IMPROVING UPON INADMISSIBLE ESTIMATORS IN DISCRETE EXPONENTIAL FAMILIES

by

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CHAPTER I

INTRODUCTION

Section 1.1. Stating the Problem

Consider the problem of estimating $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ under the loss function $L_m(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} \theta_i^m(\theta_i-a_i)^2$, when the observations X_i, $i=1,\ldots,p$, are independently from discrete exponential families with density $\varphi_i(\theta_i)t_i(x_i)\theta_i^{x_i}$. The usual estimator is typically admissible for one dimension ($p=1$), but is often inadmissible for higher dimensions ($p > 1$) and can hence be improved upon. In this thesis, the problem of improving upon inadmissible estimators is reduced to the study of difference inequalities. Typical difference inequalities are presented and solved. (Special cases had earlier been solved by M.L. Clevenson and J.V. Zidek (1975), J.C. Peng (1975), H.W. Hudson (1978), and K.W. Tsui and S.J. Press (1977)). Also, theorems are obtained which establish the inadmissibility of certain broad classes of estimators.

In Section 1.2, the notation and definitions are discussed. Section 1.3 gives a review of related results obtained by other statisticians. Section 1.4 summarizes the results in this thesis.

Section 1.2. Definitions and Notation

In this section, we briefly discuss the definitions and notation that are used throughout this thesis.

Let $X_1, \ldots, X_p$ be p independent random variables, and assume the probability density of X_i with respect to some measure μ_i is $f_i(x_i|\theta_i)$, i=1,...,p, where $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ is some unknown parameter. We use the notation

$$X_i \overset{\text{indep.}}{\sim} f_i(x_i|\theta_i) \quad i=1,\ldots,p \tag{1.2.1}$$

to indicate this. The measure μ_i is assumed to be Lebesgue measure when X_i has an absolutely continuous distribution, and is taken to be the counting measure on nonnegative integers when X_i has a discrete distribution. For most of this thesis, it is assumed that the densities are from the discrete exponential family,

$$f_i(x_i|\theta_i) = \varphi_i(\theta_i)t_i(x_i)\theta_i^{x_i}, \quad x_i=0,1,\ldots. \tag{1.2.2}$$

where $\theta_i > 0$, and θ_i belongs to some subset Ω_i of R (the set of real numbers) i=1,...,p. Note that θ_i is not the natural parameter of the exponential family. However, in many situations, θ_i is the interesting parameter to estimate.

Some important special cases of the density in (1.2.2) are the Poisson distribution, the negative binomial distribution and the logrithmic distribution. Denote the Poisson distribution with mean θ by Po(θ). Also, let NB(r,θ) denote the negative binomial distribution having the following density

$$f(x|\theta) = \binom{r+x-1}{r-1}(1-\theta)^r\theta^x, \quad x=0,1,\ldots \tag{1.2.3}$$

where $0 < \theta < 1$ and r is a known positive integer.

It is desired to estimate $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ on the basis of $\underset{\sim}{X} = (X_1,\ldots,X_p)$. The parameter space is clearly $\Theta = \Omega_1 x \Omega_2 \ldots x \Omega_p \subset R^p$. ($R^p$ is p-dimensional Euclidean space.) Let $\underset{\sim}{a} = (a_1,\ldots,a_p)$ be an available action (i.e. an estimate of $\underset{\sim}{\theta}$) and assume that the action space is $\mathscr{A}$, and $R^p \supset \mathscr{A} \supset \overline{\Theta}$. ($\overline{\Theta}$ is the closure of Θ.) When action $\underset{\sim}{a}$ is taken and $\underset{\sim}{\theta}$ is the true parameter value, it is asssumed that a loss $L(\underset{\sim}{\theta},\underset{\sim}{a})$ is incurred, where $L(\underset{\sim}{\theta},\underset{\sim}{a})$ is a real valued function defined on $\Theta \times \mathscr{A}$. Usually, we assume $L(\underset{\sim}{\theta},\underset{\sim}{a})$ has the following form

$$L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} \theta_i^{m_i}(\theta_i - a_i)^2, \tag{1.2.4}$$

where $\underset{\sim}{m} = (m_1,\ldots,m_p)$ and $m_1,\ldots,m_p$ are integers. When $m_i = m$, $i = 1,\ldots,p$, $L_{\underset{\sim}{m}}$ is denoted by L_m;

$$L_m(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} \theta_i^{m}(\theta_i - a_i)^2. \tag{1.2.5}$$

A (nonrandomized) estimator $\underset{\sim}{\delta}(\underset{\sim}{X}) = (\delta_1(\underset{\sim}{X}),\ldots,\delta_p(\underset{\sim}{X}))$ is a function from the sample space to $\mathscr{A}$, which estimates $\underset{\sim}{\theta}$ by $\underset{\sim}{\delta}(\underset{\sim}{X})$ when $\underset{\sim}{X}$ is observed. The risk function $R(\underset{\sim}{\theta},\underset{\sim}{\delta})$ of an estimator $\underset{\sim}{\delta}$ is defined to be

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}) = \int L(\underset{\sim}{\theta},\underset{\sim}{\delta}(\underset{\sim}{x})) \prod_{i=1}^{p} f_i(x_i|\theta_i)d\mu_i(x_i)$$

$$= E_{\underset{\sim}{\theta}} L(\underset{\sim}{\theta},\underset{\sim}{\delta}(\underset{\sim}{X})),$$

where, as usual, $E_{\underset{\sim}{\theta}}$ denotes expectation. The subscript $\underset{\sim}{\theta}$ might be dropped when there is no ambiguity.

An estimator $\underset{\sim}{\delta}^*$ is defined to be *as good as* $\underset{\sim}{\delta}$ if

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) \leq R(\underset{\sim}{\theta},\underset{\sim}{\delta}) \tag{1.2.6}$$

for all $\underset{\sim}{\theta} \in \Theta$. The estimator $\underset{\sim}{\delta}^*$ is said to be *better than* $\underset{\sim}{\delta}$ (or *dominates* $\underset{\sim}{\delta}$) if, in addition to (1.2.6),

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) < R(\underset{\sim}{\theta},\underset{\sim}{\delta}) \tag{1.2.7}$$

for some $\underset{\sim}{\theta} \in \Theta$. The estimator $\underset{\sim}{\delta}$ is *admissible* if there exists no better estimator, and is *inadmissible* otherwise.

For any vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$ and any real number F, define

$$\underset{\sim}{g} + \underset{\sim}{h} = (g_1 + h_1,\ldots,g_p + h_p), \tag{1.2.8}$$

$$\underset{\sim}{g}\underset{\sim}{h} = (g_1h_1,\ldots,g_ph_p), \tag{1.2.9}$$

and

$$F\underset{\sim}{g} = (Fg_1,\ldots,Fg_p). \tag{1.2.10}$$

Let $\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_p$ denote the unit vectors in R^p, i.e.

$$\underset{\sim}{e}_i = (0,\ldots,0,1,0,\ldots 0), \quad i=1,\ldots,p \tag{1.2.11}$$

↳ ith component

For any function $F(\underset{\sim}{x})$, denote the ith partial difference of $F(\underset{\sim}{x})$ by $\Delta_i F(\underset{\sim}{x})$, i.e.

$$\Delta_i F(\underset{\sim}{x}) = F(\underset{\sim}{x}) - F(\underset{\sim}{x}-\underset{\sim}{e}_i). \tag{1.2.12}$$

Also, for any number g, define

$$g^+ = \begin{cases} g & \text{if } g \geq 0 \\ 0 & \text{if } g < 0. \end{cases} \tag{1.2.13}$$

Section 1.3. History

In the following, some previous results concerning the problem of improving upon standard estimators will be discussed. It would be too difficult to list all known results, so we will only mention the ones that are closely related to the problems considered here.

Subsection 1.3.1. Stein's Result

Let $X_i \overset{\text{indep.}}{\sim} N(\theta_i,1)$, $i=1,\ldots,p$, i.e. $X_1,\ldots,X_p$ are independent normal random variables with means $\theta_1,\ldots,\theta_p$ and variance 1. In Stein (1955), the problem of estimating $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ based on $\underset{\sim}{X} = (X_1,\ldots,X_p)$ under the loss function $L_0(\underset{\sim}{\theta},\underset{\sim}{a})$ was considered. (Recall from (1.2.5) that

$$L_0(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} (\theta_i - a\ )^2.\quad).$$

Stein proved the surprising result that the usual estimator $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$ is inadmissible when $p \geq 3$. A better estimator $\underset{\sim}{\delta}^*$ was found in James and Stein (1960), which has the form

$$\underset{\sim}{\delta}^*(\underset{\sim}{X}) = \left(1 - \frac{p-2}{\sum_{i=1}^{p} X_i^2}\right)\underset{\sim}{X}\ .$$

Since then, a considerable amount of work by a number of authors (see the references) has gone into finding significant improvements upon $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$ in more general settings. For the normal distribution, the results in the most general setting obtained so far can be found in Berger, et. al. (1976) and Gleser (1979). In their paper, $\underset{\sim}{X}$ is assumed to be a multivariate normal vector with unknown mean $\underset{\sim}{\theta}$ and unknown covariance matrix Σ. The loss function they considered was $L_M(\underset{\sim}{\theta},\underset{\sim}{a}) = (\underset{\sim}{a}-\underset{\sim}{\theta})M(\underset{\sim}{a}-\underset{\sim}{\theta})^t$, where $(\underset{\sim}{a}-\underset{\sim}{\theta})^t$ denotes the transpose of $(\underset{\sim}{a}-\underset{\sim}{\theta})$ and M is a known pxp positive definite matrix. If an estimator W of Σ is available and W has a Wishart distribution with parameter Σ , new estimators were obtained which dominate the usual estimator $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$ for $p \geq 3$. Brown (1966) has also shown for a wide class of loss functions and densities that the estimator $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$ can be improved when $p \geq 3$. All the estimators mentioned above correct the usual estimator by shrinking toward the origin.

In Stein (1973), an identity (proven by integration by parts) was developed which has proven to be a powerful tool in the problem of improving upon the standard estimators. In searching for an estimator, $\underset{\sim}{\delta}^*(\underset{\sim}{X})$, better than $\underset{\sim}{\delta}^o(\underset{\sim}{X})$, Stein wrote $\underset{\sim}{\delta}^*(\underset{\sim}{X})$ as $\underset{\sim}{\delta}^o(\underset{\sim}{X}) + \underset{\sim}{\Phi}(\underset{\sim}{X})$ and used the identity to obtain the representation

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E_{\underset{\sim}{\theta}}[\mathscr{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))],$$

where $\mathscr{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))$ is an expression that does not involve $\underset{\sim}{\theta}$. $\mathscr{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))$ involves partial derivatives of $\Phi_i(\underset{\sim}{X})$, $i=1,\ldots,p$. (For the discrete case, $\mathscr{D}(\underset{\sim}{\Phi}(X))$ will involve partial differences of $\Phi_i(\underset{\sim}{X})$). The idea of Stein was then to find $\underset{\sim}{\Phi}(\underset{\sim}{X})$ so that $\mathscr{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) < 0$. If a solution exists, then for such $\underset{\sim}{\Phi}(\underset{\sim}{X})$ and $\underset{\sim}{\delta}^*(\underset{\sim}{X})$,

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E_{\underset{\sim}{\theta}}[\mathscr{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))] < 0,$$

and it follows that $\underset{\sim}{\delta}^*$ is better than $\underset{\sim}{\delta}^o$. The original example of Stein's illustrates this idea.

<u>Example 1.1.</u> Let $\underset{\sim}{X} = (X_1,\ldots,X_p)$, $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$, and

$$X_i \overset{\text{indep.}}{\sim} N(\theta_i,1), \quad i=1,\ldots,p.$$

Under the loss function L_0, and estimator better than the maximum likelihood estimator $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$ can be obtained by the following procedures when $p \geq 3$.

(i) Write the new estimtor $\underset{\sim}{\delta}^*$ as $\underset{\sim}{\delta}^*(\underset{\sim}{X}) = \underset{\sim}{\delta}^o(\underset{\sim}{X}) + \underset{\sim}{\Phi}(\underset{\sim}{X})$.

Now,

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}^*(\underset{\sim}{X})) - L(\underset{\sim}{\theta},\underset{\sim}{\delta}^o(\underset{\sim}{X})) = \sum_{i=1}^{p}\{(\theta_i-\delta_i^*(\underset{\sim}{X}))^2 - (\theta_i-\delta_i^o(\underset{\sim}{X}))^2\}$$

$$= \sum_{i=1}^{p}\{2\Phi_i(\underset{\sim}{X})(\delta_i^o(\underset{\sim}{X})-\theta_i) + \Phi_i^2(\underset{\sim}{X})\}$$

$$= \sum_{i=1}^{p}\{2\Phi_i(\underset{\sim}{X})(X_i-\theta_i) + \Phi_i^2(\underset{\sim}{X})\}$$

It follows that

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E_{\underset{\sim}{\theta}} \sum_{i=1}^{p} \{2\Phi_i(\underset{\sim}{X})(X_i-\theta_i) - \Phi_i^2(\underset{\sim}{X}_i)\}. \qquad (1.3.1)$$

(ii) An identity derived by Stein (1973) shows that, if $\Phi_1,\ldots,\Phi_p$ satisfy some regularity conditions, then

$$E_{\underset{\sim}{\theta}}[(X_i-\theta_i)\Phi_i(\underset{\sim}{X})] = E_{\underset{\sim}{\theta}}[\frac{\partial}{\partial X_i}\Phi_i(\underset{\sim}{X})]. \qquad (1.3.2)$$

Define

$$\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) = \sum_{i=1}^{p}\{2\frac{\partial}{\partial X_i}\Phi_i(\underset{\sim}{X}) + \Phi_i^2(\underset{\sim}{X})\}. \qquad (1.3.3)$$

Under the regularity conditions, it follows from (1.3.1), (1.3.2), and (1.3.3) that

$$E_{\underset{\sim}{\theta}}[\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))] = R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\theta,\underset{\sim}{\delta}^o)$$

(iii) Letting $\Phi_i(\underset{\sim}{X}) = -(p-2)X_i/\sum_{i=1}^{p} X_i^2$, it is easy to check that $\Phi_1(\underset{\sim}{X}),\ldots,\Phi_p(\underset{\sim}{X})$ satisfy the needed regularity conditions when $p \geq 3$. A straightforward calculation yields

$$\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) = \sum_{i=1}^{p}[2\frac{\partial}{\partial X_i}\Phi_i(\underset{\sim}{X}) + \Phi_i^2(\underset{\sim}{X})]$$

$$= \frac{-(p-2)^2}{\sum_{i=1}^{p} X_i^2} < 0.$$

Therefore $\underset{\sim}{\delta}^*$ is better than $\underset{\sim}{\delta}^o$.

Note in the above, that the main problem was reduced to the study of a differential inequality. The importance of the relation between such differential inequalities and inadmissibility has also been emphasized in Brown (1974). See also Stein (1965), Brown (1971 and 1974) and Berger (1976a, 1976b and 1976c).

Subsection 1.3.2. Exponential Families

Stein's phenomenon has also been observed for many other distributions. In the following, we will briefly describe some of these other cases.

(i) The Poisson distribution with the loss function

$$L_{-1}(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} \theta_i^{-1}(\theta_i - a_i)^2.$$

Let $\underset{\sim}{X} = (X_1,\ldots,X_p)$ and $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i=1,\ldots,p$. Under the loss function L_{-1}, Clevenson and Zidek (1975) obtained an estimator which dominates the usual estimator $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}$, when $p \geq 2$. The new estimator $\underset{\sim}{\delta}^C$ has the form

$$\underset{\sim}{\delta}^C(\underset{\sim}{X}) = \left(1 - \frac{c(\sum_{i=1}^{p} X_i)}{p-1+\sum_{j=1}^{p} X_j}\right)\underset{\sim}{X}, \qquad (1.3.4)$$

where $c(\cdot)$ is any nondecreasing function such that $0 \leq c(\cdot) \leq 2(p-1)$. Using the property that, conditioning on $\sum_{j=1}^{p} X_j$, $\underset{\sim}{X}$ has a multinomial distribution, Clevenson and Zidek obtained an expression for the difference of the risk of $\underset{\sim}{\delta}^C$ and that of $\underset{\sim}{\delta}^o$. This expression involves only $\sum_{j=1}^{p} X_j$ and $\sum_{j=1}^{p} \theta_j$ and was shown to be negative valued. Thus $\underset{\sim}{\delta}^C$ is better than $\underset{\sim}{\delta}^o$.

(ii) The Poisson distribution with the loss function

$$L_0(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p}(\theta_i - a_i)^2.$$

Peng (1975) considered the same problem as in (i), except under the different loss function L_0. Essentially, Peng tackled this problem, of improving upon the usual estimator $\underset{\sim}{\delta}^0(\underset{\sim}{X}) = \underset{\sim}{X}$, by Stein's technique. An identity derived by Stein for the Poisson distribution says that, for any function g defined on R^p,

$$\theta_i E_{\underset{\sim}{\theta}}(g(\underset{\sim}{X})) = E_{\underset{\sim}{\theta}}[X_i g(\underset{\sim}{X}-\underset{\sim}{e}_i)], \tag{1.3.5}$$

where, recalling (1.2.11), $\underset{\sim}{e}_i$ represents the unit vector pointing along the ith direction in R^p. By writing the competitor to $\underset{\sim}{\delta}^0(\underset{\sim}{X})$ as $\underset{\sim}{\delta}^0(\underset{\sim}{X}) + \underset{\sim}{\Phi}(\underset{\sim}{X})$ and using the identity (1.3.5), the expression

$$\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) = \sum_{i=1}^{p} (2X_i \Delta_i \Phi_i(\underset{\sim}{X}) + \Phi_i^2(\underset{\sim}{X})) \tag{1.3.6}$$

was obtained where $E_{\underset{\sim}{\theta}}[\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X}))] = R(\underset{\sim}{\theta},\underset{\sim}{\delta}^0 + \underset{\sim}{\Phi}) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^0)$. Recall from (1.2.12) that $\Delta_i \Phi_i(\underset{\sim}{X})$ denotes $\Phi_i(\underset{\sim}{X}) - \Phi_i(\underset{\sim}{X}-\underset{\sim}{e}_i)$. Therefore, the problem is reduced to the search for $\Phi_1,\ldots,\Phi_p$ for which $\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) \leq 0$. By using the interchangeability of the indices, (1.3.6) can be rewritten in another form which depends on $\underset{\sim}{X}$ only through N_i, the number of indices j so that $X_j = i$. After a considerable effort, Peng found a solution to $\mathcal{D}(\underset{\sim}{\Phi}(\underset{\sim}{X})) \leq 0$ when $p \geq 3$ with $E_{\underset{\sim}{\theta}}\mathcal{D}(\Phi(\underset{\sim}{X})) < 0$ for all $\underset{\sim}{\theta}$. Hence, a better estimator was obtained.

