$$\prod_{i=1}^{p_2} \Gamma(n/2+h-i/2) \prod_{i=1}^{p_2} (h-(i-1)/2)_{k_i}/\Gamma(p_2(h+n/2)+k+j) \tag{3.20}$$

where $A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})$ is defined by (3.16). Now using lemma (2.4), take the Mellin integral transform on both sides of (3.20), we get the density of $L_{vc}$ in the form

$$p(L_{vc}) = C(p_2,n,\underset{\sim}{\Sigma})\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} (L_{vc})^{-(h+1)} \tag{3.21}$$

$$p_2^{p_2 h} \prod_{i=1}^{p_2} (h-(i-1)/2)_{k_i} \frac{\prod_{i=1}^{p_2} \Gamma(n/2+h-i/2)}{\Gamma(p_2(h+n/2)+k+j)} \, dh$$

We now need Gauss-Legendre's multiplication formula given by

$$\prod_{r=1}^{n} \Gamma(z+(r-1)/n) = 2\pi^{(n-1)/2} n^{1/2-nz} \Gamma(nz) \tag{3.22}$$

Applying the transformation $h \to h+p_2/2$ and using (3.22) in $\Gamma(p_2(h+n/2)k+j)$, (3.21) can be written as

$$p(L_{vc}) = C_1(p_2,n,\underset{\sim}{\Sigma})(L_{vc})^{-(p_2/2+1)} \sum_{k=0}^{\infty}\sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) \; p_2^{-(k+j)} \; U(j,k) \tag{3.23}$$

where $U(j,k)$ is the following integral

$$(2\pi i)^{-1}\int_{C_1-i\infty}^{C_1+i\infty}(L_{vc})^{-h}\frac{\prod_{i=1}^{p_2}\Gamma(h+(p_2-i+1)/2+k_i)\prod_{i=1}^{p_2}\Gamma(h+(n+p_2-i)/2)}{\prod_{i=1}^{p_2}\Gamma(h+(p-i+1)/2)\prod_{i=1}^{p_2}\Gamma(h+(p_2+n)/2+(k+j+i-1)/p_2)}\,dh$$

where $C_1=C+p_2/2$ and $C_1(p_2,n,\Sigma)$ is given by (3.16). Now, (3.23) can also be written as

$$p(L_{vc}) = C_1(p_2,n,\Sigma)(L_{vc})^{-(p_2/2+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}p_2^{-(k+j)}A(J,\kappa,p_2,n,\Sigma) \tag{3.24}$$

$$(2\pi i)^{-1}\int_{C_1-i\infty}^{C_1+i\infty}(L_{vc})^{-h}\frac{\prod_{i=1}^{p_2}\Gamma(h+a_i)\prod_{i=1}^{p_2}\Gamma(h+b_i)}{\prod_{i=1}^{p_2}\Gamma(h+c_i)\prod_{i=1}^{p_2}\Gamma(h+d_i)}\,dh$$

where $a_i$, $b_i$, $c_i$ and $d_i$ $i=1,2,\ldots,p_2$ are defined in (3.17). Noticing that the integrals in (3.24) are in the form of Meijer's G-functions, we can write the density of $L_{vc}$ in the form given in (3.15).

We now discuss special cases for $p_2=1$ and 2.

<u>$p_2=1$.</u> Putting $p_2=1$ in (3.15), we get

$$p(L_{vc}) = \frac{(L_{vc})^{-3/2}}{\Gamma((n-1)/2)}\sum_{k=0}^{\infty}\frac{\Gamma(n/2+k)}{k!}(-\rho^2/(1-\rho^2))^k G_{2\,2}^{2\,0}\left[L_{vc}\middle|\begin{matrix}(n+1)/2+k, & 1/2\\ n/2, & k+1/2\end{matrix}\right] \tag{3.25}$$

where $\Sigma = \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$. Now using (2.5), (3.25) can be written as

$$p(L_{vc}) = \frac{(L_{vc})^{(n-1)/2-1}(1-L_{vc})^{1/2-1}}{\Gamma((n-1)/2)\Gamma(1/2)}\sum_{k=0}^{\infty}\Gamma(n/2+k)(-\rho^2/(1-\rho^2))^k\ {}_2F_1(n/2, \tag{3.26}$$

$$-k,\ 1/2,\ 1-L_{vc})\ ,\quad 0<L_{vc}<1$$

In particular, under the null hypothesis $H_1$: $\rho=0$, the null density is given by

$$p_1(L_{vc}) = \Gamma(n/2)/(\Gamma((n-1)/2)\Gamma(1/2))(L_{vc})^{(n-1)/2-1}(1-L_{vc})^{1/2-1} \qquad (3.27)$$

$$0<L_{vc}<1$$

$p_2=2$. In this case,

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & c\rho_{13} \\ \rho_{12} & 1 & c\rho_{23} \\ c\rho_{13} & c\rho_{23} & c^2 \end{bmatrix}, \quad c = \sigma_3/\sigma_2$$

Putting $p_2=2$ in (3.15), we get

$$p(L_{vc}) = \frac{\pi 2^{1-n}\Gamma(n/2)}{\Gamma_2(n/2)\Gamma((n-2)/2)|\underset{\sim}{\Sigma}_{22}|^{n/2}} (L_{vc})^{-2} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J} 2^{-(k+j)}(n/2)_J \qquad (3.28)$$

$$\Gamma(n+k+j)\ C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1})C_J(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2.1}^{-1}+\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}'\beta)/k!j!$$

$$G_{4\ 4}^{4\ 0}\left[L_{vc}\left|\begin{matrix} 1+(n+k+j)/2, & 1+(n+k+j+1)/2, & 1/2,\ 1 \\ n/2, & (n+1)/2, & k_2+1/2,\ k_1+1 \end{matrix}\right.\right]$$

In particular, under the null hypothesis $H_1$: $\rho_{12}=\rho_{13}=\rho_{23}=0$ and $c=1$, the null density is given by

$$p_1(L_{vc}) = \pi 2^{1-n}\Gamma(n/2)/[\Gamma_2(n/2)\Gamma((n-2)/2)](L_{vc})^{-2}\Gamma(n) \qquad (3.29)$$

$$G_{2\ 2}^{2\ 0}\left[L_{vc}\left|\begin{matrix} 1+n/2, & (3+n)/2 \\ n/2, & (n+1)/2 \end{matrix}\right.\right]$$

Now using Legendre's duplication formula, namely $\Gamma(2s) = \Gamma(s)\Gamma(s+1/2)2^{2s-1}/\pi^{1/2}$ and the well-known result

$$(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} x^{-s}\Gamma(s)/\Gamma(s+v)ds = (1-x)^{v-1}/\Gamma(v) \quad 0\le x\le 1,\ C\ge 0 \qquad (3.30)$$

(see Titchmarsh [22]), we can write (3.29) in the form

$$p_1(L_{vc}) = \Gamma(n)/(2\Gamma(n-2))(L_{vc})^{(n-3)/2}(1-(L_{vc})^{1/2})(L_{vc})^{1/2} \qquad (3.31)$$

$$0<L_{vc}<1$$

as was derived by Wilks [25].

## 4. THE EXACT NON-NULL DISTRIBUTION OF $L_{vc}$ THROUGH CONTOUR INTEGRATION.

From (3.21) of section 3, we have the distribution of $L_{vc}$ in the form

$$p(L_{vc}) = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty}(L_{vc})^{-(h+1)} \qquad (4.1)$$

$$p_2^{p_2h}\prod_{i=1}^{p_2}(h-(i-1)/2)_{k_i}\prod_{i=1}^{p_2}\Gamma(h+(n-i)/2)/\Gamma(p_2(h+n/2)+k+j)$$

For simplification, make use of the transformation $h+n/2 \to h$. Then (4.1) can be written as

$$p(L_{vc}) = C(p_2,n,\underset{\sim}{\Sigma})\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})(L_{vc})^{n/2-1}p_2^{-np_2/2} \qquad (4.2)$$

$$(2\pi i)^{-1}\int_{C+n/2-i\infty}^{C+n/2+i\infty}(L_{vc})^{-h}\, p_2^{p_2h}\prod_{i=1}^{p_2}(h-(n+i-1)/2)_{k_i}\prod_{i=1}^{p_2}\Gamma(h-i/2)/\Gamma(p_2h+k+j)dh$$

where

$$C^{-1}(p_2,n,\underset{\sim}{\Sigma}) = \prod_{i=1}^{p_2}\Gamma((n-i)/2)|\Sigma_{22}|^{n/2} \qquad (4.3)$$

$$A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) = (n/2)_J\ (np_2/2+k+j)C_\kappa(\underset{\sim}{I}-\Sigma_{2.1}^{-1})C_J(\underset{\sim}{I}-\Sigma_{2.1}^{-1}+\Sigma_{1.2}^{-1}\beta'\beta)/k!j!$$

Let

$$L_1 = L_{vc}/p_2^{p_2} \tag{4.4}$$

Then (4.2) can be written as

$$p(L_{vc}) = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})(L_{vc})^{n/2-1} \quad p_2^{-np_2/2} f_{j,k}(L_{vc}) \tag{4.5}$$

$$f_{j,k}(L_{vc}) = (2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} G_{j,k}(h)\, dh, \quad C_1 = C+n/2 \tag{4.6}$$

$$G_{j,k}(h) = (L_1)^{-h} \prod_{i=1}^{p_2} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{p_2} \Gamma(h-i/2)/\Gamma(p_2h+k+j) \tag{4.7}$$

Throughout the rest of this paper, functions $f_{j,k}(\cdot)$ and $G_{j,k}(\cdot)$ will be written as $f$ and $G$ respectively. We now start with a special case $\underline{p_2=2}$. We have from (4.7)

$$G(h) = (L_1)^{-h} \prod_{i=1}^{2} (h-(n+i-1)/2)_{k_i} \Gamma(h-1/2)\Gamma(h-1)/\Gamma(2h+k+j) \tag{4.8}$$

Using the duplication formula for gamma function in (4.8), we obtain

$$G(h) = (L_{vc})^{-h}\, D \prod_{i=1}^{2} (h-(n+i-1)/2)_{k_i} \; \Gamma(2h-2)/\Gamma(2h+k+j) \tag{4.9}$$

where $D=8(\pi)^{1/2}$. The integral in (4.6) will be evaluated by contour integration. The poles of the integrand (4.9) are at the points

$$h = -\ell/2, \quad \ell=-2,-1,0,1,2,3,\ldots\ \ldots \tag{4.10}$$

The residue at these poles can be found by putting $h=t-\ell/2$ in the integrand (4.9) and taking the residue of the integrand at $t=0$. Substituting $h=t-\ell/2$ in (4.9), we obtain

$$G(t-\ell/2) = (L_{vc})^{-t+\ell/2}\, D \prod_{i=1}^{2} (t-(\ell+n+i-1)/2)_{k_i}\, \Gamma(2t-\ell-2)/\Gamma(2t-\ell+k+j) \quad (4.11)$$

To evaluate the integral (4.6), we need to consider separately the cases (A) $\ell \geq 0$ and (B) $\ell < 0$.

Case A: Let $c=k+j-\ell$. We consider two subcases (A1) $c\leq 0$ and (A2) $c>0$.

Subcase A1: $\ell\geq 0$ and $c\leq 0$. In this case, the integrand (4.11), after expanding the gamma functions can be written as

$$G(t-\ell/2) = (L_{vc})^{-t+\ell/2}\, D \prod_{i=1}^{2} (t-(\ell+n+i-1)/2)_{k_i} \prod_{\delta=1}^{-c} (2t-\delta) \Big/ \prod_{i=1}^{\ell+2} (2t-i), \quad (4.12)$$

$$\ell \geq k+j$$

The integrand (4.12) does not have any pole at $t=0$. Therefore integral (4.6) will be 0 for $\ell\geq k+j$.

Subcase A2: $\ell\geq 0$ and $c>0$. In this case after expanding the gamma functions (4.11) can be written as

$$G(t-\ell/2) = (L_{vc})^{-t+\ell/2}(D/2) \prod_{i=1}^{2} (t-(\ell+n+i-1)/2)_{k_i}\, \Gamma(2t+1)(-1)^{\ell} \Big/ \quad (4.13)$$

$$(t\, \Gamma(2t+c) \prod_{i=1}^{\ell+2} (i-2t)) , \quad \ell=0,1,2,\ldots,k+j-1 \; .$$

The integrand in (4.13) has a simple pole of first order at $t=0$ and the residue at this point is given by

$$R_\ell = \lim_{t \to 0} t\, G_{j,k}(t-\ell/2) \tag{4.14}$$

$$R_\ell = (-1)^\ell (D/2)(L_{vc})^{\ell/2} \prod_{i=1}^{2} (-(\ell+n+i-1)/2)_{k_i} / (\Gamma(k+j-\ell)(\ell+2)!) \;,$$
$$\ell=0,1,2,\ldots,k+j-1 \;.$$

Case B. $\ell<0$. Here $\ell=-2,-1$ and the integrands are

$$G(t+1) = (L_{vc})^{-t-1}(1/t)(D/2)\prod_{i=1}^{2}(t+1-(n+i-1)/2)_{k_i}\Gamma(2t+1)/\Gamma(2t+k+j+2) \tag{4.15}$$

and

$$G(t+1/2) = (L_{vc})^{-t-1/2}(-1/t)(D/2)\prod_{i=1}^{2}(t+1-(n+i)/2)_{k_i}\Gamma(2t+1)/\Gamma(2t+k+j+1) \cdot (1-2t)^{-1} \tag{4.16}$$

Thus for $\ell=-1$ and $\ell=-2$, we have a simple pole of first order at $t=0$, and the residue at these poles are given by

$$R_{-2} = (L_{vc})^{-1}(D/2) \prod_{i=1}^{2}((3-n-i)/2)_{k_i}/\Gamma(2+k+j) \tag{4.17}$$

and

$$R_{-1} = (L_{vc})^{-1/2}(D/2)\prod_{i=1}^{2}(1-(n+i)/2)_{k_i}/\Gamma(1+k+j) \tag{4.18}$$

Hence finally from (4.14), (4.17) and (4.18)and using Cauchy's residue theorem the integral (4.6) for this case is given by

$$f_{j,k}(L_{vc}) = R_{-1}+R_{-2}+\sum_{\ell=0}^{k+j+1} R_\ell \;.$$

$$f_{j,k}(L_{vc}) = 4\pi^{1/2}[(L_{vc})^{-1} \prod_{i=1}^{2} ((3-n-i)/2)_{k_i}/\Gamma(2+k+j)+(L_{vc})^{-1/2}(-1)$$
$$\prod_{i=1}^{2} (1-(n+i)/2)_{k_i}/\Gamma(1+k+j)+ \sum_{\ell=0}^{k+j-1} (-1)^{\ell}(L_{vc})^{\ell/2} \prod_{i=1}^{2} (-(\ell+n+i-1)/2)_{k_i}$$
$$/((\ell+2)!\Gamma(k+j-\ell))] \tag{4.19}$$

Hence from (4.5) and using (4.19), the non-null density of $L_{vc}$ for $p_2=2$ is given by

$$p(L_{vc}) = C(p_2,n,\Sigma)(L_{vc})^{n/2-1}p_2^{-np_2/2} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma)f_{j,k}(L_{vc}) \tag{4.20}$$

where $f_{j,k}(L_{vc})$ is as in (4.19).

This form of the density is useful for power computations and power computed from (4.20) are given in Table (1.2). The null density of $L_{vc}$ from (4.20) reduces to that given in (3.31).

Now for finding the density of $L_{vc}$ for $p_2 \geq 3$, we still use the method of contour integration but the density will involve psi functions and their derivatives. We will make use of the following lemma due to Nair [15] in this connection.

Lemma 4.1. Let $(a_i)$ be a sequence of numbers, finite or infinite and let

$$F(x;t,a_2,a_3,\ldots) = \exp(xt+a_2t^2/2!+a_3t^3/3!+\ldots) \tag{4.21}$$

Then the n-th derivative of $F(x;t,a_2,a_3,\ldots)$ at $t=0$ is

$$D_n(x,a) = \begin{vmatrix} x & -1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & x & -1 & 0 & 0 & \cdots & 0 \\ a_3 & 2a_2 & x & -1 & 0 & \cdots & 0 \\ a_4 & 3a_3 & 3a_2 & x & -1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & \binom{n-1}{1}a_{n-1} & \binom{n-1}{2}a_{n-2} & \cdots & \cdots & \cdots & x \end{vmatrix} \qquad (4.22)$$

Now we proceed to derive the densities of $L_{vc}$ for the following two cases separately, namely (i) $p_2$ = even and (ii) $p_2$ = odd. We specify here that all the empty products in the following derivation will be interpreted as unity and all empty sums will be regarded as 0.

Case (i): $p_2=2r,\ r\geq 1$. Now starting with (4.7) with $p_2=2r$, we have the integrand given by

$$G(h) = (L_1)^{-h} \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{2r} \Gamma(h-i/2)/\Gamma(2rh+k+j) \qquad (4.23)$$

Using duplication formula of gamma function, (4.23) can be written as

$$G(h) = (L_1)^{-h}\, 2^{-2rh}\, D \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{r} \Gamma(2h-2i)/\Gamma(p_2h+k+j) \qquad (4.24)$$

where $D=\pi^{r/2}\, 2^{r(r+2)}$. Let $L=L_1 2^{2r}=L_{vc}\, 2^{2r} p_2^{-p_2}$

$$G(h) = L^{-h}\, D \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{r} \Gamma(2h-2i)/\Gamma(p_2h+k+j) \qquad (4.25)$$

The poles of the integrand $G(h)$ are at the points

$$h=-\ell/2,\ \ell=-2r,-2r+1,\ldots,-2,-1,0,1,2,\ldots,r\geq 1 \qquad (4.26)$$

and the residue at these points is equal to the residue of $G(t-\ell/2)$ at $t=0$. Now (4.25) can be written as

$$G(t-\ell/2) = L^{\ell/2} D \prod_{i=1}^{p_2} (t-(\ell+n+i-1)/2)_{k_i} L^{-t} GP(t), \tag{4.27}$$

where

$$GP(t) = \prod_{i=1}^{r} \Gamma(2t-\ell-2i)/\Gamma(p_2 t+C), \quad \ell=-2r,-2r+1,\ldots 0,1,2,\ldots \tag{4.28}$$

and $C=k+j-r\ell$. Three cases arise: (A) $\ell\geq 0$, (B) $\ell<0$, $\ell$=even, and (C) $\ell<0$, $\ell$=odd.

<u>Case A</u>: <u>$\ell\geq 0$</u>. Two subcases: (A1) $C\leq 0$ and (A2) $C>0$ .

<u>Subcase A1</u>: <u>$\ell\geq 0$ and $C\leq 0$</u> i.e., $k+j\leq r\ell$.

Expanding the gamma functions in (4.28), we have

$$GP(t) = (\Gamma(2t+1))^r \prod_{i=1}^{-C} (p_2 t-i)t^{-(r-1)} p_2/(\Gamma(p_2 t+1) \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (2t-\delta)2^r) \tag{4.29}$$

Thus for $\ell\geq 0$, and $k+j\leq r\ell$, the pole of $G(t-\ell/2)$ is of order $r-1$.

Rewriting (4.29), we have

$$GP(t) = (-1)^{k+j} p_2 t^{-(r-1)} (\Gamma(2r+1))^r (-C)! \prod_{i=1}^{-C} (1-p_2 t/i)/(\Gamma(p_2 t+1) \prod_{i=1}^{r} (\ell+2i)! \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (1-2t/\delta)2^r) \tag{4.30}$$

Hence from (4.27), we have

$$G(t-\ell/2) = L^{\ell/2} D\, p_2(-1)^{k+j}(-C)! \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i} /(2^r \prod_{i=1}^{r} (\ell+2i)) \; t^{-(r-1)} (L)^{-t} M \tag{4.31}$$

where

$$M = \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r \prod_{i=1}^{-C}(1-p_2t/i)/ \left(\Gamma(p_2t+1)\prod_{i=1}^{r}\prod_{\delta=1}^{\ell+2i}(1-2t/\delta)\right)$$

This can be written as

$$G(t-\ell/2) = L^{\ell/2}\, D\, p_2\, a_0\, 2^{-r}\, t^{-(r-1)} \exp(\log A(t)) \tag{4.32}$$

where

$$a_0 = (-1)^{k+j} \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i}\ (-C)!/\prod_{i=1}^{r}(\ell+2i) \tag{4.33}$$

$$A(t) = L^{-t} \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r \prod_{i=1}^{-C}(1-p_2t/i)/(\Gamma(p_2t+1)\prod_{i=1}^{r}\prod_{\delta=1}^{\ell+2i}(1-2t/\delta)) \tag{4.34}$$

Now the residue at the pole $t=0$ of order $r-1$ is given by

$$R_\ell = L^{\ell/2}\, D\, p_2\, a_0/(2^r\Gamma(r-1))\left(\frac{d}{dt}\right)^{r-2}_{t=0} \exp(\log A(t)) \tag{4.35}$$

Using the formulae (see Erdelyi, [5])

$$\log\Gamma(x+a) = \log\Gamma(a) + x\psi(a) + x^2\psi_1(a)/2! + x^3\psi_2(a)/3! + \ldots \tag{4.36}$$

where

$$\psi(a) = \frac{d}{dx}\log\Gamma(x)|_{x=a} \quad \text{and} \quad \psi_j(a) = \left(\frac{d}{dx}\right)^j \psi(x)|_{x=a} \tag{4.37}$$

and

$$\log(1+z) = \sum_{n=0}^{\infty}(-1)^n z^{n+1}/(n+1) \quad \text{for} \quad |z|<1 . \tag{4.38}$$

log A(t) can be written as

$$\log A(t) = a_1 t + a_2 t^2/2! + a_3 t^3/3! + \ldots \tag{4.39}$$

where

$$a_1 = -\log L + (2r-p_2)\psi(1) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i+n+\ell-1)/2) - \sum_{i=1}^{-C} (p_2/i) + \sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta) \tag{4.40}$$

and for $s \geq 2$, we have

$$a_s = (r2^s - p_2^s)\,\psi_{s-1}(1) + (s-1)!\left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}/(\alpha-(i+n+\ell-1)/2)^s - \sum_{i=1}^{-C} (p_2/i)^s + \sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)^s\right]$$

Using (4.39) in (4.35) and lemma (4.1), we get

$$R_\ell = L^{\ell/2}\, D\, p_2\, a_0/(2^r \Gamma(r-1)) D_{r-2}\,(L;a), \tag{4.41}$$

where

$$D_{r-2}(L;a) = \begin{vmatrix} a_1 & -1 & 0 & \ldots & 0 \\ a_2 & a_1 & -1 & \ldots & 0 \\ a_3 & 2a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ a_{r-2} & \binom{r-3}{1}a_{r-3} & \binom{r-4}{2}a_{r-4} & \ldots & a_1 \end{vmatrix} \tag{4.42}$$

where a's are defined in (4.40).

Subcase A2: $\ell \geq 0$ and $C > 0$ i.e., $k+j > r\ell$.

Expanding the gamma function in (4.28), we get

$$GP(t) = (\Gamma(2t+1))^r (2t)^{-r} / (\Gamma(p_2 t+C) \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (2t-\delta)) \qquad (4.43)$$

Thus for $C>0$, we have a pole of order $r$ at $t=0$ and from (4.27) and (4.43), we have

$$G(t-\ell/2) = L^{\ell/2}\, D\, b_0'\, 2^{-r}\, t^{-r} \exp(\log F(t)) \qquad (4.44)$$

where

$$b_0' = (-1)^{r\ell} \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i} / \prod_{i=1}^{r} (\ell+2i)! \qquad (4.45)$$

and

$$F(t) = L^{-t} \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r / [\Gamma(p_2 t+C) \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (1-2t/\delta)] \qquad (4.46)$$

The residue at the pole $t=0$ is given by

$$R_\ell = [L^{\ell/2}\, D\, b_0'\, 2^{-r}/\Gamma(r)] \left(\frac{d}{dt}\right)_{t=0}^{r-1} \exp(\log F(t)) \qquad (4.47)$$

Using (4.36), (4.37) and (4.38), $\log F(t)$ can be written as

$$\log F(t) = b_0'' + b_1 t + b_2 t^2/2! + b_3 t^3/3! + \ldots \qquad (4.48)$$

Using (4.48) in (4.47) and lemma (4.1), we obtain

$$R_\ell = L^{\ell/2}\, D\, 2^{-r}\, b_0/\Gamma(r)\, D_{r-1}(L;b) \text{ for } \ell\geq 0 \text{ s.t. } r\ell<k+j \qquad (4.49)$$

where

$$b_0 = b_0' b_0'' , \quad b_0'' = -\log \Gamma(C) \text{ and } b_0' \text{ is given in (4.45)} \qquad (4.50)$$

$$b_1 = -\log L + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i+n+\ell-1)/2) + 2r\psi(1) - p_2\psi(C)$$

$$+ \sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)$$

and for $s \geq 2$, we have

$$b_s = r\, 2^s\, \psi_{s-1}(1) - p_2^s\, \psi_{s-1}(C) + (s-1)! \left[\sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)^s\right]$$
$$+\left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}(s-1)!/(\alpha-(i-1+n+\ell)/2)^s\right]$$

and the determinant $D_{r-1}(L;b)$ is equal to the determinant on the right hand side of (4.22) with $x$ replaced by $b_1$, $n$ by $r-1$ and $a_s'^s$ by $b_s'^s$; $s=1,2,\ldots,r-1$.