To describe Peng's estimator, recall that g^+ denotes $\max\{g,0\}$. Also denote the number of indices i such that $X_i > 0$ by $\#(\underset{\sim}{X})$. Peng's estimtor, $\underset{\sim}{\delta}^P$, has the form

$$\delta_i^P(\underset{\sim}{X}) = X_i - \frac{(\#(\underset{\sim}{X}) - 2)^+ h(X_i)}{\sum_{j=1}^{p} h^2(X_j)}, \quad i=1,\ldots,p, \qquad (1.3.7)$$

where $h(x_i)$ is defined to be

$$h(x_i) = \sum_{k=1}^{X_i} \frac{1}{k} \qquad X_i = 1,2,\ldots$$
$$= 0. \qquad \text{otherwise}$$

(iii) The Poisson distribution with the loss function

$$L_m = \sum_{i=1}^{p} \theta_i^m(\theta_i - a_i)^2, \quad m=-2,-3,\ldots \quad .$$

Tsui and Press (1977) obtained estimators which dominate the usual estimator $\underset{\sim}{\delta}^0$ for $p \geq 2$, under the loss function L_m, $m \leq -2$. Essentially, they followed Stein's technique. As in (i) and (ii), the new estimators correct $\underset{\sim}{\delta}^0$ by shrinking toward the origin (i.e. each component of the correction term $\underset{\sim}{\Phi}(\underset{\sim}{X})$ is nonpositive).

Hudson (1978) generalized the technique of Stein, and applied it to improve upon uniform minimum variance unbiased estimators under the loss function L_0, when the observations are independently from the exponential (discrete and continuous) family. Here we will mainly discuss the discrete case. Suppose

$$X_i \overset{\text{indep.}}{\sim} f(x_i|\theta_i) = \varphi(\theta_i)t(x_i)\theta_i^{x_i}, \quad i=1,\ldots,p.$$

Denote the uniformly minimum variance unbiased estimator of θ_i by $a(X_i)$. (i.e. $a(X_i) = t(X_i-1)/t(X_i)$.) Hudson then established an identity,

$$\theta_i E_{\underset{\sim}{\theta}}(g(\underset{\sim}{X})) = E_{\underset{\sim}{\theta}}[a(X_i)g(\underset{\sim}{X}-\underset{\sim}{e}_i)],$$

which is the generalized form of (1.3.5). By use of this identity, the problem of improving upon the estimator $(a(X_1),\ldots,a(X_p))$ of $(\theta_1,\ldots,\theta_p)$ was reduced to the study of the following difference inequality:

$$\sum_{i=1}^{p} [2a(x_i)\Delta_i\Phi_i(\underset{\sim}{x}) + \Phi_i^2(\underset{\sim}{x})] \le 0 \qquad (1.3.8)$$

Following an argument similar to Peng's (1978) for solving (1.3.6), a function $\underset{\sim}{\Phi}(\underset{\sim}{x}) = (\Phi_1(\underset{\sim}{x}),\ldots,\Phi_p(\underset{\sim}{x}))$ was shown to be a solution to (1.3.8) under the assumptions that $a(\cdot)$ is an increasing function and that the dimension p is big enough. Therefore, a better estimator was found under the given assumptions.

A direct application of Hudson's result to the negative binomial case (i.e. the observations $X_i \overset{\text{indep.}}{\sim} NB(r,\theta_i)$, $i=1,\ldots,p$) gives an estimator $\underset{\sim}{\delta}^H$ which dominates the usual one $(\frac{X_1}{r-1+X_1},\ldots,\frac{X_p}{r-1+X_p})$ under the loss function L_0 for $p \ge 4$. The new estimator, $\underset{\sim}{\delta}^H$, has the form (componentwise)

$$\delta_i^H(\underset{\sim}{X}) = \frac{X_i}{r-1+X_i} - \frac{(\#(\underset{\sim}{X}) - 3)^+ h(X_i)}{\sum_{j=1}^{p} h^2(X_j)}, \qquad (1.3.9)$$

where $\#(\underset{\sim}{x})$ denotes the number of indices j for which $x_j > 0$, and $h(x_i) = \sum_{k=1}^{x_i} (r-1+k)/k$, if $x_i > 0$, and $h(x_i) = 0$, otherwise.

Subsection 1.3.3. The Gamma Distribution

Berger (1978) obtained solutions to a general differential inequality and applied them to the gamma distribution. To demonstrate the idea, consider the special case that the observations

$$X_i/\theta_i \overset{\text{indep.}}{\sim} \chi_n^2, \quad i=1,\ldots,p.$$

(χ_n^2 stands for chi square distribution with n degrees of freedom.) Berger obtained a better estimator than the standard one, $\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \underset{\sim}{X}/(n+2)$, under the loss function L_{m-2}. (When referring to this problem, L_{m-2} is used instead of L_m. The reason for this is to facilitate the comparison between the results for the gamma distribution and those for the Poisson distribution that will be developed.)

In search of a better estimator $\underset{\sim}{\delta}^B$, Berger wrote $\underset{\sim}{\delta}^B$ as (componentwise)

$$\delta_i^B(\underset{\sim}{x}) = \delta_i^o(\underset{\sim}{x})(1 + \phi_i(\underset{\sim}{x})), \quad i=1,\ldots,p.$$

By use of an identity derived in Hudson (1978), the problem was reduced to the search for $\phi_1,\ldots,\phi_p$ satisfying a key differential inequality which involves partial derivatives of $\phi_i(\underset{\sim}{x})$ of different orders. In general, the differential inequality involves many terms. However, it was shown that higher order differential terms can usually be neglected. This led to the consideration of the following inequality invloving only the first order differential terms and the square term of ϕ_i:

$$\mathscr{D}_m = \sum_{i=1}^{p} \{x_i^{m+1} \frac{\partial}{\partial x_i} \phi_i(\underset{\sim}{x}) + b_i x_i^m \phi_i^2(\underset{\sim}{x})\} < 0, \qquad (1.3.10)$$

where $b_1,\ldots,b_p$ are positive constants depending on m. Berger obtained solutions to the more general differential inequality,

$$\mathscr{D}^B(\underset{\sim}{\phi}(\underset{\sim}{x})) = \sum_{i=1}^{p} \{v_i(x_i) \frac{\partial}{\partial x_i} \phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})\} < 0. \qquad (1.3.11)$$

Berger's solutions to (1.3.11) are described below.

Let

$$g_i(x_i) = \int^{x_i} \frac{1}{v_i(t)}\, dt \qquad \text{(indefinite integral)}$$

Suppose it is possible to choose nonnegative constants, $\beta_1, \ldots, \beta_p$, $d_1, \ldots, d_p$, and b such that

$$\frac{w_i(\underset{\sim}{x}) g_i^2(x_i)}{b + \sum_{j=1}^{p} d_j |g_j(x_j)|^{\beta_j}} \leq K < \infty, \quad i=1,\ldots,p. \tag{1.3.12}$$

then for $p > \max_{1 \leq j \leq p} \beta_j$ and

$0 < c < (p - \max_{1 \leq j \leq p} \beta_j)/pK$,

$$\phi_i(\underset{\sim}{x}) = \frac{-c g_i(x_i)}{b + \sum_{j=1}^{p} d_j |g_j(x_j)|^{\beta_j}}, \quad i=1,\ldots,p \tag{1.3.13}$$

is a solution to (1.3.11). It follows that a solution to (1.3.10) is

$$\phi_i^{(m)}(\underset{\sim}{x}) = \begin{cases} \dfrac{c x_i^{-m}}{m[b + \sum_{j=1}^{p} b_j x_j^{-m}]} & \text{when } m \neq 0,\ p \geq 2 \\ \\ \dfrac{-c \ln x_i}{b + \sum_{j=1}^{p} b_j (\ln x_j)^2} & \text{when } m = 0,\ p \geq 3, \end{cases} \tag{1.3.14}$$

for some small enough constant $c > 0$. Heuristically, $\underset{\sim}{\delta}^0$ is thus dominated by $\underset{\sim}{\delta}^B$ defined by

$$\delta_i^B(\underset{\sim}{X}) = \frac{X_i}{n+2} (1 + \phi_i^{(m)}(\underset{\sim}{X})).$$

This is actually shown to be the case for m=0 and $p \geq 3$, and for m=-1, 1 and $p \geq 2$.

Two surprising phenomena exhibited in the above situation are:

(i) The correction terms (i.e. $\phi_i^{(m)}(\underset{\sim}{X})X_i/(n+2)$, $i=1,\ldots,p$) might be positive or negative depending upon the loss function L_{m-2}. For $m > 0$, the correction is positive. ($\underset{\sim}{\delta}^0$ is pulled towards $(\infty,\ldots,\infty)$!) For $m < 0$, the correction terms are always negative. ($\underset{\sim}{\delta}^0$ is pulled towards the origin.) For $m = 0$, $\underset{\sim}{\delta}^0$ is pulled towards a point.

(ii) The dimension needed for inadmissibility of $\underset{\sim}{\delta}^0$ depends on the loss function. In most cases, $\underset{\sim}{\delta}^0$ is inadmissible if and only if $p \geq 2$.

Brown (1978) also observed Berger's phenomena in the problem of estimating a normal mean under the loss function

$$L_r(\underset{\sim}{\theta},\underset{\sim}{a}) = \sum_{i=1}^{p} e^{r\theta_i}(\theta_i - a_i)^2 .$$

It was shown that, under L_r, $\underset{\sim}{\delta}^0$ is inadmissible if and only if $p \geq 2$ when $r \neq 0$. For $r = 0$, this reduces to Stein's case, so that $\underset{\sim}{\delta}^0$ is inadmissible if and only if $p \geq 3$. (The admissibility of $\underset{\sim}{\delta}^0$ for $p = 1$ was established in Hodges and Lehmann (1951), and for $p \geq 2$ in Stein (1955).) Brown's improved estimators also pull $\underset{\sim}{\delta}^0$ towards $(\infty,\ldots,\infty)$ for $r > 0$ and towards $(-\infty,\ldots,-\infty)$ for $r < 0$. For $r = 0$, the better estimator (the James-Stein estimator) pulls $\underset{\sim}{\delta}^0$ towards a point.

Section 1.4. Summary of Results Obtained in this Thesis

So far, all the discrete cases considered here have been concerned with improving upon unbiased estimators. Since many reasonable estimators are not unbiased, it is also interesting to see if such estimators can be improved upon. To improve upon such estimators under the loss function $L_{\underset{\sim}{m}}$ (cf. (1.2.4)), we follow the steps of Stein's technique as described in subsection 1.3.1. Instead of reducing the problem to the study of a difference inequality, an inequality of a more general type is encountered. (cf. (2.1.8) and (2.1.9)). This is dealt with in Chapter II by writing a competitor, $\underset{\sim}{\delta}^*$, of $\underset{\sim}{\delta}^0$ as $\underset{\sim}{\delta}^*(\underset{\sim}{x}) = \underset{\sim}{\delta}^0(\underset{\sim}{x}) + \underset{\sim}{g}(\underset{\sim}{x})\underset{\sim}{\phi}(\underset{\sim}{x})$, where

$$g(\underset{\sim}{x}) = (q_1(\underset{\sim}{x}),\ldots,q_p(\underset{\sim}{x})),\ \underset{\sim}{\phi}(\underset{\sim}{x}) = (\phi_1(\underset{\sim}{x}),\ldots,\phi_p(\underset{\sim}{x})),$$

and $\underset{\sim}{g}(\underset{\sim}{x})\underset{\sim}{\phi}(\underset{\sim}{x}) = (q_1(\underset{\sim}{x})\phi_1(\underset{\sim}{x}),\ldots,q_p(\underset{\sim}{x})\phi_p(\underset{\sim}{x}))$. By choosing a suitable $\underset{\sim}{q}(\underset{\sim}{x})$, the inequality of the more general type is then transformed into a difference inequality of the following form:

$$\sum_{i=1}^{p} \{v_i(\underset{\sim}{x})\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})\} \leq 0. \qquad (1.4.1)$$

A similar transformation was developed in Berger (1978) for the continuous case and the squared error loss. Note that for estimators that are unbiased it will be seen to be sufficient to choose $q_i(\underset{\sim}{x}) = 1$, $i=1,\ldots,p$. For such a situation the new estimator is then $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ which corresponds to the earlier work.

For the special case that $v_i(\underset{\sim}{x})$ depends on x_i only, $i=1,\ldots,p$, a class of nontrivial solutions to (1.4.1) is found in Chapter II. This special case occurs, for example, when the ith component of $\underset{\sim}{\delta}^0(\underset{\sim}{x})$ depends solely on x_i, $i=1,\ldots,p$.

In Chapter III, typical applications of the general theorems developed in Chapter II are given. For each specific application, a broad class of estimators are given that dominate the standard estimator. These classes of estimators include those obtained by Clevenson and Zidek (1975) and Tsui and Press (1977). Also, Peng (1975) obtained an estimator which is similar to one of our estimators. For the negative binomial distribution, the uniformly minimum variance unbiased estimator is shown to be dominated by a class of estimators under the loss function $L_{\underset{\sim}{m}}$ for $p \geq 3$. (Recall that Hudson found a better estimator under the loss L_0 only for $p \geq 4$.)

Chapter IV contains inadmissibility results for some broad classes of estimators. By choosing appropriate $\underset{\sim}{q}(\underset{\sim}{x})$, the theorems in Chapter II can be applied even to cases in which the ith component of $\underset{\sim}{\delta}^0$ depends on the entire $\underset{\sim}{X}$. In Section 4.1, a theorem is thus developed which proves the inadmissibility of certain general types of estimators. Further in Section 4.2, another theorem is established, with the aid of $\underset{\sim}{q}(\underset{\sim}{x})$ functions, which essentially states that if an estimator $\underset{\sim}{\delta}^0$ can be improved upon by the theorems in Chapter II, and if the better estimator $\underset{\sim}{\delta}^*$ pulls $\underset{\sim}{\delta}^0$ towards the origin, then any other estimator $\underset{\sim}{\delta}'(\underset{\sim}{x})$, which has ith component greater than or equal to that of $\underset{\sim}{\delta}^0(\underset{\sim}{x})$ for sufficiently large $\underset{\sim}{x}$ and all $i=1,\ldots,p$, is inadmissible. (In this sense, $\underset{\sim}{\delta}^0$ is an "upper bound" for the class of admissible estimators.) It can be concluded that the estimators considered in Section 4.1 are upper bounds of the class of admissible estimators. Some interesting conclusions are that Hudson's estimator $\underset{\sim}{\delta}^H$ (cf. (1.3.9)) and some of the Peng type estimators and Tsui type estimators are inadmissible. Clevenson's estimator $\underset{\sim}{\delta}^C$ (cf. (1.3.4)) is also inadmissible if $c(\cdot) \leq \ell < p-1$, for some constant ℓ, which proves a conjecture of Brown(1974) concerning the inadmissibility of some estimators similar to (1.3.4).

In Chapter V, some miscellaneous problems are considered. Section 5.1 exposes the special role played by discreteness in the problem of improving upon standard estimators. A theorem is given which implies essentially that, under the loss function L_m, $m > 0$, it is impossible to improve upon a standard estimator in certain discrete problems by always expanding it. This explains why the first aspect of Berger's phenomena is not observed in the problem of improving upon a standard estimator of Poisson means, while the second aspect of Berger's phenomena is observed. The problem of improving upon the standard estimator of Poisson means under the loss function L_m, m a positive integer, is also compared to the related gamma estimation problem. The similarity between the inequalities involved in these two problems, the theorem described above, and Berger's results (1978), seem to suggest that the standard estimator is admissible in this particular case.

In Section 5.2, we consider the question of whether the lack of a solution (except the zero solution) to the key difference inequality encountered, for a particular estimator $\underset{\sim}{\delta}^0$ implies that $\underset{\sim}{\delta}^0$ is admissible. An example is given to show that the answer to this question is negative.

In Section 5.3, an example is considered in which it is desired to estimate the unknown parameter $\underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3)$ based on three independent observations X_1, X_2 and X_3, with $X_1 \sim Po(\theta_1)$, $X_2 \sim N(\theta_2, 1)$, and $X_3/\theta_3 \sim \chi_n^2$. Under the loss function $L_{\underset{\sim}{m}}$, $\underset{\sim}{m} = (0,0,-1)$, an estimator dominating the standard estimator $(X_1, X_2, X_2/(n+2))$ is obtained. The implications of this example are discussed.

Some useful generalizations are given in Section 5.4.

CHAPTER II

THE DIFFERENCE INEQUALITY AND SOLUTIONS

In the first section of this chapter, it is shown how the problem of improving upon an estimator can be reduced to the study of a difference inequality. This reduction follows essentially the steps described in Example 1.1 of Subsection 1.3.1.

In Section 2.2, solutions to a fairly general type of difference inequality are given. In the remainder of this thesis, unless otherwise stated, x_i and X_i will denote an integer and an integer valued random variable, respectively.

Section 2.1. Derivation of the Difference Inequality

Let X be a one dimensional random variable having discrete density

$$f(x|\theta) = \varphi(\theta)t(x)\theta^x, \quad x=0,1,\ldots \quad . \tag{2.1.1}$$

For convenience, t(x) is defined to be zero when $x < 0$. The following two lemmas are the keys to obtaining the difference inequality. For the case m = 1, these lemmas were proven in Hudson (1975) by using changes of variables. The same method can be used to prove the lemmas for the case where m is a positive integer. When m is a negative integer, a special case of the lemmas for which X has Poisson distribution has been established in Tsui (1977). The proof below follows essentially Hudson's.

Lemma 2.1. Assume that X has density (2.1.1) with $t(x) > 0$ for $x = 0,1,\ldots$. For any function g, defined on R, for which

$$E_\theta(|g(X)|) < \infty,$$

the following are true:

(1) (Hudson 1975)

$$E_\theta[\theta g(X)] = E_\theta[g(X-1)\,\frac{t(X-1)}{t(X)}].$$

(2)
$$E_\theta[\theta^m g(X)] = E_\theta[g(X-m)\frac{t(X-m)}{t(X)}], \qquad (2.1.2)$$

for any nonnegative integer m.

(3) Equation (2.1.2) is true for negative integer m, if $g(x) = 0$ whenever $x < -m$.

Proof. From (2.1.1) and by change of variables, we have

$$E_\theta[\theta^m g(X)] = \sum_{k=0}^{\infty} \theta^{m+k} g(k)\varphi(\theta)t(k)$$

$$= \sum_{s=m}^{\infty} g(s-m)t(s-m)\varphi(\theta)\theta^s. \qquad (2.1.3)$$

If $m > 0$, then $t(s-m) = 0$ for $s < m$. Hence (2.1.3) can be written as

$$E_\theta[\theta^m g(X)] = \sum_{s=0}^{\infty} g(s-m)\,\frac{t(s-m)}{t(s)}\,\varphi(\theta)t(s)\theta^s$$

$$= E_\theta[g(X-m)\,\frac{t(X-m)}{t(X)}\,] .$$

For $m < 0$, the assumption on $g(x)$ in (3) implies $g(s-m) = 0$ when $s < 0$. Thus by dropping the trivial terms of the summation in (2.1.3), it follows that

$$E_\theta[\theta^m g(X)] = \sum_{s=0}^{\infty} g(s-m)\,\frac{t(s-m)}{t(s)}\,\varphi(\theta)t(s)\theta^s$$

$$= E_\theta[g(X-m)\,\frac{t(X-m)}{t(X)}] \quad . \qquad \text{Q.E.D.}$$

In Lemma 2.2, recall that $\underset{\sim}{e}_i$ denotes the ith coordinate vector in R^p as in (1.2.11).

Lemma 2.2. Let $\underset{\sim}{X} = (X_1,\ldots,X_p)$, where $X_i \overset{\text{indep.}}{\sim} f_i(x_i|\theta_i)$ and f_i is given in (1.2.2) with $t_i(x_i) > 0$ for $x_i = 0,1,\ldots,$ and $i = 1,\ldots,p$. Then for any real-valued function $g(\underset{\sim}{x})$, defined on R^p, for which $E_{\underset{\sim}{\theta}}(|g(\underset{\sim}{X})|) < \infty$, the following equation is true for any nonnegative integer m:

$$E_{\underset{\sim}{\theta}}[\theta_i^m g(\underset{\sim}{X})] = E_{\underset{\sim}{\theta}}[g(\underset{\sim}{X}-m\underset{\sim}{e}_i)\,\frac{t_i(X_i-m)}{t_i(X_i)}\,]. \tag{2.1.4}$$

Furthermore, when m is a negative integer, (2.1.4) is true, if $g(x_1,\ldots,x_i\ldots,x_p) = 0$ whenever $x_i < -m$.

Proof: By Lemma 2.1 and conditioning on $X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_p$, (2.1.4) is easily obtained.

Now consider the loss function $L_{\underset{\sim}{m}}$, and let $\underset{\sim}{\delta}^o = (\delta_1^o,\ldots,\delta_p^o)$ be an estimator we hope to improve upon. Write a competitor $\underset{\sim}{\delta}^*$ of $\underset{\sim}{\delta}^o$ as $\underset{\sim}{\delta}^* = \underset{\sim}{\delta}^o + \underset{\sim}{\Phi}$, $\underset{\sim}{\Phi} = (\Phi_1,\ldots,\Phi_p)$. Suppose that $E_{\underset{\sim}{\theta}}(\delta_i^o(\underset{\sim}{X}))^2 < \infty$ and $E_{\underset{\sim}{\theta}}[\Phi_i^2(\underset{\sim}{X})] < \infty$ for all $\underset{\sim}{\theta}$ and $i = 1,\ldots,p$, so that $R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o)$ and $R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*)$ are both finite, and

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E[L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{\delta}^*(\underset{\sim}{X})) - L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{\delta}^o(\underset{\sim}{X}))], \tag{2.1.5}$$

where

$$L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{\delta}^*(\underset{\sim}{X})) - L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{\delta}^o(\underset{\sim}{X}))$$

$$= \sum_{i=1}^{p} \theta_i^{m_i}\{[\delta_i^o(\underset{\sim}{X})+\Phi_i(\underset{\sim}{X})-\theta_i]^2 - [\delta_i^o(\underset{\sim}{X})-\theta_i]^2\}$$

$$= \sum_{i=1}^{p} \{2\theta_i^{m_i}(\delta_i^o(\underset{\sim}{X})-\theta_i)\Phi_i(\underset{\sim}{X}) + \theta_i^{m_i}\Phi_i^2(\underset{\sim}{X})\}. \tag{2.1.6}$$

If $\phi_i(x_1,\ldots,x_i,\ldots,x_p) = 0$ when $x_i < -m_i$, then from Lemma 2.2, we have

$$E_{\underset{\sim}{\theta}}\{2\theta_i^m(\delta_i^o(\underset{\sim}{X})-\theta_i)\phi_i(\underset{\sim}{X}) + \theta_i^m\phi_i^2(\underset{\sim}{X})\} = E_{\underset{\sim}{\theta}}[\mathcal{D}_i(\phi_i(\underset{\sim}{X}))], \qquad (2.1.7)$$

where

$$\mathcal{D}_i(\phi_i(\underset{\sim}{x})) = 2\delta_i^o(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\frac{t_i(x_i-m_i)}{t_i(x_i)}$$

$$- 2\phi_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\frac{t_i(x_i-m_i-1)}{t_i(x_i)}$$

$$+ \phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\frac{t_i(x_i-m_i)}{t_i(x_i)} \qquad (2.1.8)$$

Define

$$\mathcal{D}(\underset{\sim}{\phi}(\underset{\sim}{x})) = \sum_{i=1}^{p} \mathcal{D}_i(\phi_i(\underset{\sim}{x})). \qquad (2.1.9)$$

From (2.1.5), (2.1.6), (2.1.7), (2.1.8) and (2.1.9), we obtain

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E_{\underset{\sim}{\theta}}[\mathcal{D}(\underset{\sim}{\phi}(\underset{\sim}{X}))]. \qquad (2.1.10)$$

Thus $\underset{\sim}{\delta}^*$ is better than $\underset{\sim}{\delta}^o$, if for all $\underset{\sim}{x}$, $\underset{\sim}{\phi}(\underset{\sim}{x})$ is a solution to

$$\mathcal{D}(\underset{\sim}{\phi}(\underset{\sim}{x})) \leq 0, \qquad (2.1.11)$$

and for some set of $\underset{\sim}{x}$ with positive probability for some $\underset{\sim}{\theta}$, strict inequality in (2.1.11) actually holds.

It seems difficult to find a solution to (2.1.11). If, however, we write $\underset{\sim}{\phi}(\underset{\sim}{x})$ as $\underset{\sim}{q}(\underset{\sim}{x})\underset{\sim}{\phi}(\underset{\sim}{x})$, where $q_i(\underset{\sim}{x}) \geq 0$ for all $\underset{\sim}{x}$, then with appropriate choice of $\underset{\sim}{q}(\underset{\sim}{x})$, $\mathcal{D}_i(\phi_i(\underset{\sim}{x}))$ can be reduced to an expression which involves only a partial difference term and a square term of ϕ_i. We thus end up with only a partial difference inequality to deal with. (This method was first introduced for a special differential inequality in Berger (1978).) Theorem 2.1 describes explicitly how this can be done.

Theorem 2.1. Let $\underset{\sim}{X}$ be as in Lemma 2.2, and let $\underset{\sim}{\delta}^o$ be any estimator of $\underset{\sim}{\theta}$ such that $R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) < \infty$ for all $\underset{\sim}{\theta}$, under the loss function $L_{\underset{\sim}{m}}$. Define, for all $\underset{\sim}{x}$ with $x_i \geq 0$,

$$\mathscr{D}_i'(\phi_i(\underset{\sim}{x})) = \frac{2t_i(x_i-m_i-1)}{t_i(x_i)} q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ \frac{t_i(x_i-m_i)}{t_i(x_i)} q_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i), \qquad (2.1.12)$$

where q_i and ϕ_i are functions defined on I^p(I is the set of all integers.) Denote $\sum_{i=1}^{p} \mathscr{D}_i'(\phi_i(\underset{\sim}{x}))$ by $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x}))$. Under the loss function $L_{\underset{\sim}{m}}$, the estimator $\underset{\sim}{\delta}^* = \underset{\sim}{\delta}^o + \underset{\sim}{q}\underset{\sim}{\phi}$ dominates $\underset{\sim}{\delta}^o$, if $\underset{\sim}{q}$ and $\underset{\sim}{\phi}$ satisfy the following four conditions:

(i) $E_{\underset{\sim}{\theta}}(q_i(\underset{\sim}{X})\phi_i(\underset{\sim}{X}))^2 < \infty, \quad i = 1,\ldots,p;$

(ii) $x_i < -m_i$ implies that $\phi_i(\underset{\sim}{x}) = 0, \quad i = 1,\ldots,p;$

(iii) $q_i(\underset{\sim}{x}) \geq 0$ and

$$\{\delta_i^o(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)t_i(x_i-m_i)q_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$-t_i(x_i-m_i-1)q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\}\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) \leq 0; \qquad (2.1.13)$$

(iv) $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) \leq 0, \qquad (2.1.14)$

for all $\underset{\sim}{x}$, with strict inequality holding on some set of $\underset{\sim}{x}$ of positive probability for some $\underset{\sim}{\theta}$.

Proof: From (2.1.8), we have

$$\mathcal{D}_i = \{2 \frac{t_i(x_i-m_i)}{t_i(x_i)} \delta_i^o(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)q_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$- 2 \frac{t_i(x_i-m_i-1)}{t_i(x_i)} q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\}\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ 2 \frac{t_i(x_i-m_i-1)}{t_i(x_i)} q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ q_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) \frac{t_i(x_i-m_i)}{t_i(x_i)}\} .$$

From condition (iii), we have $\mathcal{D}_i \leq \mathcal{D}'_i$, $i = 1,\ldots,p$. It follows that

$$\mathcal{D}(\underset{\sim}{\phi}) = \sum_{i=1}^{p} \mathcal{D}_i(\phi_i) \leq \sum_{i=1}^{p} \mathcal{D}'_i(\phi_i) = \mathcal{D}'(\underset{\sim}{\phi}).$$

Together with conditions (i) and (ii), (2.1.10) and (2.1.14), this implies that

$$\begin{aligned} R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) &= E_{\underset{\sim}{\theta}}\mathcal{D}(\underset{\sim}{\phi}(\underset{\sim}{X})) \\ &\leq E_{\underset{\sim}{\theta}}\mathcal{D}'(\underset{\sim}{\phi}(\underset{\sim}{X})) \\ &\leq 0. \end{aligned} \qquad (2.1.15)$$

Also, $E_{\underset{\sim}{\theta}}\mathcal{D}'(\underset{\sim}{\phi}(\underset{\sim}{X})) < 0$ for some $\underset{\sim}{\theta}$, so that $\underset{\sim}{\delta}^*$ is better than $\underset{\sim}{\delta}^o$. Q.E.D.

Note 1. Conditions (i) and (ii) are easy to check. To apply the theorem, we will choose $q_1,\ldots,q_p$, independent of $\phi_1,\ldots,\phi_p$, so that (2.1.13) is satisfied, and then concentrate on finding solutions to (2.1.14).

Note 2. For the situations considered by Clevenson and Zidek (1977), Peng (1975), Tsui and Press (1977) and Hudson (1978), it is sufficient to choose $q_i(x) = 1$, $i = 1,\ldots,p$. This is due to the fact that the ith component of the estimator to be improved upon is

$$\delta_i^o(\underset{\sim}{X}) = t_i(X_i-1)/t_i(X_i)$$

which is the uniformly minimum variance unbiased estimator of θ_i. Therefore, the left hand side of (2.1.13) is always zero for $q_i(\underset{\sim}{x}) = 1$, no matter what m_i's are.

Note 3. By iteration, it is possible to determine a $\underset{\sim}{q}(\underset{\sim}{x})$ so that equality in (2.1.13) is actually achieved. For the case that $\delta_i^0(\underset{\sim}{x})$ depends on x_i only, $\mathcal{D}_i'$ will have the following form:

$$\mathcal{D}_i'(\phi_i(\underset{\sim}{x})) = v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) + w_i(x_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i). \qquad (2.1.16)$$

For such $\mathcal{D}_i'$, nontrivial solutions to $\mathcal{D}' \leq 0$ are given under certain conditions in Section 2.2.

Note 4. In more general cases, δ_i^0 depends on the entire $\underset{\sim}{x}$. The choice of $q_i(\underset{\sim}{x})$ by iteration, as in Note 3, will then give a $\mathcal{D}_i$ similar to (2.1.16) except that the v_i and W_i will now depend on the entire $\underset{\sim}{x}$. Unfortunately, solutions to $\mathcal{D}' \leq 0$ for such general $\mathcal{D}_i'$ are very hard to find. If, however, it is assumed that $\phi_i(\underset{\sim}{x}) \leq 0$, $i = 1,\ldots,p$, we can choose a simpler $q(\underset{\sim}{x})$, independent of $\underset{\sim}{\phi}(\underset{\sim}{x})$, which satisfies inequality (2.1.13) (but not necessarily equality). The simple $\underset{\sim}{q}(\underset{\sim}{x})$ will give a $\mathcal{D}_i'$ of the form

$$\mathcal{D}_i'(\phi_i(\underset{\sim}{x})) = F(\underset{\sim}{x})v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x}), \qquad (2.1.17)$$

for which nonpositive solutions $\phi_i(\underset{\sim}{x})$, $i = 1,\ldots,p$, to $\mathcal{D}' \leq 0$ can be found. In Chapter IV, it will be shown in detail how such $\underset{\sim}{q}(\underset{\sim}{x})$ can be chosen.

Note 5. In either of the situations described in Notes 3 and 4, the following difference inequality is encountered:

$$\sum_{i=1}^{p} [F(\underset{\sim}{x})v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})] \leq 0. \qquad (2.1.18)$$

If $F(\underset{\sim}{x}) > 0$, dividing both sides of (2.1.18) by $F(\underset{\sim}{x})$, it shows that (2.1.18) can be reduced to

$$\sum_{i=1}^{p} [v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})] \leq 0, \qquad (2.1.19)$$

where $w_i(\underset{\sim}{x})$ is, of course, different. (When $F(\underset{\sim}{x})$ can equal zero, a similar reduction can be worked out with a little modification.) The inequality (2.1.19) is similar to (1.3.11), although the former is a partial difference inequality while the latter is a partial differential inequality. Solving (2.1.19) is unfortunately not so easy as solving (1.3.11). The solutions obtained in the next section, however, are similar to Berger's solution (1.3.13) to the inequality (1.3.11).

An easy corollary follows from the proof of Theorem 2.1. This corollary will be needed in section 4.2.

Corollary 2.1.1. Suppose that all the notation and conditions in Theorem 2.1 remain the same with the exception that condition (iv) is now replaced by the following condition (iv)'

$$\text{(iv)}' \quad \mathcal{L}(\underset{\sim}{\phi}(\underset{\sim}{x})) \leq 0 \quad \text{for all } \underset{\sim}{x}. \qquad (2.1.20)$$

Then $\underset{\sim}{\delta}^*$ is as good as $\underset{\sim}{\delta}^o$.

Section 2.2. Solutions to the Difference Inequality

As pointed out in Section 2.1, the key to the problem of improving upon inadmissible estimators is to find a solution to $\mathcal{L}'(\underset{\sim}{\phi}) \leq 0$, where

$$\mathcal{L}'(\underset{\sim}{\phi}) = \sum_{i=1}^{p} \{v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) + W_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})\} \leq 0. \qquad (2.2.1)$$

In this section, it is assumed that for

$$i = 1,\ldots,p, \quad v_i(x_i) \geq 0, \; w_i(\underset{\sim}{x}) \geq 0,$$

and that there exist integers $\alpha_1,\ldots,\alpha_p$ such that

$$v_i(x_i) > 0 \quad \text{if} \quad x_i \geq \alpha_i \; . \tag{2.2.2}$$

These conditions will be satisifed by problems normally encountered. (Assuming $v_i(x_i)$ to be nonnegative does not lose very much generality, since the sign of v_i can be changed by replacing ϕ_i by $-\phi_i$ in (2.2.1).)

In the following, h_i is taken to be

$$\begin{aligned} h_i(x_i) &= \sum_{k=\alpha_i}^{x_i} \frac{1}{v_i(k)} \qquad x_i \geq \alpha_i \\ &= 0 \qquad\qquad\qquad \text{otherwise}, \end{aligned} \tag{2.2.3}$$

and $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ is defined as

$$\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) = \text{the number of } \{i\colon x_i \geq \alpha_i\}, \tag{2.2.4}$$

where $\alpha = (\alpha_1,\ldots,\alpha_p)$.

We will interpret 0/0 as 0 in the remainder of this paper. The following theorems provide solutions to (2.2.1) under varying conditions.

Theorem 2.2. Let $d_1(\cdot),\ldots,d_p(\cdot)$ be nondecreasing functions, defined on the set of integers, such that for $i = 1,\ldots,p$, the following conditions are satisfied:

(i) $d_i(x_i) > 0$ if $x_i \geq \alpha_i$, and $d_i(x_i) \geq 0$ for all x_i;

(ii) there exist positive constants $\beta_1,\ldots,\beta_p$, such that

$$v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i) \leq \beta_i d_i(x_i-1), \tag{2.2.5}$$

for all x_i;

(iii) for all $\underset{\sim}{x} = (x_1,\ldots,x_p)$ such that $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) > \max_{1\le j\le p} \beta_j$, implies

$$\frac{\sum_{i=1}^{p} w_i(\underset{\sim}{x})h_i^2(x_i)}{\sum_{j=1}^{p} d_j(x_j)} \le K < \infty \tag{2.2.6}$$

for some $K > 0$.

Define $D = \sum_{j=1}^{p} d_j(x_j)$, and assume that $c(\underset{\sim}{x})$ is a function which is nondecreasing in each coordinate and which satisfies

$$0 \le c(\underset{\sim}{x}) \le (\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - \max_{1\le j\le p} \beta_j)^+/K. \tag{2.2.7}$$

If $p > \max_{1\le j\le p} \beta_j$, the function $\underset{\sim}{\phi}(\underset{\sim}{x})$, defined by

$$\phi_i(\underset{\sim}{x}) = \frac{-c(\underset{\sim}{x})h_i(\underset{\sim}{x})}{D}, \quad i = 1,\ldots,p, \tag{2.2.8}$$

is a solution to (2.2.1). Furthermore,

$$\mathscr{D}'(\underset{\sim}{\phi}) \le -c(\underset{\sim}{x})(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - \max_{1\le j\le p} \beta_j - Kc(\underset{\sim}{x}))^+/D, \tag{2.2.9}$$

with strict inequality for those $\underset{\sim}{x}^o = (x_1^o,\ldots,x_p^o)$ for which

$$h_i(x_i^o-1)\Delta_i d_i(x_i^o) > 0 \tag{2.2.10}$$

for at least two i's and $c(\underset{\sim}{x}^o) \ne 0$.

Proof. Because of the monotonicity of $c(\underset{\sim}{x})$ with respect to each coordinate,

$$\Delta_i\phi_i(\underset{\sim}{x}) = \Delta_i\left(\frac{-c(\underset{\sim}{x})h_i(x_j)}{D}\right)$$

$$\le c(\underset{\sim}{x})\Delta_i\left(\frac{-h_i(x_j)}{D}\right) .$$

Define $D_i = d_i(x_i-1) + \sum_{j\ne i} d_j(x_j)$. Then,

$$\Delta_i\left(\frac{-h_i(x_i)}{D}\right) = \frac{-\Delta_i h_i(x_i)}{D} + \frac{h_i(x_i-1)\Delta_i D}{DD_i} \quad .$$

It follows that

$$\sum_{i=1}^{p} v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) \le \frac{c(\underset{\sim}{x})}{D}\left(-\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) + \sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i D}{D_i}\right)$$

$$\le \frac{c(\underset{\sim}{x})}{D}\left(-\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) + \sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i)}{D'}\right), \tag{2.2.11}$$

where D' denotes $\sum_{j=1}^{p} d_j(x_j-1)$. In the last transition, the inequality is actually strict for those $\underset{\sim}{x}$ for which $c(\underset{\sim}{x}) \neq 0$ and two of the x_i's satisfy $h_i(x_i-1)\Delta_i d_i(x_i) > 0$. It follows from (2.2.11) and (2.2.5) that

$$\sum_{i=1}^{p} v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}) \le \frac{c(\underset{\sim}{x})}{D}\left(\max_{1\le j\le p}\beta_j - \#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})\right). \tag{2.2.12}$$

By (2.2.5), we have $\sum_{j=1}^{p} w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x}) \le Kc^2(\underset{\sim}{x})/D$, which, together with (2.2.12), implies that

$$\mathcal{L}'(\underset{\sim}{\phi}) \le \frac{c(\underset{\sim}{x})}{D}\left(Kc(\underset{\sim}{x}) + \max_{1\le j\le p}\beta_j - \#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})\right).$$

Since,by (2.2.7),

$$c(\underset{\sim}{x})\left(Kc(\underset{\sim}{x}) + \max_{1\le j\le p}\beta_j - \#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})\right)$$

$$= - c(\underset{\sim}{x})\left(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - \max_{1\le j\le p}\beta_j - Kc(\underset{\sim}{x})\right)^+,$$

(2.2.9) is established. Q.E.D.