Case B: $\ell<0$ and $\ell=-2u$, $u=1,2,\ldots,r$, with $p_2=2r$. For this case we can write (4.28) as

$$GP(t) = \Gamma(2t+2u-2i)/\Gamma(p_2t+C),\ u=1,2,\ldots,r \text{ and } C=k+j-r\ell>0 \quad (4.51)$$

expanding the gamma function in (4.51), we obtain

$$GP(t) = (\Gamma(2t+1))^{r-u+1} \prod_{i=1}^{u-1} \Gamma(2t+2u-2i)(2t)^{-(r-u+1)}/[\Gamma(p_2t+C)$$
$$\prod_{i=u+1}^{r} \prod_{\delta=1}^{2i-2u} (2t-\delta)] \quad (4.52)$$

(All empty products are treated as 1 and empty sums as 0.) It is clear from (4.52) that we have a pole of order $r-u+1$ at $t=0$, $u=1,2,\ldots,r$. It is easy to check that $G(t-\ell/2)$ can be written as

$$G(t+u) = L^{-u}\, D\, C_0'\, (2t)^{-(r-u+1)}\, \exp(\log H(t)) \qquad (4.53)$$

where after using (4.36), (4.37) and (4.38), $\log H(t)$ can be written as

$$\log H(t) = C_0'' + C_1 t + C_2 t^2/2! + C_3 t^3/3! + \ldots \qquad (4.54)$$

Now using (4.54) in (4.53) and appealing to the lemma (4.1), we get the residue as

$$R_u = L^{-u}\, D\, C_0\, 2^{-(r-u+1)}/\Gamma(r-u+1)\, D_{r-u}(L;C), \qquad (4.55)$$
$$u=1,2,\ldots,r;\ r\geq 1$$

where the determinant $D_{r-u}(L;C)$ can be obtained from the right hand side of (4.22) with $x$ replaced by $C_1$, $n$ by $r-u$ and $a_s'^{s}$ by $C_s'^{s}$, $s=1,2,\ldots,r-u$. The coefficients $C_s'$s are given by

$$C_0' = \prod_{i=1}^{p_2} (u-(n+i-1)/2)_{k_i} / \prod_{i=u+1}^{r} (2i-2u)!$$
$$(4.56)$$
$$C_0'' = \prod_{i=1}^{u-1} \Gamma(2u-2i)/\Gamma(C), \quad C_0 = C_0' C_0''$$

$$C_1 = -\log L + 2(r-u+1)\psi(1) - p_2\psi(C) + 2\sum_{i=1}^{u-1} \psi(2u-2i) + \sum_{i=u+1}^{r} \sum_{\delta=1}^{2i-2u} (2/\delta) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n)/2+u)$$

and for $s\geq 2$

$$C_s = (r-u+1)2^s\, \psi_{s-1}(1) - p_2^s\, \psi_{s-1}(C) + \sum_{i=1}^{u-1} 2^s\, \psi_{s-1}(2u-2i) + (s-1)! \left[\sum_{i=u+1}^{r} \sum_{\delta=1}^{2i-2u} (2/\delta)^s + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}/(\alpha-(i-1+n)/2+u)^s\right]$$

Case C: $\ell<0$ and $\ell=-2v+1$, $v=1,2,\ldots,r$. Now (4.28) can be written as

$$GP(t) = \prod_{i=1}^{r} \Gamma(2t+2v-1-2i)/\Gamma(p_2 t+C), \quad v=1,2,\ldots,r \tag{4.57}$$

After the expansion of gamma functions, one obtains

$$GP(t) = (\Gamma(2t+1))^{r-v+1}(2t)^{-(r-v+1)} \prod_{i=1}^{v-1} \Gamma(2t+2u-2i-1)/(\Gamma(p_2 t+C) \prod_{i=v}^{r} \prod_{\delta=1}^{1+2i-2v} (2t-\delta)) \tag{4.58}$$

Thus, here we have a pole of order $r-v+1$ at $t=0$, $v=1,2,\ldots,r$. Proceeding as before, we have $G(t-\ell/2)$ in the form

$$G(t+v-1/2) = (L)^{-v+1/2}\, D\, d_0' \, (2t)^{-(r-v+1)} \exp(\log I(t)) , \tag{4.59}$$

where

$$\log I(t) = d_0''+d_1 t+d_2 t^2/2!+d_3 t^3/3!+\ldots.. \tag{4.60}$$

Using (4.60) in (4.59) and applying lemma (4.1), we have the residue $R_v$ given by

$$R_v = (L)^{-v+1/2}\, D\, d_0\, 2^{-(r-v+1)} D_{r-v}(L;d)/\Gamma(r-v+1), \quad v=1,2,\ldots,r \tag{4.61}$$

where $D_{r-v}(L;d)$ is equal to the determinant on the right hand side of (4.22) with $x$ replaced by $d_1$, $n$ by $r-v$ and $a_q'$s by $d_q'$s, $q=1,2,\ldots,r-v$. The coefficients $d_q'$s are given by

$$d_0' = (-1)^{r-v+1} \prod_{i=1}^{p_2} (v-(n+i)/2)_{k_i} / \prod_{i=v}^{r} (1+2i-2v)!$$

$$d_0'' = \prod_{i=1}^{v-1} \Gamma(2v-2i-1)/\Gamma(C) \quad \text{and} \quad d_0 = d_0' d_0''$$

$$d_1 = -\log L + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(\ell+n+i-1)/2) + 2\sum_{i=1}^{v-1} \psi(2v-2i-1) - p_2\psi(C)$$

$$+ 2(r-v+1)\psi(1) + \sum_{i=v}^{r} \sum_{\delta=1}^{1+2i-2v} (2/\delta)$$

and for $s \geq 2$, we have

$$d_s = \sum_{i=1}^{v-1} 2^s \psi_{s-1}(2v-2i-1) - p_2^s \psi_{s-1}(C) + (r-v+1)2^s\psi_{s-1}(1)$$

$$+ (s-1)! \left[ \sum_{i=v}^{r} \sum_{\delta=1}^{1+2i-2v} (2/\delta)^s + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}/(\alpha+v-(n+i)/2)^s \right]$$

Hence, for the case $p_2$=even, we have from (4.5) and Cauchy's residue theorem, the non-null density of $L_{vc}$ in the form

$$p(L_{vc}) = C(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma)(L_{vc})^{n/2-1} p_2^{-np_2/2} \tag{4.62}$$

$$\left[ \sum_{\substack{\ell \geq 0 \\ r\ell \geq k+j}} R_\ell + \sum_{\substack{\ell \geq 0 \\ r\ell < k+j}} R_\ell + \sum_{u=1}^{r} R_u + \sum_{v=1}^{r} R_v \right]$$

where $R_\ell$, $R_u$, $R_v$ are given in (4.41), (4.49), (4.55) and (4.61) respectively. In particular, if we put $p_2=2$ in (4.62), we get (4.20).

Case (ii): $p_2=2s+1$, $s \geq 0$ ($s=0$, covers the case $p_2=1$). Once again in the following discussion, all empty products will be interpreted as unity and empty sums as 0. The functions $f_{j,k}$, $G_{j,k}$, $GP_{j,k}$ and $R_{j,k}$ will be written as f, G, GP and R respectively. Now starting with (4.7) and using the duplication formula for gamma functions, we have

$$G(h) = L^{-h} D \prod_{i=1}^{p_2} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{s} \Gamma(2h-2i)\Gamma(h-s-1/2)/\Gamma(p_2h+k+j) \tag{4.63}$$

where

$$D=\pi^{s/2}\, 2^{s(s+2)} \qquad \text{and} \qquad L=L_{vc}\, 2^{2s}/p_2^{p_2h} \tag{4.64}$$

The poles of the integrand G(h) are at the points

$$h = -\ell/2\ , \quad \ell = -2s-1,-2s,\ldots,0,1,2,\ldots \tag{4.65}$$

and the residue of G(h) at these points can be obtained by finding the residue of $G(t-\ell/2)$ at $t=0$. Now

$$G(t-\ell/2)= L^{\ell/2} D \prod_{i=1}^{p_2} (t-(n+\ell+i-1)/2)_{k_i} L^{-t} GP(t), \tag{4.66}$$

where

$$GP(t) = \prod_{i=1}^{s} \Gamma(2t-(\ell+2i))\Gamma(t-s-(\ell+1)/2)/\Gamma(p_2t+k+j-p_2\ell/2) \tag{4.67}$$

We have to consider separately the cases (A) $\ell\geq0$, $\ell$=even, (B) $\ell\geq0$, $\ell$=odd, (C) $\ell<0$, $\ell$=even and (D) $\ell<0$, $\ell$=odd. Let $d=k+j-p_2\ell/2$. Now (4.66) can be written as

$$G(t-\ell/2) = L^{\ell/2}\, D \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2) \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(n+\ell+i-1)/2))GP(t) \tag{4.68}$$

Case A: Two subcases arise (A1) $d\le 0$ and (A2) $d>0$.

Subcase A1: $\ell\ge 0$, $\ell=2u_2$, $u_2=0,1,2,\ldots,d\le 0$.

After expanding the gamma functions in (4.67), we have

$$GP(t) = (\Gamma(2t+1))^s\, \Gamma(t+1/2)\, p_2\, 2^{-s}\, t^{-(s-1)} \prod_{\delta=1}^{-d} (p_2t-\delta)/(\Gamma(p_2t+1) \prod_{\delta=0}^{u_2+s} (t-\delta-1/2)) \tag{4.69}$$

So we have a pole of order $(s-1)$ at $t=0$. Proceeding as before, we have

$$G(t-u_2) = L^{u_2}\, D\, p_2\, 2^{-s}\, t^{-(s-1)}\, f_0'\, \exp(\log P(t)) \tag{4.70}$$

and the residue $R_{u_2}$ is given by

$$R_{u_2} = L^{u_2}\, D\, p_2\, 2^{-s}\, f_0\, \mathcal{D}_{s-2}(L;f)/\Gamma(s-1),\ p_2u_2\ge k+j, \quad u_2=0,1,2,\ldots \tag{4.71}$$

where the determinant $D_{s-2}(L;f)$ is same as the one in (4.22) with $n$ replaced by $s-2$, $x$ by $f_1$ and $a_q's$ by $f_q's$, $q=1,2,\ldots,s-2$ and the coefficients $f_q's$ are given by

$$f_0' = (-1)^{k+j+s+1}(-d)! \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} / (\prod_{i=1}^{s} (\ell+2i)! \prod_{\delta=1}^{s+u_2} (\delta+1/2)) \tag{4.72}$$

and

$$f_0 = f_0' \Gamma(1/2)$$

$$f_1 = -\log L+\psi(1/2)+\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (1/(\alpha-(i+n+\ell-1)/2))-\psi(1)-\sum_{\delta=1}^{-d} (p_2/\delta)$$

$$+\sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta)+\sum_{\delta=0}^{u_2+s} 1/(\delta+1/2)$$

and for $q\geq 2$, we have

$$f_q = \psi_{q-1}(1/2)+\psi_{q-1}(1)[s2^q-p_2^q]+(q-1)! \left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q \right.$$

$$\left. -\sum_{\delta=1}^{-d} (p_2/\delta)^q + \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta)^q + \sum_{\delta=0}^{u_2+s} (1/(\delta+1/2))^q \right]$$

Subcase A2: $\ell\geq 0$, $\ell=2u_2$, $d>0$, $u_2=0,1,2,\ldots$

Expanding the gamma product in (4.67), we have

$$GP(t) = (\Gamma(2t+1))^s\, \Gamma(t+1/2)(2t)^{-s}/(\Gamma(p_2t+d) \prod_{i=1}^{s} \prod_{\delta=1}^{\ell+2i} (2t-\delta) \prod_{\delta=0}^{s+u_2} (t-\delta-1/2)) \tag{4.73}$$

In this case, we have a pole of order $s$ at $t=0$. Following the same procedure as earlier, we have

$$G(t-u_2) = L^{u_2}\, D\, g_0' \exp(\log Q(t))/(2t)^s , \tag{4.74}$$

where $\log Q(t)=g_0''+g_1t+g_2t^2/2!+\ldots.$ and the residue $R_{u_2}$ is given by

$$R_{u_2} = (\Gamma(s))^{-1} L^{u_2} D\, 2^{-s} g_0\, D_{s-1}(L;g),\ u_2=0,1,2,\ldots,\ \text{s.t. } p_2u_2<k+j \tag{4.75}$$

where the determinant $D_{s-1}(L;g)$ is similar to the determinant on the right hand side of (4.22) having (s-1) rows and the elements $a_q^{s'}$ replaced by $g_q^{'s}$ and x by $g_1$, where $g_q$'s are

$$g_0' = (-1)^{u_2+s+1} \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} / [\prod_{\delta=1}^{s} (\ell+2i)! \prod_{\delta=0}^{u_2+s} (\delta+1/2)] \tag{4.76}$$

$$g_0'' = \Gamma(1/2)/\Gamma(d) \quad \text{and} \quad g_0 = g_0' g_0''$$

$$g_1 = -\log L + 2s\psi(1) + \psi(1/2) - p_2\psi(d) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\ell-(i+n+\ell-1)/2)$$

$$+ \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta) + \sum_{\delta=1}^{u_2+s} (\delta+1/2)^{-1}$$

and for $q \geq 2$, we have

$$g_q = (q-1)! \left[ \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q + \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta)^q \right.$$
$$\left. + \sum_{\delta=0}^{u_2+s} (\delta+1/2)^{-q} \right] + \psi_{q-1}(1/2) - p_2^q \, \psi_{q-1}(d) + 2^q s \, \psi_{q-1}(1)$$

Case B: $\ell \geq 0$, $\ell = 2v_2+1$, $v_2 \geq 0$. The gamma product in (4.67) can be written as

$$GP(t) = \prod_{i=1}^{s} \Gamma(2t-(\ell+2i)) \, \Gamma(t-v_2-s-1)/\Gamma(p_2 t+d-1/2), \tag{4.77}$$

where

$$d = k+j-s-p_2 v_2 \quad . \tag{4.78}$$

Two subcases arise (B1) $d \le 0$, and (B2) $d > 0$.

Subcase B1: $\ell \ge 0$, $\ell = 2v_2+1$, $v_2 \ge 0$ s.t. $p_2 v_2 \ge k+j-s$ .

Now (4.77) can be written as

$$GP(t) = (\Gamma(2t+1))^s \, \Gamma(t+1) 2^{-s} t^{-(s+1)} \prod_{\delta=0}^{-d} (p_2 t - \delta - 1/2) / (\Gamma(p_2 t + 1/2) \prod_{\delta=1}^{v_2+s+1} (t-\delta) \prod_{i=1}^{s} \prod_{\delta=1}^{\ell+2i} (2t-\delta)) \tag{4.79}$$

So we have a pole of order $s+1$ at $t=0$. As before, we have

$$G(t-v_2-1/2) = L^{v_2+1/2} D \, m_0' \, 2^{-s} \exp(\log R(t))/t^{s+1} \tag{4.80}$$

and using lemma (4.1), the residue $R_{v_2}$ is given by

$$R_{v_2} = D(L)^{v_2+1/2} m_0 \, 2^{-s} D_s(L;m)/\Gamma(s+1), \quad v_2 \ge 0 \text{ s.t. } p_2 v_2 \ge k+j-s \tag{4.81}$$

with $D_s(L;m)$ being the determinant of order $s$ and can be obtained from (4.22) by replacing $x$ by $m_1$ and $a_q'$s by $m_q'$s , where $m_q'$s are given by

$$m_0' = (-1)^{k+j-s} \prod_{i=1}^{p_2} ((-n-\ell-i+1)/2)_{k_i} \prod_{\alpha=0}^{d} (\alpha+1/2) / ((v_2+s+1)! \prod_{i=1}^{s} (\ell+2i)!)$$

$$m_0 = m_0'/\Gamma(1/2) \tag{4.82}$$

$$m_1 = -\log L + p_2\psi(1) - p_2\psi(1/2) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n+\ell)/2) - \sum_{\alpha=0}^{-d} p_2/(\alpha+1/2) + \sum_{\delta=1}^{v_2+s+1} (1/\delta) + \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta)$$

and for $q \geq 2$, we have

$$m_q = \psi_{q-1}(1)[1+s2^q] - p_2^q\ \psi_{q-1}(1/2) + (q-1)!\left[\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/\right.$$

$$\left.(\alpha-(i+n+\ell-1)/2)^q - \sum_{\alpha=0}^{-d}(p_2/(\alpha+1/2))^q + \sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)^q + \sum_{\delta=1}^{v_2+s+1}(1/\delta)^q\right]$$

Subcase B2: $\ell \geq 0$, $\ell = 2v_2+1$, $v_2 \geq 0$, $p_2 v_2 < k+j-s$.

Now (4.77) can be written as

$$GP(t) = \Gamma(t+1)(\Gamma(2t+1))^s\ 2^{-s}/[\Gamma(p_2 t+d-1/2)\prod_{i=1}^{s}\prod_{\delta=1}^{\ell+2i}(2t-\delta)\prod_{\delta=1}^{v_2+s+1}(t-\delta)\ t^{s+1}] \tag{4.83}$$

Here also we have a pole of order $s+1$ at $t=0$, and as earlier using lemma (4.1), we have

$$G(t-v_2-1/2) = L^{v_2+1/2}\ D\ n_0'\ 2^{-s}\ t^{-(s+1)}\ \exp(\log S(t)) \tag{4.84}$$

and

$$R_{v_2} = D\ L^{v_2+1/2}\ n_0\ D_s(L;n)/(2^s\,\Gamma(s+1)),\ v_2 \geq 0 \text{ s.t. } p_2 v_2 < k+j-s \tag{4.85}$$

where the determinant $D_s(L;n)$ is defined similarly as in (4.81) with $m$'s replaced by $n$'s and the coefficients $n$'s are defined as

$$n_0' = (-1)^{-v_2+1} \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} / [\prod_{i=1}^{s} (\ell+2i)!(v_2+s+1)!] ,$$

$$n_0 = n_0'/\Gamma(d-1/2) \tag{4.86}$$

and

$$n_1 = -\log L + p_2\psi(1) - p_2\psi(d-1/2) + \sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n+\ell)/2)$$
$$+ \sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta) + \sum_{\delta=1}^{v_2+s+1}(1/\delta)$$

and for $q \geq 2$

$$n_q = \psi_{q-1}(1)[1+s2^q] - p_2^q\, \psi_{q-1}(d-1/2) + (q-1)!\left[\sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)^q + \sum_{\delta=1}^{v_2+s+1}(1/\delta)^q + \sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/(\alpha-(i-1+n+\ell)/2)^q\right]$$

Case C. $\ell<0$, $\ell=-2u$, $u=1,2,3,\ldots,s$. For this case the gamma product in (4.67) can be expanded as

$$GP(t) = \frac{\prod_{i=1}^{u-1}\Gamma(2t+2u-2i)(\Gamma(2t+1))^{s-u+1}\Gamma(t+1/2)}{(2t)^{s-u+1}\Gamma(p_2t+p_2u+k+j)\prod_{\alpha=0}^{s-u}(t-\alpha-1/2)\prod_{i=u+1}^{s}\prod_{\delta=1}^{2u-2i}(2t-\delta)} \tag{4.87}$$

We have a pole of order $s-u+1$, $u=1,2,\ldots,s$. Proceeding as before, we have

$$G(t+u) = L^{-u}\, D\, y_0'\, \exp(\log V(t))/(2t)^{s-u+1} \tag{4.88}$$

and the residue $R_u$ is given by

$$R_u = L^{-u}\, D\, y_0\, D_{s-u}(L;y)/(2^{s-u+1}\Gamma(s-u+1)) \;, u=1,2,\ldots,s \quad (4.89)$$

where the determinant $D_{s-u}(L;y)$ is equal to the R.H.S. of (4.22) with $s-u$ rows and $x$ replaced by $y_1$ and $a_q'$s by $y_q'$s, $q=1,2,\ldots,s-u$, and the coefficients $y_q'$s are given by

$$y_0' = \prod_{i=1}^{p_2}(-(n+\ell+i-1)/2)_{k_i}\;(-1)^{s-u+1}/(\prod_{\alpha=0}^{s-u}(\alpha+1/2)\prod_{i=u+1}^{s}(2u-2i)!) \quad (4.90)$$

$$y_0 = y_0' \prod_{i=1}^{u-1}\Gamma(2u-2i)\Gamma(1/2)/\Gamma(p_2u+k+j)$$

$$y_1 = -\log L+2\sum_{i=1}^{u-1}\psi(2u-2i)+2(s-u+1)\psi(1)+\psi(1/2)-p_2\psi(p_2u+k+j)$$

$$+\sum_{\alpha=0}^{s-u}(\alpha+1/2)^{-1}+\sum_{i=u+1}^{s}\sum_{\delta=1}^{2u-2i}(2/\delta)+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}1/(\alpha-(i+n+\ell-1)/2)$$

and for $q\geq 2$, we have

$$y_q = \sum_{i=1}^{u-1}2^q\,\psi_{q-1}(2u-2i)+(s-u+1)\,2^q\psi_{q-1}(1)+\psi_{q-1}(1/2)-p_2^q\,\psi_{q-1}(p_2u+k+j)$$

$$+(q-1)!\left[\sum_{\alpha=0}^{s-u}(\alpha+1/2)^{-q}+\sum_{i=u+1}^{s}\sum_{\delta=1}^{2u-2i}(2/\delta)^q+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/(\alpha-(n+\ell+i-1)/2)^q\right]$$

Case D: $\ell<0$, $\ell=-2v+1$, $v=1,2,\ldots,s,\ s+1$. The gamma product in (4.67) can be written as

$$GP(t) = \frac{2\Gamma(t+1)(\Gamma(2t+1))^{s-v+1}\prod_{i=1}^{v-1}\Gamma(2t+2v-2i-1)}{(2t)^{s-v+2}\Gamma(p_2t+k+j+p_2(v-1/2))\prod_{\delta=1}^{s+1-v}(t-\delta)\prod_{i=v}^{s}\prod_{\delta=1}^{1+2i-2v}(2t-\delta)} \quad (4.91)$$

So here we have a pole of order $s-v+2$ at $t=0$, $v=1,2,\ldots,s+1$, and as earlier, we have

$$G(t+v-1/2) = 2\, L^{-v+1/2}\, D\, z_0'\, \exp(\log W(t))/(2t)^{s-v+2} \qquad (4.92)$$

and using lemma (4.1), the residue $R_v$ is given by

$$R_v = D\, L^{-v+1/2}\, z_0\, D_{s-v+1}(L;z)/(2^{s-v+1}\Gamma(s-v+2)),\ v=1,2,\ldots,s+1 \qquad (4.93)$$

where the determinant $D_{s-v+1}(L;z)$ can be obtained from (4.22) by replacing $n$ by $s-v+1$, $x$ by $z_1$ and $a_q'$s by $z_q'$s , where the $z_q'$s are given by

$$z_0' = \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i} / ((s-v+1)! \prod_{i=v}^{s} (1+2i-2v)!)$$

$$z_0 = z_0' \prod_{i=1}^{v-1} \Gamma(2v-2i-1)/\Gamma(k+j-p_2\ell/2) \qquad (4.94)$$

$$z_1 = -\log L+(2(s-v+1)+1)\psi(1)+2\sum_{i=1}^{v-1} \psi(2v-2i-1)-p_2\psi(k+j-p_2\ell/2)$$
$$+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n+\ell)/2) + \sum_{\delta=1}^{s+v-1} 1/\delta + \sum_{i=v}^{s}\sum_{\delta=1}^{1+2i-2v} (2/\delta)$$

and for $q\geq 2$, we have

$$z_q = [1+2^q(s-v+1)]\psi_{q-1}(1)+2^q\sum_{i=1}^{v-1}\psi_{q-1}(2v-2i-1)-p_2^q\psi_{q-1}(k+j-p_2\ell/2)$$
$$+(q-1)!\left[\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q + \sum_{\delta=1}^{s+v-1}(1/\delta)^q + \sum_{i=v}^{s}\sum_{\delta=1}^{1+2i-2v}(2/\delta)^q\right]$$

Hence, when $p$ is odd, the density of $L_{vc}$ is given by

$$p(L_{vc}) = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) \, (L_{vc})^{n/2-1} \, p_2^{-np_2/2}$$

$$\left[ \sum_{\substack{u_2=0 \\ p_2u_2 \geq k+j}}^{\infty} R_{u_2} + \sum_{\substack{u_2=0 \\ p_2u_2 < k+j}}^{\infty} R_{u_2} + \sum_{\substack{v_2=0 \\ p_2v_2 \geq k+j-s}}^{\infty} R_{v_2} + \sum_{\substack{v_2=0 \\ p_2v_2 < k+j-s}}^{\infty} R_{v_2} + \sum_{u=1}^{s} R_u + \sum_{v=1}^{s+1} R_v \right], \tag{4.95}$$

where $R_{u_2}$, $R_{v_2}$, $R_u$, $R_v$ are given in (4.71), (4.75), (4.81), (4.85), (4.89) and (4.93) respectively.

Remark. Putting $\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \underset{\sim}{I}_{p_2} \end{bmatrix}$ in (4.62) and (4.95), we can deduce the results of Nagarsenker [17] and Wilks [25].

## 5. DISTRIBUTION OF $L_{vc}$ AS A CHI-SQUARE SERIES.

In this section we express the density of $L_{vc}$ as a chi-square series using methods similar to those of Box [2].

Let $\lambda=(L_{vc})^{n/2}$ and $\lambda^*=-2\rho\log\lambda$ where $\rho$ is chosen so that the rate of convergence of the resulting series can be controlled, $0\leq\rho\leq1$. Let $\phi(t)$ be the characteristic function of $\lambda^*$. Then

$$\phi(t) = E(L_{vc})^{-it\rho n} \tag{5.1}$$

In section 3, we obtained the non-null moments $E[L_{vc}]^h$ for integral values of $h$. But the result (3.20) can be extended to any complex number $h$ by analytic continuation. So we have for any complex number $h$

$$E[L_{vc}]^h = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma})\, p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma((n-i)/2+h) \prod_{i=1}^{p_2} (h-(i-1)/2)_{k_i} / \Gamma(p_2(h+n/2)+k+j) \tag{5.2}$$

Now using (5.2) in (5.1), we obtain

$$\phi(t) = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(j,\kappa,p_2,n,\underset{\sim}{\Sigma})\, p_2^{-it\rho np_2} \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)-\delta)/2) \prod_{\delta=1}^{p_2} ((1-\delta-2it\rho n)/2)_{k_\delta} / \Gamma(np_2(1-2it\rho)/2+k+j) \tag{5.3}$$

For $t=0$, we have $\phi(t)=1$ using $\underset{\sim}{\Sigma}_{22}^{-1} = \underset{\sim}{\Sigma}_{2.1}^{-1} - \underset{\sim}{\Sigma}_{1.2}^{-1}\, \underset{\sim}{\beta}'\underset{\sim}{\beta}$ and for $t \neq 0$ (5.3) can be written as

$$\phi(t) = C(p_2,n,\underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) \exp(\log G(t)) \tag{5.4}$$

where $G_{j,k}(t)$ is denoted by $G(t)$ and is given by

$$G(t) = \frac{p_2^{-it\rho np_2} \prod_{\delta=1}^{p_2} \Gamma(n(1-2it\rho)-\delta)/2) \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)+1-\delta-n)/2+k_\delta)}{\Gamma(np_2(1-2it\rho)/2+k+j) \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)+1-\delta-n)/2)} \tag{5.5}$$

In the following derivation, functions G, W, w, R, all depend upon j and k; for simplicity of notation the subscripts or the superscripts j,k will not be explicitly given unless necessary. From (5.5) taking logarithm on both sides, we get

$$\log G(t) = -it\rho np_2 \log p_2 + \sum_{\delta=1}^{p_2} \log \Gamma(n(1-2it\rho)-\delta)/2) \tag{5.6}$$

$$- \log \Gamma(np_2(1-2it\rho)/2+k+j) + \sum_{\delta=1}^{p_2} \log \Gamma((n(1-2it\rho)+1-\delta-n)/2+k_\delta)$$

$$- \sum_{\delta=1}^{p_2} \log \Gamma(n(1-2it\rho)+1-\delta-n)/2)$$

We now need the following expansion for gamma function (see Anderson [1]).

$$\log \Gamma(x+h) = \log \sqrt{2\pi} + (x+h-1/2)\log x - x - \sum_{r=1}^{m} (-1)^r B_{r+1}(h)/(r(r+1)x^r) + R_{m+1}(x) \tag{5.7}$$

Where $R_{m+1}$ is the remainder such that $|R_{m+1}(x)| \leq A/|x^{m+1}|$, $A$ is a constant independent of $x$ and $B_r(h)$ is the Bernoulli polynomial of degree $r$ and order unity defined by

$$\frac{\zeta e^{h\zeta}}{e^{\zeta}-1} = \sum_{r=0}^{\infty} \zeta^r B_r(h)/r!$$

where the polynomials are given by

$$B_0(h) = 1,\ B_1(h) = h-1/2,\ B_2(h) = h^2-h+1/6,\ \ B_3(h) = h^3-3h^2/2+h/2$$

and in general we have

$$B_r(h) = \sum_{\ell=0}^{r} \binom{r}{\ell} B_\ell\, h^{r-\ell} \quad ,$$

where $B_\ell$ are the Bernoulli numbers and $\binom{r}{\ell} = r!/((r-\ell)!\ell!)$.