Note 1. Theorem 2.2 is true even when $p \le \max_{1\le j\le p}\beta_j$. However, in this case, the ϕ_i in (2.2.8) are always zero, which gives an uninteresting solution to (2.2.1). Therefore $p > \max_{1\le j\le p}\beta_j$ is assumed in Theorem 2.2.

Note 2. It is easy to choose the $d_i(\cdot)$, so that they are nondecreasing, nonnegative, and satisfy (2.2.5). Indeed, the following is such a choice:

$$d_i(x_i) = \prod_{k=\alpha_i+1}^{x_i} \left(1 + \frac{\beta_i}{v_i(k)h_i(k-1)}\right) \qquad x_i \geq \alpha_i + 1$$

$$= 1 \qquad x_i = \alpha_i$$

$$= 0 \qquad \text{otherwise} \qquad (2.2.13)$$

Unfortunately, this choice is too complicated to be useful. Hence, in the following corollaries, efforts will be made to obtain simpler $d_i(\cdot)$ for special cases.

Note 3. From (2.2.5), it is clear that a larger β_i allows a larger d_i, and hence (2.2.6) is easier to satisfy. But then the dimension, p, required for nontrivial solutions to the inequality (2.2.1), is higher. Also, a larger $d_i(\cdot)$ gives a larger upper bound in (2.2.9); and the corresponding new estimator will have a smaller improvement in risk. (cf (2.2.9) and (2.1.15)) For these two reasons, we will choose β_i as small as possible.

The following corollaries and examples illustrate the use of Theorem 2.2 in solving the difference inequalities. In each case, d_i is given explicitly. The first corollary is applicable when the v_i's are increasing functions. It also tells more about how the choice of the d_i can be made.

Corollary 2.2.1 Let $\mathcal{L}'$, v_i, w_i and α_i denote as in (2.2.1) and (2.2.2). Suppose that $v_1,\ldots,v_p$ are increasing functions and that $\beta_1,\ldots,\beta_p$ are positive integers. If

$$d_i(x_i) = b_i h_i(x_i) h_i(x_i+1) \ldots h_i(x_i+\beta_i-1) + b_0, \qquad (2.2.14)$$

for $i = 1,\ldots,p$ and some constants $b_0 \geq 0$ and $b_i > 0$, then (2.2.5) is satisfied. If, in addition, (2.2.6) is satisfied for this choice of the d_i, then $\phi_1,\ldots,\phi_p$, as in (2.2.8), is a solution to the inequality (2.2.1), providing $p > \max\limits_{1 \leq j \leq p} \beta_j$.

Proof: Clearly $d_i(x_i)$ is increasing and greater than zero for $x_i \geq \alpha_i$. Therefore, by Theorem 2.2, it is only necessary to show that (2.2.5) is satisfied. Now (2.2.5) is trivial for $x_i \leq \alpha_i$. For $x_i \geq \alpha_i + 1$, $\beta_i \geq 2$, we have

$$\Delta_i d_i(x_i) = b_i[h_i(x_i+\beta_i-1)-h_i(x_i-1)]h_i(x_i)\ldots.h_i(x_i+\beta_i-2)$$

$$\leq \frac{b_i\beta_i}{v_i(x_i)} h_i(x_i)\ldots h_i(x_i+\beta_i-2).$$

Consequently, for $\beta_i \geq 2$,

$$v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i) \leq \beta_i b_i h_i(x_i-1)h_i(x_i)\ldots h_i(x_i+\beta_i-2)$$

$$\leq \beta_i d_i(x_i-1). \qquad (2.2.15)$$

For $\beta_i = 1$, it is clear that

$$v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i) = b_i h_i(x_i-1) \leq d_i(x_i-1). \qquad \text{Q.E.D.}$$

Note that the d_i in (2.2.14) are similar to the functions $b_i h_i^{\beta_i}(x_i) + b_0$. In all the problems discussed in this thesis, the d_i can be chosen to be (or at least to be similar to) a polynomial function of h_i so that (2.2.5) is satisfied.

In applying Theorem 2.2, we assume at first that the d_i have the form $h_i^{\beta_i}(x_i)$, and choose β_i as small as possible so that (2.2.6) is satisfied. For such β_i, we modify the d_i(by using the form (2.2.14) or by adding extra positive terms of the form $h_i^{k_i}$, $k_i < \beta_i$) so that

(2.2.5) is satisfied. Solutions to (2.2.1) are then given in (2.2.8). If, no matter how large β_i is, (2.2.6) is never satisfied, by this theorem, it seems unlikely that solutions to (2.2.1) can be found. Indeed all admissible estimators have a difference inequality for which (2.2.6) is never satisfied for $d_i(x_i) = h_i^{\beta_i}(x_i)$ and any β_i.

For the case $0 \le \beta_i \le 1$, Theorem 2.2 can be reduced to the following simple corollary.

Corollary 2.2.2 Let $\mathscr{D}'$, v_i, w_i and α_i denote the same as in (2.2.1) and (2.2.2). For any constants b_0, b_i, and β_i such that $b_0 \ge 0, b_i > 0$ and $0 \le \beta_i \le 1$, $i = 1,\ldots,p$, the function

$$d_i(x_i) = b_0 + b_i h_i^{\beta_i}(x_i) \qquad (2.2.16)$$

satisfies (2.2.5). If, in addition, (2.2.6) is satisfied for this choice of the d_i, then $\underset{\sim}{\phi}$, with ϕ_i as in (2.2.8), is a solution to the inequality (2.2.1), provided $p > \max_{1 \le j \le p} \beta_j$.

Proof: Again, it is only necessary to show that d_i satisfies (2.2.5) for $x_i \ge \alpha_i + 1$. Now $\beta_i \le 1$, which, together with mean value theorem implies

$$\begin{aligned} v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i) &= v_i(x_i)h_i(x_i-1)\Delta_i(b_i h_i^{\beta_i}(x_i)) \\ &\le \beta_i b_i v_i(x_i) h_i^{\beta_i}(x_i-1)\Delta_i h_i(x_i) \\ &= \beta_i b_i h_i^{\beta_i}(x_i-1) \\ &\le \beta_i d_i(x_i-1). \end{aligned}$$

Q.E.D.

Example 2.1. In Chapter III, it will be shown (See (3.1.9), (3.1.10) and (3.1.11)) that the problem that Clevenson and Zidek (1973) considered (refer to Section 1.3.2) can be reduced to the study of the following inequality:

$$\sum_{i=1}^{p} \{\Delta_i \phi_i(\underset{\sim}{x}) + \frac{1}{2(x_i+1)} \phi_i^2(\underset{\sim}{x})\} \leq 0. \qquad (2.2.17)$$

Using the notation of Theorem 2.2, note that $v_i(x_i) = 1$, $w_i(\underset{\sim}{x}) = 1/2(x_i + 1)$, and $\alpha_i = 0$, $i = 1,\ldots,p$.

Hence

$$h_i(x_i) = x_i + 1 \qquad x_i \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

Since, for $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) > 1$, we have

$$\sum_{i=1}^{p} \frac{1}{2} \frac{h_i^2(x_i)/(x_i+1)}{\sum_{j=1}^{p} h_j(x_j)} = \frac{1}{2},$$

(2.2.6) is satisfied with $\beta_i = 1$ and $K = \frac{1}{2}$. It follows from Corollary 2.2.2 that if $p > 1$, then

$$\phi_i(\underset{\sim}{x}) = \frac{-c(\underset{\sim}{x})(x_i+1)}{b_0 + \sum_{j=1}^{p} (x_j+1)}, \quad i = 1,\ldots,p,$$

is a solution to (2.2.17) for any nonnegative number b_0 and any function $c(\underset{\sim}{x})$ increasing in each coordinate which satisfies

$$0 \leq c(\underset{\sim}{x}) \leq 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})-1)^+ .$$

Note that $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ is equal to p if it can be assumed that $x_i \geq 0$, $i = 1,\ldots,p$.

It happens quite often especially for the negative binomial distribution that we encounter a difference inequality of the form (2.2.1), with $1/v_i(x_i) \leq M_i$ for all $x_i \geq \alpha_i$. In such a situation, and for any integer β_i, d_i can be chosen to be a polynomial function of $h_i(x_i)$, which has order β_i and positive coefficients, so that (2.2.5) is satisfied. However, since the applications in the following chapters involve only the case $\beta_i = 2$, the following corollary is restricted to deal only with this case.

Corollary 2.2.3 Let $\mathscr{L}'$, v_i, w_i and α_i denote the same as in (2.2.1) and (2.2.2). If, for some constant $M_i > 0$,

$$\frac{1}{v_i(x_i)} \leq M_i \quad \text{for all} \quad x_i \geq \alpha_i + 1, \tag{2.2.18}$$

then

$$d_i(x_i) = h_i^2(x_i) + b_i h_i(x_i) + b_0$$

satisfies (2.2.5) for any constants, $b_0, b_1, \ldots, b_p$ such that $b_i \geq M_i$ and $b_0 \geq 0$. If, in addition, (2.2.6) is satisfied for this choice of the $d_i(x_i)$, then $\phi_1, \ldots, \phi_p$, as in (2.2.8), is a solution to the inequality (2.2.1), provided $p > 2$.

Proof: Again, it is only necessary to prove that (2.2.5) is satisfied for $x_i \geq \alpha_i + 1$. Now,

$$\Delta_i d_i(x_i) = \frac{1}{v_i(x_i)} [h_i(x_i) + h_i(x_i - 1)] + \frac{b_i}{v_i(x_i)} \quad .$$

It follows that

$$\begin{aligned} h_i(x_i-1) v_i(x_i) \Delta_i d_i(x_i) &= [h_i(x_i-1) h_i(x_i) + h_i^2(x_i-1)] + b_i h_i(x_i-1) \\ &= 2h_i^2(x_i-1) + \frac{h_i(x_i-1)}{v_i(x_i)} + b_i h_i(x_i-1) \\ &\leq 2h_i^2(x_i-1) + (M_i + b_i) h_i(x_i-1) \\ &\leq 2d_i(x_i-1). \end{aligned}$$

Q.E.D.

Example 2.2 Assume $X_i \overset{\text{indep.}}{\sim} NB(r, \theta_i)$. The uniformly minimum variance unbiased estimator of θ_i is then $X_i/(r-1+X_i)$. It will be shown in the next chapter, that, for improving upon

$$\left(\frac{X_i}{r-1+X_1}, \ldots, \frac{X_p}{r-1+X_p}\right)$$

under the loss function L_0, the key difference inequality to solve is

$$\sum_{i=1}^{p} \left\{ \frac{x_i}{r-1+x_i} \Delta_i \phi_i(\underset{\sim}{x}) + \frac{1}{2} \phi_i^2(\underset{\sim}{x}) \right\} \leq 0. \qquad (2.2.19)$$

Here, $v_i(x_i) = x_i/(r-1+x_i)$, $w_i(\underset{\sim}{x}) = \frac{1}{2}$, $\alpha_i = 1$ and $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ denotes number of i such that $x_i \geq 1$. Also,

$$h_i(x_i) = \sum_{k=1}^{x_i} \frac{k+r-1}{k} \qquad x_i \geq 1$$

$$= 0 \qquad \text{otherwise} .$$

Since $\frac{1}{v_i(x_i)} = \frac{x_i+r-1}{x_i} \leq \frac{r+1}{2}$, for $x_i \geq 2$, it is clear that (2.2.18) is satisfied for $M = (r+1)/2$. Let

$$d_i(x_i) = h_i^2(x_i) + b\ h_i(x_i) + b_0 ,$$

where $b \geq M$ and $b_0 \geq 0$ are constants. Clearly,

$$\frac{1}{2} \frac{\sum_{i=1}^{p} h_i^2(x_i)}{\sum_{i=1}^{p} d_i(x_i)} \leq \frac{1}{2} .$$

Hence, by corollary 2.2.3, for any function $c(\underset{\sim}{x})$ which is nondecreasing in each coordinate and which satisfies $0 \leq c(\underset{\sim}{x}) \leq 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})-2)^+$, it follows that

$$\phi_i(\underset{\sim}{x}) = \frac{-c(\underset{\sim}{x})h_i(x_i)}{\sum_{j=1}^{p} \{h_j^2(x_j) + b\ h_j(x_j) + b_0\}}, \qquad i = 1,\ldots,p ,$$

is a solution to (2.2.19), provided $p \geq 3$.

CHAPTER III

TYPICAL APPLICATIONS

In this chapter, the theorems in Chapter II are used to improve upon certain standard estimators under losses of the form $L_{\underset{\sim}{m}}$. To be more precise, let $\underset{\sim}{X}$ be as in Lemma 2.2, and let

$$\underset{\sim}{\delta}^{0}(\underset{\sim}{X}) = (\delta_1^0(X_1),\ldots,\delta_p^0(X_p)),$$

where $\delta_i^0(X_i)$ is the usual unbiased estimator of θ_i (i.e. $\delta_i^0(X_i) = t_i(X_i-1)/t_i(X_i)$, where recall 0/0 is interpretated as 0.). Under the loss function $L_{\underset{\sim}{m}}$, we will develop classes of vector functions $\underset{\sim}{q}(\underset{\sim}{x}) = (q_1(\underset{\sim}{x}),\ldots,q_p(\underset{\sim}{x}))$ and $\underset{\sim}{\phi}(\underset{\sim}{x}) = (\phi_1(\underset{\sim}{x}),\ldots,\phi_p(\underset{\sim}{x}))$ which satisfy the four assumptions in Theorem 2.1. It will follow that $\underset{\sim}{\delta}^0 + \underset{\sim}{q}\underset{\sim}{\phi}$ dominates $\underset{\sim}{\delta}^0$ and hence a class of better estimators will have been found.

Assumption (iii) in Theorem 2.1 indicates that for

$$\delta_i^0(x_i) = t_i(x_i-1)/t_i(x_i), \tag{3.1.1}$$

it is sufficient to choose $q_i(\underset{\sim}{x}) = 1$, since, by plugging such $\delta_i^0(x_i)$ into (2.1.13), it is clear that the right hand side of (2.1.13) is always zero if $q_i(\underset{\sim}{x}) = 1$, no matter which loss function $L_{\underset{\sim}{m}}$ is assumed.

To satisfy assumption (iv) in Theorem 2.1, it is necessary to find a nontrivial solution $\phi_1,\ldots,\phi_p$ to

$$\mathcal{D}'(\underset{\sim}{\phi}) = \sum_{i=1}^{p} \mathcal{D}_i'(\phi_i) \le 0, \tag{3.1.2}$$

where, from (2.1.12) and the fact that $q_i(x) = 1$, $\mathcal{L}_i'$ now has the form

$$\mathcal{L}_i'(\phi_i(\underset{\sim}{x})) = \frac{2t_i(x_i-m_i-1)}{t_i(x_i)} \Delta_i\phi_i(\underset{\sim}{x}-m_ie_i)$$

$$+ \frac{t_i(x_i-m_i)}{t_i(x_i)} \phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i). \qquad (3.1.3)$$

In the following two sections, solutions to (3.1.2) are obtained by applying the theorems in Section 2.2. All the solutions below will be bounded, so that assumption (i) in Theorem 2.1 is automatically satisfied. Furthermore, it can be easily checked that all the $\phi_1,\ldots,\phi_p$ below satisfy assumption (ii) in Theorem 2.1. Therefore, in applying Theorem 2.1, we only describe the difference inequality and how the solutions are obtained. Also, in the following we will look for solutions to $\frac{1}{2}\mathcal{D}'(\phi) \le 0$ which is, of course, equivalent to $\mathcal{D}'(\phi) \le 0$.

In Section 3.1, it is assumed that the X_i's are from Poisson families, while in Section 3.2, the X_i's are from negative binomial families.

Section 3.1 Poisson Distributions

In this section, it is assumed that $X_i \overset{\text{indep.}}{\sim} P_o(\theta_i)$, i=1,...,p, and hence that $t_i(x_i) = 1/x_i!$ (using the notation of Theorem 2.1).

Under the loss function L_m, where m is some negative integer, $\underset{\sim}{\delta}^o$ can also be improved. This will follow from Theorem 3.1, in which the loss function is assumed to be $L_{\underset{\sim}{m}}$. In the remainder of the thesis, let γ_{ij} denote Kronecker constant, i.e.

$$\gamma_{ij} = 1 \quad i = j$$
$$= 0 \quad \text{otherwise.} \qquad (3.1.4)$$

Furthermore, for any function $c(\underset{\sim}{x})$, we will use, "$c(\underset{\sim}{x}) \not\equiv 0$" to denote that $c(\underset{\sim}{x})$ is not identically zero.

Theorem 3.1 Let $\underset{\sim}{m} = (m_1,\ldots,m_p)$, where the m_i are nonpositive integers, and let n_i and α_i denote $-m_i$ and $(m_i + 1)^+$ respectively. Assume that $p > \max_{1\leq j\leq p} (\alpha_i + 1)$. Under the loss function $L_{\underset{\sim}{m}}$, the usual stimator $\underset{\sim}{\delta}^0(\underset{\sim}{x}) = \underset{\sim}{x}$ is inadmissible. Indeed a better estimator can be described as follows: Define

$$h_i(x_i) = \begin{cases} \sum_{k=1}^{x_i} \frac{1}{k} & x_i \geq 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } n_i = 0 \qquad (3.1.5)$$

and

$$h_i(x_i) = \begin{cases} \frac{1}{n_i}(x_i + 1)\ldots(x_i + n_i) & x_i \geq 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } n_i > 0.$$

Also, define, for some constant $b_0 \geq 0$,

$$d_i(x_i) = h_i(x_i)h_i(x_i + 1) + b_0 \quad \text{if } n_i = 0$$
$$= \frac{1}{n_i} h_i(x_i) + b_0 \quad \text{if } n_i > 0 \qquad (3.1.6)$$

Let $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ denote the number of indices i for which $x_i \geq \alpha_i$. For any nonnegative number b_0 and any function $c(\underset{\sim}{x})$ which is nondecreasing in each coordinate and satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - \max_{1\leq j\leq p} (\alpha_i + 1))^+, \qquad (3.1.7)$$

$\underset{\sim}{\delta}^o$ is dominated by $\underset{\sim}{\delta}^o(\underset{\sim}{x}) + \underset{\sim}{\phi}(\underset{\sim}{x})$, where

$$\phi_i(\underset{\sim}{x}) = \frac{-c(\underset{\sim}{x}-n_i\underset{\sim}{e}_i)h_i(x_i-n_i)}{\sum_{j=1}^{p} d_j(x_j-n_i\gamma_{ij})} . \tag{3.1.8}$$

Proof: By (3.1.1) and (3.1.2), we have to obtain a solution $\phi_1,\ldots,\phi_p$ to the following difference inequality:

$$\frac{1}{2}\,\mathcal{S}''\,(\underset{\sim}{\psi}(\underset{\sim}{x})) = \sum_{i=1}^{p}[v_i(x_i)\Delta_i\psi_i(\underset{\sim}{x}) + w_i(x_i)\psi_i^2(\underset{\sim}{x})] \le 0, \tag{3.1.9}$$

where

$$\psi_i(\underset{\sim}{x}) = \phi_i(x_i + n_i\underset{\sim}{e}_i), \tag{3.1.10}$$

and

$$\left.\begin{aligned} v_i(x_i) &= x_i \\ w_i(x_i) &= 1/2 \end{aligned}\right\} \quad \text{if } n_i = 0 \tag{3.1.11}$$

or

$$\left.\begin{aligned} v_i(x_i) &= \frac{x_i!}{(x_i+n_i-1)!} \\ w_i(x_i) &= \frac{x_i!}{2(x_i+n_i)!} \end{aligned}\right\} \quad \text{if } n_i > 0.$$

To find a solution to (3.1.9), we will use Theorem 2.2 as a guide. Note that $v_i(x_i) > 0$ if $x_i \ge \alpha_i$, and it follows from (3.1.5) and (3.1.11) that

$$\begin{aligned} h_i(x_i) &= \sum_{k=\alpha_i}^{x_i} \frac{1}{v_i(k)} \qquad x_i \ge \alpha_i \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

When $n_i = 0$, $v_i(x_i)$ is increasing. By Corollary 2.2.1, $d_i(x_i)$

satisfies (2.2.5) with $\beta_i = 2$. Also by Corollary 2.2.2, $d_i(x_i)$ satisfies (2.2.5) with $\beta_i = 1$ when $n_i > 0$. Therefore assumption (ii) of Theorem 2.2 is satisfied with $\beta_i = \alpha_i + 1$. It is also clear that

$$W_i(x_i)h_i^2(x_i) = h_i^2(x_i)/2 \leq d_i(x_i)/2 \quad \text{if } n_i = 0$$

and

$$W_i(x_i)h_i^2(x_i) = h_i^2(x_i)/2n_i \leq d_i(x_i)/2 \quad \text{if } n_i > 0.$$

Therefore, assumption (iii) of Theorem 2.2 is satisfied, and hence a solution to (3.1.9) is $\psi_1,\ldots,\psi_p$, where

$$\psi_i(x) = \frac{-c(\underset{\sim}{x})h_i(x_i)}{\sum_{j=1}^{p} d_j(x_j)}, \qquad i = 1,\ldots,p, \tag{3.1.12}$$

or equivalently

$$\phi_i(\underset{\sim}{x}) = \frac{-c(\underset{\sim}{x}-n_i\underset{\sim}{e}_i)h_i(x_i-n_i\underset{\sim}{e}_i)}{\sum_{j=1}^{p} d_j(x_j)}, \qquad i = 1,\ldots,p. \tag{3.1.13}$$

Also, it is clear that

$$E_{\underset{\sim}{\theta}}\mathcal{D}''(\underset{\sim}{\psi}) < 0$$

for all $\underset{\sim}{\theta}$. Theorem 2.1, thus implies that $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ dominates $\underset{\sim}{\delta}^0$. Q.E.D.

Corollary 3.1.1 (Clevenson and Zidek 1975) Assume that $p \geq 2$. Under the loss function L_{-1}, $\underset{\sim}{\delta}^0(\underset{\sim}{X}) = \underset{\sim}{X}$ is inadmissible. Indeed for any constant $b_0 \geq 0$, and any function $c(\underset{\sim}{x})$ which is nondecreasing in each coordinate and satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2(p-1), \tag{3.1.14}$$

$\underset{\sim}{\delta} + \underset{\sim}{\phi}$ dominates $\underset{\sim}{\delta}^0$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X}-e_i)X_i}{b_0 + p-1 + \sum_{j=1}^{p} X_j}. \tag{3.1.15}$$

Proof: Clearly $m_i = -1$, $n_i = 1$ and $\alpha_i = 0$, $i = 1,\ldots,p$. Hence $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ is the number of indices i for which $x_i \geq 0$. It is clear that ϕ_i in (3.1.8) is reduced to the form in (3.1.15) with a different b_0. To complete the proof, it is only necessary to show that the condition (3.1.7) is equivalent to (3.1.14). Now, for each $c(\underset{\sim}{x})$ satisfying (3.1.14), we will design a version $c'(x)$ of $c(x)$ so that (3.1.7) is satisfied ($c'(x)$ is also nondecreasing in each coordinate and not identically zero) and $c'(\underset{\sim}{x}-\underset{\sim}{e}_i)x_i = c(\underset{\sim}{x}-\underset{\sim}{e}_i)x_i$, $i = 1,\ldots,p$ with probability one. Therefore $\phi_1,\ldots,\phi_p$ remain the same with probability one even if c is replaced by c' in (3.1.13). This proves that (3.1.14) is in fact equivalent to (3.1.7). Indeed let $A = \{(x_1,\ldots,x_p):x_i \geq 0\}$ and

$$c'(\underset{\sim}{x}) = c(\underset{\sim}{x})I_A(\underset{\sim}{x}). \qquad (3.1.16)$$

Since with probability one, $x_i \geq 0$ $i = 1,\ldots,p$, it is clear that with probability one,

$$c(\underset{\sim}{x}-\underset{\sim}{e}_j)x_j = c'(\underset{\sim}{x}-\underset{\sim}{e}_j)x_j \quad j = 1,\ldots,p.$$

Now $c'(\underset{\sim}{x})$ is nondecreasing in each coordinate and is not identically zero. Furthermore $c'(\underset{\sim}{x})$ satisfies (3.1.7) hence we are done. Q.E.D.

The better estimators obtained in Clevenson and Zidek (1975) correspond to those $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ with $c(\underset{\sim}{x})$ depending on $\underset{\sim}{X}$ only through $\sum_{i=1}^{p} X_i$.

The main idea in the proof in Corollary 3.2.1 will be used in other occasions. Therefore a more general lemma is established here.

Lemma 3.1. Assume that $c(\underset{\sim}{x})$ is defined on I^p (I is the set of all integer.) which satisfies the following conditions:

(i) $c(\underset{\sim}{x}) \not\equiv 0$;

(ii) $c(\underset{\sim}{x})$ is nondecreasing in each coordinate;

(iii) $0 \leq c(\underset{\sim}{x}) \leq n_0(p-\beta)$, where n_0 and β are some positive constants.

Let $h_i(x_i)$ be as in (2.2.3) with $\underset{\sim}{\alpha} = (0,\ldots,0)$, and let $c'(\underset{\sim}{x})$ denote $c(\underset{\sim}{x})I_A(\underset{\sim}{x})$, where $A = \{(x_1,\ldots,x_p): x_i \geq 0\}$. Then, conditions (i) and (ii) above are satisfied when $c(\underset{\sim}{x})$ is replaced by $c'(\underset{\sim}{x})$. Besides, the following are true:

(iii)' $0 \leq c'(\underset{\sim}{x}) \leq n_0(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})-\beta)^+$ for all $\underset{\sim}{x} \in I^p$;

(iv) $c(\underset{\sim}{x}-ne_j)h_j(x_j-n) = c'(\underset{\sim}{x}-n\underset{\sim}{e}_j)h_j(x_j-n)$, $j = 1,\ldots,p$ for all $\underset{\sim}{x} \in A$ and any integer n.

Proof: It is obvious that $c'(\underset{\sim}{x})$ satisfies conditions (i) and (ii). If $\underset{\sim}{x} \in A$, then $c'(\underset{\sim}{x}) = c(\underset{\sim}{x})$ and $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) = p$, condition (iii)' is thus equivalent to condition (iii) and hence is satisfied. If $\underset{\sim}{x} \notin A$, then $c'(\underset{\sim}{x}) = 0$, and condition (iii) is clearly satisfied. Therefore condition (iii)' is satisfied for all $\underset{\sim}{x} \in I^p$. To prove condition (iv), assume that $\underset{\sim}{x} \in A$. For any index j, if $x_j > n$, then $(\underset{\sim}{x}-ne_j) \in A$ and condition (iv) is true since $c'(\underset{\sim}{x}-ne_j) = c(\underset{\sim}{x}-ne_j)$. If $x_j < n$, then $h_j(x_j-n) = 0$ and hence condition (iv) is trivially satisfied. Therefore, condition (iv) holds for all $\underset{\sim}{x} \in A$. Q.E.D.

Corollary 3.1.2. (Tsui and Press 1977) Assume that $p \geq 2$. Under the loss function L_m, where $m = -n$ is some negative integer, $\delta^0(\underset{\sim}{X})=\underset{\sim}{X}$ is inadmissible. Indeed, for any nonnegative number b_0 and any function $c(\underset{\sim}{x})$ which is nondecreasing in each coordinate and satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2n(p-1), \qquad (3.1.17)$$

$\underset{\sim}{\delta}^0$ is dominated by $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ with

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X}-n\underset{\sim}{e}_i)h(X_i-n)}{b_0 + \sum_{j=1}^{p} h(X_j-n\gamma_{ij})}, \qquad (3.1.18)$$

and

$$h(x) = (x+1)\ldots(x+n) \quad \text{if} \quad x \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

Proof: Clearly $\alpha_i = 0$, $i = 1,\ldots,p$. Again the main step of the proof is to show that the following assumption about $c(\underset{\sim}{x})$,

$$0 \leq c(\underset{\sim}{x}) \leq 2n(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - 1)^+, \qquad (3.1.19)$$

can be replaced by (3.1.17). However, this is an immediate result of Lemma 3.1.

Q.E.D.

The better estimators obtained in Tsui and Press (1977) correspond to those $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ with $c(\underset{\sim}{X})$ in (3.1.18) depending on $\underset{\sim}{X}$ only through $\sum_{i=1}^{p} X_i$.

For the loss function L_0, the following corollary is a direct result from Theorem 3.1.

Corollary 3.1.3. Assume $p \geq 3$, under the loss function L_0, $\delta^0(\underset{\sim}{X}) = \underset{\sim}{X}$ is inadmissible. Indeed, a better estimator can be described as follows: Denote the number of indices i for which $x_i \geq 1$ by $\#_1(\underset{\sim}{x})$. Define

$$h(x_i) = \sum_{k=1}^{x_i} \frac{1}{k} \qquad x_i \geq 1$$
$$= 0 \qquad \text{otherwise}$$

Let $c(\underset{\sim}{x})$ be any function which is increasing in each coordinate and satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2(\#_1(\underset{\sim}{x}) - 2)^+$$

Then, for any constant b_0, the new estimator $\underset{\sim}{\delta}^*$ with the ith component,

$$\delta_i^*(\underset{\sim}{X}) = X_i - \frac{c(\underset{\sim}{X})h(X_i)}{b_0 + \sum_{j=1}^{p} h(X_j)h(X_j+1)} \tag{3.1.20}$$

is a better than δ^0.

The improved estimator obtained in Peng (1975) (refer to (1.3.7)) is very similar to $\underset{\sim}{\delta}^*$ with $c(\underset{\sim}{x}) = (\#_1(\underset{\sim}{x})-2)^+$ and $b_0 = 0$.

As another interesting application of Theorem 3.1, consider the situation $p = 3$ and $\underset{\sim}{m} = (0,-1,-2)$. By Theorem 3.2, $\underset{\sim}{\delta}^0(\underset{\sim}{X}) = \underset{\sim}{X}$ is inadmissible and is dominated by $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ with ϕ_i as in (3.1.8). The functions h_1, h_2, h_3 and d_1, d_2, d_3, which determine the form of the correction terms ϕ_i, are given in (3.15) and (3.1.6). It is clear that h_1 and d_1 are similar to $h(x_i)$ and $h^2(x_i)$ in Peng's estimator (cf. (1.3.7)). Also, h_2 and d_2 have the same form as in Clevenson's estimator (cf. (3.1.15)), and h_3 and d_3 are as in Tsui's estimator (cf. (3.1.18)). Note that the choice of h_i and d_i depends only upon m_i (and not m_j, $j \neq i$). A similar property is also observed in the negative binomial case (Section 3.1), as well as in the more general case in which the densities of the X_i are not of the same form (Section 5.3). In this general situation, the choice of h_i and d_i depends on m_i and the density of X_i(i.e. $t_i(x_i)$), and not on the other coordinates.

In the earlier work for the Poisson case, the proofs that the estimators presented (for example in Peng (1975), Clevenson and Zidek (1973) and Hudson (1978)) are better than $\delta^0(\underset{\sim}{X}) = \underset{\sim}{X}$ are heavily based on the symmetry of the problems. (i.e. the m_i are all equal, and the X_i's have the same type of distribution.) Theorem 3.2 shows, however, that Stein's effect seems to be a property more basic than symmetry. For the continuous case, this was also observed in Berger (1978).

Section 3.2. Negative Binomial Distribution

In this section, assume that $X_i \overset{\text{indep.}}{\sim} NB(r_i,\theta_i)$, where r_i is some positive integer, $i = 1,\ldots,p$, and hence that

$$t_i(x_i) = \binom{r_i + x_i - 1}{r_i - 1}, \qquad x_i = 0,1,\ldots .$$

One of the classical estimators of $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ is

$$\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \left(\frac{X_1}{r_1-1+X_1}, \ldots, \frac{X_p}{r_p-1+X_p}\right) ,$$

where $X_i/(r_i-1+X_i)$ is the uniformly minimum variance unbiased estimator of θ_i. For $p = 1$, $\underset{\sim}{\delta}^o$ is admissible under the square error loss. (See Blackwell and Girshick (1954) p. 307) and is hence admissible under any loss function of the form $L(\theta,a) = v(\theta)(a-\theta)^2$ with $v(\theta) > 0$ for all θ.

Our goal here is to improve upon $\underset{\sim}{\delta}^o$ under the loss function $L_{\underset{\sim}{m}}$. Therefore, a solution to $\mathcal{D}'(\underset{\sim}{\phi}) \le 0$ (defined in (3.1.2) and (3.1.3)). Since $t_i(x_i) = \binom{r_i+x_i-1}{r_i-1}$, the difference inequality $\frac{1}{2}\mathcal{D}'(\underset{\sim}{\phi}) \le 0$ has the following form:

$$\frac{1}{2}\mathcal{D}'(\underset{\sim}{\phi}) = \sum_{i=1}^{p}[v_i(x_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) + w_i(x_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)] \le 0, \quad (3.2.1)$$

where

$$v_i(x_i) = \binom{r_i+x_i-m_i-2}{r_i-1} \Big/ \binom{r_i+x_i-1}{r_i-1}, \text{ if } \quad x_i \geq (m_i+1)^+$$

$$= 0 \qquad\qquad , \text{ otherwise}, \qquad (3.2.2)$$

and

$$w_i(x_i) = \binom{r_i+x_i-m_i-1}{r_i-1} \Big/ 2\binom{r_i+x_i-1}{r_i-1} \text{ if } \quad x_i \geq m_i^+$$

$$= 0 \qquad\qquad \text{otherwise}. \qquad (3.2.3)$$

When $m_i = 0$ and $r_i = r$, $i = 1,\ldots,p$, $v_i(x_i) = x_i/(x_i+r_i-1)$ and $w_i(x_i) = 1/2$. The corresponding difference inequality (3.2.1) was solved in Example 2.2 by applying Corollary 2.2.3. The same corollary is also applicable to the general case as seen in the following theorem.

In the theorem below, define for $i = 1,\ldots,p$,

$$\alpha_i = (m_i + 1)^+, \qquad (3.2.4)$$

$$h_i(x_i) = \sum_{k=\alpha_i}^{x_i} \binom{r_i+k-1}{r_i-1} \Big/ \binom{r_i+k-m_i-2}{r_i-1} \text{ if } \quad x_i \geq \alpha_i$$

$$= 0 \qquad\qquad \text{otherwise}, \qquad (3.2.5)$$

$$M_i = \frac{1}{r_i}\binom{r_i+\alpha_i}{r_i-1} \qquad (3.2.6)$$

$$N_i = 1 \qquad \text{if} \quad m_i \geq 0$$
$$= \binom{r_i - m_i - 1}{r_i - 1} \qquad \text{otherwise,} \tag{3.2.7}$$

and

$$K = \max_{1 \leq j \leq p} \{N_i\} . \tag{3.2.8}$$

Theorem 3.2. Assume that $p \geq 3$. Let $\underset{\sim}{m} = (m_1, \ldots, m_p)$, with m_i being any integer. Under the loss function $L_{\underset{\sim}{m}}$, the estimator

$$\underset{\sim}{\delta}^o(\underset{\sim}{X}) = \left(\frac{X_1}{r_1 + X_1 - 1}, \ldots, \frac{X_p}{r_p + X_p - 1}\right)$$

is inadmissible. Indeed, a better estimator can be described as follows: Let $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$ denote the number of indices i for which $x_i \geq \alpha_i$. Furthermore, let $b_0 \geq 0$, $b_j \geq M_j$, $j = 1, \ldots, p$, and $c(\underset{\sim}{x})$ be any function that is nondecreasing in each coordinate and satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x}) - 2)^+/K. \tag{3.2.9}$$

Then, $\underset{\sim}{\delta}^o$ is dominated by $\underset{\sim}{\delta}^o + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1, \ldots, \phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X} + m_i \underset{\sim}{e}_i) h_i(X_i + m_i)}{b_0 + \sum_{j=1}^{p} \{h_j^2(X_j + m_i \gamma_{ij}) + b_j h_j(X_j + m_i \gamma_{ij})\}} \tag{3.2.10}$$

Proof: In applying Theorem 2.1, it remains only to show that $\underset{\sim}{\phi}$ is a solution to (3.2.1). Corollary 2.2.3 is applicable, since it is clear that, for $m_i \geq 0$,

$$\max_{x_i \geq \alpha_i + 1} \frac{1}{v_i(x_i)} = \max_{x_i \geq \alpha_i + 1} \frac{(\gamma_i + x_i - m_i - 1) \cdots (\gamma_i + x_i - 1)}{(x_i - m_i) \cdots \cdot x_i}$$

$$= \frac{1}{r_i} \binom{r_i + m_i + 1}{r_i - 1} ,$$

and for $m_i < 0$ and $x_i \geq \alpha_i$,

$$\frac{1}{v_i(x_i)} \leq 1 .$$

It thus follows from (3.2.6) that

$$\frac{1}{v_i(x_i)} \leq M_i \qquad (3.2.11)$$

for $x_i \geq \alpha_i + 1$, and so (2.2.18) is satisfied. Similarly, an upper bound on $w_i(x_i)$ in (3.2.3) is

$$w_i(x_i) \leq \begin{cases} 1/2 & \text{if} \quad m_i \geq 0 \\ \binom{r_i - m_i - 1}{r_i - 1} \Big/ 2 & \text{otherwise,} \end{cases}$$

or equivalently,

$$w_i(x_i) \leq N_i/2 . \qquad (3.2.12)$$

Let

$$\psi_i(x) = \frac{-c(\underset{\sim}{x}) h_i(x_i)}{D} , \qquad (3.2.13)$$

where $D = b_0 + \sum_{j=1}^{p} \{h_j^2(x_j) + b_j h_j(x_j)\}$. It is clear that

$$\frac{\sum_{i=1}^{p} w_i(x_i) h_i^2(x_i)}{D} \leq \frac{1}{2} \max_{1 \leq j \leq p} \{N_j\} \frac{\sum_{i=1}^{p} h_i^2(x_i)}{D}$$

$$\leq \frac{1}{2} K ,$$

so that (2.2.6) is satisfied. Hence, by Corollary 2.2.3, $\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) = \psi_i(\underset{\sim}{x})$ satisfies (3.2.1), with strict inequality on a set of $\underset{\sim}{x}$ of positive probability. Q.E.D.