Now using (5.7) in (5.6), we obtain

$$\log G(t) = (p-1)/2 \log 2\pi - ((np_2-1)/2+k+j)\log p_2$$
$$- ((p_2-1)/2+p_2(p_2+1)/4+j)\log(n(1-2it\rho)/2) \qquad (5.8)$$
$$+ \sum_{r=1}^{m} (n(1-2it\rho)/2)^{-r} w_r + R^0_{m+1}(n,t) \quad ,$$

where the coefficients $w_r$ are given by

$$w_r = \left[ \sum_{\delta=1}^{p_2} [B_{r+1}(1-\delta-n)/2) - B_{r+1}((1-\delta-n)/2+k_\delta)] + B_{r+1}(k+j)/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(-\delta/2) \right] (-1)^r/(r(r+1))$$

Therefore $G(t)$ can be written as

$$G(t) = (2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2-(k+j)} (n(1-2it\rho)/2)^{-u}$$
$$\sum_{r=0}^{\infty} W_r((1-2it\rho)n/2)^{-r} + R'_{m+1}(n,t) \quad , \qquad (5.9)$$

where $W_r$ is the coefficient of $((1-2it\rho)n/2)^{-r}$ in the expansion of $\exp(\sum_{r=1}^{m}((1-2it)n/2)^{-r}w_r)$ and

$u=(p_2-1)/2+p_2(p_2+1)/4+j$. Then (5.9) can be put in the form

$$G(t) = (2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2-(k+j)} \sum_{r=0}^{\infty} W_r((1-2it\rho)n/2)^{-(r+u)} \qquad (5.10)$$
$$+ R'_{m+1}(n,t)$$

Hence the characteristic function of $\lambda^*$ is given by

$$\phi(t) = C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J, \kappa, p_2, n, \Sigma) p_2^{-(k+j)} \tag{5.11}$$

$$\sum_{r=0}^{\infty} (W_r((1-2it\rho)n/2)^{-(r+u)} + R''_{m+1}(n,t)$$

where

$$C_1(p_2,n,\Sigma) = C(p_2,n,\Sigma)(2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2} .$$

Since $(1-i\beta t)^{-\alpha}$ is the characteristic function of gamma density $g_\alpha(\beta,x)$ where

$$g_\alpha(\beta,x) = [\beta^\alpha \Gamma(\alpha)]^{-1} x^{\alpha-1} e^{-x/\beta} \tag{5.12}$$

The density of $\lambda^*$ can be derived from (5.11) in the form

$$p(\lambda^*) = C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J, \kappa, p_2, n, \Sigma) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (2/n)^{r+u} W_r \, g_{r+u}(2\rho, \lambda^*) + R'''_{m+1}(n) \tag{5.13}$$

Hence the probability that $\lambda^*$ is larger than any value, say $\lambda_0$ is

$$P[\lambda^*>\lambda_0] = C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J, \kappa, p_2, n, \Sigma) p_2^{-(k+j)} .$$

$$\sum_{r=0}^{\infty} (2/n)^{r+u} W_r \, G_{r+u}(2\rho, \lambda_0) + R_{m+1}(n) \tag{5.14}$$

where

$$G_{r+u}(2\rho, \lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2\rho,x)\, dx \tag{5.15}$$

and

$$R_{m+1}(n) = (2\pi)^{-1} C_1(p_2,n,\underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\underset{\sim}{\Sigma}) p_2^{-(k+j)}$$

$$\int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda^*} \sum_{r=0}^{\infty} W_r (2/n)^{r+u} (1-2it\rho)^{-(r+u)} [\exp(R'''_{m+1}(n))-1] dt\, d\lambda^* \qquad (5.16)$$

From (5.14), we obtain the distribution of $\lambda^*$ as a series of gamma distributions. In particular taking $\rho=1$, we see that the distribution of $\lambda^*$ may be expressed as a series of chi-square distributions. Now

$$P[\lambda^*>\lambda_0] = P[-2\rho \log L_{vc}^{n/2}>\lambda_0] = P[L_{vc}<\exp(-\lambda_0/n\rho)] \qquad (5.17)$$

Therefore, once we know the distribution of $\lambda^*$, the distribution of $L_{vc}$ can be obtained by using (5.17).

In particular, the null distribution of $L_{vc}$ is given by

$$p_1(\lambda^*) = C_1(p_2,n,\underset{\sim}{\Sigma}) \Gamma(np_2/2) \sum_{r=0}^{\infty} (2/n)^{r+u_0} W_{0,r} g_{r+u}(2\rho, \lambda^*) \qquad (5.18)$$

$$+ R_{0,m+1}(n) \quad ,$$

where

$$C_1(p_2,n,\underset{\sim}{\Sigma}) = (2\pi)^{(p_2-1)/2} \, p_2^{(1-np_2)/2} \Big/ \prod_{i=1}^{p_2} \Gamma((n-i)/2) \qquad (5.19)$$

which is same as the one obtained by Nargarsenker [17].

$W_{0,r}$ being the coefficient of $((1-2it\rho)n/2)$ in the expansion of

$\exp[\sum_{r=1}^{m}((1-2it\rho)n/2)^{-r}w_{0,r}]$, where $u_0=(p_2-1)/2+p_2(p_2+1)/4$ and

$$w_{0,r} = [B_{r+1}(0)/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(-\delta/2)](-1)^r/(r(r+1)) \tag{5.20}$$

$R_{0,m+1}(n)$ is defined similarly as in (5.16) with $j=k=0$.

## 6. POWER COMPUTATIONS OF $L_{vc}$ CRITERION.

The distributions obtained in sections 3, 4, 5 were used to study the power behavior of the Wilks' $L_{vc}$ criterion. Powers have been computed for $p=2$ using (3.26) and for $p=3$ using (4.20) and (5.14) which have been presented in tables (1.1) and (1.2) respectively. The computations involve zonal polynomials of degree 0 to 9 (see [9]). The lower 5 percent points of $L_{vc}$ criterion (see Wilks [25]) have been used for our computations. All the computations were carried out on a CDC 6500 computer at the Purdue University Computing Center. Before computing the power for specific values of the parameters, the total probability for that case has been computed and the number of decimals included in the tables were determined depending upon the number of places of accuracy obtained in the total probabilities. The accuracy of the results have been checked by comparing the powers for specific values of the parameters based on (4.20) and (5.14).

From Table (1.1), we observe that power increases with the sample size $N$ as well as the only parameter involved, $\rho$. For the case $p=3$, we observe from Table (1.2) the power increases with $N$, each of the parameters $c$, $\rho_{12}$ and $\rho_{13}$, but decreases with $\rho_{23}$.

Table 1.1

Power Computations For Wilks' $L_{vc}$ Criterion

$p = 2$

| N \ $\rho^2$ | $.0^{4}1$ | $.0^{3}1$ | $.0^{2}1$ | $.0^{2}5$ | .01 | .05 | .1 |
|---|---|---|---|---|---|---|---|
| 3 | .0500005 | .050005 | .05005 | .05025 | .05050 | .05258 | .05534 |
| 4 | .050001 | .050014 | .05014 | .05070 | .05140 | .05727 | .06523 |
| 5 | .050002 | .05002 | .05024 | .05123 | .05247 | .06288 | .07721 |
| 6 | .050004 | .05004 | .05036 | .05179 | .05359 | .06883 | .08999 |
| 7 | .050005 | .05005 | .05047 | .05235 | .05474 | .07493 | .1031 |
| 8 | .050006 | .05006 | .05058 | .05293 | .05590 | .08111 | .1165 |
| 9 | .050007 | .05007 | .05070 | .05350 | .05706 | .08735 | .1299 |
| 10 | .050008 | .05008 | .05081 | .05408 | .05822 | .09362 | .1434 |
| 15 | .050014 | .05014 | .05138 | .05698 | .06409 | .1254 | .2115 |
| 20 | .050020 | .05020 | .05196 | .05989 | .06999 | .1576 | .2789 |
| 25 | .050025 | .05025 | .05253 | .06281 | .07593 | .1899 | .3443 |
| 30 | .050031 | .05031 | .05311 | .06573 | .08190 | .2223 | .4070 |
| 40 | .05004 | .05042 | .05426 | .07161 | .09392 | .2864 | .5217 |
| 60 | .05007 | .05065 | .05657 | .08347 | .1182 | .4086 | .7030 |
| 70 | .05008 | .05077 | .05773 | .08944 | .1305 | .4654 | .7704 |
| 80 | .05009 | .05088 | .05889 | .09544 | .1429 | .5188 | .8243 |
| 110 | .05012 | .05123 | .06237 | .1136 | .1802 | .6566 | .9254 |
| 120 | .05013 | .05134 | .06353 | .1197 | .1926 | .6951 | .9448 |
| 140 | .05016 | .05157 | .06586 | .1319 | .2176 | .7617 | .9703 |
| 200 | .05023 | .05226 | .07289 | .1689 | .2917 | .8926 | .9959 |

Table 1.1 (Continued)

| N \ $\rho^2$ | .15 | .2 | .25 | .3 | .35 | .4 | .45 |
|---|---|---|---|---|---|---|---|
| 3 | .05831 | .06153 | .06503 | .06886 | .07308 | .07777 | .0830 |
| 4 | .07400 | .08371 | .09450 | .1066 | .1202 | .1356 | .1533 |
| 5 | .09320 | .1111 | .1311 | .1537 | .1792 | .2080 | .2408 |
| 6 | .1137 | .1404 | .1702 | .2037 | .2411 | .2829 | .3295 |
| 7 | .1349 | .1704 | .2100 | .2539 | .3024 | .3555 | .413 |
| 8 | .1562 | .2006 | .2496 | .3034 | .3616 | .4240 | .4899 |
| 9 | .1777 | .2307 | .2887 | .3513 | .4179 | .4875 | .559 |
| 10 | .1992 | .2606 | .3270 | .3975 | .4709 | .5459 | .621 |
| 15 | .3052 | .4029 | .5012 | .5962 | .6846 | .763 | |
| 20 | .4051 | .5283 | .6417 | .7403 | .821 | .9 | |
| 25 | .4966 | .6340 | .7489 | .8383 | .903 | | |
| 30 | .5782 | .7203 | .827 | .902 | | | |
| 40 | .7114 | .8427 | .923 | .993 | | | |
| 60 | .8753 | .9556 | .99 | | | | |
| 70 | .9206 | .9774 | | | | | |
| 80 | .9502 | .9889 | | | | | |
| 110 | .9887 | .99 | | | | | |
| 120 | .993 | | | | | | |

Table 1.2

Power Computations For Wilks' $L_{vc}$ Criterion

p = 3

| n | c | 1.0 | 1.0 | 1.0 | 1.0 | 1.025 | 1.025 |
|---|---|---|---|---|---|---|---|
| | $\rho_{12}$ | .05 | .05 | .3 | .4 | .005 | .05 |
| | $\rho_{13}$ | .05 | .1 | .3 | .3 | .005 | .05 |
| | $\rho_{23}$ | .05 | .2 | .0 | .0 | .05 | .05 |
| 3 | | .0502 | .052 | .057 | .061 | .05006 | .0502 |
| 4 | | .0505 | .057 | .067 | .076 | .0502 | .0506 |
| 5 | | .0509 | .062 | .079 | .093 | .0504 | .0510 |
| 6 | | .0511 | .067 | .087 | .109 | .0506 | .0513 |
| 7 | | .0516 | .080 | .103 | .134 | .0508 | .0518 |
| 8 | | .0519 | .092 | .117 | .157 | .0511 | .052 |
| 10 | | .053 | .125 | .146 | .207 | .0516 | .053 |
| 17 | | .055 | .210 | .207 | .289 | .053 | .058 |
| 22 | | .057 | .292 | .261 | .371 | .056 | .061 |

| n | c | 1.025 | 1.025 | 1.025 | 1.025 | 1.05 | 1.05 |
|---|---|---|---|---|---|---|---|
| | $\rho_{12}$ | .05 | .1 | .3 | .3 | .005 | .05 |
| | $\rho_{13}$ | .1 | .15 | .3 | .3 | .005 | .05 |
| | $\rho_{23}$ | .2 | .2 | .05 | .0 | .05 | .05 |
| 3 | | .0516 | .0522 | .057 | .057 | .0501 | .0502 |
| 4 | | .054 | .0542 | .067 | .067 | .0502 | .0506 |
| 5 | | .057 | .058 | .078 | .079 | .0504 | .0510 |
| 6 | | .059 | .061 | .086 | .087 | .0507 | .0514 |
| 7 | | .065 | .066 | .102 | .103 | .0512 | .0518 |
| 8 | | .069 | .069 | .115 | .117 | .0514 | .052 |
| 10 | | .077 | .079 | .142 | .145 | .0517 | .053 |
| 17 | | .14 | .153 | .22 | .24 | .055 | .059 |
| 22 | | .17 | .19 | .29 | .31 | .059 | .067 |

Table 1.2 (Continued)

| n | c | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 |
|---|---|---|---|---|---|---|---|
| | $\rho_{12}$ | .05 | .1 | .1 | .2 | .25 | .0 |
| | $\rho_{13}$ | .1 | .15 | .1 | .2 | .25 | .3 |
| | $\rho_{23}$ | .2 | .2 | .1 | .2 | .25 | .0 |
| 3 | | .051 | .052 | .051 | .054 | .056 | .054 |
| 4 | | .052 | .054 | .052 | .059 | .065 | .058 |
| 5 | | .056 | .058 | .053 | .065 | .075 | .063 |
| 6 | | .058 | .060 | .055 | .068 | .082 | .066 |
| 7 | | .061 | .066 | .057 | .077 | .092 | .074 |
| 8 | | .064 | .070 | .059 | .084 | .109 | .079 |
| 10 | | .070 | .078 | .063 | .098 | .12 | .091 |
| 17 | | .14 | .151 | .077 | .15 | .20 | .13 |
| 22 | | .17 | .186 | .097 | .21 | .25 | .16 |

| n | c | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 |
|---|---|---|---|---|---|---|
| | $\rho_{12}$ | .3 | .3 | .0 | .4 | .4 |
| | $\rho_{13}$ | .3 | .3 | .4 | .3 | .4 |
| | $\rho_{23}$ | .0 | .3 | .0 | .0 | .0 |
| 3 | | .057 | .058 | .056 | .0607 | .064 |
| 4 | | .068 | .069 | .066 | .076 | .086 |
| 5 | | .079 | .082 | .076 | .093 | .111 |
| 6 | | .087 | .090 | .083 | .109 | .134 |
| 7 | | .103 | .102 | .097 | .133 | .170 |
| 8 | | .118 | .113 | .108 | .155 | .202 |
| 10 | | .15 | .14 | .13 | .20 | .27 |
| 17 | | .26 | .22 | .22 | .37 | .50 |
| 22 | | .34 | .29 | .28 | .50 | .60 |

Table 1.2 (Continued)

| n | | | | | | |
|---|---|---|---|---|---|---|
| | c | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 |
| | $\rho_{12}$ | .005 | .1 | .05 | .1 | .2 |
| | $\rho_{13}$ | .005 | .1 | .1 | .15 | .2 |
| n | $\rho_{23}$ | .05 | .1 | .2 | .2 | .2 |
| 3 | | .0507 | .053 | .053 | .054 | .058 |
| 4 | | .055 | .057 | .059 | .060 | .069 |
| 5 | | .059 | .063 | .065 | .067 | .081 |
| 6 | | .061 | .068 | .070 | .073 | .091 |
| 7 | | .066 | .078 | .081 | .084 | .108 |
| 8 | | .081 | .087 | .090 | .094 | .12 |
| 10 | | .098 | .105 | .108 | .114 | .15 |
| 17 | | .18 | .20 | .22 | .23 | .25 |
| 22 | | .21 | .22 | .25 | .26 | .28 |

# CHAPTER II

# ON THE EXACT NON-NULL DISTRIBUTION OF WILKS' $L_{vc}$ CRITERION IN THE COMPLEX CASE

## 1. INTRODUCTION AND SUMMARY

Let $\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_N$ be independent complex normal random p-vectors with mean vector $\tilde{\xi}$ and covariance matrix $\tilde{\Sigma}$, i.e., $\tilde{Z}_i \sim CN(\tilde{\xi}, \tilde{\Sigma})$. Let $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_N)$. Then $\tilde{Z} \sim CN(\tilde{Z}; \tilde{\mu}, \tilde{\Sigma})$, (see Goodman [6]) where the complex multivariate normal distribution is defined by

$$CN(\tilde{Z}; \tilde{\mu}, \tilde{\Sigma}) = (\pi)^{-pN}|\tilde{\Sigma}|^{-N}\exp(-tr\tilde{\Sigma}^{-1}(\tilde{Z} - \tilde{\mu})(\overline{\tilde{Z} - \tilde{\mu}})') \tag{1.1}$$

and $\tilde{\mu} = (\tilde{\xi}, \tilde{\xi}, \ldots, \tilde{\xi})$ is a $p \times N$ complex matrix. Let us define

$$\tilde{Z}_0 = N^{-1}\sum_{i=1}^{N} \tilde{Z}_i \quad \text{and} \quad \tilde{S} = \sum_{i=1} (\tilde{Z}_i - \tilde{Z}_0)(\overline{\tilde{Z}_i - \tilde{Z}_0})' . \tag{1.2}$$

Then $N^{-1/2}(\tilde{Z}_0 - \tilde{\xi}) \sim CN(\tilde{0}, \tilde{\Sigma})$ and $S$ has an independent complex Wishart distribution which is defined by

$$CW(\tilde{S}; \tilde{p}, \tilde{N}, \tilde{\Sigma}) = [\Gamma_p(n)]^{-1}|\tilde{\Sigma}|^{-n}|\tilde{S}|^{n-p}\exp(-tr\tilde{\Sigma}^{-1}\tilde{S}) \tag{1.3}$$

with $n = N - 1$ and $\tilde{\Gamma}_p(n)$ is defined in the next section. $\tilde{\Sigma}$ and $\tilde{S}$ are Hermitian positive definite matrices of order p. In this chapter, in order to study the structure of the covariance matrices of the complex multivariate normal populations, we derive the exact non-null moments and distribution of the Wilks' [25] $L_{vc}$ criterion for testing

H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$, $\sigma$ and $\rho$ unknown against the alternative $A \neq H$; $\underset{\sim}{\mu}$ unknown and $\underset{\sim}{e}' = (1, 1, \ldots, 1)$. We derive the distribution of $L_{vc}$ in three series forms and compute powers for $p = 2$ for various values of N and the parameters involved for 5% significance level based on the null distribution and the percentage points of $L_{vc}$ obtained in Chapter III. In Section 2, we give some definitions and lemmas which are needed in our derivation. In Section 3, we obtain the non-null density of $L_{vc}$ as a series of Meijer's [14] G-functions using Mellin [19] integral transform. Some special cases have also been discussed which are used to compute powers for the case $p = 2$. In Section 4, we obtain the density in an alternative series form using the method of contour integration (i.e., see [18]) and in Section 5, the non-null moments of the criterion are used to obtain the distribution as a chi-square series employing methods similar to those of Box[2]. In Section 6, we tabulate the powers for various values of N and $\rho$ for the case $p = 2$.

## 2. SOME DEFINITIONS AND RESULTS

We now give some definitions and lemmas of interest for the following derivation.

Definitions: Let k be a non-negative integer and let $\kappa = (k_1, k_2, \ldots, k_p)$ be a partition of k such that $k_1 \geq k_2 \geq \cdots \geq k_p \geq 0$, $\sum_{i=1}^{p} k_i = k$ and let

$$[a]_\kappa = \prod_{i=1}^{p} (a - i + 1)_{k_i} = \tilde{\Gamma}_p(a, \kappa)/\tilde{\Gamma}_p(a) \tag{2.1}$$

$$(a)_k = (a)(a+1) \cdots (a+k-1) \quad \text{and} \tag{2.2}$$

$$\tilde{\Gamma}_p(a) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(a-i+1) = \int_{\bar{S}'=S>0} |S|^{a-p} \exp(-\mathrm{tr}S)dS \tag{2.3}$$

Also the hypergeometric function of a matrix variate is defined by (see James [8]),

$$_p\tilde{F}_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; Z) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa} \frac{\tilde{C}_\kappa(Z)}{k!} \tag{2.4}$$

Where $\tilde{C}_\kappa(Z)$ denotes the zonal polynomial, a symmetric function in the characteristic roots of the hermitian matrix $Z$ (see James [8]) of degree $k$ corresponding to the partition $\kappa$. In particular we have

$$_0\tilde{F}_0(Z) = \exp(\mathrm{tr}Z) \quad \text{and} \quad _1\tilde{F}_0(a; Z) = |I - Z|^{-a} \tag{2.5}$$

Lemmas: We now give some lemmas which will be used in the sequel.

Lemma 2.1. Let $R$ be a complex symmetric matrix whose real part is positive definite and let $T$ be an arbitrary complex symmetric matrix. Then

$$\int_{S=\bar{S}'>0} \exp(-\mathrm{tr}RS)|S|^{t-m}\tilde{C}_\kappa(ST)dS = \tilde{\Gamma}_m(t, \kappa)|R|^{-t} \tilde{C}_\kappa(TR^{-1}) .$$

the integration being taken over the space of positive definite Hermitian (p.d.h.) $m \times m$ matrices. (See James [8].)

We now define the Laplace transform of a function $f(S)$ of the p.d.h. $m \times m$ matrix $S$

$$g(Z) = \int_{S=\bar{S}'>0} \exp(-\mathrm{tr}SZ)f(S)dS \quad \text{where} \quad Z = X + iY, \tag{2.6}$$

is a complex symmetric matrix; $X$ and $Y$ are real and it is assumed that the integral converges in the "half-plane" $R(Z) = X > X_0$ for some positive definite $X_0$. (See Constantine [4]). The following theorem will also be needed.

Convolution Theorem. If $g_1(Z)$, $g_2(Z)$ are the Laplace transforms of $f_1(S)$ and $f_2(S)$, then $g_1(Z)g_2(Z)$ is the Laplace transform of

$$f(R) = \int_{S=\bar{S}'>0}^{R} f_1(S)f_2(R-S)dS,$$

the integration being over the space of all $S$ for which $0 < S < R$

Lemma 2.2. If $R$ and $S$ are $m \times m$ p.d.h. matrices, then

$$\int_{S=S'>0}^{I} |S|^{t-m}|I-S|^{u-m}\,\tilde{C}_\kappa(RS)dS = \tilde{\Gamma}_m(t,\kappa)\tilde{\Gamma}_m(u)\tilde{C}_\kappa(R)/\tilde{\Gamma}_m(t+u,\kappa)$$

Proof: Let

$$F(R) = \int_{S=\bar{S}'>0}^{I} |S|^{t-m}|I-S|^{u-m}\tilde{C}_\kappa(RS)dS \tag{2.7}$$

then $F(R)$ is a symmetric function of $R$, i.e., $F(R) = F(\bar{U}'RU)$ for all $U$ s.t. $U\bar{U}' = I$. Therefore, we have

$$F(R) = F(I)\tilde{C}_\kappa(R)/\tilde{C}_\kappa(I) \tag{2.8}$$

In order to complete the proof, we need to show that

$$F(I)/\tilde{C}_\kappa(I) = \tilde{\Gamma}_m(t,\kappa)\tilde{\Gamma}_m(u)/\tilde{\Gamma}_m(t+u,\kappa)$$

Make the transformation $S \to R^{-1/2}TR^{-1/2}$. The Jacobian of the transformation is $|R|^m$. Under this transformation, we have from (2.7)

$$F(\underset{\sim}{R})|\underset{\sim}{R}|^{t+u-m} = \int_{\underset{\sim}{T}=\underset{\sim}{\bar{T}}'>\underset{\sim}{0}}^{\underset{\sim}{R}} |\underset{\sim}{\bar{T}}|^{t-m}|\underset{\sim}{R} - \underset{\sim}{T}|^{u-m} \tilde{C}_{\kappa}(\underset{\sim}{T})d\underset{\sim}{T} \qquad (2.9)$$

Taking the Laplace transform on both sides of (2.9) we have

$$\int_{\underset{\sim}{R}=\underset{\sim}{\bar{R}}'>0} F(\underset{\sim}{R})|\underset{\sim}{R}|^{t+u-m}\exp(-\mathrm{tr}\underset{\sim}{R}\underset{\sim}{Z})d\underset{\sim}{R} =$$

$$\int_{\underset{\sim}{R}=\underset{\sim}{\bar{R}}'>\underset{\sim}{0}} \left[\int_{\underset{\sim}{T}=\underset{\sim}{\bar{T}}'>\underset{\sim}{0}}^{\underset{\sim}{R}} |\underset{\sim}{T}|^{t-m}|\underset{\sim}{R} - \underset{\sim}{T}|^{u-m}\tilde{C}_{\kappa}(\underset{\sim}{T})d\underset{\sim}{T}\right]\exp(-\mathrm{tr}\underset{\sim}{R}\underset{\sim}{Z})d\underset{\sim}{R} \qquad (2.10)$$

After using (2.8) and lemmas (2.1), L.H.S. of (2.10) is given by

$$\text{L.H.S.} = F(\underset{\sim}{I})/C_{\kappa}(\underset{\sim}{I})\tilde{\Gamma}_p(t+u,\ \kappa)|\underset{\sim}{Z}|^{-(t+u)}\tilde{C}_{\kappa}(\underset{\sim}{Z}^{-1}) \qquad (2.11)$$

Let $f_1(\underset{\sim}{T}) = |\underset{\sim}{T}|^{t-m}\tilde{C}_{\kappa}(\underset{\sim}{T})$ and $f_2(\underset{\sim}{T}) = |\underset{\sim}{T}|^{u-m}$ and $g_1(\underset{\sim}{Z})$, $g_2(\underset{\sim}{Z})$ be the Laplace transforms of $f_1(\underset{\sim}{T})$ and $f_2(\underset{\sim}{T})$ respectively, then using (2.3), lemma (2.1) and the covolution theorem, we have R.H.S. of (2.10) in the form

$$\text{R.H.S.} = g_1(\underset{\sim}{Z})g_2(\underset{\sim}{Z}) = \tilde{\Gamma}_m(t,\ \kappa)\tilde{\Gamma}_m(u)|\underset{\sim}{Z}|^{-(t+u)}\tilde{C}_{\kappa}(\underset{\sim}{Z}^{-1}) \qquad (2.12)$$

which proves the lemma.

## 3. EXACT NON-NULL DISTRIBUTION OF $L_{vc}$

In this section we derive the non-null density of $L_{vc}$ as a series of Miejer's G-functions [14] using Mellin-integral transform [19]. As in Chapter I, using lemma (2.1) of Chapter I, the test of $H: \underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$ reduces to that of $H: \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2\underset{\sim}{I}_{p_2} \end{bmatrix}$,

$\sigma_1, \sigma_2 > 0$ and unknown, against the alternatives $A \neq H$ ; $p_2 = p - 1$. The likelihood ratio criterion is based on the statistic

$$L_{vc} = |\underset{\sim}{S}|/[s_{11}(\mathrm{tr}\underset{\sim}{S}_{22}/p_2)^{p_2}] \tag{3.1}$$

as given in Chapter I for the real case, where

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & \underset{\sim}{S}_{12} \\ \bar{\underset{\sim}{S}}'_{12} & \underset{\sim}{S}_{22} \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix} \quad \text{with} \quad n = N - 1,$$

$N$ being the size of the random sample from $CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma} = \bar{\underset{\sim}{\Sigma}}' > \underset{\sim}{0}$. Furthermore, we make use of the transformation $\begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{bmatrix} \to \begin{bmatrix} \underset{\sim}{x}_1/\sigma_1 \\ \underset{\sim}{x}_2/\sigma_2 \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix}$

Under this transformation the problem of testing $H$ versus $A$ reduces to the problem of testing $H_1$: $\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & \underset{\sim}{I}_{p_2} \end{bmatrix}$ versus $A_1 \neq H_1$, where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \underset{\sim}{\Sigma}_{12}/\sigma_1\sigma_2 \\ \underset{\sim}{\Sigma}'_{12}/\sigma_1\sigma_2 & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix} \qquad \sigma_1, \sigma_2 \text{ positive and unknown.}$$

From now on we assume that this has been done and we are testing $H_1$ versus $A_1$. We now define

$$T = s_{11}^{-1/2}\, \underset{\sim}{S}_{12}\, \underset{\sim}{S}_{22}^{-1}\, \bar{\underset{\sim}{S}}'_{12}\, \bar{s}_{11}^{-1/2} \tag{3.2}$$

Then the statistic $L_{vc}$ can be written as

$$L_{vc} = |\underset{\sim}{S}_{22}|(1 - T)/(\mathrm{tr}\underset{\sim}{S}_{22}/p_2)^{p_2} \tag{3.3}$$

We now need the following lemma.

Lemma 3.1. The joint p.d.f. of $\underset{\sim}{T}$, $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$ is given by

$$f(\underset{\sim}{T}, \underset{\sim}{S}_{11}, \underset{\sim}{S}_{22}) = U(p_1, p_2, n, \underset{\sim}{\Sigma})|\underset{\sim}{S}_{11}|^{n-p_1}|\underset{\sim}{S}_{22}|^{n-p_2}\exp(-tr\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{11})$$

$$\exp(-tr\underset{\sim}{\Sigma}_{2.1}^{-1}\underset{\sim}{S}_{22})|\underset{\sim}{T}|^{p_2-p_1}|\underset{\sim}{I}-\underset{\sim}{T}|^{n-p_1-p_2}{}_0F_1(p_2, (\bar{\underset{\sim}{S}}_{11}^{1/2})'$$

$$\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'(\bar{\underset{\sim}{\Sigma}}_{1.2})^{-1}\underset{\sim}{S}_{11}^{1/2}\underset{\sim}{T}) \tag{3.4}$$

where

$$\underset{\sim}{\Sigma}_{1.2} = \underset{\sim}{\Sigma}_{11} - \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\bar{\underset{\sim}{\Sigma}}_{12}'$$

$$\underset{\sim}{\Sigma}_{2.1} = \underset{\sim}{\Sigma}_{22} - \bar{\underset{\sim}{\Sigma}}_{12}'\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12}$$

$$\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \bar{\underset{\sim}{\Sigma}}_{12}' & \underset{\sim}{\Sigma}_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix} \qquad \text{and} \qquad \underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \bar{\underset{\sim}{S}}_{12}' & \underset{\sim}{S}_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix}$$

and $p_1 + p_2 = p$, $p_2 \geq p_1 \geq 1$ without loss of generality.

$$\underset{\sim}{T} = \underset{\sim}{S}_{11}^{-1/2}\ S_{12}\ S_{22}^{-1}\ \bar{S}_{12}'(\bar{S}_{11}^{-1/2})'$$

$$U^{-1}(p_1, p_2, n, \underset{\sim}{\Sigma}) = \tilde{\Gamma}_{p_2}(n)\tilde{\Gamma}_{p_1}(n-p_2)|\underset{\sim}{\Sigma}_{1.2}|^n|\underset{\sim}{\Sigma}_{22}|^n\tilde{\Gamma}_{p_1}(p_2) \quad .$$

$\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$ and $\underset{\sim}{T}$ are p.d.h. and $\underset{\sim}{0} < \underset{\sim}{T} < \underset{\sim}{I}$.

Proof. Let $\underset{\sim}{S}_{1.2} = \underset{\sim}{S}_{11} - \underset{\sim}{S}_{12}\ \underset{\sim}{S}_{22}^{-1}\ \bar{\underset{\sim}{S}}_{12}'$. It is easy to prove that $\underset{\sim}{S}_{1.2}$ and $(\underset{\sim}{S}_{12}, \underset{\sim}{S}_{22})$ are independently distributed and

$\underset{\sim}{S}_{1.2} \sim CW(\underset{\sim}{S}_{1.2}; p_1, n-p_2, \underset{\sim}{\Sigma}_{1.2})$. Also

$$\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}_{12}' \sim CW(\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{S}_{12}'; p_1, p_2, \underset{\sim}{\Sigma}_{1.2}, \underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}') \text{ given } \underset{\sim}{S}_{22}, \text{ i.e.,}$$

$S_{12}S_{22}\bar{S}'_{12}$ has noncentral complex Wishart distribution with mean matrix $\beta F^{-1}$, where $F$ is s.t. $F^{-1}(\bar{F}')^{-1} = S_{22}$, given $S_{22}$, where non-central Wishart density is given by (see James [8])

$$CW(S_{12}S_{22}^{-1}\bar{S}'_{12};\ p_1,\ p_2,\ \Sigma_{1.2},\ \beta S_{22}\bar{\beta}') = \exp(-\mathrm{tr}\Sigma_{1.2}^{-1}\beta S_{22}\bar{\beta}')$$

$${}_0\tilde{F}_1(p_2;\ \Sigma_{1.2}^{-1}\beta S_{22}\bar{\beta}'\Sigma_{1.2}^{-1}S_{12}S_{22}^{-1}\bar{S}'_{12})\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}S_{12}S_{22}^{-1}\bar{S}'_{12})$$

$$|S_{12}S_{22}^{-1}\bar{S}'_{12}|^{p_2-p_1}/[|\Sigma_{1.2}|^n\tilde{\Gamma}_{p_1}(p_2)] \tag{3.5}$$

Now, the joint conditional distribution of $S_{1.2}$ and $S_{12}S_{22}^{-1}\bar{S}'_{12}$ given $S_{22}$ is given by

$$dH|S_{22} = U_1(p_1,\ p_2,\ n,\ \Sigma)\,{}_0\tilde{F}_1(p_2;\ \Sigma_{1.2}^{-1}\beta S_{22}\bar{\beta}'\Sigma_{1.2}^{-1}S_{12}S_{22}^{-1}\bar{S}'_{12})$$

$$|S_{1.2}|^{n-p_1-p_2}\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}S_{1.2})\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}\beta S_{22}\bar{\beta}')|S_{12}S_{22}^{-1}\bar{S}'_{12}|^{p_2-p_1}$$

$$\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}S_{12}S_{22}^{-1}\bar{S}'_{12})d(S_{1.2})d(S_{12}S_{22}^{-1}\bar{S}'_{12}) \tag{3.6}$$

$$U_1^{-1}(p_1,\ p_2,\ n;\ \Sigma) = |\Sigma_{1.2}|^n\tilde{\Gamma}_{p_1}(p_2)\tilde{\Gamma}_{p_1}(n-p_2) \tag{3.7}$$

We now make the following transformation

$$\begin{aligned} S_{11} &= S_{1.2} + S_{12}S_{22}^{-1}\bar{S}'_{12} \\ T &= S_{11}^{-1/2}S_{12}S_{22}^{-1}\bar{S}'_{12}(\bar{S}'_{11})^{-1/2} \end{aligned} \tag{3.8}$$

The Jacobian of the transformation is $|S_{11}|^{p_1}$ (see Khatri [11]). Hence, the joint conditional density of $T$, and $S_{11}$ given $S_{22}$ is given by

$$h(\underset{\sim}{S}_{11}, \underset{\sim}{T}|\underset{\sim}{S}_{22}) = U_1(p_1, p_2, n, \underset{\sim}{\Sigma})\,{}_0\tilde{F}_1(p_2; (\bar{\underset{\sim}{S}}'_{11})^{1/2}\underset{\sim}{\Sigma}^{-1}_{1.2}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'\underset{\sim}{\Sigma}^{-1}_{1.2}\underset{\sim}{S}^{1/2}_{11}\underset{\sim}{T})$$

$$\exp(-tr\underset{\sim}{\Sigma}^{-1}_{1.2}\underset{\sim}{S}_{11})|\underset{\sim}{S}_{11}|^{n-p_1}|\underset{\sim}{I} - \underset{\sim}{T}|^{n-p_1-p_2}|\underset{\sim}{T}|^{p_2-p_1}\exp(-tr\bar{\underset{\sim}{\beta}}'\underset{\sim}{\Sigma}^{-1}_{1.2}\underset{\sim}{\beta}\underset{\sim}{S}_{22})$$

Also $\underset{\sim}{S}_{22} \sim CW(n, p_2, \underset{\sim}{\Sigma}_{22})$. If $g(\underset{\sim}{S}_{22})$ denotes the density of $\underset{\sim}{S}_{22}$, then the joint density of $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$, and $\underset{\sim}{T}$ is $h(\underset{\sim}{S}_{11}, \underset{\sim}{T}|\underset{\sim}{S}_{22})g(\underset{\sim}{S}_{22})$ which will be the same as (3.4) after using the identity $\underset{\sim}{\Sigma}^{-1}_{22} + \bar{\underset{\sim}{\beta}}'\underset{\sim}{\Sigma}^{-1}_{1.2}\underset{\sim}{\beta} = \underset{\sim}{\Sigma}^{-1}_{2.1}$.

Now we need the following theorem in order to derive $E(L_{vc})^h$.

Theorem 3.1.

$$E[\exp(-t\ tr\underset{\sim}{S}_{22})|\underset{\sim}{S}_{22}|^h(1-T)^h] = U_3(p_2, n, \underset{\sim}{\Sigma}, h)\sum_{j=0}^{\infty}\sum_{J}\sum_{k=0}^{\infty}\sum_{\kappa}$$

$$(t+1)^{-p_2(h+n)+k+j)}[h]_\kappa[n]_J\,\tilde{C}_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1}_{2.1})\,\tilde{C}_J(\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1}_{2.1} + (\bar{\Sigma}_{1.2})^{-1}\bar{\underset{\sim}{\beta}}'\underset{\sim}{\beta})/k!j! \tag{3.9}$$

where

$$U_3(p_2, n, \underset{\sim}{\Sigma}, h) = \tilde{\Gamma}_{p_2}(n-1+h)/[\tilde{\Gamma}_{p_2}(n-1)|\underset{\sim}{\Sigma}_{22}|^n] \quad . \tag{3.10}$$

Proof. Let $\underset{\sim}{V} = \exp(-t\ tr\underset{\sim}{S}_{22})|\underset{\sim}{S}_{22}|^h(1-T)^h$. Now using lemma (3.1) with $p_1 = 1$, we obtain

$$E[V] = U(1, p_2, n, \underset{\sim}{\Sigma})\int_{s_{11}=\bar{s}_{11}>0}\int_{\bar{\underset{\sim}{S}}'_{22}=\underset{\sim}{S}_{22}>0}\int_{\bar{\underset{\sim}{T}}'=\underset{\sim}{T}>0}^{\underset{\sim}{I}}(s_{11})^{n-1}|\underset{\sim}{S}_{22}|^{n+h-p_2}$$

$$\exp(-tr\Sigma^{-1}_{1.2}s_{11})\exp(-tr(\underset{\sim}{\Sigma}^{-1}_{2.1} + t\,\underset{\sim}{I})\underset{\sim}{S}_{22})|\underset{\sim}{T}|^{p_2-p_1}|\underset{\sim}{I} - \underset{\sim}{T}|^{n-p_1-p_2+h}$$

$$\sum_{k=0}\sum_{\kappa}\tilde{C}_\kappa(\bar{s}^{1/2}_{11}\,\Sigma^{-1}_{1.2}\,\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'\,\Sigma^{-1}_{1.2}\,s^{1/2}_{11}\underset{\sim}{T})/([p_2]_\kappa k!)ds_{11}d\underset{\sim}{S}_{22} \tag{3.11}$$

Using the monotone convergence theorem, the interchange of the integral and summation signs is valid. Now using lemma (2.2) in order to integrate with respect to $\underset{\sim}{T}$, we get from (3.11)

$$E[V] = U_2 \sum_{k=0}^{\infty} \sum_{\kappa} \int_{s_{11}>0} \int_{\bar{\underset{\sim}{S}}'_{22}=\underset{\sim}{S}_{22}} s_{11}^{n-1} |S_{22}|^{n+h-p_2} \exp(-\mathrm{tr}\Sigma_{1.2}^{-1} s_{11})$$

$$\exp(-\mathrm{tr}(t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1})\underset{\sim}{S}_{22})\tilde{C}_{\kappa}(s_{11}\bar{\underset{\sim}{\beta}}'\underset{\sim}{\beta}\Sigma_{1.2}^{-2}\underset{\sim}{S}_{22})/(k![n+h]_{\kappa})ds_{11}d\underset{\sim}{S}_{22} \qquad (3.12)$$

where

$$U_2 = U\,(1, p_2, n, \underset{\sim}{\Sigma})\tilde{\Gamma}(p_2)\tilde{\Gamma}(n - p_2 + h)/\tilde{\Gamma}(h + n) \qquad (3.13)$$

Now using lemma (2.1) to integral with respect to $S_{22}$ and then in turn using monotone covergence theorem and the relation (2.5), we get

$$E[V] = U_4\ |t\ I + \underset{\sim}{\Sigma}_{2.1}^{-1}|^{-(n+h)} \int_{s_{11}>0} s_{11}^{n-1} \exp(-(\Sigma_{1.2}^{-1} - \underset{\sim}{\beta}(t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1})^{-1}$$

$$\bar{\underset{\sim}{\beta}}'\Sigma_{1.2}^{-2})s_{11})ds_{11} \qquad (3.14)$$

where $U_4 = U_2\tilde{\Gamma}_{p_2}(n + h)$. Now integrating with respect to $s_{11}$ and using relation (2.5), we get

$$E[V] = U_3(p_2, n, \underset{\sim}{\Sigma}, h)|t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1}|^{-h}|t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1} - \bar{\underset{\sim}{\beta}}'\underset{\sim}{\beta}\underset{\sim}{\Sigma}_{1.2}^{-1}|^{-n} \qquad (3.15)$$

where $U_3(p_2, n, \underset{\sim}{\Sigma}, h)$ is given by (3.10).

Now adding and substracting $\underset{\sim}{I}$ inside each of the two determinants and using (2.5), we have

$$E[V] = U_3(p_2, n, \underset{\sim}{\Sigma}, h)(t+1)^{-p_2(h+n)}\ {}_1\tilde{F}_0(h;\ (t+1)^{-1}(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1}))$$

$${}_1\tilde{F}_0(n;\ (t+1)^{-1}(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1} + \underset{\sim}{\Sigma}_{1.2}^{-1}\ \bar{\underset{\sim}{\beta}}'\underset{\sim}{\beta})). \qquad (3.16)$$

which can be expressed as (3.9) after using (2.4).

Theorem 3.2. For any finite $p$ the p.d.f. of $L_{vc}$ is given by

$$p(L_{vc}) = D_1(p_2, n, \Sigma)(L_{vc})^{-(p_2+1)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} n_2^{-(k+j)}$$

$$B(J, k, p_2, n, \Sigma) G_{2p_2\ \ 2p_2}^{2p_2\ \ 0}\left[L_{vc}\left| \begin{matrix} c_1, c_2, \ldots, c_{p_2}; d_1, d_2, \ldots, d_{p_2} \\ a_1, a_2, \ldots, a_{p_2}; b_1, b_2, \ldots, b_{p_2} \end{matrix}\right.\right] \quad (3.17)$$

where

$$D_1(p_2, n, \Sigma) = (2\pi)^{(p_2-1)/2}\, p_2^{\frac{1}{2}-np_2} / \left(\prod_{i=1}^{p_2} \Gamma(n-i) |\Sigma_{22}|^n\right)$$

$$B(J,k,p_2,n,\Sigma) = \Gamma(np_2 + k + j)\tilde{C}_\kappa(I - \Sigma_{2.1}^{-1})\tilde{C}_J(I - \Sigma_{2.1}^{-1} + \Sigma_{1.2}^{-1}\bar{\beta}'\beta)/k!j! \quad (3.18)$$

$$a_i = p_2 + n - i \ , \quad b_i = p_2 - i + 1 + k_i \quad (3.19)$$

$$c_i = p_2 + 1 - i \ , \quad d_i = p_2 + n + (k + j + i - 1)p_2^{-1}; \quad i = 1, 2, \ldots, p_2 \ .$$

Proof: First, we evaluate the h-th moment of $L_{vc}$ as the method of derivation of the density of $L_{vc}$ depends on lemma (2.4) of Chapter I, concerning the Mellin transform. Integrating both sides of (3.9) with respect to $t$, $p_2h$ times under the integral sign and putting $t = 0$ in the final result, we get

$$E[L_{vc}]^h = U_3(p_2, n, \Sigma, h)\, p_2^{p_2h} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} [n]_J [h]_\kappa \tilde{C}_\kappa(I - \Sigma_{2.1}^{-1})\tilde{C}_J(I - \Sigma_{2.1}^{-1}$$

$$+ \Sigma_{1.2}^{-1}\bar{\beta}'\beta)/((np_2 + k + j)_{hp_2} k!j!) \quad (3.20)$$

Let

$$D(p_2, n, \Sigma) = 1/\left(|\Sigma_{22}|^n \prod_{i=1}^{p_2} \Gamma(n-i)\right) , \quad (3.21)$$

then

$$E[L_{vc}]^h = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma(h+n-i)$$

$$\prod_{i=1}^{p_2} (h-i+1)_{k_i} / \Gamma(p_2(h+n)+k+j) \qquad (3.22)$$

where $B(J, k, p_2, n, \underset{\sim}{\Sigma})$ is defined by (3.18). Now using Mellin integral transform on both sides of (3.22) (see lemma (2.4), Chapter I), we get the density of $L_{vc}$ in the form

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$(2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_{vc})^{-(h+1)} p_2^{p_2 h} \prod_{i=1}^{p_2} \frac{\Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i}}{\Gamma(p_2(h+n)+k+j)} dh. \quad (3.23)$$

Now applying the transformation $h \to h + p_2$ and using Gauss - Legendre's multiplication theorem (see (3.22), Chapter I) on $\Gamma(p_2(h+n)+k+j)$ we get

$$p(L_{vc}) = D_1(p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{-(p_2+1)} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} (2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} (L_{vc})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h+p_2-i+1+k_i) \prod_{i=1}^{p_2} \Gamma(h+n+p_2-i)}{\prod_{i=1}^{p_2} \Gamma(h+p_2-i+1) \prod_{i=1}^{p_2} \Gamma(h+p_2+n+(k+j+i-1)/p_2)} \cdot dh$$

$$(3.24)$$

where $C_1 = C + p_2$ and $D_1(p_2, n, \underset{\sim}{\Sigma})$ is given by (3.18). (3.24) can also be written as

$$p(L_{VC}) = D_1(p_2, n, \Sigma)(L_{VC})^{-(p_2+1)} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \Sigma)$$

$$p_2^{-(k+j)}(2\pi i)^{-1} \int_{C_i - i\infty}^{C_1 + i\infty} (L_{VC})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h+a_i) \prod_{i=1}^{p_2} \Gamma(h+b_i)}{\prod_{i=1}^{p_2} \Gamma(h+c_i) \prod_{i=1}^{p_2} \Gamma(h+d_i)} \, dh \qquad (3.25)$$

$a_i^{s'}$, $b_i^{s'}$, $c_i^{s'}$, and $d_i^{s'}$ being defined in (3.19). Noticing that the integrals in (3.25) are in the form of Meijer's G-function (see (2.4) of Chapter I), we can write (3.25) in the form (3.17).

Special Cases. We now discuss the cases $p_2 = 1$ and $p_2 = 2$.

$p_2 = 1$. Putting $p_2 = 1$ in (3.17), we obtain

$$p(L_{VC}) = \frac{(L_{VC})^{-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2))^k G_{2\,2}^{2\,0}\left[L_{VC}\Big|\begin{matrix} 1 & n+k+1 \\ n & k+1 \end{matrix}\right] \qquad (3.26)$$

where $\Sigma = \begin{bmatrix} 1 & \rho \\ \bar{\rho} & 1 \end{bmatrix}$, $|\rho|^2 = \rho\bar{\rho}$ .

Now using (2.5) of Chapter I, (3.26) can be put in the form

$$p(L_{VC}) = \frac{(L_{VC})^{n-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2))^k$$

$${}_2F_1(n, -k, 1; 1 - L_{VC}), \; 0 < L_{VC} < 1. \qquad (3.27)$$

In particular, under the null hypothesis, $H_1$: $\rho = 0$, the null density of $L_{VC}$ is given by

$$p_1(L_{VC}) = (L_{VC})^{n-2}\Gamma(n)/\Gamma(n-1), \quad 0 < L_{VC} < 1. \qquad (3.28)$$

$\underline{p_2 = 2}$. In this case $\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & C\rho_{13} \\ \bar{\rho}_{12} & 1 & C\rho_{23} \\ C\bar{\rho}_{13} & C\bar{\rho}_{23} & C^2 \end{bmatrix}$ , $C = \sigma_3/\sigma_2$

Now putting $p_2 = 2$ in (3.17), we obtain

$$p(L_{VC}) = \frac{\Gamma(n)|\underset{\sim}{\Sigma}_{22}|^{-n}}{\Gamma(n-2)\tilde{\Gamma}_2(n)} 2^{1-2n}(\pi)^{3/2}(L_{VC})^{-3} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} 2^{-(k+j)}$$

$$[n]_J \Gamma(2n+k+j)\tilde{C}_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1})\tilde{C}_J(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-2} + \underset{\sim}{\Sigma}_{1.2}^{-1}\, \bar{\underset{\sim}{\beta}}'\underset{\sim}{\beta})/k!j!$$

$$G^{4\;0}_{4\;4}\left[L_{VC}\middle|\begin{matrix} c_1, c_2; d_1, d_2 \\ a_1, a_2; b_1, b_2 \end{matrix}\right] \tag{3.29}$$

where

$$a_1 = n+1, \quad a_2 = n; \quad b_1 = 2+k_1, \quad b_2 = 1+k_2$$

$$c_1 = 2, \quad c_2 = 1; \quad d_1 = 2+n+(k+j)/2, \quad d_2 = 2+n+(k+j+1)/2$$

Also under the null hypothesis we have

$$p_1(L_{VC}) = \pi^{3/2}\, 2^{1-2n}\Gamma(2n)/[\Gamma(n-2)\tilde{\Gamma}_2(n)](L_{VC})^{-3} G^{2\;0}_{2\;2}\left[L_{VC}\middle|\begin{matrix} 2+n, & n+3/2 \\ n, & n+1 \end{matrix}\right] \tag{3.30}$$

which after using the duplication formula of gamma functions and (2.5) of Chapter I, can be written as

$$p_1(L_{VC}) = \frac{\Gamma(n)\Gamma(n+1/2)}{\Gamma(n-1)\Gamma(n-2)\Gamma(7/2)} (L_{VC})^{n-3}(1-L_{VC})^{5/2}\, {}_2F_1(3/2, 1, 7/2; 1-L_{VC}) \tag{3.31}$$

$$0 < L_{VC} < 1$$

Using the relation ${}_2F_1(a, b, C; 1) = \Gamma(C)\Gamma(C-a-b)/\Gamma(C-a)T(C-b)$ (see Erdelyi (5)), it can be checked that

$$\int_0^1 p_1(L_{vc})dL_{vc} = 1.$$

## 4. THE EXACT NON-NULL DISTRIBUTION OF $L_{vc}$ CRITERION THROUGH CONTOUR INTEGRATION

Starting from (3.23) of Section 3, we have

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{J=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$(2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_{vc})^{-(h+1)} p_2^{p_2 h} \frac{\prod_{i=1}^{p_2} \Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i}}{\Gamma(p_2(h+n)+k+j)} dh \qquad (4.1)$$

For simplifications, make the transformation $h+n \to h$. Then (4.1) can be written as

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{n-1}$$

$$p_2^{-np_2} (2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} (L_{vc}/p_2^{p_2})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h-i) \prod_{i=1}^{p_2} (h-n-i+1)_{k_i}}{\Gamma(p_2 h+k+j)} dh \qquad (4.2)$$

where $C_1 = C+n$ and

$$D^{-1}(p_2, n, \underset{\sim}{\Sigma}) = |\underset{\sim}{\Sigma}_{22}|^n \prod_{i=1}^{p_2} \Gamma(n-i)$$

$$B(J, k, p_2, n, \underset{\sim}{\Sigma}) = [n]_J \Gamma(np_2+k+j) \tilde{C}_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1}) \qquad (4.3)$$

$$\tilde{C}_J(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1} + \underset{\sim}{\Sigma}_{1.2}^{-1}\bar{\beta}'\beta)/k!j!$$

Let

$$L_1 = L_{vc}/p_2^{p_2}, \tag{4.4}$$

then (4.2) can be written as

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{n-1} p_2^{-np_2} f(L_{vc}). \tag{4.5}$$

where

$$f_{j,k}(L_{vc}) = (2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} G(h)dh \tag{4.6}$$

and

$$G_{j,k}(h) = (L_1)^{-h} \prod_{i=1}^{p_2} \Gamma(h-i) \prod_{i=1}^{p_2} (h-n-i+1)_{k_i} / \Gamma(p_2 h + k + j) \tag{4.7}$$

For ease in typing, the functions $f_{j,k}$, $G_{j,k}$, $R_{j,k}$ will be written as f, G, R respectively throughout this Chapter.

We now consider a special case.

$\underline{p_2 = 1}$. We have from (4.2)

$$p(L_{vc}) = \frac{1}{\Gamma(n-1)} (L_{vc})^{n-1} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2))^k$$

$$(2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} (L_{vc})^{-h} \Gamma(h-1)(h-n)_k / \Gamma(h+k) dh \tag{4.8}$$

The integral in (4.8) will be evaluated by contour integration. The poles of the integrand are at points

$$h = -\ell, \quad \ell = -1, 0, 1, 2, \ldots \tag{4.9}$$

The residue at these points can be found by putting $h = t - \ell$ in (4.8) and taking the residue of the integrand at $t = 0$. The integrand is given by

$$G(t-\ell) = (L_{vc})^{-t+\ell}\Gamma(t-\ell-1)(t-\ell-n)_k/\Gamma(t-\ell+k). \qquad (4.10)$$

To evaluate the integral in (4.8), we need to consider separately, the cases (A) $\ell < k$ (B) $\ell \geq k$.

CASE A: $\ell < k$; $\ell = -1, 0, 1, \ldots, k-1$. In this case, after expanding the gamma functions (4.10) can be written as

$$G(t-\ell) = (L_{vc})^{-t+\ell}\Gamma(t+1)(t-\ell-n)_k/(t\prod_{\delta=1}^{\ell+1}(t-\delta)\Gamma(t+k-\ell)). \qquad (4.11)$$

The integrand $G(t-\ell)$ in (4.11) has a simple pole of first order at $t=0$, and the residue at this point is given by

$$R_\ell = \lim_{t\to 0} t\ G(t-\ell) \quad ,$$

and

$$R_\ell = (L_{vc})^\ell(-\ell-n)_k{}^{(-1)^{\ell+1}}/((\ell+1)!\Gamma(k-\ell)). \qquad (4.12)$$

CASE B: $\ell \geq k$; $\ell = k, k+1, \ldots$ . After expanding the gamma functions in (4.10), we get

$$G(t-\ell) = (L_{vc})^{-t+\ell}(t-\ell-n)_k \prod_{\delta=1}^{\ell-k}(t-\delta)/\prod_{\delta=1}^{\ell+1}(t-\delta). \qquad (4.13)$$

The integrand in (4.13) does not have any pole at $t=0$.

Thus from (4.12) and (4.13) and using Cauchy's residue theorem, the integral in (4.8) for this case is given by

$$f(L_{vc}) = \sum_{\ell=1}^{k-1} R_\ell \ , \qquad (4.14)$$

and the density (4.8) is given by

$$p(L_{vc}) = \frac{(L_{vc})^{n-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} \left[\frac{-|\rho|^2}{1-|\rho|^2}\right]^k \sum_{v=0}^{k} \frac{(-L_{vc})^v(-v-n+1)_k}{v!\Gamma(k+1-v)} ,$$

$$0 < L_{vc} < 1 , \qquad (4.15)$$

which after using Vandermonde's theorem (see Erdély; [5])

$$_2F_1(-n, b; c; 1) = (c-b)_n/(c)_n \quad c \neq 0, -1, -2, \ldots \qquad (4.16)$$

and for other b and c, reduces to (3.27) of Section 3. This form of the density has been used for power computations, which are presented in Table (2.1).

Now for finding the density of $L_{vc}$ for $p_2 \geq 2$, we still use the method of contour integration but the density now will involve psi functions and their derivative. We will make use of lemma (4.1) of Chapter I in this connection. Throughout the rest of this Chapter all empty products $\prod_{i=m}^{n}(\cdot)$ and empty sums $\sum_{i=m}^{n}(\cdot)$ for $m > n$ will be treated as 1 and 0 respectively.

Now from (4.7), the poles of the integrand $G(h)$ are at points

$$h = -\ell, \ \ell = -p_2, -p_2+1, \ldots, -1, 0, 1, 2, \ldots \qquad (4.17)$$

To compute the residue at these poles, we put $h = t - \ell$ in (4.7) and find the residue at $t = 0 \ \forall \ \ell$. Now, (4.7) can be written as

$$G(t-\ell) = (L_1)^{-t+\ell} \prod_{i=1}^{p_2} (t-\ell-n-i+1)_{K_i} \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)+k+j) \qquad (4.18)$$

Let $c = k+j-p_2\ell$. Two cases arise: (A) $\ell \geq 0$ (B) $\ell < 0$.

Let

$$GP(t) = \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)+k+j) \tag{4.19}$$

The poles of the integrand in (4.18) are the poles of (4.19)

CASE A: $\ell \ge 0$. Two subcases: (A1) $c \le 0$ and (A2) $c > 0$.

SUBCASE A1: $\ell \ge 0$ and $c \le 0$. Expanding the gamma functions in (4.19) we obtain

$$GP(t) = p_2(\Gamma(t+1))^{p_2} t^{-(p_2-1)} \prod_{\delta=1}^{-c} (tp_2-\delta)/(\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (t-\delta)) \tag{4.20}$$

Thus for $\ell \ge 0$ and $k+j \le p_2\ell$, the pole of the integrand $G(t-\ell)$ is of order $p_2-1$.