The corollaries below follow immediately from Theorem 3.2.

Corollary 3.2.1 Assume that $p \geq 3$. Under the loss function L_0, $\underset{\sim}{\delta}^0$ is inadmissible. Indeed, let $\#_1(\underset{\sim}{x})$ denote the number of indices i for which $x_i \geq 1$, and define

$$h(x_i) = \sum_{k=1}^{x_i} \frac{r_i-1+k}{k} \qquad x_i \geq 1$$

$$= 0 \qquad \text{otherwise.}$$

Furthermore, let $b_0 \geq 0$, $b_j \geq \frac{1+r_j}{2}$ and $c(\underset{\sim}{x})$ be any function nondecreasing in each coordinate, which satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and $0 \leq c(\underset{\sim}{x}) \leq 2(\#_1(\underset{\sim}{x})-2)^+$. Then, $\underset{\sim}{\delta}^0$ is dominated by $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X})h(X_i)}{b_0 + \sum_{j=1}^{p}\{h^2(X_j) + b_j h(X_j)\}} \quad . \tag{3.2.14}$$

Note that Hudson (1978) proved that $\underset{\sim}{\delta}^0$ is inadmissible under the loss funciton L_0 when $p \geq 4$. His improved esimator was given in (1.3.9).

Corollary 3.2.2 Assume that $p \geq 3$. Under the loss function L_{-1}, $\underset{\sim}{\delta}^0$ is inadmissible. Indeed, let $b_0 \geq 0$, $b_j \geq 1$, $j = 1,\ldots,p$, and $c(\underset{\sim}{x})$ be any function nondecreasing in each coordinate, which satisfies $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq \frac{2(p-2)}{\max\{r_1,\ldots,r_p\}} \quad . \tag{3.2.15}$$

Then, $\underset{\sim}{\delta}^0$ is dominated by $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X}-\underset{\sim}{e}_i)X_i}{b_0 + \sum_{j=1}^{p}\{(X_j+1-\gamma_{ij})^2 + b_j(X_j+1-\gamma_{ij})\}} \,. \qquad (3.2.16)$$

Proof: By Theorem 3.2, $\underset{\sim}{\delta}^o + \underset{\sim}{\phi}$ dominates $\underset{\sim}{\delta}^o$ if $c(\underset{\sim}{x})$ satisfies

$$0 \le c(\underset{\sim}{x}) \le 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})-2)^+/\max\{r_i,\ldots,r_p\}.$$

where $\underset{\sim}{\alpha} = (0,\ldots,0)$. Lemma 3.1 then implies that this condition is equivalent to (3.2.15). Q.E.D.

Corollary 3.2.3 Assume $p \ge 3$. Under the loss function L_m, $m = -2,-3,\ldots$, δ^o is inadmissible. Indeed, define

$$n = -m$$

$$K = \max_{1\le i\le p} \binom{r_i+n_i-1}{r_i-1}$$

and, for $i = 1,\ldots,p$,

$$h_i(x_i) = \sum_{k=0}^{x_i} \frac{(k+1)\cdots(k+n-1)}{(k+r_i)\cdots(k+n+r_i-2)} \qquad x_i \ge 0,$$

$$= 0 \qquad \text{otherwise.}$$

Let $b_0 \ge 0$, $b_j \ge 1$, $j = 1,\ldots,p$, and $c(\underset{\sim}{x})$ be any function nondecreasing in each coordinate such that $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \le c(\underset{\sim}{x}) \le 2(p-2)/K. \qquad (3.2.17)$$

Then $\underset{\sim}{\delta}^o$ is dominated by $\underset{\sim}{\delta}^o + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-(c\underset{\sim}{X}-ne_i)h_i(X_i-ne_i)}{b_0 + \sum_{j=1}^{p}\{h_j^2(X_j-n\gamma_{ij}) + b_jh_j(X_j-n\gamma_{ij})\}} \qquad (3.2.18)$$

Proof: An argument similar to the proof of Corollary 3.2.2 shows that (3.2.9) is equivalent to (3.2.17). Q.E.D.

Corollary 3.2.4 Assume $p \geq 3$. Under the loss function L_m, $m = 1,2,\ldots$, $\underset{\sim}{\delta}^o$ is inadmissible. Indeed, let $\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})$, $\underset{\sim}{\alpha}=(m+1,\ldots,m+1)$, denote the number of indices i such that $x_i \geq m+1$, and define

$$h_i(x_i) = \sum_{k=m+1}^{x_i} \frac{(r_i-1+k-m)\cdots(r_i-1+k)}{(k-m)\cdots k} \qquad x_i \geq m+1$$

$$= 0 \qquad \text{otherwise.}$$

Furthermore, let $b_0 \geq 0$, $b_j \geq \frac{(r_i+1)\cdots(m+r_1+1)}{(m+2)!}$, $j = 1,\ldots,p$, and $c(\underset{\sim}{x})$ be any function, nondecreasing in each coordinate such that $c(\underset{\sim}{x}) \not\equiv 0$ and

$$0 \leq c(\underset{\sim}{x}) \leq 2(\#_{\underset{\sim}{\alpha}}(\underset{\sim}{x})-2)^+.$$

Then $\underset{\sim}{\delta}^o$ is dominated by $\underset{\sim}{\delta}^o + \underset{\sim}{\phi}$, where $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ and

$$\phi_i(\underset{\sim}{X}) = \frac{-c(\underset{\sim}{X}+m\underset{\sim}{e}_i)h_i(X_i+m)}{b_0 + \sum_{j=1}^{p}\{h_j^2(X_j+m\gamma_{ij}) + b_jh_j(X_j+m\gamma_{ij})\}} \quad . \qquad (3.2.19)$$

CHAPTER IV

GENERAL INADMISSIBILITY RESULTS

In the last Chapter, it was shown that by choosing $q_i(x) = 1$ (i.e. write the new estimator $\underset{\sim}{\delta}^*$ as $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$), the problems of improving upon the uniformly minimum variance unbiased estimators can be reduced to the study of a difference inequality of the form (2.2.1). However, since many reasonable estimators are not unbiased, a more sophisticated choice of the q_i's is needed when one tries to improve upon such general estimators. In this chapter, it will be shown how the theorems in Chapter II can be applied to improve upon other estimators by choosing appropriate q_i's.

As pointed out at the end of Section 2.1, it is not difficult to choose q_i's so that assumption (iii) of Theorem 2.1 is satisfied. For instance, assume that $X_i \overset{\text{indep.}}{\sim} NB(1,\theta_i)$, $i = 1,\dots,p$. The uniformly minimum variance unbiased estimator of θ_i is

$$\begin{aligned} \delta(x_i) &= 1 \qquad x_i \geq 1 \\ &= 0 \qquad x_i = 0, \end{aligned}$$

(refer to Section 3.2.) which is not an interesting estimator to improve upon. A more appealing estimator is $\underset{\sim}{\delta}^G(\underset{\sim}{x})=(\delta_1^G(x_1),\dots,\delta_p^G(x_p))$, with $\underset{\sim}{\delta}_i^G(x_i) = \dfrac{x_i}{x_i + \varepsilon_i}$ $0 < \varepsilon_i < 1$. For $p = 1$, $\underset{\sim}{\delta}^G(\underset{\sim}{x})$ is an admissible generalized Bayes estimator, under L_0. However, for higher dimension, $\underset{\sim}{\delta}^0$ can be improved upon by applying Theorem 2.1. To choose q_i's so that the inequality (2.1.13) is satisfied, it is sufficient to set

$$q_i(x_i) = \prod_{k=1}^{x_i} \frac{k+\varepsilon_i}{k} \qquad x_i = 1,2,\ldots$$
$$= 1 \qquad x_i = 0$$
$$= 0 \qquad x_i < 0$$

It is easy to check that the equality in (2.1.13) holds for such a choice of q_i's. The problem is then reduced to the study of the difference in equality (2.1.14). Since q_i depends on x_i only, the difference inequality (2.1.14) is of the form (2.2.1), solutions of which are given by a generalization of Corollary 2.2.1. Better estimators are then obtained for $p > \max_{1 \leq i \leq p} \frac{2}{1-\varepsilon_i}$. This proves that under L_0, $\underset{\sim}{\delta}$ is inadmissible for such p. The detailed calculation will be reported elsewhere.

All the estimators so far considered have had $\delta_i^0(\underset{\sim}{X})$ depending on $\underset{\sim}{X}$ only through $\underset{\sim}{X}_i$. If, however, $\delta_i^0(\underset{\sim}{X})$ depends on the whole $\underset{\sim}{X}$, it is also interesting to see if improvements can be obtained. Of course, we can choose the q_i's so that equality (rather than inequality) in (2.1.13) is satisfied, as was done in the previous example. But then, the q_i's depend on the whole $\underset{\sim}{x}$, and the difference inequality that must be solved has the form of (1.4.1) with v_i depending on the whole $\underset{\sim}{x}$. To solve such a difference inequality seems to be very hard. Therefore, in Section 4.1, the q_i's will be chosen so that not only is the inequality (2.1.13) satisfied, but also the q_i's lead to a difference inequality $\sum_{i=1}^{p} \mathcal{D}_i' \leq 0$ of the type (2.1.18), to which the theorems in Section 2.2 can be applied. We thus establish a theorem which indicates the inadmissibility of many estimators obtained in Chapter III or by other statisticians.

In Section 4.2, another theorem is developed using the q_i functions, which establishes "upper bounds" on the class of admissible estimators in certain sense.

Section 4.1. Applications to the General Estimators

In this section, we will try to improve upon estimators of a general form. Before doing so, a theorem will be developed which gives solutions to the difference inequality $\mathcal{D}'(\phi) \leq 0$, defined in (2.2.1), under much weaker assumptions. Although the solution $\phi_1,\ldots,\phi_p$ takes zero value when the x_i's are small, the theorem is a useful tool in proving inadmissibility.

The following lemma will be used in establishing the theorem.

Lemma 4.1 Consider the general difference inequality

$$\mathcal{D}(\underset{\sim}{\phi}(\underset{\sim}{x})) = \sum_{i=1}^{p} v_i(\underset{\sim}{x})\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(x) \leq 0, \tag{4.1.1}$$

where $v_i(\underset{\sim}{x})$ and $w_i(\underset{\sim}{x})$ are nonnegative for all $\underset{\sim}{x}$. Suppose that

$$\underset{\sim}{\phi}^*(\underset{\sim}{x}) = (\phi_1^*(\underset{\sim}{x}),\ldots,\phi_p^*(\underset{\sim}{x}))$$

is a solution to (4.1.1) and $\phi_i^*(\underset{\sim}{x}) \leq 0$ for all $\underset{\sim}{x}$ and all i, $1 \leq i \leq p$. For any function $F(\underset{\sim}{x})$, $0 \leq F(\underset{\sim}{x}) \leq 1$, which is nondecreasing in each coordinate, $F(\underset{\sim}{x})\underset{\sim}{\phi}^*(\underset{\sim}{x})$ is also a solution to (4.1.1). (Recall from (1.2.10), that $F(\underset{\sim}{x})\underset{\sim}{\phi}^*(\underset{\sim}{x}) = (F(\underset{\sim}{x})\phi_1^*(\underset{\sim}{x}),\ldots,F(\underset{\sim}{x})\phi_p^*(\underset{\sim}{x}))$.) Furthermore,

$$\mathcal{D}(F(\underset{\sim}{x})\underset{\sim}{\phi}^*(\underset{\sim}{x})) \leq F(\underset{\sim}{x})\mathcal{D}(\underset{\sim}{\phi}^*(\underset{\sim}{x})). \tag{4.1.2}$$

Proof: It is sufficient to show that (4.1.2) is satisfied.

Now

$$\mathcal{L}\,(F(\underset{\sim}{x})\underset{\sim}{\phi}^*(\underset{\sim}{x})) = \sum_{i=1}^{p}\{v_i(\underset{\sim}{x})\Delta_i(F(\underset{\sim}{x})\phi_i^*(\underset{\sim}{x})) + w_i(\underset{\sim}{x})(F(\underset{\sim}{x})\phi_i^*(\underset{\sim}{x}))^2\}$$

$$\leq \sum_{i=1}^{p}\{v_i(\underset{\sim}{x})F(\underset{\sim}{x})\Delta_i\phi_i^*(\underset{\sim}{x}) + F(\underset{\sim}{x})w_i(\underset{\sim}{x})(\phi_i^*(\underset{\sim}{x}))^2\}$$

$$= F(\underset{\sim}{x})\,\mathcal{L}\,(\underset{\sim}{\phi}^*(\underset{\sim}{x}))$$

Q.E.D.

A direct application of Lemma 4.1 gives Lemma 4.2 which illustrates a key idea that is used in proving Theorem 4.1. In the remainder of this thesis, let $I_A(x)$ denote the indicator function, i.e.

$$I_A(\underset{\sim}{x}) = 1 \quad \text{if} \quad \underset{\sim}{x} \in A$$
$$= 0 \quad \text{if} \quad \underset{\sim}{x} \notin A.$$

Lemma 4.2 Consider two difference inequalities

$$\mathcal{L}\,(\underset{\sim}{\phi}) = \sum_{i=1}^{p}\{v_i(\underset{\sim}{x})\Delta_i\phi_i(\underset{\sim}{x}) + w_i(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})\} \leq 0, \tag{4.1.3}$$

and

$$\mathcal{L}'\,(\underset{\sim}{\phi}) = \sum_{i=1}^{p}\{v_i'(\underset{\sim}{x})\Delta_i\phi_i(\underset{\sim}{x}) + w_i'(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x})\} \leq 0. \tag{4.1.4}$$

Let $A = \{(x_1,\ldots,x_p)\colon x_i \geq \alpha\}$, where α is some number. Suppose that $v_i(\underset{\sim}{x}) = v_i'(\underset{\sim}{x})$ and $w_i(\underset{\sim}{x}) = w_i'(\underset{\sim}{x})$ for $\underset{\sim}{x} \in A$ and $i = 1,\ldots,p$. If

$$\underset{\sim}{\phi}^* = (\phi_1^*,\ldots,\phi_p^*), \text{ with } \phi_i^*(\underset{\sim}{x}) \leq 0, \quad i = 1,\ldots,p,$$

is a solution to (4.1.3) then $\underset{\sim}{\phi}^* I_A$ is also a solution to (4.1.4).

Proof: For $\underset{\sim}{x} \in A$, it is clear that $\mathcal{L}'(\underset{\sim}{\phi}^*(\underset{\sim}{x})) = 0$. For $\underset{\sim}{x} \notin A$, it follows from Lemma 4.1 that $\mathcal{L}(\underset{\sim}{\phi}^*(\underset{\sim}{x})) \leq 0$. But since $\mathcal{L}\,(\underset{\sim}{\phi}^*(\underset{\sim}{x})) = \mathcal{L}'\,(\underset{\sim}{\phi}^*(\underset{\sim}{x}))$ for $\underset{\sim}{x} \in A$, it can be concluded that $\mathcal{L}'\,(\underset{\sim}{\phi}^*(\underset{\sim}{x})) \leq 0$. Q.E.D.

The implications of Lemma 4.2 are interesting. In solving the difference inequality (4.1.3) using negative ϕ_i's, Lemma 4.2 states that the functional forms of v_i and w_i are unimportant for all small x_i. To be more precise,we will say that a statement is true for sufficiently large $\underset{\sim}{x}$ when there exists an M such that the statement is true for all $\underset{\sim}{x} = (x_1,\ldots,x_p)$ with $x_i \geq M$, $i = 1,\ldots,p$. Therefore, if v_i and w_i equal v_i' and w_i' respectively for sufficiently large $\underset{\sim}{x}$, then the class of solutions to $\mathcal{D}(\underset{\sim}{\phi})$ and $\mathcal{D}'(\underset{\sim}{\phi})$ are the same for sufficiently large $\underset{\sim}{x}$. In proving inadmissibility, we therefore need be concerned only with $\underset{\sim}{x}$ which are sufficiently large. The next theorem makes use of this idea to obtain solutions to $\mathcal{D}(\underset{\sim}{\phi}) \leq 0$ under weak assumptions.

Theorem 4.1 Let $\mathcal{D}'(\underset{\sim}{\phi})$ be as in (2.2.1). Also let the assumptions about $v_i(x_i)$, $w_i(\underset{\sim}{x})$ and α_i be the same. Define h_i as in (2.2.3). Suppose, for $i = 1,\ldots,p$, exists nonnegative constants b_0 and λ and positive constants β_i, U_i, K, and $\alpha_i', \alpha_i' \geq \alpha_i + 1$, such that

$$\text{(i)} \quad h(x_i)/h(x_i-1) \leq U_i \quad \text{for} \quad x_i \geq \alpha_i', \tag{4.1.5}$$

$$\text{(ii)} \quad \frac{\sum_{i=1}^{p} w_i(\underset{\sim}{x})h_i^2(x_i)}{b_0 + D^\lambda} \leq K \quad \text{for} \quad x_i \geq \alpha_i', \quad i = 1,\ldots,p, \tag{4.1.6}$$

where $D = \sum_{i=1}^{p} h_i^{\beta_i}(x_i)$.

Let β' denote

$$\lambda\{\max_{1\leq i\leq p} \beta_i U_i^{(\beta_i-1)^+}\}\{\max_{1\leq i\leq p} U_i^{\beta_i(\lambda-1)^+}\}$$

and define

$$A = \{(x_1,\ldots,x_p)\colon x_i \geq \alpha_i' \quad i = 1,\ldots,p\}.$$

Then for $p > \beta'$, $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ with

$$\phi_i(\underset{\sim}{x}) = \frac{-ch_i(x_i)}{b_0 + D^\lambda} I_A(\underset{\sim}{x}), \tag{4.1.7}$$

is a solution to $\mathscr{L}'(\phi) \leq 0$, providing c is any constant satisfying $0 < c < (p-\beta')/K$. Furthermore

$$\mathscr{L}'(\underset{\sim}{\phi}) \leq \frac{-c(p-\beta'-cK)}{b_0 + D^\lambda} I_A(\underset{\sim}{x}). \tag{4.1.8}$$

Proof. Note that $p-\beta'-cK > 0$. Therefore it is sufficient to prove that (4.1.8) holds. Let $\psi = (\psi_1,\ldots,\psi_p)$, where

$$\psi_i(x) = \frac{-ch_i(x_i)}{b_0 + D} . \tag{4.1.9}$$

Clearly $\phi_i = \psi_i I_A$. For $\underset{\sim}{x} \notin A$, (4.1.8) is trivial. If $\underset{\sim}{x} \in A$, we will show that

$$\mathscr{L}'(\underset{\sim}{x}) \leq \frac{-c(p-\beta'-cK)}{b_0 + D^\lambda} . \tag{4.1.10}$$

Together with Lemma 4.1, this will show that (4.1.8) holds. For $\underset{\sim}{x} \in A$, let

$$D_j = h_i^{\beta_i}(x_i-1) + \sum_{\substack{j\neq i \\ 1\leq i\leq p}} h_i^{\beta_i}(x_j)$$

and

$$D' = \sum_{j=1}^{p} h_j^{\beta_j}(x_j-1) .$$

Now it is clear that

$$\sum_{i=1}^{p} v_i(x_i)\Delta_i\psi_i(\underset{\sim}{x}) = -c\sum_{i=1}^{p} v_i(x_i)\Delta_i\left[\frac{h_i(x_i)}{b_0+D^{\lambda}}\right]$$

$$= -c\sum_{i=1}^{p} v_i(x_i)\left[\frac{\Delta_i h_i(x_i)}{b_0+D^{\lambda}} - \frac{h_i(x_i-1)\Delta_i D^{\lambda}}{(b_0+D^{\lambda})(b_0+D_i^{\lambda})}\right]$$

$$= c\left[\frac{-p}{b_0+D^{\lambda}} + \frac{1}{b_0+D^{\lambda}}\sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i D^{\lambda}}{b_0+D_i^{\lambda}}\right] .$$

It follows that

$$\sum_{i=1}^{p} v_i(x_i)\Delta_i\psi_i(\underset{\sim}{x}) \le c\left[\frac{-p}{b_0+D^{\lambda}} + \frac{1}{b_0+D^{\lambda}}\sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i D^{\lambda}}{b_0+(D')^{\lambda}}\right], \text{ for } \underset{\sim}{x}\in A. \tag{4.1.11}$$

If it can be proven that for all $\underset{\sim}{x} \in A$,

$$\sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i D^{\lambda}}{b_0+(D')^{\lambda}} \le \beta' . \tag{4.1.12}$$

then, together with (4.1.11) and (4.1.6), (4.1.10) follows.