In the following the functions A, GP, B, C, G, R depend upon j and k, but for the ease of typing the subscripts j, k will be supressed. Now using (4.20), (4.18) can be written as

$$G(t-\ell) = (L_1)^{\ell} a_0' t^{-(p_2-1)} A_{j,k}(t) \tag{4.21}$$

where

$$a_0' = (-1)^{k+j-p_2(p_2+1)/2} \cdot p_2(-c)! \prod_{i=1}^{p_2} (-\ell-n-i+1)_{k_i} / \prod_{i=1}^{p_2} (\ell+1)! \tag{4.22}$$

$$A(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=0}^{k_i-1} (1+t/(\delta-\ell-n-i+1))(\Gamma(t+1))^{p_2} \prod_{\delta=1}^{-c} (1-tp_2/\delta)/ (\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)) \tag{4.23}$$

The residue of order $p_2-1$ at $t=0$ is given by,

$$R_{\ell} = (L_1)^{\ell} a_0'/\Gamma(p_2-1) \left(\frac{d}{dt}\right)_{t=0}^{p_2-2} \exp(\log A(t)) . \tag{4.24}$$

Using (4.36), (4.37), (4.38) of Chapter I, we can write $\log A(t)$ as

$$\log A(t) = a_1 t + a_2 t^2/2! + a_3 t^3/3! + \cdots , \tag{4.25}$$

where

$$a_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} 1/(\delta - n - \ell - i + 1) - \sum_{\delta=1}^{-c} (p_2/\delta) + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \tag{4.26}$$

and for $q \geq 2$, we have

$$a_q = (p_2 - p_2^q)\psi_{q-1}(1) + (q-1)! \left[ \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} (-1)^{q+1}/(\delta - n - \ell - i + 1)^q - \sum_{i=1}^{-c} (p_2/\delta)^q + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \right] .$$

Using (4.25) in (4.24) and lemma (4.1) of Chapter I, we get

$$R_\ell = (L_1)^\ell a_0' D_{p_2-2}(L_1;\, a)/\Gamma(p_2 - 1) \tag{4.27}$$

where

$$D_{p_2-2}(L_1;\, a) = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & a_1 & -1 & \cdots & \\ a_3 & 2a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{p_2-2} & \binom{p_2-3}{1} a_{p_2-3} & \binom{p_2-4}{2} a_{p_2-4} & \cdots & a_1 \end{vmatrix} \tag{4.28}$$

where $a_q$'s are defined in (4.26).

<u>SUBCASE A2: $\ell \geq 0$ and $c > 0$</u> i.e., $k + j > p_2 \ell$. Expanding the gamma functions in (4.19), we get

$$G(P(t)) = (\Gamma(t+1))^{p_2}\, t^{-p_2} / \left( \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (t-\delta)\Gamma(p_2 t + c) \right) \tag{4.29}$$

Thus in this case we have a pole of order $p_2$ at $t = 0$. Using (4.29) in (4.18), we have

$$G(t-\ell) = (L_1)^{\ell}\, t^{-p_2}\, b_0'\, \exp(\log B(t)) \tag{4.30}$$

where

$$b_0' = (-1)^{\ell p_2 + p_2(p_2+1)/2} \prod_{i=1}^{p_2} (-\ell - n - i + 1)_{k_i} \Big/ \prod_{i=1}^{p_2} (\ell + i)! \tag{4.31}$$

and

$$B(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=1}^{k_i - 1} (1 + t/(\delta - \ell - n - i + 1))(\Gamma(t+1))^{p_2} / (\Gamma(tp_2 + c) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1 - t/\delta)) .$$

Using (4.36), (4.37), and (4.38) of Chapter I, $\log B(t)$ can be written as

$$\log B(t) = -\log \Gamma(c) + b_1 t + b_2 t^2/2! + b_3 t^3/3! + \cdots , \tag{4.32}$$

where

$$b_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=1}^{k_i - 1} 1/(\delta - n - \ell - i + 1) + p_2(\psi(1) - \psi(c)) + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \tag{4.33}$$

and for $q \geq 2$, we have

$$b_q = p_2 \psi_{q-1}(1) - p_2^q \psi_{q-1}(c) + (q-1)! \left[ \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i - 1} (-1)^{q+1} / (\delta - n - \ell - i + 1)^q + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \right]$$

using (4.32) and lemma (4.1) of Chapter I, the residue at $t = 0$ is given by

$$R_\ell = (L_1)^\ell b_0 D_{p_2-1}(L_1;\ b)/\Gamma(p_2)\ , \tag{4.34}$$

where

$$b_0 = b_0'/\Gamma(c) \tag{4.35}$$

and $b_0'$ is given by (4.31) and $b_q'$s are given by (4.33). The determinant $D_{p_2-1}(L_1;\ b)$ is equal to the determinant on the right hand side of (4.28) with $p_2-1$ rows and $a_q'$s replaced by $b_q'$s; $q = 1, 2, \ldots, p_2-1$ .

<u>CASE B:</u> $\ell < 0$ i.e., $\ell = -p_2, -p_2+1, \ldots, -2, -1$ . For this case, (4.19) after the expansion of gamma functions can be written as

$$GP(t) = (t)^{-(p_2+\ell+1}(\Gamma(t+1))^{p_2+\ell+1}\prod_{i=1}^{-\ell-1}\Gamma(t-\ell-i)/(\Gamma(tp_2+c)\prod_{i=1}^{p_2}\prod_{\delta=1}^{\ell+i}(t-\delta)) \tag{4.36}$$

Thus in this case, we have a pole of order $p_2 + \ell + 1$ at $t = 0$. Using (4.36) in (4.18), we have

$$G(t-\ell) = (L_1)^\ell C_0'(t)^{-(p_2+\ell+1)}C(t) \tag{4.37}$$

where

$$C_0' = (-1)^{(p_2+\ell)(p_2+\ell+1)/2}\prod_{i=1}^{p_2}(-\ell-n-i+1)_{k_i}/\prod_{i=-\ell}^{p_2}(\ell+i)! \tag{4.38}$$

$$C(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=0}^{k_i-1} (1 + t/(\delta - \ell - n - i + 1)) \prod_{i=1}^{-\ell-1} \Gamma(t - \ell - i)$$

$$(\Gamma(t+1))^{p_2+\ell+1} / (\Gamma(tp_2 + c) \prod_{i=-\ell}^{p_2} \prod_{\delta=1}^{\ell+i} (1 - t/\delta)) \qquad (4.39)$$

Thus the residue at $t = 0$ is given by

$$R_\ell = (L_1)^\ell C_0 \left(\frac{d}{dt}\right)_{t=0}^{p_2+\ell} \exp(\log C(t))/\Gamma(p_2 + \ell + 1) \qquad (4.40)$$

where after using (4.36), (4.37), and (4.38) of Chapter I, $C(t)$ can be written as

$$\log C(t) = C_0'' + C_1 t + C_2 t^2/2! + C_3 t^3/3! + \cdots \qquad (4.41)$$

where

$$C_0'' = \log( \prod_{i=1}^{-(\ell+1)} \Gamma(-\ell - i)/\Gamma(c), \quad \text{where} \quad c = k + j - p_2\ell \qquad (4.42)$$

$$C_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} 1/(\delta - \ell - n - i + 1) + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} 1/\delta$$

$$+ \sum_{i=1}^{-(\ell+1)} \psi(-i - \ell) - p_2\psi(c) + (p_2 + \ell + 1)\psi(1)$$

and for $q \geq 2$, we have

$$C_q = \sum_{i=1}^{-(\ell+1)} \psi_{q-1}(-\ell - i) - p_2^q \psi_{q-1}(c) + (p_2 + \ell + 1)\psi_{q-1}(1) +$$

$$(q-1)! \left[ \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} (-1)^{q+1}/(\delta - \ell - n - i + 1)^q + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \right]$$

and let

$$C_0 = C_0' \cdot \exp(C_0'')$$

Now appealing to lemma (4.1) of Chapter I, and using (4.41) in (4.40), we have

$$R_\ell = (L_1)^\ell C_0 D_{p_2+\ell}(L_1;\, \mathbf{c})/\Gamma(p_2+\ell+1) \tag{4.43}$$

where the determinant $D_{p_2+\ell}(L_1;\mathbf{c})$ is equal to the determinant on the right hand side of (4.28) with $a_q's$ replaced by $C_q's$, $q = 1, 2, \ldots, p_2+\ell$ and have $p_2+\ell$ rows. Hence, for any $p_2 \geq 1$, we have from (4.5), (4.6) and Cauchy's residue theorem, the non-null density of $L_{vc}$ in the form

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) (L_{vc})^{n-1} p_2^{-np_2} \left[ \sum_{\substack{\ell\geq 0 \\ k+j\leq p_2\ell}} R_\ell + \sum_{\substack{\ell\geq 0 \\ k+j>p_2\ell}} R_\ell + \sum_{\ell=1}^{-p_2} R_\ell \right] \tag{4.44}$$

where $R_\ell's$ are given in (4.27), (4.34), and (4.43). If we put $p_2 = 1$ in (4.44), we get (4.15).

## 5. DISTRIBUTION OF $L_{vc}$ AS A CHI-SQUARE SERIES

In this section, we express the density of $L_{vc}$ as a chi-square series using methods similar to those of Chapter I.

Let $\lambda = (L_{vc})^n$ and $\lambda^* = -2\rho\log\lambda$, where $\rho$ is chosen so that the rate of convergence of the resulting series can be controlled, $\rho \geq 0$. Let $\phi(t)$ be the characteristic function of $\lambda^*$. Then

$$\phi(t) = E(L_{vc})^{-2it\rho n} \tag{5.1}$$

In Section 3, we obtained the non-null moments $E[L_{vc}]^h$ for integral values of $h$. But the result (3.22) can be extended to any complex number $h$ by analytic continuation. So, we have for any complex number $h$

$$E[L_{vc}]^h = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i} / \Gamma(p_2(h+n)+k+j) \tag{5.2}$$

where $B(J, k, p_2, n, \underset{\sim}{\Sigma})$ is defined by (3.18). Using (5.2), (5.1) can be written as

$$\phi(t) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-2np_2\rho it} \prod_{\delta=1}^{p_2} (1-2it\rho n-\delta)_{k_\delta} \prod_{\delta=1}^{p_2} \Gamma(n(1-2it\rho)-\delta)/\Gamma(np_2(1-2\rho it)+k+j) \tag{5.3}$$

Note that $\phi(0) = 1$ (using $\underset{\sim}{\Sigma}_{22}^{-1} = \Sigma_{2.1}^{-1} - \underset{\sim}{\Sigma}_{1.2}^{-1} \underset{\sim}{\beta}'\underset{\sim}{\beta}$) and for $t \neq 0$, (5.3) can be written as

$$\phi(t) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) \exp(\log G(t)) \tag{5.4}$$

where $G_{j,k}(t)$ is denoted by $G(t)$ and is given by

$$G(t) = \frac{p_2^{-2np_2 it} \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)-\delta+n(1-\rho)) \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)+k_\delta+1-\delta-n\rho)}{\Gamma(np_2\rho(1-2it)+k+j+p_2 n(1+\rho)) \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)+1-\delta-n\rho)} \tag{5.5}$$

Throughout this section functions G, W, w, R all depend upon j and k, but for simplicity of notation the subscripts or the superscripts j, k will not be explicitly given unless necessary. From (5.5) taking logarithm on both sides, we get

$$\log G(t) = -2np_2 it \log p_2 + \sum_{\delta=1}^{p_2} \log \Gamma(n\rho(1-2it) - \delta + n(1-\rho))$$
$$+ \sum_{\delta=1}^{p_2} \log \Gamma(n\rho(1-2it) + k_\delta + 1 - \delta - n\rho) - \log \Gamma(np_2\rho(1-2it) + k + j$$
$$+ p_2 n(1-\rho)) - \sum_{\delta=1}^{p_2} \log \Gamma(n\rho(1-2it) + 1 - \delta - n\rho) \qquad (5.6)$$

Using the expansion (5.7) of Chapter I, for each of the gamma functions in (5.6), we obtain

$$\log G(t) = (p_2 - 1)/2 \log 2\pi - (k + j + p_2 n - \tfrac{1}{2}) \log p_2$$
$$- (j + p_2 + (p_2^2 - 1)/2) \log (n\rho(1-2it)) + \sum_{r=1}^{m} (\rho n(1-2it))^r w_r$$
$$+ R_{m+1}^{0}(n, t) \quad , \qquad ((5.7)$$

where the coefficients $w_r$ are given by

$$w_r = \left[\sum_{\delta=1}^{p_2} B_{r+1}(1 - \delta - n\rho) - \sum_{\delta=1}^{p_2} B_{r+1}(1 + k_\delta - \delta - n\rho) + B_{r+1}(k + j + p_2 n(1-\rho))/p_2^r\right.$$
$$\left. - \sum_{\delta=1}^{p_2} B_{r+1}(n(1-\rho) - \delta)\right] (-1)^r/(r(r+1)) \qquad (5.8)$$

Thus $G(t)$ is given by

$$G(t) = (2\pi)^{(p_2-1)/2}(n\rho(1-2it))^{-(j+p_2+(p_2^2-1)/2)} p_2^{-(k+j+p_2n-\frac{1}{2})}$$

$$\sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-r} + R'_{m+1}(n,t) \tag{5.9}$$

where $W_r$ is the coefficient of $((1-2it)\rho n)^{-r}$ in the expansion of $\exp(\sum_{r=1}^{m}((1-2it)\rho n)^{-r}w_r)$.

Let $u = p_2 + p_2^2/2 + j - \frac{1}{2}$. Then (5.9) can be written as

$$G(t) = (2\pi)^{(p_2-1)/2} p_2^{-(k+j+p_2n-\frac{1}{2})} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-(r+u)} + R'_{m+1}(n,t) \tag{5.10}$$

Hence the characteristic function of $\lambda^*$ is given by

$$\phi(t) = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-(r+u)} + R''_{m+1}(n,t). \tag{5.11}$$

where

$$D_1(p_2, n, \underset{\sim}{\Sigma}) = D(p_2, n, \underset{\sim}{\Sigma})(2\pi)^{(p_2-1)/2} p_2^{(\frac{1}{2}-np_2)}$$

Since $(1-i\beta t)^{-\alpha}$ is the characteristic function of the gamma density $g_\alpha(\beta, x)$, where

$$g_\alpha(\beta, x) = [\beta^\alpha \Gamma(\alpha)]^{-1} x^{\alpha-1} e^{-x/\beta} \tag{5.12}$$

Thus the density of $\lambda^*$ can be derived from (5.11) in the form

$$p(\lambda^*) = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r g_{r+u}(2, \lambda^*) + R'''_{m+1}(n) \tag{5.13}$$

Hence the probability that $\lambda^*$ is larger than any value, say $\lambda_0$ is

$$p[\lambda^* > \lambda_0] = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r G_{r+u}(2, \lambda_0) + R_{m+1}(n), \tag{5.14}$$

where

$$G_{r+u}(2, \lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2, x)dx \tag{5.15}$$

and

$$R_{m+1}(n) = (2\pi)^{-1} D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} \int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda^*} \sum_{r=0}^{\infty} W_r (\rho n)^{-(r+u)} (1 - 2it)^{-(r+u)} [\exp(R'''_{m+1}(n)) - 1] dt d\lambda^* \tag{5.16}$$

From (5.14), we get the distribution of $\lambda^*$ as a series of chi-square distributions. Now

$$P[\lambda^* > \lambda_0] = P[-2\rho \log(L_{vc})^n > \lambda_0] = P[L_{vc} < \exp(-\lambda_0/2n\rho)] \tag{5.17}$$

Therefore, once we know the distribution of $\lambda^*$, the distribution of $L_{vc}$ can be obtained by using (5.17).

## 6. POWER COMPUTATIONS OF $L_{vc}$ CRITERION

Powers have been computed for $p = 2$ using (3.27) and (4.15) which have been tabulated in Table (2.1). The computations were carried out on CDC 6500 computer at Purdue University Computing Center. Before computing the power for specific values of the parameter the total probability for that case has been computed and the number of decimals included in the tables were determined depending upon the number of places of accuracy obtained in the total probabilities. From Table (2.1), we observe that power increases with the sample size N as well as the parameter $|\rho|$.

Table 2.1

Power Computations For Wilks' $L_{vc}$ Criterion

$p = 2$

| $N \backslash \lvert\rho\rvert^2$ | $.0^{4}1$ | $.0^{3}1$ | $.0^{2}1$ | $.0^{2}5$ | .01 | .1 | .15 |
|---|---|---|---|---|---|---|---|
| 3 | .05000095 | .0500095 | .050095 | .05048 | .05096 | .06047 | .06655 |
| 4 | .050002 | .050023 | .05023 | .05117 | .05236 | .07670 | .9301 |
| 5 | .050004 | .050038 | .05038 | .05191 | .05385 | .0947 | .1228 |
| 6 | .050005 | .05005 | .05053 | .05266 | .05537 | .1136 | .1542 |
| 7 | .050007 | .050068 | .05068 | .05342 | .05691 | .1331 | .1866 |
| 8 | .050008 | .050083 | .05083 | .05418 | .05846 | .1531 | .2195 |
| 9 | .0500097 | .050098 | .05098 | .05494 | .06002 | .1735 | .2528 |
| 10 | .050011 | .05011 | .05113 | .05571 | .06158 | .1942 | .2861 |
| 15 | .050019 | .05019 | .05188 | .05956 | .06952 | .2993 | .4469 |
| 20 | .050026 | .05026 | .05264 | .06347 | .07762 | .4026 | .5877 |
| 25 | .050034 | .05034 | .05339 | .06741 | .08589 | .4992 | .7022 |
| 30 | .050041 | .05041 | .05415 | .07140 | .09429 | .5864 | .7905 |
| 40 | .050056 | .05056 | .05568 | .07950 | .1115 | .7284 | .9028 |
| 50 | .05007 | .05071 | .05721 | .08774 | .1292 | .8290 | .9579 |
| 60 | .05009 | .05086 | .05874 | .09614 | .1473 | .8960 | .9827 |

Table 2.1 (Continued)

| N \ $\lvert\rho\rvert^2$ | .2 | .25 | .3 | .35 | .4 | .45 |
|---|---|---|---|---|---|---|
| 3 | .07331 | .08089 | .08944 | .09916 | .1103 | .1232 |
| 4 | .1117 | .1332 | .1579 | .1864 | .2192 | .2572 |
| 5 | .1553 | .1926 | .2352 | .2835 | .3377 | .3980 |
| 6 | .2010 | .2543 | .3138 | .3794 | .4502 | .525 |
| 7 | .2477 | .3160 | .3905 | .4695 | .5512 | .63 |
| 8 | .2946 | .3765 | .4631 | .5515 | .6387 | .73 |
| 9 | .3409 | .4347 | .5306 | .6245 | .712 | |
| 10 | .3863 | .4901 | .5924 | .6883 | .77 | |
| 15 | .5891 | .7141 | .8147 | .88 | | |
| 20 | .7409 | .8524 | .923 | .99 | | |
| 25 | .8441 | .9284 | .99 | | | |
| 30 | .9097 | .967 | | | | |
| 35 | .9494 | | | | | |
| 40 | .9724 | | | | | |

# CHAPTER III

# EXACT DISTRIBUTION OF WILKS' $L_{vc}$ CRITERION AND ITS PERCENTAGE POINTS IN THE COMPLEX CASE

## 1. INTRODUCTION

Let $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$ be independent complex normal random p-vectors with unknown mean vector $\underset{\sim}{\xi}$ and positive definite hermitian (p.d.h.) covariance matrix $\underset{\sim}{\Sigma}$, i.e., $\underset{\sim}{Z}_i \sim CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$. Let $\underset{\sim}{Z} = (\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N)$. Then $\underset{\sim}{Z} \sim CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, (see Goodman [6]), where the complex multivariate normal distribution is defined by (as can be seen from Chapter II, but repeated here for convenience)

$$CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma}) = (\pi)^{-pN}|\underset{\sim}{\Sigma}|^{-N}\exp(-tr\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{Z} - \underset{\sim}{\mu})(\overline{\underset{\sim}{Z} - \underset{\sim}{\mu}})') \tag{1.1}$$

and $\underset{\sim}{\mu} = (\underset{\sim}{\xi}, \underset{\sim}{\xi}, \ldots, \underset{\sim}{\xi})$ is a $p \times N$ complex matrix Let us define

$$\underset{\sim}{Z}_0 = N^{-1}\sum_{i=1}^{N}\underset{\sim}{Z}_i \quad \text{and} \quad \underset{\sim}{S} = \sum_{i=1}^{N}(\underset{\sim}{Z}_i - \underset{\sim}{Z}_0)\ (\overline{\underset{\sim}{Z}_i - \underset{\sim}{Z}_0})'. \tag{1.2}$$

Then $N^{-1/2}(\underset{\sim}{Z}_0 - \underset{\sim}{\xi}) \sim CN(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ and $\underset{\sim}{S}$ has an independent complex Wishart distribution which is defined by

$$CW(\underset{\sim}{S}; \underset{\sim}{p}, \underset{\sim}{N}, \underset{\sim}{\Sigma}) = [\Gamma_p(n)]^{-1}|\underset{\sim}{\Sigma}|^{-n}|\underset{\sim}{S}|^{n-p}\exp(-tr\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S}) \tag{1.3}$$

with $n = N - 1$ and

$$\tilde{\Gamma}_p(n) = (\pi)^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(n - i + 1) \tag{1.4}$$

and $\underset{\sim}{S}$ is a p.d.h. matrix of order p. In this chapter, we obtain the exact null distribution of Wilks' $L_{vc}$ criterion for testing $H: \underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$, $\sigma > 0$, $\sigma$ and $\rho$ unknown against the alternative $A \neq H$ where $\underset{\sim}{e}' = (1, 1, \ldots, 1)$. In Section 2, we present the distribution of $L_{vc}$ in terms of Meijer's [14] G-function, where as in Section 3, using the methods similar to those of Chapter II, we obtain the distribution of $L_{vc}$ in two series forms which are useful to compute the percentage points of $L_{vc}$ to a desirable degree of accuracy. The percentage points of $L_{vc}$ have been tabulated for $p = 2(1)8$ and for various values of the significance level in Table (3.1).

## 2. DERIVATION OF THE DISTRIBUTION OF $L_{vc}$

In this section, we obtain the null moments and the exact distribution of $L_{vc}$ in terms of Meijer's [14] G-function using Mellin's integral [19] as special cases of the results in Chapter II.

As in Chapter II, the test of $H: \underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$ reduces to that of $H: \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \underset{\sim}{I}_{p_2} \end{bmatrix}$, $\sigma_1 > 0$, $\sigma_2 > 0$

and unknown, against the alternative $A \neq H$; $p_2 = p - 1$. The likelihood ratio criterion for testing H versus A, can be expressed in terms of the following statistic

$$L_{vc} = |\underset{\sim}{S}| / [s_{11}(tr(\underset{\sim}{S}_{22}/p_2))^{p_2}] \tag{2.1}$$

where

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}_{12}' & \underset{\sim}{S}_{22} \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix} \quad \text{with } n = N-1,$$

$N$ being the size of the random sample from $CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$.

The following lemmas are direct consequences of theorem (3.2) of Chapter II.

Lemma 2.1. The h-th moment of $L_{vc} = |\underset{\sim}{S}|/[s_{11}(tr(S_{22}/p_2))^{p_2}]$ under the null hypothesis $H$ is given by

$$E[L_{vc}]^h = p_2^{p_2 h}\,\Gamma(np_2) \prod_{i=1}^{p_2} \Gamma(h+n-i)/[\Gamma(p_2(h+n)) \prod_{i=1}^{p_2} \Gamma(n-i)] \qquad (2.2)$$

where $h$ is any complex number.

Lemma 2.2. The null density of $L_{vc}$ is given by

$$p(L_{vc}) = D_1(p_2, n)(L_{vc})^{-(p_2+1)}\, G^{p_2\ \ 0}_{p_2\ \ p_2}\left[L_{vc}\middle|\begin{matrix} a_1, a_2, \ldots, a_{p_2} \\ b_1, b_2, \ldots, b_{p_2}\end{matrix}\right] \qquad (2.3)$$

where

$$D_1(p_2, n) = (2\pi)^{(p_2-1)/2}\, p_2^{\frac{1}{2}-np_2}\, \Gamma(np_2)/\prod_{i=1}^{p_2} \Gamma(n-i) \qquad (2.4)$$

and

$$\begin{aligned} a_i &= p_2 + n + (i-1)p_2^{-1}\,, \\ b_i &= p_2 + n - i \qquad ; \qquad i = 1, 2, \ldots, p_2\,. \end{aligned} \qquad (2.5)$$

Special Cases In particular for $p_2 = 1$ and $p_2 = 2$, respectively, we have

$$p(L_{vc}) = (L_{vc})^{n-2}\Gamma(n)/\Gamma(n-1) \qquad (2.6)$$

and

$$p(L_{vc}) = \pi^{3/2} 2^{1-2n}\Gamma(2n)/[\Gamma(n-2)\tilde{\Gamma}_2(n)](L_{vc})^{-3}\, G^{20}_{22}\left[L_{vc}\middle|\begin{matrix} 2+n, & n+3/2 \\ n\ , & n+1\end{matrix}\right] \qquad (2.7)$$

Using the duplication formula of gamma functions and (2.5) of

Chapter I, (2.7) can be written as

$$p(L_{vc}) = \frac{\Gamma(n)\Gamma(n+\frac{1}{2})}{\Gamma(n-1)\Gamma(n-2)\Gamma(\frac{7}{2})} (L_{vc})^{3/2} (1 - L_{vc})^{5/2} {}_2F_1(3/2, 1, 7/2; 1 - L_{vc}) \quad (2.8)$$

## 3. EXACT DISTRIBUTION OF $L_{vc}$ IN TWO SERIES FORMS

This section has two parts (a) and (b). In part (a) we obtain the distribution of $L_{vc}$ using method of contour integration as in Chapter II. This form of the density is well suited for the computation of percentage points of $L_{vc}$ for small values of N, the sample size. In part (b), we obtain the distribution of $L_{vc}$ as a gamma series. This form of the density has been used for the computations of percentage points for large values of N.

### (a) Distribution of $L_{vc}$ through Contour Integration

Using Mellin Integral transform on (2.2), we have the density of $L_{vc}$ in the form

$$p(L_{vc}) = D(p_2, n)\, p_2^{-p_2 n} (L_{vc})^{n-1} (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_1)^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h-i)}{\Gamma(p_2 h)}\, dh \quad (3.1)$$

where

$$D(p_2, n) = \Gamma(np_2) / \prod_{i=1}^{p_2} \Gamma(n-i) \quad (3.2)$$

and

$$L_1 = L_{vc}/p_2^{p_2} .$$

The poles of the integrand are at points

$$h = -\ell, \ \ell = -p_2, \ldots, -1, 0, 1, 2, \ldots . \quad (3.3)$$

The residue at these points can be obtained by $h = t - \ell$ in (3.1) and then finding the residue at $t=0$. Making this transformation, the integrand in (3.1) can be written as

$$G(t-\ell) = (L_1)^{\ell-t} \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)),\ \ell = -p_2, \dots, -1, 0, 1, \dots \quad (3.4)$$

Two cases arise (A) $\ell \geq 0$, (B) $\ell < 0$.

CASE A: $\ell \geq 0$. After expanding the gamma function in (3.4), we have

$$G(t-\ell) = a_0 (L_1)^{\ell} \cdot (t)^{-(p_2-1)} A(t), \quad (3.5)$$

where

$$a_0 = (-1)^{p_2(p_2+1)/2} \cdot p_2 (p_2 \ell)! / \prod_{i=1}^{p_2} (\ell+i)! \quad (3.6)$$

and

$$A(t) = (L_1)^{-t} (\Gamma(t+1))^{p_2} \prod_{\delta=1}^{p_2 \ell} (1 - p_2 t/\delta) / [\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1 - t/\delta)] \quad (3.7)$$

From (3.5), we note that we have a pole of order $(p_2-1)$ at $t=0$. Using (4.36), (4.37), (4.38) of Chapter I, we can write log A(t) as

$$\log A(t) = a_1 t + a_2 t^2/2! + a_3 t^3/3! + \dots \quad (3.8)$$

where

$$a_1 = \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) - \log L_1 - \sum_{\delta=1}^{p_2 \ell} (p_2/\delta) \quad (3.9)$$

and for $q \geq 2$

$$a_q = \psi_{q-1}^{(1)} [p_2 - p_2^q] + (q-1)! [\sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q - \sum_{\delta=1}^{p_2 \ell} (p_2/\delta)^q]$$

Now using (3.8) in (3.5) and lemma (4.1) of Chapter I, we get the residue $R_\ell$ given by

$$R_\ell = (L_1)^\ell a_0 D_{p_2-2}(L_1;\, a)/\Gamma(p_2-1) \tag{3.10}$$

where $D_{p_2-2}$ is the same as the right hand side of (4.28) of Chapter II with $a_q^{\prime}s$ defined by (3.9).