Now, for $\lambda \ge 1$, applying mean value theorem, we have

$$v_i(x_i)h_i(x_i-1)\Delta_i D^{\lambda} \le \lambda D^{\lambda-1} v_i(x_i)h_i(x_i-1)\Delta_i h_i^{\beta_i}(x_i). \tag{4.1.13}$$

If $\beta_i \ge 1$,

$$v_i(x_i)h_i(x_i-1)\Delta_i h_i^{\beta_i}(x_i) \le \beta_i h_i^{\beta_i-1}(x_i)h_i(x_i-1)$$

$$\le \beta_i U_i^{\beta_i-1} h_i^{\beta_i}(x_i-1), \tag{4.1.14}$$

where the last inequality follows from assumption (i). Similarly for $\beta_i < 1$,

$$v_i(x_i)h_i(x_i-1)\Delta_i h_i^{\beta_i}(x_i) \le \beta_i h_i^{\beta_i}(x_i-1). \qquad (4.1.15)$$

Therefore (4.1.14) and (4.1.15) imply that

$$v_i(x_i)h_i(x_i-1)\Delta_i h_i^{\beta_i}(x_i) \le \beta_i U_i^{(\beta_i-1)^+} h_i^{\beta_i}(x_i-1). \qquad (4.1.16)$$

Thus, by (4.1.13) and (4.1.16), it is clear that for $\lambda \ge 1$,

$$v_i(x_i)h_i(x_i-1)\Delta_i D^\lambda \le \lambda\beta_i U_i^{(\beta_i-1)^+} h_i^{\beta_i}(x_i-1)D^{\lambda-1}. \qquad (4.1.17)$$

For $\lambda < 1$, (4.1.13) again holds if $D^{\lambda-1}$ is replaced by $(D')^{\lambda-1}$. Together with (4.1.16), this implies that

$$v_i(x_i)h_i(x_i-1)\Delta_i D^\lambda \le \lambda\beta_i U_i^{(\beta_i-1)^+} h_i^{\beta_i}(x_i-1)(D')^{\lambda-1} \qquad (4.1.18)$$

Hence, (4.1.17) and (4.1.18) give

$$v_i(x_i)h_i(x_i-1)\Delta_i D^\lambda \le \lambda\beta_i U_i^{(\beta_i-1)^+} h_i^{\beta_i}(x_i-1)(D')^{\lambda-1}(D/D')^{(\lambda-1)^+}$$

Summing over all i, we get

$$\sum_{i=1}^{p} v_i(x_i)h_i(x_i-1)\Delta_i D^\lambda \le \lambda \max_{1\le i\le p} \beta_i U_i^{(\beta_i-1)^+} (D')^\lambda (D/D')^{(\lambda-1)^+}. \qquad (4.1.19)$$

By assumption (i),

$$D/D' \le \max_{1\le i\le p} U_i^{\beta_i}, \qquad (4.1.20)$$

which, together with (4.1.19), implies (4.1.12). Q.E.D.

The above theorem appears complicated at first glance, especially because of β'. In most situations, however, the limit of $h_i(x_i+1)/h_i(x_i)$ is one as $x_i\to\infty$. Then β' can be chosen as close to $\lambda \max_{1\le j\le p} \beta_j$ as one wishes. Solutions to $\mathscr{B}'(\phi) \le 0$ can hence be found if $p > \lambda \max_{1\le j\le p} \beta_j$ and (4.1.6) is satisfied for some nonnegative constant b_0 and some positive constants K, β_i and λ. This is stated in the following corollary.

Corollary 4.1.1. Let $\mathscr{D}'$, v_i, w_i, α_i, h_i and D be as in Theorem 4.1. Assume that $h_i(x_i)/h_i(x_i-1)$ approaches 1 as $x_i \to \infty$, and that (4.1.6) is satisfied for some nonnegative constant b_0 and some positive constants K, β_i, and λ. Suppose $p > \lambda \max_{1\le j\le p} \beta_j$, then there is a solution to $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) \le 0$ with strict inequality for sufficiently large $\underset{\sim}{x}$. Indeed, let $U > 1$ be any number such that

$$p > \lambda\{\max_{1\le i\le p} \beta_i U^{(\beta_i-1)^+}\}\{\max_{1\le i\le p} U^{\beta_i(\lambda-1)^+}\}. \qquad (4.1.21)$$

Denote the expression on the right hand side of the inequality by β'. Furthermore let $\alpha_i' = \alpha_i + 1$ be such that $h_i(x_i)/h_i(x_i-1) < U$ for all $x_i \ge \alpha_i'$. For any constant c, $0 < c < (p-\beta')/K$ and $A = \{(x_1,\ldots,x_p): x_i \ge \alpha_i\}$, $\phi_1,\ldots,\phi_p$ given in (4.1.7) is a solution to $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) \le 0$ with strict inequality holding for sufficiently large $\underset{\sim}{x}$.

Proof. There certainly exists $U > 1$, such that (4.1.21) is satisfied, since β' approaches $\lambda \max_{1\le i\le p} \beta_i$ as $U \to 1$, and $p > \lambda \max_{1\le i\le p} \beta_i$. Theorem 4.1, then completes the proof. Q.E.D.

An application of Corollary 4.1.1 will be seen in the next theorem.

Now assume that $\underset{\sim}{x}$, $f_i(x_i|\theta_i)$ and t_i are as in Lemma 2.2. For any function $g(x)$ defined on R, let $\Delta g(x)$ denote $g(x)-g(x-1)$. Under the loss function $\underset{\sim}{L}_m$, the following theorem describes how Corollary 4.1.1 and Theorem 2.1 can be applied to improve upon a complicated estimator $\underset{\sim}{\delta}^{0}(\underset{\sim}{X})$ of the form (componentwise)

$$\delta_i^0(\underset{\sim}{X}) = \frac{t_i(X_i-1)}{t_i(X_i)} s_i(X_i+m_i)\left(1 - \frac{\ell_0 B_i H_i^{B_i-1}(X_i+m_i)\Delta_i H_i(X_i+m_i)}{\sum_{j=1}^{p} H_j^{B_j}(X_j+m_j\gamma_{ij})}\right)^+ \qquad (4.1.22)$$

where ℓ_0, B_i are some constants and s_i and H_i are some functions defined on the set of all integers.

One of the major problems encountered in the proof of Theorem 4.2, is that of choosing the q_i functions so that(2.1.13)is satisfied and the difference inequality $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) \leq 0$ (see (2.1.14)) will be of form (2.1.18). Here we will describe a heuristic argument which guides us to such a choice. The q_i's which satisfy

$$q_i(\underset{\sim}{x}-m_i+1)\underset{\sim}{e}_i) = q_i'(x_i)Q(\underset{\sim}{x}),$$

will be considered, since these q_i's will lead us to the difference inequality of the form (2.1.18). Assuming that $\phi_i(\underset{\sim}{x}) \leq 0$, $i=1,\ldots,p$, it can be seen that (2.1.13) will be satisfied for such q_i if

$$\frac{q_i'(x_i+1)}{q_i'(x_i)} \geq \frac{1}{s_i(x_i)} \qquad (4.1.23)$$

and

$$\frac{Q(\underset{\sim}{x}+\underset{\sim}{e}_i)}{Q(\underset{\sim}{x})} \geq \left(\frac{1}{1 - \frac{\ell_0 B_i H_i^{B_i-1}\Delta_i H_i(x_i)}{\sum_{j=1}^{p} H_j^{B_j}(x_j)}}\right)^+ \qquad (4.1.24)$$

It is relatively easy to construct q_i of the desired form which satisfy (4.1.23). To choose Q, let

$$D_0 = \sum_{j=1}^{p} H_j^{B_i}(x_j) \ . \qquad (4.1.25)$$

Under the conditions in Theorem 4.2, (4.1.24) is approximately equivalent to

$$\frac{\Delta_i Q(\underset{\sim}{x} + \underset{\sim}{e}_i)}{Q(\underset{\sim}{x})} \geq \frac{\ell_0 \Delta_i D_0}{D_0} \qquad (4.1.26)$$

which suggests seeking a solution to the differential equation

$$\frac{dQ}{Q} = \frac{\ell_0 dD_0}{D_0} .$$

The suggested choice of Q is thus $D_0^{\ell_0}$. In Theorem 4.2, Q is chosen to be $D_0^{\ell_1}$, where $\ell_1 > \ell_0$, and it is shown that (4.1.24) is satisfied for sufficiently large $\underset{\sim}{x}$. Modifying Q so that it takes zero value when the x_i are small, Q is thus shown to satisfy (4.1.24).

In the following, define, for any function g,

$$\prod_{\mu=0}^{x} g(u) = g(0)\ldots g(x) \qquad x = 0,1,\ldots$$

$$= 1 \qquad x = -1,-2,\ldots \quad .$$

Theorem 4.2 Let $\underset{\sim}{X}$, $f_i(x_i|\theta_i)$ and t_i be as in Lemma 2.2. Consider the loss function $L_{\underset{\sim}{m}}$, $\underset{\sim}{m} = (m_1,\ldots,m_p)$, and δ^0 defined in (4.1.22) with $B_i > 0$ for all i. Assume that for i = 1,...,p,

(i) $E_{\underset{\sim}{\theta}}(\delta_i^0(\underset{\sim}{X}))^2 < \infty$,

(ii) $H_i(x_i) \to \infty$ as $x_i \to \infty$, $H_i(x_i) \geq 0$ for all x_i and $H_i(x_i) > 0$ for $x_i > 0$.

(iii) $\dfrac{H_i(x_i+1)}{H_i(x_i)} \to 1$ as $x_i \to \infty$,

(iv) for $x_i > 0$ $H_i(x_i)$ is strictly increasing and

$\dfrac{\Delta H_i(x_i+1)}{\Delta H_i(x_i)} \to 1$ as $x_i \to \infty$,

and

(v) $s_i(x_i) \geq 0$ for all x_i and $s_i(x_i) > 0$ for $x_i \geq 0$.

Define

$$h_i(x_i) = \sum_{k=(m_i+1)}^{x_i} \frac{t_i(k)}{t_i(k-m_i-1)} \prod_{\mu=0}^{k-1} s_i(u) \qquad \text{if} \quad x_i \geq (m_i+1)^+$$

$$= 0 \qquad\qquad \text{otherwise} \qquad (4.1.27)$$

Let $D_0(\underset{\sim}{x})$ be as in (4.1.25) and $D_1(\underset{\sim}{x})$ denote $\sum_{i=1}^{p} h_i^{\beta_i}(x_i)$, for some positive numbers $\beta_1,\ldots,\beta_p$. If without ambiguity, $D_0(\underset{\sim}{x})$ and $D_1(\underset{\sim}{x})$ will be denoted D_0 and D_1. Furthermore define, for some constant $\ell_1 > \ell_0$

$$q_i^0(\underset{\sim}{x}) = D_0^{\ell_1}(\underset{\sim}{x}+(m_i+1)\underset{\sim}{e}_i) \prod_{\mu=0}^{x_i+m_i} \frac{1}{s_i(u)}$$

and

$$\phi_i^0(\underset{\sim}{x}) = \frac{h_i(x_i+m_i)}{D_1^{\lambda}(\underset{\sim}{x}+m_i\underset{\sim}{e}_i)} \quad .$$

Assume that the following conditions hold:

(vi) $E(q_i^0(\underset{\sim}{X})\phi_i^0(\underset{\sim}{X}))^2 < \infty \qquad i = 1,\ldots,p;$

(vii) $h_i(x_i)/h_i(x_i-1) \to 1 \quad$ as $x_i \to \infty$; (4.1.28)

(viii) For some constants $\lambda \geq 0$ and $K > 0$,

$$\sum_{i=1}^{p} h_i^2(x_i) \frac{t_i(x_i-m_i)}{t_i(x_i)} \Big[\prod_{\mu=0}^{x_i} \frac{1}{s_i(u)} \Big]^2 D_0^{\ell_1}/D_1^{\lambda} \leq K < \infty. \qquad (4.1.29)$$

then if $p > \lambda \max_{1\leq i\leq p} \beta_i$, $\underset{\sim}{\delta}^0$ is inadmissible. Indeed better estimators can be described as following: Let $A_\alpha=\{(x_1,\ldots,x_p):x_i \geq \alpha, i=1,\ldots,p\}$ for some number α. For some constants $\alpha^0 > 1$, $\alpha^* > 1$ and $c > 0$, define

$$q_i(\underset{\sim}{x}) = q_i^0(\underset{\sim}{x}) I_{A_{\alpha^0}}(\underset{\sim}{x}+(m_i+1)\underset{\sim}{e}_i) \qquad (4.1.30)$$

$$\phi_i(\underset{\sim}{x}) = -c\phi_i^0(\underset{\sim}{x}) I_{A_{\alpha^*}}(\underset{\sim}{x}+m_i\underset{\sim}{e}_i). \qquad (4.1.31)$$

Then for some positive constants c, small enough, and α^0 and α^*, big enough, $\underset{\sim}{\delta}^0$ is dominated by $\underset{\sim}{\delta}^*$ with $\delta_i^* = \delta_i^0 + q_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x}), i=1,\ldots,p$.

Proof: We will verify the conditions of Theorem 2.1. To verify condition (ii) of Theorem 2.1, assume for a moment that $\phi_i(\underset{\sim}{x}) \le 0$, $i = 1,\ldots,p$. (This will be seen to be true later). We first show that there exists $\alpha^0 > 1$ such that q_i in (4.1.30) satisfies (2.1.13). Note that

$$q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i) = D_0^{\ell_1} \prod_{\mu=0}^{x_i-1} \frac{1}{s_i(u)} I_{A_{\alpha 0}}(\underset{\sim}{x}). \tag{4.1.32}$$

It is sufficient to show that

$$\delta_i^0(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)t_i(x_i-m_i)q_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)-t_i(x_i-m_i-1)q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i) \ge 0 \tag{4.1.33}$$

plugging δ_i^0 and q_i into (4.1.33), it is clear that (4.1.33) will follow from the inequality

$$\left(1 - \frac{\ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{D_0}\right)^+ D_0^{\ell_1}(\underset{\sim}{x}+\underset{\sim}{e}_i) \prod_{\mu=0}^{x_i-1} \frac{1}{s_i(u)} I_{A_\alpha 0}(\underset{\sim}{x}+\underset{\sim}{e}_i)$$

$$\ge D_0^{\ell_1} \prod_{\mu=0}^{x_i-1} \frac{1}{s_i(u)} I_{A_\alpha 0}(\underset{\sim}{x}). \tag{4.1.34}$$

Since

$$I_{A_\alpha 0}(\underset{\sim}{x}+\underset{\sim}{e}_i) \ge I_{A_\alpha 0}(\underset{\sim}{x}),$$

it is sufficient to choose α^0 such that

$$D_0^{\ell_1}(\underset{\sim}{x}+\underset{\sim}{e}_i)\left(1 - \frac{\ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i)}{D_0}\right)^+ I_{A_\alpha 0}(\underset{\sim}{x}) \ge D_0^{\ell_1} I_{A_\alpha 0}(\underset{\sim}{x}). \tag{4.1.35}$$

Note that, for sufficiently large $\underset{\sim}{x}$,

$$1 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i)/D_0 > 0. \tag{4.1.36}$$

This follows from the observations that

$$0 \le \frac{H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{\sum_{j=1}^{p} H_j^{B_j}(x_j)} \le \frac{\Delta H_i(x_i)}{H_i(x_i)} = 1 - \frac{H_i(x_i-1)}{H_i(x_i)}$$

and (by condition (iii))

$$\lim_{x_i \to \infty} [1 - \frac{H_i(x_i-1)}{H_i(x_i)}] = 0 .$$

Now let $\alpha^1 > 1$ be the number such that $\underset{\sim}{x} \in A_{\alpha^1}$, implies that

$$H_i^{B_i}(x_i) - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i) > 0.$$

Clearly, (4.1.36) holds for $\underset{\sim}{x} \in A_{\alpha^1}$. To choose α^0 so that (4.1.34) is satisfied, it is only necessary to choose $\alpha^0 \ge \alpha^1$ so that, for $\underset{\sim}{x} \in A_{\alpha^0}$ and $i = 1,\ldots,p$,

$$\frac{D_0(\underset{\sim}{x}+\underset{\sim}{e}_i)}{D_0}^{\ell_1} \ge \frac{1}{1 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)/D_0}$$

$$= 1 + \frac{\ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{D_0 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)} , \qquad (4.1.37)$$

or equivalently

$$\frac{\Delta_i D_0^{\ell_1}(x+e_i)}{D_0^{\ell_1}} \ge \frac{\ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i)}{D_0 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i)} . \qquad (4.1.38)$$

By mean value theorem, there exists a D_i^*, $D_0 < D_i^* < D_0(x+e_i)$, so that

$$\Delta_i D_i^{\ell_1}(x+\ell_i) = \ell_1 (D_i^*)^{\ell_1-1} \Delta H_i^{B_i}(x_i+1). \qquad (4.1.39)$$

Hence (4.1.38) is equivalent to

$$\ell_1 (D_i^*)^{\ell_1 - 1} \Delta H_i^{B_i}(x_i+1)/D_0^{\ell_1} \geq \frac{\ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i)}{D_0 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta_i H_i(x_i).} \qquad (4.1.40)$$

Now we must separately consider three cases (a) $\ell_1 \geq 0 \geq \ell_0$, (b) $\ell_1 > \ell_0 > 0$ and (c) $0 > \ell_1 > \ell_0$. Case (a) is trivial, since by choosing α^0 equal to α^1, the left hand side of (4.1.38) is nonnegative and the right hand side is nonpositive.