CASE B. $\ell < 0$. As before, after the expansion of the gamma functions in (3.4), we have

$$G(t-\ell) = b_0^{\prime}(L_1)^\ell (t)^{-(p_2+\ell+1)} B(t) \tag{3.11}$$

where

$$b_0^{\prime} = (-1)^{(p_2+\ell)(p_2+\ell+1)/2} \Big/ \prod_{i=-\ell}^{p_2} (\ell+i)! \tag{3.12}$$

and

$$B(t) = (L_1)^{-t}(\Gamma(t+1))^{p_2+\ell+1} \prod_{i=1}^{-\ell-1} \Gamma(t-\ell-i)/[\Gamma(p_2(t-\ell)) \prod_{i=-\ell}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)] \tag{3.13}$$

From (3.11) we notice that we have a pole of order $(p_2+\ell+1)$ at $t=0$ and as before using (4.36), (4.37), and (4.38) of Chapter I, $\log B(t)$ can be written as

$$\log B(t) = \log b_0^{\prime\prime} + b_1 t + b_2 t^2/2! + \ldots , \tag{3.14}$$

where

$$b_0^{\prime\prime} = \prod_{i=1}^{-\ell-1} \Gamma(-\ell-i)/\Gamma(-p_2\ell)$$

$$b_1 = -\log L_1 + (p_2+\ell+1)\psi(1) - p_2\psi(-p_2\ell) + \sum_{i=1}^{-\ell-1} \psi(-\ell-i) + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \tag{3.15}$$

and for $q \geq 2$, we have

$$b_q = \sum_{i=1}^{-\ell-1} \psi_{q-1}(-\ell - i) + (p_2 + \ell + 1)\psi_{q-1}(1) - p_2^q \psi_{q-1}(-p_2\ell) + (q-1)! \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \; .$$

Now using (3.14) in (3.11) and appealing to lemma (4.1) of Chapter I, we have the residue $R_\ell$ in the form

$$R_\ell = b_0(L_1)^\ell D_{p_2+\ell}(L_1;\, b)/\Gamma(p_2 + \ell + 1) \tag{3.16}$$

where $b_0 = b_0' \cdot b_0''$ and the determinant $D_{p_2+\ell}(L_1;\, b)$ is of order $p_2 + \ell$ and is the same as the one on the R.H.S. of (4.28) of Chapter II, with elements $a_q's$ replaced by $b_q's$, where $b_q's$ are defined by (3.15). Hence for any $p_2 \geq 1$, we have from (3.1) and (3.2) and Cauchy's residue theorem, the exact distribution of $L_{vc}$ in the form

$$p(L_{vc}) = D(p_2,\, n) p_2^{-np_2} (L_{vc})^{n-1} \left[ \sum_{\ell \geq 0} R_\ell + \sum_{\ell=-p_2}^{-1} R_\ell \right] \tag{3.17}$$

where $R_\ell's$ are given in (3.10) and (3.16) respectively.

(b) Distribution of $L_{vc}$ as a gamma series

We shall now obtain the distribution of $L_{vc}$ in a gamma series form. Let $\lambda = (L_{vc})^n$ and $\lambda^* = -2\rho \log \lambda$, where $\rho$ is chosen so that the rate of convergence of the resulting series can be controlled,

$0 \le \rho \le 1$. Let $\phi(t)$ be the characteristic function of $\lambda^*$. Then

$$\phi(t) = E(L_{vc})^{-2it\rho n} \tag{3.18}$$

Now using (2.2), (3.18) can be written as

$$\phi(t) = D(p_2, n)\exp(\log G(t)), \text{ where} \tag{3.19}$$

$$G(t) = p_2^{-2np_2 it} \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it) + n(1-\rho) - \delta)/\Gamma(np_2\rho(1-2it) + p_2 n(1-\rho)) \tag{3.20}$$

and $D(p_2, n)$ is given by (3.2).

Taking logarithm on both sides of (3.20) and using the expansion (5.7) of Chapter I for each of the gamma functions involved in (3.20), we obtain,

$$\log G(t) = (p_2 - 1)/2\log 2 - (p_2 n - \tfrac{1}{2})\log p_2 - (p_2 + (p_2^2 - 1)/2)\log(n\rho(1-2it)) + \sum_{r=1}^{m} (\rho n(1-2it))^r w_r + R_{m+1}^0(n, t) \tag{3.21}$$

where the coefficients $w_r$ are given by

$$w_r = \left[B_{r+1}(np_2(1-\rho))/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(n(1-\rho) - \delta)\right](-1)^r/(r(r+1)). \tag{3.22}$$

Thus $G(t)$ can be written as

$$G(t) = (2\pi)^{(p_2-1)/2} (n\rho(1-2it))^{-(p_2+(p_2^2-1)/2)} p_2^{-(p_2 n - \frac{1}{2})} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-r} + R'_{m+1}(n, t) \tag{3.23}$$

where $W_r$ is the coefficient of $((1-2it)\rho n)^{-r}$ in the expansion of $\exp(\sum_{r=1}^{m} ((1-2it)\rho n)^{-r} w_r)$.

Let $u = p_2 + p_2^2/2 - ½$. Now from (3.19) and (3.23), we have

$$\phi(t) = D_1(p_2, n) \sum_{r=0}^{\infty} ((1-2it)\rho n)^{-(r+u)} W_r + R''_{m+1}(n, t) \tag{3.24}$$

where $D_1(p_2, n) = D(p_2, n)(2\pi)^{(p_2-1)/2} p_2^{(½-np_2)}$.

Now $(1-i\beta t)^{-\alpha}$ being the characteristic function of the gamma density $g_\alpha(\beta, x)$, we have from (5.12) of Chapter II and (3.24) above, the density of $L_{vc}$ in the form

$$p(L_{vc}) = D_1(p_2, n) \sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r g_{r+u}(2, \lambda^*) + R'''_{m+1}(n) \tag{3.25}$$

Hence the probability that $\lambda^*$ is larger than any value, say $\lambda_0$ is

$$P[\lambda^* > \lambda_0] = D_1(p_2, n) \sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r G_{r+u}(2, \lambda_0) + R_{m+1}(n) \tag{3.26}$$

where $$G_{r+u}(2, \lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2, x)dx \quad \text{and} \tag{3.27}$$

$$R_{m+1}(n) = (2\pi)^{-1} D_1(p_2, n) \int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda^*} \sum_{r=0}^{\infty} W_r (\rho n)^{-(r+u)} (1-2\,it)^{-(r+u)} \cdot [\exp(R'''_{m+1}(n)) - 1] dt\, d\lambda^*. \tag{3.28}$$

The choice of $\rho = 1$ does not give rapid convergence of the series in (3.26) for small values of N. Therefore, we chose $\rho$ such that $w_1 = 0$ which is obtained by taking $\rho$ as

$$\rho = 1 - [(2p_2^2(p_2+1)(p_2+2) + p_2^2 - 1)/(6np_2(p_2^2 + 2p_2 - 1))] . \qquad (3.29)$$

Thus from (3.26) we obtain the distribution of $\lambda^*$ as a series of gamma distributions.

## 4. COMPUTATIONS OF PERCENTAGE POINTS

In this section, we tabulate the .005, .01, .025, .05, .1, and .25 percentage points of $L_1 = L^{2/N}$ for $p = 2(1)8$ and various values of $N$ using (3.17), (3.26) and (3.29). These percentage points have been presented in Table (3.1) upto four significant digits. All the computations were carried out on CDC 6500 computer at the Purdue University Computing Center. The accuracy of the results have been checked by computing the percentage points for the case $p = 3$ in two ways (i) using the exact distribution of $L_{vc}$ given by (3.17) and (ii) using the chi-square series form of the distribution of $L_{vc}$ given by (3.26). The results obtained were in complete agreement at least upto four decimal places.

Table 3.1
Percentage Points of Wilk's $L_{VC}$ Criterion (Complex Case)
p = 2

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 3 | $.0^2 5000$ | .01000 | .02500 | .05000 | .1000 | .2500 |
| 4 | .07071 | .1000 | .1581 | .2236 | .3162 | .5000 |
| 5 | .1710 | .2154 | .2924 | .3684 | .4642 | .6299 |
| 6 | .2659 | .3162 | .3976 | .4729 | .5623 | .7071 |
| 7 | .3466 | .3981 | .4782 | .5493 | .6310 | .7579 |
| 8 | .4135 | .4642 | .5407 | .6070 | .6813 | .7937 |
| 9 | .4691 | .5180 | .5904 | .6518 | .7197 | .8203 |
| 10 | .5157 | .5623 | .6306 | .6877 | .7499 | .8409 |
| 11 | .5551 | .5995 | .6637 | .7169 | .7743 | .8572 |
| 12 | .5887 | .6310 | .6915 | .7411 | .7943 | .8706 |
| 13 | .6178 | .6579 | .7151 | .7616 | .8111 | .8816 |
| 14 | .6431 | .6813 | .7354 | .7791 | .8254 | .8909 |
| 15 | .6653 | .7017 | .7530 | .7942 | .8377 | .8989 |
| 16 | .6849 | .7197 | .7684 | .8074 | .8483 | .9057 |
| 17 | .7024 | .7356 | .7820 | .8190 | .8577 | .9117 |
| 18 | .7181 | .7499 | .7941 | .8293 | .8660 | .9170 |
| 19 | .7322 | .7627 | .8049 | .8384 | .8733 | .9217 |
| 20 | .7450 | .7743 | .8147 | .8467 | .8799 | .9259 |
| 22 | .7673 | .7943 | .8316 | .8609 | .8913 | .9330 |
| 24 | .7860 | .8111 | .8456 | .8727 | .9006 | .9389 |
| 25 | .7942 | .8185 | .8518 | .8779 | .9047 | .9415 |
| 26 | .8019 | .8254 | .8575 | .8827 | .9085 | .9439 |
| 28 | .8156 | .8377 | .8677 | .8912 | .9152 | .9481 |
| 30 | .8276 | .8483 | .8766 | .8985 | .9211 | .9517 |
| 35 | .8517 | .8697 | .8942 | .9132 | .9326 | .9589 |
| 40 | .8699 | .8859 | .9075 | .9242 | .9412 | .9642 |
| 45 | .8841 | .8984 | .9178 | .9327 | .9479 | .9683 |
| 50 | .8955 | .9085 | .9260 | .9395 | .9532 | .9715 |
| 55 | .9049 | .9168 | .9328 | .9450 | .9575 | .9742 |
| 60 | .9127 | .9237 | .9384 | .9497 | .9611 | .9764 |
| 65 | .9193 | .9295 | .9431 | .9536 | .9641 | .9782 |
| 70 | .9250 | .9345 | .9472 | .9569 | .9667 | .9798 |
| 75 | .9300 | .9389 | .9507 | .9598 | .9690 | .9812 |
| 80 | .9343 | .9427 | .9538 | .9623 | .9709 | .9824 |
| 85 | .9382 | .9460 | .9565 | .9646 | .9726 | .9834 |
| 90 | .9416 | .9490 | .9590 | .9665 | .9742 | .9844 |
| 95 | .9446 | .9517 | .9611 | .9683 | .9755 | .9852 |
| 100 | .9474 | .9541 | .9631 | .9699 | .9768 | .9860 |

Table 3.1 (Continued)

p = 3

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 4 | $.0^2 1011$ | $.0^2 2028$ | $.0^2 5128$ | .01045 | .02167 | .06003 |
| 5 | .02051 | .02951 | .04824 | .07077 | .1055 | .1869 |
| 6 | .06311 | .08122 | .1144 | .1497 | .1984 | .2978 |
| 7 | .1158 | .1407 | .1835 | .2260 | .2814 | .3863 |
| 8 | .1702 | .1995 | .2479 | .2940 | .3519 | .4566 |
| 9 | .2222 | .2542 | .3055 | .3532 | .4113 | .5131 |
| 10 | .2703 | .3039 | .3565 | .4043 | .4615 | .5594 |
| 11 | .3143 | .3485 | .4013 | .4485 | .5043 | .5978 |
| 12 | .3541 | .3884 | .4408 | .4871 | .5410 | .6302 |
| 13 | .3901 | .4243 | .4758 | .5208 | .5729 | .6579 |
| 14 | .4227 | .4564 | .5069 | .5506 | .6007 | .6817 |
| 15 | .4523 | .4854 | .5346 | .5769 | .6252 | .7025 |
| 16 | .4792 | .5116 | .5595 | .6005 | .6469 | .7207 |
| 17 | .5037 | .5354 | .5820 | .6216 | .6662 | .7369 |
| 18 | .5261 | .5571 | .6023 | .6406 | .6836 | .7513 |
| 19 | .5467 | .5769 | .6208 | .6579 | .6993 | .7642 |
| 20 | .5656 | .5950 | .6377 | .6735 | .7135 | .7758 |
| 22 | .5992 | .6271 | .6674 | .7010 | .7383 | .7960 |
| 24 | .6280 | .6546 | .6926 | .7243 | .7591 | .8128 |
| 25 | .6410 | .6669 | .7039 | .7346 | .7684 | .8203 |
| 26 | .6531 | .6783 | .7144 | .7442 | .7769 | .8271 |
| 28 | .6750 | .6991 | .7332 | .7614 | .7923 | .8394 |
| 30 | .6944 | .7173 | .7498 | .7765 | .8057 | .8500 |
| 35 | .7341 | .7546 | .7835 | .8070 | .8327 | .8713 |
| 40 | .7648 | .7832 | .8092 | .8303 | .8531 | .8874 |
| 45 | .7891 | .8059 | .8294 | .8485 | .8691 | .8998 |
| 50 | .8089 | .8243 | .8458 | .8632 | .8819 | .9098 |
| 55 | .8253 | .8395 | .8593 | .8753 | .8925 | .9180 |
| 60 | .8391 | .8523 | .8707 | .8855 | .9013 | .9248 |
| 65 | .8509 | .8632 | .8803 | .8941 | .9088 | .9306 |
| 70 | .8611 | .8727 | .8886 | .9015 | .9152 | .9355 |
| 75 | .8700 | .8809 | .8959 | .9079 | .9208 | .9398 |
| 80 | .8778 | .8881 | .9022 | .9136 | .9257 | .9436 |
| 85 | .8848 | .8945 | .9078 | .9186 | .9300 | .9469 |
| 90 | .8910 | .9001 | .9129 | .9230 | .9339 | .9499 |
| 95 | .8965 | .9053 | .9174 | .9270 | .9373 | .9525 |
| 100 | .9015 | .9099 | .9214 | .9306 | .9404 | .9549 |

Table 3.1 (Continued)

p = 4

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 5 | $.0^{3}3016$ | $.0^{3}5898$ | $.0^{2}1467$ | $.0^{2}2989$ | $.0^{2}6277$ | .01822 |
| 6 | $.0^{2}6977$ | .01016 | .01696 | .02546 | .03915 | .07392 |
| 7 | .02530 | .03307 | .04778 | .06409 | .08769 | .1400 |
| 8 | .05264 | .06498 | .08690 | .1097 | .1407 | .2044 |
| 9 | .08515 | .1014 | .1291 | .1567 | .1929 | .2634 |
| 10 | .1199 | .1393 | .1713 | .2023 | .2419 | .3162 |
| 11 | .1551 | .1769 | .2120 | .2453 | .2869 | .3631 |
| 12 | .1897 | .2131 | .2505 | .2852 | .3280 | .4046 |
| 13 | .2229 | .2476 | .2864 | .3220 | .3652 | .4414 |
| 14 | .2544 | .2800 | .3197 | .3557 | .3990 | .4742 |
| 15 | .2843 | .3104 | .3506 | .3867 | .4297 | .5036 |
| 16 | .3123 | .3388 | .3792 | .4151 | .4577 | .5300 |
| 17 | .3387 | .3652 | .4056 | .4413 | .4832 | .5539 |
| 18 | .3633 | .3899 | .4301 | .4653 | .5065 | .5755 |
| 19 | .3865 | .4130 | .4527 | .4875 | .5279 | .5952 |
| 20 | .4082 | .4345 | .4738 | .5080 | .5476 | .6131 |
| 22 | .4476 | .4734 | .5117 | .5447 | .5826 | .6448 |
| 24 | .4825 | .5076 | .5447 | .5764 | .6127 | .6717 |
| 25 | .4984 | .5232 | .5596 | .5908 | .6262 | .6837 |
| 26 | .5135 | .5379 | .5736 | .6042 | .6388 | .6948 |
| 28 | .5411 | .5647 | .5992 | .6286 | .6617 | .7150 |
| 30 | .5659 | .5887 | .6220 | .6502 | .6819 | .7327 |
| 35 | .6177 | .6388 | .6692 | .6947 | .7233 | .7686 |
| 40 | .6587 | .6781 | .7060 | .7293 | .7552 | .7960 |
| 45 | .6919 | .7098 | .7355 | .7569 | .7806 | .8177 |
| 50 | .7192 | .7359 | .7597 | .7794 | .8012 | .8352 |
| 55 | .7422 | .7577 | .7799 | .7981 | .8183 | .8496 |
| 60 | .7617 | .7762 | .7969 | .8139 | .8327 | .8618 |
| 65 | .7784 | .7921 | .8115 | .8275 | .8450 | .8721 |
| 70 | .7930 | .8059 | .8242 | .8391 | .8556 | .8810 |
| 75 | .8058 | .8180 | .8352 | .8494 | .8648 | .8887 |
| 80 | .8171 | .8287 | .8450 | .8583 | .8730 | .8955 |
| 85 | .8272 | .8382 | .8536 | .8663 | .8802 | .9015 |
| 90 | .8362 | .8467 | .8614 | .8735 | .8866 | .9069 |
| 95 | .8443 | .8543 | .8684 | .8799 | .8924 | .9117 |
| 100 | .8517 | .8612 | .8747 | .8857 | .8976 | .9160 |

Table 3.1 (Continued)

p = 5

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 6 | $.0^{3}1234$ | $.0^{3}2238$ | $.0^{3}5162$ | $.0^{2}1014$ | $.0^{2}2091$ | $.0^{2}6124$ |
| 7 | $.0^{2}2510$ | $.0^{2}3677$ | $.0^{2}6227$ | $.0^{2}9493$ | .01492 | .02947 |
| 8 | .01029 | .01361 | .02006 | .02743 | .03845 | .06424 |
| 9 | .02379 | .02975 | .04061 | .05224 | .06865 | .1041 |
| 10 | .04188 | .05053 | .06563 | .08115 | .1022 | .1454 |
| 11 | .06322 | .07435 | .09323 | .1120 | .1368 | .1858 |
| 12 | .08659 | .09990 | .1220 | .1435 | .1712 | .2245 |
| 13 | .1111 | .1262 | .1510 | .1746 | .2046 | .26088 |
| 14 | .1359 | .1526 | .1795 | .2049 | .2365 | .2949 |
| 15 | .1607 | .1787 | .2073 | .2339 | .2668 | .3265 |
| 16 | .1851 | .2041 | .2340 | .2616 | .2954 | .3558 |
| 17 | .2089 | .2287 | .2597 | .2879 | .3222 | .3830 |
| 18 | .2319 | .2524 | .2841 | .3128 | .3474 | .4082 |
| 19 | .2541 | .2750 | .3073 | .3363 | .3711 | .4316 |
| 20 | .2754 | .2967 | .3293 | .3585 | .3933 | .4534 |
| 22 | .3153 | .3371 | .3700 | .3993 | .4337 | .4926 |
| 24 | .3519 | .3737 | .4067 | .4356 | .4695 | .5268 |
| 25 | .3689 | .3908 | .4236 | .4523 | .4858 | .5423 |
| 26 | .3852 | .4070 | .4396 | .4681 | .5013 | .5568 |
| 28 | .4157 | .4372 | .4694 | .4973 | .5296 | .5834 |
| 30 | .4435 | .4648 | .4963 | .5236 | .5550 | .6070 |
| 35 | .5033 | .5237 | .5535 | .5790 | .6083 | .6560 |
| 40 | .5521 | .5713 | .5994 | .6232 | .6503 | .6943 |
| 45 | .5924 | .6105 | .6369 | .6592 | .6844 | .7250 |
| 50 | .6262 | .6433 | .6681 | .6890 | .7125 | .7502 |
| 55 | .6550 | .6711 | .6944 | .7140 | .7360 | .7711 |
| 60 | .6797 | .6949 | .7169 | .7353 | .7560 | .7888 |
| 65 | .7011 | .7156 | .7364 | .7538 | .7732 | .8040 |
| 70 | .7199 | .7336 | .7533 | .7698 | .7881 | .8172 |
| 75 | .7364 | .7495 | .7683 | .7839 | .8012 | .8287 |
| 80 | .7512 | .7637 | .7815 | .7963 | .8128 | .8388 |
| 85 | .7644 | .7763 | .7933 | .8074 | .8231 | .8478 |
| 90 | .7762 | .7876 | .8039 | .8174 | .8324 | .8559 |
| 95 | .7870 | .7979 | .8135 | .8264 | .8407 | .8632 |
| 100 | .7967 | .8072 | .8222 | .8345 | .8482 | .8697 |

Table 3.1 (Continued)

p = 6

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 8 | $.0^{3}9465$ | $.0^{2}1384$ | $.0^{2}2354$ | $.0^{2}3620$ | $.0^{2}5772$ | .01177 |
| 9 | $.0^{2}4201$ | $.0^{2}5607$ | $.0^{2}8393$ | .01165 | .01665 | .02886 |
| 10 | .01062 | .01342 | .01864 | .02437 | .03266 | .05135 |
| 11 | .02019 | .02462 | .03254 | .04087 | .05245 | .07720 |
| 12 | .03249 | .03862 | .04925 | .06009 | .07473 | .1048 |
| 13 | .04697 | .05476 | .06796 | .08111 | .09847 | .1331 |
| 14 | .06308 | .07243 | .08798 | .1032 | .1229 | .1613 |
| 15 | .08035 | .09111 | .1088 | .1257 | .1475 | .1889 |
| 16 | .09835 | .1104 | .1299 | .1484 | .1718 | .2157 |
| 17 | .1168 | .1299 | .1510 | .1708 | .1956 | .2414 |
| 18 | .1354 | .1495 | .1719 | .1928 | .2187 | .2661 |
| 19 | .1539 | .1689 | .1924 | .2142 | .2410 | .2896 |
| 20 | .1723 | .1880 | .2125 | .2350 | .2625 | .3119 |
| 22 | .2081 | .2249 | .2509 | .2745 | .3030 | .3533 |
| 24 | .2423 | .2598 | .2868 | .3111 | .3401 | .3907 |
| 25 | .2586 | .2765 | .3038 | .3282 | .3574 | .4079 |
| 26 | .2745 | .2926 | .3202 | .3447 | .3739 | .4244 |
| 28 | .3048 | .3232 | .3511 | .3758 | .4049 | .4548 |
| 30 | .3332 | .3517 | .3797 | .4043 | .4332 | .4842 |
| 35 | .3961 | .4147 | .4422 | .4662 | .4942 | .5410 |
| 40 | .4493 | .4674 | .4941 | .5172 | .5438 | .5881 |
| 45 | .4944 | .5119 | .5376 | .5596 | .5849 | .6266 |
| 50 | .5331 | .5499 | .5744 | .5954 | .6194 | .6587 |
| 55 | .5664 | .5825 | .6060 | .6260 | .6487 | .6858 |
| 60 | .5955 | .6109 | .6333 | .6523 | .6739 | .7089 |
| 65 | .6210 | .6357 | .6571 | .6752 | .6957 | .7289 |
| 70 | .6435 | .6576 | .6781 | .6954 | .7149 | .7463 |
| 75 | .6636 | .6771 | .6967 | .7132 | .7318 | .7617 |
| 80 | .6815 | .6945 | .7132 | .7290 | .7468 | .7753 |
| 85 | .6977 | .7101 | .7281 | .7432 | .7602 | .7874 |
| 90 | .7123 | .7243 | .7415 | .7560 | .7723 | .7983 |
| 95 | .7255 | .7371 | .7537 | .7676 | .7832 | .8082 |
| 100 | .7377 | .7488 | .7648 | .7782 | .7932 | .8171 |

Table 3.1 (Continued)

p = 7

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 9 | $.0^{3}3774$ | $.0^{3}5468$ | $.0^{3}9234$ | $.0^{2}1420$ | $.0^{2}2278$ | $.0^{2}4745$ |
| 10 | $.0^{2}1725$ | $.0^{2}2315$ | $.0^{2}3504$ | $.0^{2}4921$ | $.0^{2}7143$ | .01275 |
| 11 | $.0^{2}4689$ | $.0^{2}5981$ | $.0^{2}8427$ | .01117 | .01522 | .02465 |
| 12 | $.0^{2}9547$ | .01176 | .01577 | .02008 | .02620 | .03971 |
| 13 | .01631 | .01957 | .02533 | .03132 | .03959 | .05711 |
| 14 | .02482 | .02921 | .03678 | .04446 | .05482 | .07611 |
| 15 | .03485 | .04039 | .04974 | .05906 | .07139 | .09612 |
| 16 | .04615 | .05280 | .06387 | .07472 | .08887 | .1167 |
| 17 | .05845 | .06616 | .07884 | .09111 | .1069 | .1374 |
| 18 | .07153 | .08024 | .09441 | .1080 | .1252 | .1581 |
| 19 | .08518 | .09481 | .1103 | .1250 | .1436 | .1784 |
| 20 | .09922 | .1097 | .1264 | .1422 | .1619 | .1984 |
| 22 | .1279 | .1399 | .1587 | .1761 | .1976 | .2369 |
| 24 | .1568 | .1699 | .1903 | .2090 | .2319 | .2730 |
| 25 | .1710 | .1846 | .2057 | .2250 | .2484 | .2901 |
| 26 | .1852 | .1992 | .2209 | .2405 | .2644 | .3067 |
| 28 | .2127 | .2275 | .2501 | .2704 | .2949 | .3379 |
| 30 | .2393 | .2545 | .2778 | .2987 | .3235 | .3669 |
| 35 | .3006 | .3166 | .3407 | .3621 | .3872 | .4304 |
| 40 | .3545 | .3708 | .3950 | .4163 | .4411 | .4832 |
| 45 | .4018 | .4179 | .4419 | .4627 | .4869 | .5275 |
| 50 | .4433 | .4591 | .4825 | .5027 | .5261 | .5651 |
| 55 | .4797 | .4952 | .5179 | .5375 | .5600 | .5973 |
| 60 | .5120 | .5269 | .5490 | .5678 | .5895 | .6252 |
| 65 | .5406 | .5551 | .5764 | .5946 | .6153 | .6495 |
| 70 | .5662 | .5802 | .6008 | .6183 | .6382 | .6709 |
| 75 | .5892 | .6028 | .6226 | .6394 | .6586 | .6899 |
| 80 | .6099 | .6230 | .6421 | .6584 | .6768 | .7068 |
| 85 | .6287 | .6414 | .6598 | .6755 | .6932 | .7220 |
| 90 | .6458 | .6581 | .6759 | .6910 | .7080 | .7358 |
| 95 | .6614 | .6733 | .6905 | .7050 | .7215 | .7482 |
| 100 | .6757 | .6872 | .7039 | .7179 | .7338 | .7595 |

Table 3.1 (Continued)

p = 8

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 10 | $.0^{3}1597$ | $.0^{3}2284$ | $.0^{3}3803$ | $.0^{3}5810$ | $.0^{3}9305$ | $.0^{2}1957$ |
| 11 | $.0^{3}7175$ | $.0^{3}9652$ | $.0^{2}1470$ | $.0^{2}2080$ | $.0^{2}3053$ | $.0^{2}5577$ |
| 12 | $.0^{2}2056$ | $.0^{2}2641$ | $.0^{2}3764$ | $.0^{2}5044$ | $.0^{2}6970$ | .01159 |
| 13 | $.0^{2}4445$ | $.0^{2}5519$ | $.0^{2}7500$ | $.0^{2}9663$ | .01279 | .01989 |
| 14 | $.0^{2}8022$ | $.0^{2}9711$ | .01273 | .01593 | .02042 | .03021 |
| 15 | .01282 | .01522 | .01940 | .02373 | .02966 | .04219 |
| 16 | .01879 | .02195 | .02738 | .03287 | .04027 | .05548 |
| 17 | .02583 | .02979 | .03648 | .04315 | .05198 | .06976 |
| 18 | .03383 | .03860 | .04654 | .05435 | .06456 | .08475 |
| 19 | .04266 | .04822 | .05738 | .06628 | .07779 | .1002 |
| 20 | .05218 | .05851 | .06884 | .07878 | .09150 | .1160 |
| 22 | .07283 | .08059 | .09307 | .1049 | .1197 | .1477 |
| 24 | .09492 | .1039 | .1183 | .1317 | .1483 | .1790 |
| 25 | .1063 | .1159 | .1310 | .1451 | .1625 | .1944 |
| 26 | .1178 | .1279 | .1438 | .1585 | .1765 | .2095 |
| 28 | .1409 | .1519 | .1691 | .1849 | .2041 | .2387 |
| 30 | .1639 | .1757 | .1940 | .2106 | .2307 | .2665 |
| 35 | .2195 | .2327 | .2528 | .2708 | .2923 | .3299 |
| 40 | .2709 | .2849 | .3059 | .3246 | .3467 | .3847 |
| 45 | .3177 | .3319 | .3534 | .3722 | .3944 | .4322 |
| 50 | .3598 | .3741 | .3956 | .4144 | .4363 | .4733 |
| 55 | .3976 | .4119 | .4332 | .4517 | .4731 | .5092 |
| 60 | .4317 | .4458 | .4667 | .4848 | .5057 | .5408 |
| 65 | .4624 | .4763 | .4967 | .5144 | .5347 | .5687 |
| 70 | .4901 | .5037 | .5237 | .5409 | .5607 | .5935 |
| 75 | .5153 | .5286 | .5481 | .5648 | .5839 | .6156 |
| 80 | .5383 | .5512 | .5701 | .5863 | .6049 | .6355 |
| 85 | .5592 | .5718 | .5902 | .6059 | .6239 | .6535 |
| 90 | .5784 | .5906 | .6085 | .6238 | .6412 | .6699 |
| 95 | .5960 | .6079 | .6253 | .6402 | .6571 | .6847 |
| 100 | .6123 | .6239 | .6408 | .6552 | .6716 | .6983 |

# CHAPTER IV

## ON THE EXACT DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR TESTING H: $\mu=\mu_0$; $\Sigma=\sigma^2 I$ .

### 1. INTRODUCTION AND SUMMARY

Let $x_1, x_2, \ldots, x_N$ be a random sample of size N from a p-variate normal distribution with unknown mean vector $\mu$ and positive definite covariance matrix $\Sigma$, i.e.,

$$x_i \sim N(\mu,\Sigma), \quad \Sigma > 0 .$$

Let

$$\bar{x} = N^{-1} \sum_{i=1}^{N} x_i \quad \text{and} \quad S = \sum_{i=1}^{N} (x_i-\bar{x})(x_i-\bar{x})'; \; n=N-1 \qquad (1.1)$$

Then the likelihood ratio criterion (LRC) for testing the hypothesis $H_0$: $\mu=\mu_0$, $\Sigma=\sigma^2 I$ against the alternative $A_0 \neq H_0$, $\sigma^2$ unknown and $\mu_0$, a given known vector can be expressed as (see Khatri and Srivastava [13])

$$L = [p^p \, |S|/[\mathrm{tr}\, S+N(\bar{x}-\mu_0)'(\bar{x}-\mu_0)]^p]^{N/2} \qquad (1.2)$$

Let

$$L_1 = L^{2/N} \quad . \qquad (1.3)$$

In this chapter, the exact null distribution of $L_1$ has been obtained in the form of Meijer's [14] G-function and also in a chisquare series form using the methods similar to those of Chapter I in order to compute percentage points of $L_1$. We also discuss the asymptotic behavior of the distribution of $-2\log L$. The percentage points of $L_1$ have been tabulated for $p=2(1)10$ and for various values of the significance level in table (4.1).

## 2. DERIVATION OF THE DISTRIBUTION OF $L_1$.

In this section, we derive the exact distribution of $L_1$. Let

$$\underset{\sim}{V} = \underset{\sim}{S}+\underset{\sim}{y}\underset{\sim}{y}' \quad\text{and}\quad u = \underset{\sim}{y}'\underset{\sim}{V}^{-1}\underset{\sim}{y} \quad\text{where}\quad \underset{\sim}{y} = N^{1/2}(\bar{\underset{\sim}{x}}-\underset{\sim}{\mu}_0) \ . \tag{2.1}$$

Under $H_0$, $\underset{\sim}{V}$ has Wishart distribution $W(\underset{\sim}{\Sigma},p,n)$. Now $\underset{\sim}{V}$ and $\underset{\sim}{w}=\underset{\sim}{V}^{1/2}\underset{\sim}{y}$ are independently distributed (see Khatri [10]) and consequently $\underset{\sim}{V}$ and $u = \underset{\sim}{w}'\underset{\sim}{w}$ are independently distributed. Now

$$1-u = (1-\underset{\sim}{y}'\underset{\sim}{V}^{-1}\underset{\sim}{y}) = |\underset{\sim}{I}-\underset{\sim}{y}\underset{\sim}{y}'\underset{\sim}{V}^{-1}| = |\underset{\sim}{V}-\underset{\sim}{y}\underset{\sim}{y}'|/|\underset{\sim}{V}| = |\underset{\sim}{S}|/|\underset{\sim}{S}+\underset{\sim}{y}\underset{\sim}{y}'| \tag{2.2}$$

$\underset{\sim}{S}$ and $\underset{\sim}{y}$ being independently distributed, $1-u$ has beta distribution with parameters $((n-p+1)/2,\ p/2)$ (see Rao [21]). Thus $u \sim \text{Beta}(p/2,\ (N-p)/2)$ and the joint distribution of $u$ and $\underset{\sim}{V}$ is given by

$$f(u,\underset{\sim}{V}) = C(p,N,\underset{\sim}{\Sigma})u^{p/2-1}(1-u)^{(N-p)/2-1}|\underset{\sim}{V}|^{(n-p)/2}\exp(-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{V}/2) \tag{2.3}$$

where

$$C^{-1}(p,N,\underset{\sim}{\Sigma}) = \beta(p/2,(N-p)/2)\ 2^{Np/2}|\underset{\sim}{\Sigma}|^{N/2}\ \Gamma_p(N/2) \tag{2.4}$$

and

$$\Gamma_p(N/2) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(N/2-(i-1)/2),$$
$$\beta(\ell,m) = \int_0^1 x^{\ell-1}(1-x)^{m-1}dx \tag{2.5}$$

In terms of $u$ and $\underset{\sim}{V}$, $L_1$ can be written as

$$L_1 = p^p|\underset{\sim}{V}|\ (1-u)/(\text{tr}\ \underset{\sim}{V})^p \tag{2.6}$$

Hence

$$E[L_1^h] = p^{ph}C(p,n,\underset{\sim}{\Sigma}) \int_{\underset{\sim}{V}>0} \int_0^1 u^{p/2-1}(1-u)^{h-1+(N-p)/2} \frac{|\underset{\sim}{V}|^{h+(n-p)/2}}{(\text{tr}\ \underset{\sim}{V})^{ph}} \exp\ (-\text{tr}\underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{V}/2)\ \ dh \tag{2.7}$$

which can be written as

$$E[L_1^h] = p^{ph}C(p,n,\underset{\sim}{\Sigma})\beta(p/2,h+(N-p)/2) \int_{\underset{\sim}{V}>\underset{\sim}{0}} |\underset{\sim}{V}|^{h+N/2-(p+1)/2} \frac{\exp(-\text{tr}\underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{V}/2)}{(\text{tr}\ \underset{\sim}{V})^{ph}}\ d\underset{\sim}{V} \tag{2.8}$$

Now, make use of the transformation $\underset{\sim}{\Sigma}^{-1/2}\underset{\sim}{V}\underset{\sim}{\Sigma}^{-1/2} = \underset{\sim}{W}$, where $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{I}$. The Jacobian of the transformation is $(\sigma^2)^{p(p+1)/2}$. under this transformation (2.8) can be written as

$$E[L_1^h] = p^{ph} C(p,n,\Sigma)\beta(p/2,h-p+N/2)(\sigma^2)^{Np/2} \int_{W>0} |W|^{h+N/2-(p+1)/2} \frac{\exp(-\mathrm{tr}\, W/2)}{(\mathrm{tr}\, W)^{ph}} dW \tag{2.9}$$

But it is well known that

$$\int_{W>0} |W|^{h+N/2-(p+1)/2} (\mathrm{tr}\, W)^{-ph} \exp(-\mathrm{tr}\, W/2) dW = \Gamma_p(h+n/2,0) \Gamma(pN/2)2^{Np/2}/\Gamma(p(h+N/2)) \tag{2.10}$$

(see Pillai and Nagarsenker [19]). Thus

$$E[L_1^h] = p^{ph}\Gamma(Np/2)\,\Gamma_p(h+(N-1)/2)/[\Gamma(p(h+N/2)\Gamma_p((N-1)/2)] \tag{2.11}$$

Now using Gauss Legendre's multiplication theorem (see (3.22) Chapter I) on $\Gamma(p(h+N/2))$ and (2.5), (2.11) becomes

$$E[L_1^h] = K(p,N) \prod_{i=1}^{p} \Gamma(h+(N-i)/2) / \prod_{i=1}^{p} \Gamma(h+N/2+(i-1)/p) \tag{2.12}$$

where

$$K(p,N) = (2\pi)^{p-1}\, p^{(1-Np)/2}\, \Gamma((N-p)/2) / \prod_{i=1}^{p} \Gamma((N-i)/2) \tag{2.13}$$

Using the Mellin integral transform on (2.12) (see Lemma (2.4), Chapter I), the distribution of $L_1$ has the form

$$f(L_1) = K(p,N)(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} (L_1)^{-(h+1)} \frac{\prod_{i=1}^{p} \Gamma((N-i)/2+h)}{\prod_{i=1}^{p} \Gamma(N/2+h+(i-1)/p)} dh \quad (2.14)$$

The integral on the right hand side can be represented in terms of Meijer's G-function [14]. Hence, we have

$$f(L_1) = K(p,N)L_1^{-1}\, G^{p\ o}_{p\ p}\left[L_1 \middle| \begin{matrix} a_1,a_2,\ldots,a_b \\ b_1,b_2,\ldots,b_p \end{matrix}\right] , \quad (2.15)$$

$$a_i = (N-i)/2, \quad b_i = N/2+(i-1)/p;\ i=1,2,3,\ldots,p \quad (2.16)$$

3. <u>DERIVATION OF THE DISTRIBUTION OF L AS A CHISQUARE SERIES</u>

Using (2.11) one obtains

$$E[L^h] = E[L_1^{Nh/2}] = p^{Nph/2}\, \Gamma(Np/2)\, \Gamma_p((N-1+Nh)/2)/[\Gamma_p((N-1)/2)\Gamma(Np(h+1)/2)] \quad (3.1)$$

Using (2.5), (3.1) can be written as

$$E[L^h] = k_1(p,N)\, p^{Nph/2} \prod_{i=1}^{p} \Gamma((N(h+1)-i)/2)/\Gamma(Np(h+1)/2) \quad (3.2)$$

where

$$k_1(p,N) = \Gamma(Np/2) \prod_{i=1}^{p} \Gamma((N-i)/2) \quad (3.3)$$

Let

$$\lambda = -2\rho \log L , \tag{3.4}$$

where $0 \le \rho \le 1$, and is chosen so that the rate of convergence of the resulting chisquare series is as rapid as possible. If $\phi(t)$ is the characteristic function of $\lambda$, then

$$\phi(t) = k_1(p,N) p^{-Nit\rho} \prod_{j=1}^{p} \Gamma(N(1-2it\rho)-j)/2) / \Gamma(Np(1-2it\rho)/2) \tag{3.5}$$

Now taking the logarithm of $\phi(t)$, we can write $\log \phi(t)$ in the form

$$\log \phi(t) = \log k_1(p,N) - p\, Nit\rho \log p + \sum_{j=1}^{p} \log \Gamma((N\rho(1-2it)+N(1-\rho)-j)/2)$$

$$- \log \Gamma((Np\rho(1-2it)+Np(1-\rho))/2) \tag{3.6}$$

Further, using expansion (5.7) of Chapter I to each of the gamma functions in (3.6), one obtains

$$\log \phi(t) = \log k_1(p,N) + (p-1)/2 \log 2\pi - s \log (N\rho(1-2it)/2)$$

$$- (Np-1)/2 \log p + \sum_{r=1}^{m} w_r (N\rho(1-2it)/2)^{-r} + R'_{m+1}(N,t) , \tag{3.7}$$

where

$$s = (p^2+3p-2)/4 \tag{3.8}$$

and the coefficients $w_r s'$ are

$$w_r = (-1)^r [B_{r+1}(Np(1-\rho)/2)/p^r - \sum_{j=1}^{p} B_{r+1}((N(1-\rho)-j)/2)]/r(r+1) \tag{3.9}$$

Thus the characteristic function of $\lambda$ can be obtained from (2.25) as

$$\phi(t) = k_2(p,N)(N\rho(1-2it)/2)^{-s} \sum_{j=0}^{\infty} W_j(N\rho/2)^{-j}(1-2it)^{-j}+R''_{m+1}(N,t) \qquad (3.10)$$

where

$$k_2(p,N) = k_1(p,N)(2\pi)^{(p-1)/2}p^{(1-Np)/2} \qquad (3.11)$$

and $W_j$ is the coefficient of $N^{-j}$ in the expansion of $\exp(\sum_{r=1}^{m} w_j N^{-j})$. Now $(1-2it)^{-a}$ being the characteristic function of a chisquare density with $2a$ degrees of freedom, say $g_{2a}(\chi^2)$, the distribution of $\lambda$ can be derived from (3.10) in the form

$$f(\lambda) = k_2(p,N)\sum_{j=0}^{\infty} W_j(N\rho/2)^{-(j+s)}g_{2(j+s)}(\chi^2)+R'''_{m+1}(N) \qquad (3.12)$$

Hence the probability that $\lambda$ is larger than any value, say $\lambda_0$, is

$$P[\lambda \geq \lambda_0] = k_2(p,N)\sum_{j=0}^{\infty} W_j(N\rho/2)^{-(j+s)}G_{2(s+j)}(\lambda_0)+R_{m+1}(N) \qquad (3.13)$$

where

$$G_{2(s+j)}(\lambda_0) = \int_{\lambda_0}^{\infty} g_{2(s+j)}(\chi^2)\, d\chi^2$$

and

$$R_{m+1}(N) = (2\pi)^{-1}k_2(p,N)\int_{\lambda_0}^{\infty}\int_{-\infty}^{\infty} \exp(-it\lambda)\sum_{j=0}^{\infty} W_j(N\rho/2)^{-(j+s)}$$

$$(1-2it)^{-(j+s)}[\exp(R'''_{m+1}(N,t))-1]dt\, d\lambda .$$

The choice of $\rho=1$ does not give rapid convergence of the series in (3.13). Thus we choose $\rho$ such that $w_1=0$, which is obtained by taking $\rho$ as follows:

$$\rho = (1-((p+1)/6Np)[(2p^3+7p^2+4(p-1))/(p^2+3p-2)] \qquad (3.14)$$

4. CHISQUARE APPROXIMATIONS TO THE DISTRIBUTION OF L.

We will now show that for large sample sizes $-2\log L$ has a chi-square distribution with $(p(p+1)/2+p-1)$ d.f.

From (3.3) and (3.5), the characteristic function of $\lambda = -2\log L$ is given by

$$\phi(t) = \frac{p^{-pNit} \prod_{i=1}^{p} \Gamma((N(1-2it)-j)/2)\Gamma(Np/2)}{\prod_{j=1}^{p} \Gamma((N-j)/2)\Gamma(Np(1-2it)/2)} \qquad (4.1)$$

Using Gauss-Legendre's multiplication theorem (see (3.22) of Chapter I)

$$\phi(t) = \prod_{j=1}^{p} \phi_j(t) \qquad (4.2)$$

where

$$\phi_j(t) = \frac{\Gamma(N/2+(j-1)/p)\,\Gamma((N(1-2it)-j)/2)}{\Gamma((N-j)/2)\Gamma(N(1-2it)/2+(j-1)/p)} \qquad (4.3)$$

Thus $-2\log L$ is distributed as the sum of $p$ independent variates, the characteristic function of the $j$th variable being given in (4.3).

Now using Stirling's approximation (see Anderson [1 ]) to each of the gamma functions in (4.3), we obtain

$$\phi_j(t) \sim \frac{\exp(-(N/2+(j-1)/p) - (N(1-2it)-j)/2)}{\exp(-(N-j)/2 - N(1-2it)/2 - (j-1)/p)}$$

$$\bullet \frac{(N/2+(j-1)/p)^{(N-1)/2+(j-1)/p}\ ((N(1-2it)-j)/2)^{(N(1-2it)-j-1)/2}}{((N-j)/2)^{(N-j-1)/2}\ (N(1-2it)/2+(j-1)/p)^{(N(1-2it)-1)/2+(j-1)/p}} \tag{4.4}$$

$$= \frac{(1-2it)^{-j/2-(j-1)/p}\ (1 - \frac{j}{N(1-2it)})^{(N(1-2it)-j-1)/2}(1+\frac{2(j-1)}{pN})^{a}}{(1 - \frac{j}{N})^{(N-j-1)/2}\ (1 + \frac{2(j-1)}{Np(1-2it)})^{(N(1-2it)-1)/2+(j-1)/p)}}$$

$$a = (N-1)/2+(j-1)/p.$$

Now as $N \to \infty$, $\phi_j(t) \to (1-2it)^{-j/2-(j-1)/p}$ the characteristic function of a $\chi^2$ variable with $j+2(j-1)/p$ degrees of freedom.

Therefore, $\lambda = -2\log L$ is asymptotically distributed as a $\chi^2$ variable with $\sum_{j=1}^{p}(j+2(j-1)/p) = p(p+1)/2+p-1$ degrees of freedom. Table (4.1) gives the percentage points of $L_1$ up to $N = 300$. For larger values of $N$, we can refer to chisquare tables. Chisquare approximations for $p = 2(1)6$ and $\alpha = .025$, $.05$ and $.1$ are given below.

Chi-Square Approximation

| p/α | .025 | .05 | .01 |
|---|---|---|---|
| 2 | .9635 | .9689 | .9744 |
| 3 | .9432 | .9496 | .9564 |
| 4 | .9209 | .9281 | .9361 |
| 5 | .8961 | .9044 | .9133 |
| 6 | .8696 | .8784 | .8882 |

## 5. COMPUTATIONS OF PERCENTAGE POINTS

In this section, we tabluate the .005, .01, .025, .05, .1 and .25 percentage points of $L_1 = L^{2/N}$ for $p = 2(1)10$ and various values of $N$ using (2.15),(3.13) and (3.14). These percentage points have been presented in Table (4.1) up to four significant digits. All the computations here were carried out on a CDC 6500 computer at the Purdue University Computing Center. The accuracy of the results have been checked by computing the percentage points for the case $p = 2$ in two ways, (i) using the exact distribution of $L_1$ in terms of Meijer's G-function and (ii) using the chisquare series form of the distribution of $L_1$. The results obtained in two ways were in complete agreement at least up to four significant digits.

Table 4.1

Percentage Points of the LRC for testing H : $\mu = \mu_0$ , $\Sigma = \sigma^2 I$

p = 2

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 3 | $.0^{5}_{2}6269$ | $.0^{4}_{2}2513$ | $.0^{3}_{2}1583$ | $.0^{3}6412$ | $.0^{2}2634$ | .01795 |
| 4 | $.0^{2}1714$ | $.0^{2}3469$ | $.0^{2}8892$ | .01832 | .03834 | .1065 |
| 5 | .01229 | .01984 | .03768 | .06180 | .1027 | .2082 |
| 6 | .03426 | .04931 | .08042 | .1174 | .1732 | .2979 |
| 7 | .06451 | .08662 | .1287 | .1749 | .2398 | .3727 |
| 8 | .09928 | .1272 | .1775 | .2297 | .2997 | .4347 |
| 9 | .1358 | .1681 | .2242 | .2802 | .3525 | .4862 |
| 10 | .1722 | .2097 | .2678 | .3259 | .3990 | .5296 |
| 11 | .2077 | .2457 | .3080 | .3670 | .4398 | .5665 |
| 12 | .2415 | .2811 | .3448 | .4040 | .4758 | .5981 |
| 13 | .2735 | .3141 | .3785 | .4373 | .5077 | .6256 |
| 14 | .3036 | .3448 | .4092 | .4674 | .5361 | .6496 |
| 15 | .3317 | .3732 | .4373 | .4946 | .5615 | .6707 |
| 16 | .3581 | .3996 | .4631 | .5193 | .5844 | .6895 |
| 17 | .3827 | .4240 | .4868 | .5418 | .6050 | .7063 |
| 18 | .4057 | .4467 | .5085 | .5623 | .6238 | .7213 |
| 19 | .4273 | .4678 | .5286 | .5812 | .6409 | .7349 |
| 20 | .4474 | .4875 | .5472 | .5985 | .6565 | .7473 |
| 22 | .4840 | .5229 | .5804 | .6293 | .6840 | .7689 |
| 24 | .5163 | .5540 | .6092 | .6558 | .7075 | .7871 |
| 25 | .5311 | .5681 | .6222 | .6677 | .7181 | .7951 |
| 26 | .5450 | .5814 | .6344 | .6788 | .7278 | .8026 |
| 28 | .5706 | .6057 | .6566 | .6989 | .7455 | .8160 |
| 30 | .5935 | .6275 | .6763 | .7167 | .7610 | .8278 |
| 35 | .6416 | .6727 | .7170 | .7533 | .7927 | .8515 |
| 40 | .6797 | .7083 | .7486 | .7815 | .8170 | .8694 |
| 45 | .7106 | .7369 | .7740 | .8040 | .8362 | .8835 |
| 50 | .7360 | .7605 | .7947 | .8223 | .8518 | .8949 |
| 55 | .7574 | .7802 | .8120 | .8375 | .8647 | .9042 |
| 60 | .7757 | .7969 | .8265 | .8503 | .8755 | .9121 |
| 65 | .7913 | .8113 | .8390 | .8612 | .8847 | .9187 |
| 70 | .8050 | .8238 | .8499 | .8707 | .8926 | .9244 |
| 75 | .8169 | .8347 | .8593 | .8789 | .8996 | .9294 |
| 80 | .8275 | .8444 | .8677 | .8862 | .9056 | .9337 |
| 85 | .8370 | .8530 | .8751 | .8926 | .9110 | .9375 |
| 90 | .8454 | .8607 | .8817 | .8983 | .9158 | .9410 |
| 95 | .8530 | .8676 | .8876 | .9035 | .9202 | .9440 |
| 100 | .8600 | .8739 | .8930 | .9082 | .9241 | .9468 |
| 150 | .9048 | .9145 | .9277 | .9381 | .9490 | .9644 |
| 200 | .9279 | .9353 | .9454 | .9533 | .9616 | .9732 |
| 250 | .9420 | .9480 | .9561 | .9625 | .9692 | .9786 |
| 300 | .9514 | .9565 | .9634 | .9687 | .9743 | .9821 |

Table 4.1 (Continued)

p = 3

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 4 | $.0^{5}1913$ | $.0^{5}6487$ | $.0^{4}3589$ | $.0^{3}1389$ | $.0^{3}5643$ | $.0^{2}4016$ |
| 5 | $.0^{3}4443$ | $.0^{3}9085$ | $.0^{2}2380$ | $.0^{2}5032$ | .01095 | .03329 |
| 6 | $.0^{2}3879$ | $.0^{2}6367$ | .01247 | .02111 | .03665 | .08134 |
| 7 | .01260 | .01848 | .03115 | .04697 | .07235 | .1355 |
| 8 | .02673 | .03660 | .05620 | .07883 | .1126 | .1892 |
| 9 | .04523 | .05908 | .08509 | .1135 | .1539 | .2398 |
| 10 | .06680 | .08430 | .1159 | .1490 | .1943 | .2863 |
| 11 | .09030 | .1110 | .1471 | .1839 | .2328 | .3286 |
| 12 | .1148 | .1382 | .1781 | .2176 | .2690 | .3668 |
| 13 | .1397 | .1653 | .2081 | .2497 | .3028 | .4014 |
| 14 | .1645 | .1919 | .2370 | .2801 | .3342 | .4328 |
| 15 | .1889 | .2177 | .2645 | .3086 | .3633 | .4613 |
| 16 | .2126 | .2426 | .2907 | .3354 | .3903 | .4872 |
| 17 | .2355 | .2664 | .3154 | .3605 | .4153 | .5109 |
| 18 | .2577 | .2892 | .3388 | .3841 | .4386 | .5325 |
| 19 | .2789 | .3109 | .3609 | .4061 | .4601 | .5524 |
| 20 | .2993 | .3316 | .3817 | .4268 | .4802 | .5708 |
| 22 | .3373 | .3700 | .4200 | .4644 | .5164 | .6034 |
| 24 | .3721 | .4048 | .4542 | .4977 | .5482 | .6315 |
| 25 | .3883 | .4209 | .4699 | .5129 | .5626 | .6441 |
| 26 | .4039 | .4362 | .4848 | .5272 | .5761 | .6559 |
| 28 | .4328 | .4648 | .5124 | .5536 | .6009 | .6774 |
| 30 | .4593 | .4908 | .5373 | .5773 | .6230 | .6964 |
| 35 | .5165 | .5463 | .5900 | .6271 | .6690 | .7353 |
| 40 | .5631 | .5913 | .6322 | .6666 | .7051 | .7655 |
| 45 | .6018 | .6284 | .6666 | .6986 | .7342 | .7896 |
| 50 | .6344 | .6594 | .6953 | .7251 | .7581 | .8091 |
| 55 | .6621 | .6857 | .7194 | .7473 | .7781 | .8254 |
| 60 | .6860 | .7083 | .7400 | .7663 | .7950 | .8391 |
| 65 | .7067 | .7279 | .7579 | .7826 | .8096 | .8508 |
| 70 | .7249 | .7450 | .7734 | .7968 | .8222 | .8610 |
| 75 | .7410 | .7601 | .7871 | .8092 | .8333 | .8698 |
| 80 | .7553 | .7736 | .7993 | .8202 | .8430 | .8776 |
| 85 | .7682 | .7856 | .8101 | .8301 | .8517 | .8845 |
| 90 | .7798 | .7964 | .8198 | .8389 | .8595 | .8907 |
| 95 | .7902 | .8062 | .8286 | .8468 | .8665 | .8962 |
| 100 | .7997 | .8151 | .8366 | .8540 | .8729 | .9012 |
| 150 | .8624 | .8733 | .8885 | .9007 | .9138 | .9334 |
| 200 | .8952 | .9036 | .9154 | .9248 | .9348 | .9498 |
| 250 | .9154 | .9223 | .9318 | .9394 | .9476 | .9597 |
| 300 | .9290 | .9349 | .9429 | .9493 | .9562 | .9663 |

Table 4.1 (Continued)

p = 4

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 5 | $.0^{6}5136$ | $.0^{5}1756$ | $.0^{5}9597$ | $.0^{4}3670$ | $.0^{3}1484$ | $.0^{2}1082$ |
| 6 | $.0^{3}1324$ | $.0^{3}2722$ | $.0^{3}7228$ | $.0^{2}1556$ | $.0^{2}3478$ | .01123 |
| 7 | $.0^{2}1328$ | $.0^{2}2207$ | $.0^{2}4421$ | $.0^{2}7666$ | .01374 | .03251 |
| 8 | $.0^{2}4860$ | $.0^{2}7241$ | .01252 | .01953 | .03079 | .06136 |
| 9 | .01138 | .01538 | .02495 | .03588 | .05288 | .09427 |
| 10 | .02088 | .02772 | .04095 | .05597 | .07818 | .1288 |
| 11 | .03300 | .04230 | .05959 | .07842 | .1052 | .1633 |
| 12 | .04723 | .05893 | .08001 | .1022 | .1329 | .1969 |
| 13 | .06305 | .07699 | .1015 | .1267 | .1606 | .2292 |
| 14 | .08000 | .09599 | .1235 | .1512 | .1878 | .2599 |
| 15 | .09769 | .1155 | .1457 | .1755 | .2142 | .2889 |
| 16 | .1158 | .1353 | .1676 | .1992 | .2396 | .3161 |
| 17 | .1341 | .1550 | .1893 | .2222 | .2640 | .3418 |
| 18 | .1524 | .1745 | .2104 | .2445 | .2872 | .3658 |
| 19 | .1706 | .1937 | .2309 | .2659 | .3094 | .3883 |
| 20 | .1884 | .2124 | .2507 | .2865 | .3305 | .4095 |
| 22 | .2231 | .2485 | .2884 | .3250 | .3696 | .4481 |
| 24 | .2561 | .2823 | .3232 | .3603 | .4048 | .4823 |
| 25 | .2718 | .2984 | .3396 | .3768 | .4212 | .4979 |
| 26 | .2871 | .3140 | .3553 | .3925 | .4367 | .5127 |
| 28 | .3163 | .3434 | .3850 | .4220 | .4657 | .5398 |
| 30 | .3435 | .3709 | .4123 | .4489 | .4919 | .5643 |
| 35 | .4042 | .4313 | .4718 | .5071 | .5480 | .6155 |
| 40 | .4556 | .4819 | .5210 | .5546 | .5932 | .6562 |
| 45 | .4993 | .5247 | .5621 | .5941 | .6304 | .6892 |
| 50 | .5369 | .5613 | .5969 | .6273 | .6615 | .7165 |
| 55 | .5694 | .5928 | .6268 | .6555 | .6879 | .7395 |
| 60 | .5977 | .6201 | .6526 | .6799 | .7105 | .7590 |
| 65 | .6227 | .6441 | .6751 | .7010 | .7300 | .7758 |
| 70 | .6448 | .6653 | .6949 | .7196 | .7471 | .7904 |
| 75 | .6645 | .6842 | .7125 | .7360 | .7622 | .8033 |
| 80 | .6821 | .7010 | .7281 | .7506 | .7756 | .8146 |
| 85 | .6981 | .7162 | .7422 | .7637 | .7876 | .8248 |
| 90 | .7125 | .7299 | .7549 | .7755 | .7983 | .8338 |
| 95 | .7256 | .7424 | .7664 | .7862 | .8081 | .8420 |
| 100 | .7376 | .7538 | .7769 | .7959 | .8169 | .8495 |
| 150 | .8175 | .8293 | .8461 | .8597 | .8747 | .8976 |
| 200 | .8602 | .8695 | .8825 | .8932 | .9048 | .9224 |
| 250 | .8867 | .8943 | .9051 | .9138 | .9232 | .9376 |
| 300 | .9048 | .9113 | .9203 | .9277 | .9357 | .9478 |

Table 4.1 (Continued)

p = 5

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 6 | $.0^{6}3114$ | $.0^{6}8901$ | $.0^{5}3956$ | $.0^{4}1337$ | $.0^{4}4963$ | $.0^{3}3438$ |
| 7 | $.0^{4}4354$ | $.0^{4}8888$ | $.0^{3}2363$ | $.0^{3}5129$ | $.0^{2}1167$ | $.0^{2}3932$ |
| 8 | $.0^{3}4727$ | $.0^{3}7935$ | $.0^{2}1618$ | $.0^{2}2858$ | $.0^{2}5251$ | .01306 |
| 9 | $.0^{2}1909$ | $.0^{2}2880$ | $.0^{2}5085$ | $.0^{2}8025$ | .01311 | .02745 |
| 10 | $.0^{2}4867$ | $.0^{2}6865$ | .01106 | .01623 | .02456 | .04593 |
| 11 | $.0^{2}9602$ | .01292 | .01952 | .02722 | .03899 | .06722 |
| 12 | .01614 | .02097 | .03018 | .04051 | .05568 | .09025 |
| 13 | .02435 | .03079 | .04268 | .05558 | .07396 | .1142 |
| 14 | .03403 | .04210 | .05662 | .07196 | .09328 | .1384 |
| 15 | .04494 | .05461 | .07163 | .08923 | .1132 | .1626 |
| 16 | .05685 | .06804 | .08740 | .1070 | .1333 | .1863 |
| 17 | .06954 | .08216 | .1037 | .1251 | .1535 | .2094 |
| 18 | .08280 | .09675 | .1202 | .1433 | .1734 | .2318 |
| 19 | .09648 | .1116 | .1368 | .1614 | .1930 | .2534 |
| 20 | .1104 | .1267 | .1535 | .1792 | .2121 | .2742 |
| 22 | .1387 | .1568 | .1862 | .2140 | .2489 | .3132 |
| 24 | .1668 | .1865 | .2178 | .2471 | .2833 | .3490 |
| 25 | .1807 | .2010 | .2331 | .2630 | .2997 | .3657 |
| 26 | .1944 | .2152 | .2481 | .2784 | .3155 | .3817 |
| 28 | .2211 | .2428 | .2768 | .3078 | .3454 | .4117 |
| 30 | .2468 | .2692 | .3039 | .3353 | .3731 | .4391 |
| 35 | .3061 | .3294 | .3650 | .3967 | .4342 | .4983 |
| 40 | .3584 | .3820 | .4175 | .4487 | .4852 | .5467 |
| 45 | .4044 | .4277 | .4626 | .4930 | .5282 | .5869 |
| 50 | .4448 | .4677 | .5017 | .5311 | .5649 | .6207 |
| 55 | .4804 | .5027 | .5357 | .5640 | .5964 | .6495 |
| 60 | .5120 | .5337 | .5655 | .5927 | .6238 | .6743 |
| 65 | .5401 | .5611 | .5919 | .6180 | .6477 | .6958 |
| 70 | .5653 | .5856 | .6153 | .6404 | .6689 | .7148 |
| 75 | .5879 | .6076 | .6362 | .6604 | .6877 | .7315 |
| 80 | .6084 | .6274 | .6550 | .6783 | .7045 | .7464 |
| 85 | .6270 | .6454 | .6720 | .6944 | .7195 | .7597 |
| 90 | .6439 | .6617 | .6874 | .7090 | .7332 | .7717 |
| 95 | .6594 | .6766 | .7015 | .7223 | .7456 | .7826 |
| 100 | .6736 | .6903 | .7143 | .7344 | .7569 | .7925 |
| 150 | .7701 | .7826 | .8005 | .8153 | .8317 | .8574 |
| 200 | .8227 | .8327 | .8469 | .8586 | .8714 | .8914 |
| 250 | .8557 | .8640 | .8758 | .8854 | .8960 | .9124 |
| 300 | .8784 | .8855 | .8955 | .9037 | .9127 | .9265 |

Table 4.1 (Continued)

p = 6

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 8 | $.0^{4}1581$ | $.0^{4}3155$ | $.0^{4}8244$ | $.0^{3}1784$ | $.0^{3}4086$ | $.0^{2}1416$ |
| 9 | $.0^{3}1732$ | $.0^{3}2924$ | $.0^{3}6035$ | $.0^{2}1081$ | $.0^{2}2026$ | $.0^{2}5246$ |
| 10 | $.0^{3}7562$ | $.0^{2}1153$ | $.0^{2}2071$ | $.0^{2}3325$ | $.0^{2}5551$ | .01212 |
| 11 | $.0^{2}2078$ | $.0^{2}2965$ | $.0^{2}4867$ | $.0^{2}7273$ | .01125 | .02192 |
| 12 | $.0^{2}4377$ | $.0^{2}5962$ | $.0^{2}9179$ | .01303 | .01907 | .03421 |
| 13 | $.0^{2}7789$ | .01024 | .01503 | .02052 | .02880 | .04849 |
| 14 | .01235 | .01580 | .02232 | .02955 | .04014 | .06425 |
| 15 | .01803 | .02257 | .03090 | .03991 | .05276 | .08105 |
| 16 | .02474 | .03041 | .04058 | .05135 | .06638 | .09854 |
| 17 | .03239 | .03919 | .05119 | .06365 | .08073 | .1164 |
| 18 | .04084 | .04877 | .06254 | .07660 | .09558 | .1344 |
| 19 | .04998 | .05901 | .07446 | .09003 | .1108 | .1524 |
| 20 | .05970 | .06977 | .08682 | .1038 | .1261 | .1703 |
| 22 | .08042 | .09244 | .1124 | .1318 | .1568 | .2051 |
| 24 | .1023 | .1160 | .1384 | .1598 | .1870 | .2384 |
| 25 | .1135 | .1279 | .1514 | .1737 | .2018 | .2544 |
| 26 | .1247 | .1399 | .1643 | .1874 | .2163 | .2699 |
| 28 | .1473 | .1637 | .1898 | .2141 | .2444 | .2996 |
| 30 | .1697 | .1871 | .2145 | .2399 | .2711 | .3274 |
| 35 | .2237 | .2429 | .2726 | .2996 | .3322 | .3895 |
| 40 | .2737 | .2938 | .3247 | .3523 | .3852 | .4421 |
| 45 | .3191 | .3398 | .3710 | .3987 | .4313 | .4870 |
| 50 | .3602 | .3809 | .4121 | .4395 | .4715 | .5255 |
| 55 | .3971 | .4178 | .4486 | .4755 | .5067 | .5589 |
| 60 | .4305 | .4509 | .4812 | .5074 | .5377 | .5880 |
| 65 | .4607 | .4807 | .5103 | .5358 | .5651 | .6136 |
| 70 | .4880 | .5076 | .5365 | .5612 | .5896 | .6363 |
| 75 | .5128 | .5320 | .5601 | .5841 | .6116 | .6565 |
| 80 | .5355 | .5542 | .5815 | .6048 | .6313 | .6747 |
| 85 | .5562 | .5744 | .6010 | .6236 | .6492 | .6910 |
| 90 | .5752 | .5929 | .6187 | .6407 | .6655 | .7058 |
| 95 | .5927 | .6100 | .6350 | .6563 | .6803 | .7192 |
| 100 | .6089 | .6257 | .6500 | .6707 | .6939 | .7315 |
| 150 | .7205 | .7336 | .7524 | .7680 | .7856 | .8134 |
| 200 | .7829 | .7935 | .8086 | .8212 | .8351 | .8571 |
| 250 | .8226 | .8315 | .8441 | .8545 | .8661 | .8842 |
| 300 | .8501 | .8577 | .8685 | .8774 | .8873 | .9027 |

Table 4.1 (Continued)

p = 7

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 9 | $.0^{5}6329$ | $.0^{4}1222$ | $.0^{4}3087$ | $.0^{4}6578$ | $.0^{3}1498$ | $.0^{3}5260$ |
| 10 | $.0^{4}6543$ | $.0^{3}1104$ | $.0^{3}2293$ | $.0^{3}4146$ | $.0^{3}7886$ | $.0^{2}2108$ |
| 11 | $.0^{3}3015$ | $.0^{3}4633$ | $.0^{3}8444$ | $.0^{2}1375$ | $.0^{2}2339$ | $.0^{2}5290$ |
| 12 | $.0^{3}8837$ | $.0^{2}1274$ | $.0^{2}2126$ | $.0^{2}3226$ | $.0^{2}5087$ | .01027 |
| 13 | $.0^{2}1976$ | $.0^{2}2721$ | $.0^{2}4261$ | $.0^{2}6144$ | $.0^{2}9168$ | .01702 |
| 14 | $.0^{2}3708$ | $.0^{2}4931$ | $.0^{2}7356$ | .01020 | .01459 | .02540 |
| 15 | $.0^{2}6163$ | $.0^{2}7972$ | .01145 | .01539 | .02129 | .03520 |
| 16 | $.0^{2}9379$ | .01186 | .01651 | .02165 | .02912 | .04616 |
| 17 | .01336 | .01659 | .02249 | .02887 | .03796 | .05806 |
| 18 | .01808 | .02209 | .02931 | .03695 | .04764 | .07070 |
| 19 | .02348 | .02832 | .03686 | .04576 | .05801 | .08388 |
| 20 | .02953 | .03519 | .04506 | .05519 | .06894 | .09744 |
| 22 | .04326 | .05057 | .06301 | .07546 | .09200 | .1252 |
| 24 | .05876 | .06763 | .08245 | .09701 | .1160 | .1531 |
| 25 | .06702 | .07663 | .09255 | .10806 | .1281 | .1670 |
| 26 | .07554 | .08585 | .1028 | .1192 | .1403 | .1807 |
| 28 | .09318 | .1048 | .1237 | .1417 | .1645 | .2075 |
| 30 | .1114 | .1241 | .1446 | .1640 | .1883 | .2334 |
| 35 | .1574 | .1725 | .1962 | .2180 | .2449 | .2935 |
| 40 | .2024 | .2190 | .2448 | .2682 | .2965 | .3466 |
| 45 | .2451 | .2627 | .2897 | .3139 | .3429 | .3934 |
| 50 | .2848 | .3030 | .3306 | .3552 | .3844 | .4346 |
| 55 | .3215 | .3399 | .3679 | .3925 | .4215 | .4709 |
| 60 | .3552 | .3738 | .4017 | .4261 | .4547 | .5031 |
| 65 | .3862 | .4047 | .4324 | .4565 | .4846 | .5318 |
| 70 | .4146 | .4330 | .4603 | .4840 | .5115 | .5575 |
| 75 | .4408 | .4590 | .4858 | .5091 | .5359 | .5806 |
| 80 | .4649 | .4828 | .5092 | .5319 | .5581 | .6015 |
| 85 | .4872 | .5047 | .5306 | .5528 | .5783 | .6205 |
| 90 | .5077 | .5250 | .5503 | .5720 | .5968 | .6377 |
| 95 | .5268 | .5437 | .5685 | .5896 | .6138 | .6535 |
| 100 | .5445 | .5611 | .5853 | .6059 | .6295 | .6680 |
| 150 | .6696 | .6830 | .7023 | .7185 | .7368 | .7663 |
| 200 | .7414 | .7524 | .7682 | .7814 | .7962 | .8198 |
| 250 | .7877 | .7970 | .8103 | .8214 | .8338 | .8535 |
| 300 | .8200 | .8280 | .8395 | .8491 | .8597 | .8765 |

Table 4.1 (Continued)

p = 8

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 11 | $.0^{4}2560$ | $.0^{4}4297$ | $.0^{4}8908$ | $.0^{3}1617$ | $.0^{3}3105$ | $.0^{3}8505$ |
| 12 | $.0^{3}1212$ | $.0^{3}1873$ | $.0^{3}3449$ | $.0^{3}5682$ | $.0^{3}9813$ | $.0^{2}2288$ |
| 13 | $.0^{3}3748$ | $.0^{3}5448$ | $.0^{3}9220$ | $.0^{2}1418$ | $.0^{2}2274$ | $.0^{2}4736$ |
| 14 | $.0^{3}8844$ | $.0^{2}1229$ | $.0^{2}1954$ | $.0^{2}2857$ | $.0^{2}4337$ | $.0^{2}8307$ |
| 15 | $.0^{2}1743$ | $.0^{2}2341$ | $.0^{2}3546$ | $.0^{2}4987$ | $.0^{2}7256$ | .01302 |
| 16 | $.0^{2}3029$ | $.0^{2}3957$ | $.0^{2}5769$ | $.0^{2}7864$ | .01106 | .01883 |
| 17 | $.0^{2}4797$ | $.0^{2}6128$ | $.0^{2}8657$ | .01150 | .01573 | .02564 |
| 18 | $.0^{2}7082$ | $.0^{2}8878$ | .01222 | .01589 | .02122 | .03336 |
| 19 | $.0^{2}9896$ | .01221 | .01643 | .02098 | .02746 | .04185 |
| 20 | .01324 | .01611 | .02127 | .02673 | .03438 | .05100 |
| 22 | .02144 | .02551 | .03262 | .03994 | .04994 | .07084 |
| 24 | .03146 | .03678 | .04586 | .05501 | .06725 | .09210 |
| 25 | .03707 | .04300 | .05306 | .06309 | .07638 | .1030 |
| 26 | .04302 | .04957 | .06058 | .07146 | .08576 | .1141 |
| 28 | .05581 | .06355 | .07636 | .08885 | .1050 | .1365 |
| 30 | .06955 | .07841 | .09289 | .1068 | .1247 | .1588 |
| 35 | .1065 | .1178 | .1358 | .1527 | .1738 | .2129 |
| 40 | .1449 | .1580 | .1786 | .1976 | .2210 | .2632 |
| 45 | .1830 | .1975 | .2199 | .2403 | .2651 | .3092 |
| 50 | .2197 | .2351 | .2588 | .2802 | .3058 | .3508 |
| 55 | .2546 | .2706 | .2951 | .3170 | .3431 | .3883 |
| 60 | .2873 | .3038 | .3288 | .3509 | .3771 | .4222 |
| 65 | .3180 | .3347 | .3599 | .3821 | .4082 | .4528 |
| 70 | .3466 | .3634 | .3886 | .4107 | .4366 | .4806 |
| 75 | .3732 | .3900 | .4151 | .4371 | .4627 | .5058 |
| 80 | .3980 | .4148 | .4397 | .4613 | .4865 | .5288 |
| 85 | .4211 | .4377 | .4624 | .4837 | .5085 | .5499 |
| 90 | .4427 | .4591 | .4834 | .5045 | .5287 | .5692 |
| 95 | .4628 | .4791 | .5030 | .5237 | .5475 | .5870 |
| 100 | .4817 | .4977 | .5212 | .5415 | .5648 | .6034 |
| 150 | .6179 | .6314 | .6509 | .6675 | .6863 | .7168 |
| 200 | .6984 | .7097 | .7260 | .7397 | .7552 | .7801 |
| 250 | .7511 | .7608 | .7747 | .7863 | .7994 | .8204 |
| 300 | .7882 | .7966 | .8087 | .8188 | .8302 | .8482 |

Table 4.1 (Continued)

p = 9

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 13 | $.0^{4}4939$ | $.0^{4}7646$ | $.0^{3}1417$ | $.0^{3}2354$ | $.0^{3}4115$ | $.0^{3}9837$ |
| 14 | $.0^{3}1588$ | $.0^{3}2325$ | $.0^{3}3980$ | $.0^{3}6192$ | $.0^{2}1008$ | $.0^{2}2157$ |
| 15 | $.0^{3}3929$ | $.0^{3}5506$ | $.0^{3}8870$ | $.0^{2}1313$ | $.0^{2}2024$ | $.0^{2}3986$ |
| 16 | $.0^{3}8105$ | $.0^{2}1098$ | $.0^{2}1687$ | $.0^{2}2402$ | $.0^{2}3549$ | $.0^{2}6544$ |
| 17 | $.0^{2}1469$ | $.0^{2}1936$ | $.0^{2}2862$ | $.0^{2}3950$ | $.0^{2}5639$ | $.0^{2}9862$ |
| 18 | $.0^{2}2416$ | $.0^{2}3114$ | $.0^{2}4460$ | $.0^{2}5999$ | $.0^{2}8326$ | .01393 |
| 19 | $.0^{2}3692$ | $.0^{2}4669$ | $.0^{2}6513$ | $.0^{2}8573$ | .01162 | .01873 |
| 20 | $.0^{2}5323$ | $.0^{2}6626$ | $.0^{2}9035$ | .01167 | .01550 | .02420 |
| 22 | $.0^{2}9713$ | .01178 | .01548 | .01940 | .02492 | .03693 |
| 24 | .01561 | .01855 | .02369 | .02900 | .03628 | .05162 |
| 25 | .01910 | .02250 | .02839 | .03441 | .04257 | .05954 |
| 26 | .02293 | .02681 | .03346 | .04018 | .04922 | .06777 |
| 28 | .03153 | .03637 | .04454 | .05266 | .06340 | .08494 |
| 30 | .04124 | .04703 | .05668 | .06614 | .07848 | .1028 |
| 35 | .06914 | .07717 | .09020 | .1026 | .1184 | .1484 |
| 40 | .1003 | .1102 | .1260 | .1408 | .1593 | .1934 |
| 45 | .1327 | .1442 | .1622 | .1788 | .1992 | .2362 |
| 50 | .1653 | .1779 | .1976 | .2155 | .2373 | .2762 |
| 55 | .1971 | .2107 | .2316 | .2504 | .2731 | .3132 |
| 60 | .2278 | .2420 | .2638 | .2833 | .3066 | .3473 |
| 65 | .2571 | .2718 | .2941 | .3140 | .3377 | .3787 |
| 70 | .2848 | .2999 | .3226 | .3428 | .3666 | .4075 |
| 75 | .3111 | .3264 | .3493 | .3695 | .3933 | .4340 |
| 80 | .3359 | .3512 | .3743 | .3945 | .4181 | .4584 |
| 85 | .3592 | .3746 | .3976 | .4177 | .4412 | .4809 |
| 90 | .3812 | .3966 | .4194 | .4394 | .4626 | .5018 |
| 95 | .4019 | .4172 | .4399 | .4596 | .4826 | .5211 |
| 100 | .4214 | .4366 | .4591 | .4786 | .5012 | .5391 |
| 150 | .5662 | .5795 | .5990 | .6157 | .6347 | .7168 |
| 200 | .6544 | .6659 | .6825 | .6966 | .7126 | .7801 |
| 250 | .7133 | .7232 | .7376 | .7497 | .7633 | .8204 |
| 300 | .7551 | .7638 | .7764 | .7870 | .7989 | .8482 |

Table 4.1 (Continued)

p = 10

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 13 | $.0^{5}_{4}4419$ | $.0^{5}_{4}7260$ | $.0^{4}_{4}1474$ | $.0^{4}_{4}2657$ | $.0^{4}_{3}5113$ | $.0^{3}_{3}1439$ |
| 14 | $.0^{4}_{4}2049$ | $.0^{4}_{4}3168$ | $.0^{4}_{3}5886$ | $.0^{4}_{3}9825$ | $.0^{3}_{3}1732$ | $.0^{3}_{3}4227$ |
| 15 | $.0^{4}_{3}6755$ | $.0^{4}_{3}9930$ | $.0^{3}_{3}1715$ | $.0^{3}_{3}2693$ | $.0^{3}_{3}4436$ | $.0^{3}_{2}9726$ |
| 16 | $.0^{3}_{3}1737$ | $.0^{3}_{3}2451$ | $.0^{3}_{3}3994$ | $.0^{3}_{2}5977$ | $.0^{3}_{2}9340$ | $.0^{2}_{2}1885$ |
| 17 | $.0^{3}_{3}3733$ | $.0^{3}_{3}5097$ | $.0^{3}_{2}7926$ | $.0^{2}_{2}1142$ | $.0^{2}_{2}1711$ | $.0^{2}_{2}3234$ |
| 18 | $.0^{3}_{2}7032$ | $.0^{3}_{2}9347$ | $.0^{2}_{2}1399$ | $.0^{2}_{2}1954$ | $.0^{2}_{2}2829$ | $.0^{2}_{2}5070$ |
| 19 | $.0^{2}_{2}1200$ | $.0^{2}_{2}1559$ | $.0^{2}_{2}2262$ | $.0^{2}_{2}3078$ | $.0^{2}_{2}4330$ | $.0^{2}7423$ |
| 20 | $.0^{2}_{2}1895$ | $.0^{2}_{2}2417$ | $.0^{2}_{2}3414$ | $.0^{2}_{2}4545$ | $.0^{2}6241$ | .01030 |
| 22 | $.0^{2}_{2}3988$ | $.0^{2}_{2}4935$ | $.0^{2}6679$ | $.0^{2}8583$ | .01134 | .01761 |
| 24 | $.0^{2}_{2}7133$ | $.0^{2}8623$ | .01129 | .01412 | .01811 | .02682 |
| 25 | $.0^{2}9121$ | .01092 | .01411 | .01744 | .02208 | .03206 |
| 26 | .01139 | .01352 | .01724 | .02110 | .02641 | .03767 |
| 28 | .01674 | .01957 | .02444 | .02939 | .03607 | .04987 |
| 30 | .02314 | .02671 | .03276 | .03881 | .04685 | .06313 |
| 35 | .04298 | .04844 | .05743 | .06615 | .07740 | .09927 |
| 40 | .06694 | .07417 | .08584 | .09690 | .1109 | .1373 |
| 45 | .09339 | .1022 | .1161 | .1291 | .1453 | .1752 |
| 50 | .1211 | .1312 | .1470 | .1616 | .1795 | .2120 |
| 55 | .1491 | .1602 | .1776 | .1934 | .2126 | .2470 |
| 60 | .1769 | .1889 | .2073 | .2240 | .2442 | .2800 |
| 70 | .2301 | .2433 | .2633 | .2812 | .3025 | .3397 |
| 75 | .2553 | .2688 | .2893 | .3076 | .3292 | .3666 |
| 80 | .2793 | .2932 | .3140 | .3324 | .3542 | .3917 |
| 85 | .3023 | .3163 | .3373 | .3558 | .3777 | .4150 |
| 90 | .3241 | .3382 | .3593 | .3779 | .3997 | .4368 |
| 95 | .3448 | .3590 | .3802 | .3987 | .4204 | .4572 |
| 100 | .3645 | .3787 | .3998 | .4183 | .4398 | .4762 |
| 150 | .5151 | .5282 | .5474 | .5638 | .5827 | .6658 |
| 200 | .6101 | .6216 | .6383 | .6526 | .6688 | .7385 |
| 250 | .6745 | .6846 | .6993 | .7117 | .7258 | .7853 |
| 300 | .7209 | .7299 | .7429 | .7538 | .7662 | .8180 |

# CHAPTER V

## SUMMARY AND CONCLUSION

The present thesis has dealt with the distribution problems of Wilks' (1946) $L_{vc}$ criterion for testing $H: \Sigma = \sigma^2[(1-\rho)I + \rho e e']$, $\rho$ and $\sigma$ unknown against the alternative $A \neq H$ and also of the likelihood ratio criterion for testing $H: \mu = \mu_0$; $\Sigma = \sigma^2 I$, $\mu_0$ specified, $\sigma^2$ unknown. The main objective of the current work has been to present the non-null distribution of $L_{vc}$ criterion in a form suitable for power computations which was not possible earlier.

In chapter I of the thesis, the non-null distribution of $L_{vc}$ has been given in a closed form as a series of Meijer's G-functions using Mellin - Integral transform. This form of the density was used to compute the powers for five percent critical points of $L_{vc}$ for the case p=2. The powers seem to increase with N, the sample size, and the only parameter $\rho$. The non-null density of $L_{vc}$ criterion has also been derived in two other series forms using contour integration and as a series of chi-square distributions.

These forms of the density were used to compute the power for the case p=3. Powers have been computed for different N and the parameters $\rho_{12}$, $\rho_{13}$, $\rho_{23}$ and $c = \sigma_3/\sigma_2$. In this case powers increase with N, each of the parameters c, $\rho_{12}$, $\rho_{13}$ but decrease with $\rho_{23}$. Power computations involve the computations of zonal polynomials which become complicated for higher values of p. Therefore, it might be of interest to investigate the asymptotic behavior of the non-null distribution of $L_{vc}$.

In chapter II, the non-null distribution of Wilks' $L_{vc}$ criterion has been discussed in multivariate complex normal case. In order to obtain the non-null distribution of $L_{vc}$, certain theorems have been proved regarding the distribution theory of multivariate complex normal distribution. In this chapter, the non-null distribution of $L_{vc}$ have been computed in three series forms using methods similar to those of chapter I. Powers have been computed for the case p=2 for different values of N and the parameter $|\rho|$.

In chapter III, the exact null distribution of Wilks' $L_{vc}$ is deduced form chapter II in the multivariate complex normal case in forms suitable for computations of percentage points. The percentage points have been computed for p=2(1)8, $\alpha$=.005, .01, .025, .05, .1, .25 and for various degrees of freedom n = N-1.

In chapter IV, the density of the likelihood ratio criterion $L_1$ for testing H: $\underset{\sim}{\mu} = \underset{\sim}{\mu}_0$ , $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}$ , $\underset{\sim}{\mu}_0$, a given known vector, and $\sigma^2$ unknown, was obtained in a closed form in terms of Meijer's G-function using Mellin - Integral transform. The distribution has also been expressed in two series forms in order to facilitate the computation of the percentage points of the criterion, using methods similar to those of chapter I. The percentage points of $L_1$ have been tabulated for p=2(1)10, values of $\alpha$ as above and for various values of N. The asymptotic distribution of $-2 \log L_1$ has also been studied. It was proved that $-2 \log L_1$ is asymptotically distributed as a chi-square variable with $p(p-1)/2 + p$ degrees of freedom.

In summary, the present work solves most of the distribution problems regarding Wilks' (1946) $L_{vc}$ criterion in the classical and complex Gaussian cases. The methods obtained in chapter I are quite general and could be used to express the non-null distributions of other well known likelihood ratio criteria in the field of multivariate analysis, in forms which are of practical use.

The following are some suggestions for future work:

1. The non-null distribution of $L_1$ available so far (See Khatri and Srivastava (1973)), is not suitable

for power computations. The methods similar to those of chapter I may be used to express the non-null distribution of $L_1$ in forms suitable for power computations. The asymptotic non-null distribution of $L_1$ may also be investigated.

2. Further work needs to be done to obtain more rapidly convergent series in all the distribution problems obtained.

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