For case (b), (4.1.40) is equivalent to

$$R_i \geq 1,$$

where

$$R_i = \frac{\ell_1 (D_i^*)^{\ell_1-1} \Delta H_i^{B_i}(x_i+1)}{\ell_0 \quad D_0^{\ell_1}} \; \frac{D_0 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)} \qquad (4.1.41)$$

Clearly,

$$R_i = \frac{\ell_1}{\ell_0}\left(\frac{D_i^*}{D_0}\right)^{\ell_1-1} \frac{D_0 - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{D_0} \; \frac{\Delta H_i^{B_i}(x_i+1)}{B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}$$

$$\geq \frac{\ell_1}{\ell_0}\left(\frac{D_0}{D_i^*}\right)^{(1-\ell_1)^+} \frac{H_i^{B_i}(x_i) - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{H_i^{B_i}(x_i)} \; \frac{\Delta H_i^{B_i}(x_i+1)}{B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)} \cdot$$

$$\geq \frac{\ell_1}{\ell_0}\left(\frac{D_0}{D_0(x+e_i)}\right)^{(1-\ell_1)^+} \frac{H_i^{B_i}(x_i) - \ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{H_i^{B_i}(x_i)}$$

$$\times \left(\frac{H_i(x_i)}{H_i(x_i+1)}\right)^{(1-B_i)^+} \frac{\Delta H_i(x_i+1)}{\Delta H_i(x_i)}$$

$$\geq \frac{\ell_1}{\ell_0}\left(\frac{H_i(x_i)}{H_i(x_i+1)}\right)^{(1-\ell_1)^+} \frac{H_i^{B_i}(x_i)-\ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{H_i^{B_i}(x_i)}$$

$$\times\left(\frac{H_i(x_i)}{H_i(x_i+1)}\right)^{(1-B_i)^+} \frac{\Delta H_i(x_i+1)}{\Delta H_i(x_i)}$$

By conditions (iii) and (iv), the last expression approaches $\frac{\ell_1}{\ell_0} > 1$ as $x_i \to \infty$. Therefore there exists $\alpha^0 > \alpha^1$, such that $R_i \geq 1$, $i = 1,\ldots,p$ if $\underset{\sim}{x} \in A_{\alpha^0}$.

For case (c), (4.1.40) is equivalent to

$$R_i \leq 1.$$

Now from (4.1.41),

$$R_i \leq \frac{\ell_1}{\ell_0} \frac{D_0-\ell_0 B_i H_i^{B_i-1}(x_i)\Delta H_i(x_i)}{D_0} \frac{H_i(x_i+1)}{H_i(x_i)} (B_i-1)^+ \frac{\Delta_i H_i(x_i+1)}{\Delta_i H_i(x_i)} .$$

Again by conditions (iii) and (iv), the last expression approaches $\frac{\ell_1}{\ell_0} < 1$ as $x_i \to \infty$. Therefore there exist $\alpha^0 > \alpha^1$ such that $\underset{\sim}{x} \in A_{\alpha^0}(\underset{\sim}{x})$ implies that $R_i \leq 1$ for all i. In conclusion (4.1.37), and hence (4.1.33) are satisfied if q_i is defined as in (4.1.30). For convenience in the following, it is also assumed that $\alpha^0 > (m_i+1)^+$, $i = 1,\ldots,p$.

To satisfy conditon (iv) of Theorem 2.1, it is necessary to solve the inequality

$$\mathcal{L}' = \sum_{i=1}^{p} \mathcal{L}_i' \leq 0 \quad \text{or} \quad \frac{1}{2}\sum_{i=1}^{p} \mathcal{L}_i' \leq 0.$$

where

$$\mathcal{S}_i' = \frac{2t_i(x_i-m_i-1)}{t_i(x_i)} \quad q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ \frac{t_i(x_i-m_i)^2}{t_i(x_i)} q_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i).$$

Clearly,

$$\frac{1}{2}\mathcal{S}' = D_0^{\ell_1} I_{A_{\alpha 0}} \mathcal{S}'' \qquad (4.1.43)$$

where

$$\mathcal{S}'' = \sum_{i=1}^{p} \frac{t_i(x_i-m_i-1)}{t_i(x_i)} \prod_{\mu=0}^{x_i-1} \frac{1}{s(u)} \Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ \sum_{i=1}^{p} \frac{1}{2} \frac{D_0^{2\ell_1}(\underset{\sim}{x}+\underset{\sim}{e}_i)}{D_0^{\ell_1}} \frac{t_i(x_i-m_i)}{t_i(x_i)} \prod_{\mu=0}^{x_i} \frac{1}{x^2(u)} \phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i).$$

$$(4.1.44)$$

Of course, a solution to $I_{A_{\alpha 0}} \mathcal{S}'' \leq 0$ is also a solution to $\mathcal{S}' \leq 0$.

Note that

$$1 \leq \left[\frac{D_0(x+e_i)}{D_0(\underset{\sim}{x})}\right] \leq \left[\frac{H_i^{B_i}(x_i+1)}{H_i^{B_i}(x_i)}\right] ,$$

for all $\underset{\sim}{x}$ such that $H_i(x_i) \neq 0$. By condition (iii), there exists K_1 so that

$$\frac{H_i^{B_i}(x_i+1)}{H_i^{B_i}(x_i)} \leq K_1$$

for all $x_i > 0$.

Thus there exists K_2 so that if $x_i > 0$, $i = 1,\ldots,p$, then

$$D_0^{2\ell_1}(\underset{\sim}{x}+\underset{\sim}{e}_i)/D_0^{\ell_1} \leq K_2 D_0^{\ell_1} . \qquad (4.1.45)$$

Therefore, by (4.1.44) and (4.1.45),

$$I_{A_{\alpha 0}} \mathscr{D}'' \le I_{A_{\alpha 0}} \mathscr{D}'''$$

where

$$\mathscr{D}''' = \sum_{i=1}^{p} \frac{t_i(x_i - m_i - 1)}{t_i(x_i)} \prod_{u=0}^{x_i - 1} \frac{1}{s(u)} \Delta_i \phi_i(\underset{\sim}{x} - m_i \underset{\sim}{e}_i)$$

$$+ \sum_{i=1}^{p} \frac{K_2}{2} D_0^{\ell_1} \frac{t_i(x_i - m_i)}{t_i(x_i)} \prod_{\mu=0}^{x_i} \frac{1}{s^2(u)} \phi_i^2(\underset{\sim}{x} - m_i \underset{\sim}{e}_i).$$

A solution to $\mathscr{D}''' \le 0$ is certainly a solution to $\mathscr{D}' \le 0$. Let h_i be as in (4.1.27). By Corollary 4.1.1, (4.1.28) and (4.1.29), there exists a solution to $\mathscr{D}''' \le 0$; namely $\phi_1, \ldots, \phi_p$, where

$$\phi_i(\underset{\sim}{x} - m_i \underset{\sim}{e}_i) = \frac{-ch_i(x_i)}{D_1^{\lambda}} I_{A_{\alpha *}}(\underset{\sim}{x})$$

for some constants $c > 0$ and $\alpha^* > \alpha^0$. Furthermore, for such ϕ_i's,

$$\mathscr{D}''' \underset{\sim}{\phi}(\underset{\sim}{x} - m_i \underset{\sim}{e}_i) < 0$$

for sufficiently large $\underset{\sim}{x}$. By Theorem 2.1, the proof is now complete.

Q.E.D.

Corollary 4.2.1. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i), i = 1, \ldots, p$. Under the loss function L_{-1}, the estimator $\underset{\sim}{\delta}^C$ of Clevenson's type given componentwise by

$$\delta_i^C(\underset{\sim}{X}) = \left(1 - \frac{\ell_0}{p - 1 + \sum_{j=1}^{p} x_j}\right)^+ X_i \tag{4.1.46}$$

is inadmissible if $\ell_0 < p-1$.

Proof: To apply Theorem 4.2, let δ_i^o be as in (4.1.22) with $m_i = -1$, $s_i(x_i) = 1$, $B_i = 1$ and $H_i(x_i) = 1 + x_i$ if $x_i \ge 0$ and $H_i(x_i) = 0$ if $x_i < 1$. Clearly $\underset{\sim}{\delta}^o = \underset{\sim}{\delta}^C$ and conditons (i) through (v) of Theorem 4.2

are satisfied. Let $h_i(x_i) = \sum_{k=0}^{x_i} 1 = x_i+1$, as in (4.1.27). Obviously condition (vii) of Theorem 4.2 is satisifed. Let ℓ_1 be any number such that if fit $\ell_0 < \ell_1 < p-1$. Futhermore let

$$D_0 = \sum_{j=1}^{p} H_j(x_j) = \sum_{j=1}^{p} h_j(x_j) = D_1 .$$

To check condition (viii), note that

$$\begin{aligned}\sum_{i=1}^{p} h_i^2(x_i) \frac{t_i(x_i+1)}{t_i(x_i)} D_0^{\ell_1}/D_1^{\lambda} &= \sum_{i=1}^{p} h_i(x_i) P_0^{\ell_1}/D_0^{\lambda} \\ &= D_0^{\ell_1+1}/D_0^{\lambda} \\ &\leq K,\end{aligned}$$

for some constant K if $\lambda = (\ell_1+1)^+$. Hence condition (viii) is satisfied with $\lambda = (\ell_1+1)^+$. Since $\ell_1 < p-1$, $(\ell_1+1)^+ < p$ and consequently, $p > \lambda$. Finally, by (4.1.30) and (4.1.31), $q_i\phi_i$ is bounded. Hence $E\, q_i^2(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x}) < \infty$ and condition (vi) is satisfied. By Theorem 4.2, $\underset{\sim}{\delta}^C$ is thus inadmissible.

Corollary 4.2.2. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_0, the estimator $\underset{\sim}{\delta}^P$ of Peng's type similar to (1.3.7) given componentwise by

$$\delta_i^P(\underset{\sim}{x}) = \left(X_i - \frac{\ell h(X_i)}{\sum_{j=1}^{p} h_j^2(X_j)} \right)^+ , \qquad (4.1.47)$$

is inadmissible if $\ell < p-2$. Recall that $h(\cdot)$ was defined by

$$\begin{aligned} h(x_i) &= \sum_{K=1}^{x_i} \frac{1}{K} \qquad x_i = 1,2,\ldots \\ &= 0 \qquad x_i < 1 .\end{aligned}$$

Proof: To apply Theorem 4.2, here let δ_i^0 be as in 4.1.22 with $m_i = 0$, $s_i(x_i) = 1$, $B_i = 2$ and $H_i(x_i) = h(x_i)$. Clearly $\underset{\sim}{\delta}^0 = \underset{\sim}{\delta}^P$ if $\ell_0 = \ell/2$. It is straightforward to show that condition (i) through (v) of Theorem 4.2 are satisfied. Let $h_i(x_i)$ be defined as in (4.1.27), then $h_i(x_i) = h(x_i)$. Again condition (vii) is satisifed. Furthermore let $\beta_i = 2$ and

$$D_0 = \sum_{j=1}^{p} h^2(x_j) = D_1 .$$

To check condition (viii), let ℓ_1 be any number so that $\ell_0 < \ell_1 < \frac{p-2}{2}$. Now

$$\sum_{i=1}^{p} h^2(x_i) \frac{t_i(x_i - m_i)}{t_i(x_i)} D_0^{\ell_1}/D_1^{\lambda} = D_0^{\ell_1+1}/D_0^{\lambda} \leqq K$$

for some constant K if $\lambda = (\ell_1+1)^+$. Note $p > 2\lambda$. Finally by (4.1.30) and (4.1.31), $q_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x})$ is bounded, hence $Eq_i^2(\underset{\sim}{x})\phi_i^2(\underset{\sim}{x}) < \infty$ and condition (vi) is satisfied. By Theorem 4.2, the proof is complete. Q.E.D.

The proofs of the following two corollaries are similar to those of the previous corollaries, and so are omitted.

Corollary 4.2.3. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$ $i = 1,\ldots,p$. Under the loss function L_m, $m = -n$ and n being a positive integer, the estimator $\underset{\sim}{\delta}^T$ of Tsui's type given componentwise by

$$\underset{\sim}{\delta}_i^T(\underset{\sim}{X}) = \left(X_i - \frac{\ell h(X_i - n)}{\sum_{j=1}^{p} h(X_j - n\gamma_{ij})} \right)^+ , \qquad (4.1.48)$$

is inadmissible if $\ell < n(p-1)$. Recall that $h(\cdot)$ was defined as

$$h(x_i) = (x_i+1)\ldots(x_i+n) \quad \text{if} \quad x_i \geq 0$$
$$= 0 \quad \text{otherwise} .$$

Corollary 4.2.4. Assume that $X_i \overset{\text{indep.}}{\sim} NB(r_i,\theta_i)$, $i = 1,\ldots,p$. Let $\underset{\sim}{\delta}^{NB}$ be the estimator of $\underset{\sim}{\theta}$ given (componentwise) by

$$\delta_i^{NB}(\underset{\sim}{X}) = \left(\frac{X_i}{r_i - 1 + X_i} - \frac{\ell h(X_i)}{\sum_{j=1}^{p} h^2(X_j)} \right)^+ \tag{4.1.49}$$

where

$$\begin{aligned} h(x_i) &= \sum_{k=1}^{x_i} \frac{r_i - 1 + k}{k} \qquad && x_i \geq 1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Under the loss function L_0, $\underset{\sim}{\delta}^{NB}$ is inadmissible if $\ell < p-2$.

Note that all the inadmissible estimators stated in the above corollaries are dominated by a corresponding $\underset{\sim}{\delta}^*$ given in Theorem 4.2. The correction terms $q_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x})$ (See (4.1.30), (4.1.31)) are nonpositive and are obtained by applying Theorem 2.1. These facts will be used in the next section.

Section 4.2. "Upper Bounds" on the Class of Admissible Estimators

As seen in the last section, to improve upon an estimator of a general form, quite complicated calculations are generally involved. In this section, a theorem is developed by which a broad class of estimators can be shown to be inadmissible. This theorem also shows what is meant by an "upper bound" on the class of admissible estimators. Again, we will use $I_A(\underset{\sim}{x})$ to denote the indicator function, i.e.

$$\begin{aligned} I_A(\underset{\sim}{x}) &= 1 \qquad && \text{if } \underset{\sim}{x} \in A \\ &= && \text{otherwise.} \end{aligned}$$

Theorem 4.3 Let $\underset{\sim}{X}$ be as in Lemma 2.2. Consider two estimators $\underset{\sim}{\delta}^0$ and $\underset{\sim}{\delta}^1$ of $\underset{\sim}{\theta}$, both of which have nonnegative components. Assume

$E[\delta_i^1(\underset{\sim}{X})]^2 < \infty$ for $i = 1,\ldots,p$. Let $A = \{(x_1,\ldots,x_p)\colon x_i \geq \alpha$, $i = 1,\ldots,p\}$ for some number α. Under the loss function $L_{\underset{\sim}{m}}$, suppose the following two conditions hold:

(i) $\delta_i^1(\underset{\sim}{x}) \geq \delta_i^0(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in A$ and $i = 1,\ldots,p$;

(ii) $\underset{\sim}{\delta}^0(\underset{\sim}{X})$ can be improved by the procedure of Theorem 2.1 with the correction terms $q_i(\underset{\sim}{X})\phi_i(\underset{\sim}{X})$ being nonpositive.

Then, the estimator $\underset{\sim}{\delta}^*$ with ith component

$$\delta_i^*(\underset{\sim}{X}) = \delta_i^1(\underset{\sim}{X}) + I_S(\underset{\sim}{X}+m_i\underset{\sim}{e}_i)q_i(\underset{\sim}{X})\phi_i(\underset{\sim}{X}),$$

is as good as $\underset{\sim}{\delta}^1$, where $S = \{\underset{\sim}{x}\colon x_i \geq \alpha+m_i^+,\ i = 1,\ldots,p\}$.

Proof: We will use Corollary 2.1.1. Thus, assuming conditions (i), (ii) and (iii) of Theorem 2.1 and condition (iv)' of Corollary 2.1.1, we need to show that these four conditions are also satisifed with δ_i^0 and ϕ_i being replaced by $\delta_i^1(\underset{\sim}{x})$ and $\phi_i(\underset{\sim}{x})I_S(\underset{\sim}{x}+m_i\underset{\sim}{e}_i)$, respectively. ($q_i$ remains unchanged).

Clearly, conditions (i) and (ii) of Theorem 2.1 are satisfied. Note that condition (iii) in this situation has the form

$$\{\delta_i^1(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)t_i(x_i-m_i)q_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)-t_i(x_i-m_i-1)q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\}$$
$$\cdot\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)I_S(\underset{\sim}{x}) \leq 0 \qquad (4.2.1)$$

To verify (4.2.1), observe that

$$\underset{\sim}{x} \in S \Rightarrow (\underset{\sim}{x}-m_i\underset{\sim}{e}_i) \in A.$$

Consequently, condition (i) of this theorem implies that

$$\delta_i^1(\underset{\sim}{x}-m_i\underset{\sim}{e}_i) \geq \delta_i^0(\underset{\sim}{x}-m_i\underset{\sim}{e}_i), \quad i = 1,\ldots,p. \qquad (4.2.2)$$

By (2.1.13), (4.2.2), and condition (ii) of this theorem, (4.2.1) is established. Furthermore, from condition (iv)' of Corollary 2.1.1, we have

$$\mathcal{D}'(\underset{\sim}{\phi}) = \sum_{i=1}^{p} \{\frac{2t_i(x_i-m_i-1)}{t_i(x_i)} q_i(\underset{\sim}{x}-(m_i+1)\underset{\sim}{e}_i)\Delta_i\phi_i(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)$$

$$+ \frac{t_i(x_i-m_i)}{t_i(x_i)} q_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\phi_i^2(\underset{\sim}{x}-m_i\underset{\sim}{e}_i)\} \le 0 \tag{4.2.3}$$

Lemma 4.1 implies that $\phi_i(\underset{\sim}{x})I_S(\underset{\sim}{x}+m_i\underset{\sim}{e}_i)$, $i = 1,\ldots,p$, also satisfies (4.2.3). Therefore, condition (iv) of Corollary 2.1.1 is satisified.

Q.E.D.

Corollary 4.3.1 Let the notation and assumptions be as in Theorem 4.3. If $I_S(\underset{\sim}{x}+m_i\underset{\sim}{e}_i)q_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x})$ is not identically zero, then $\underset{\sim}{\delta}^1$ is inadmissible.

Proof: Clearly $\underset{\sim}{\delta}^*$ is as good as $\underset{\sim}{\delta}^1$, and $\underset{\sim}{\delta}^* \not\equiv \underset{\sim}{\delta}^1$. Because $L_{\underset{\sim}{m}}$ is a strictly convex loss,

$$R(\underset{\sim}{\theta},\frac{1}{2}(\underset{\sim}{\delta}^*+\underset{\sim}{\delta}^1)) < \frac{1}{2}R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) + \frac{1}{2}R(\underset{\sim}{\theta},\underset{\sim}{\delta}^1) \le R(\underset{\sim}{\theta},\underset{\sim}{\delta}^1).$$ Q.E.D.

If the correction term $q_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x})$ is nonzero for all sufficiently large $\underset{\sim}{x}$, Corollary 4.3.1 implies that any estimator $\underset{\sim}{\delta}^1$, with $\delta_i^1(\underset{\sim}{x}) \ge \delta_i^0(\underset{\sim}{x})$ for sufficiently large $\underset{\sim}{x}$, is inadmissible. In this sense, $\underset{\sim}{\delta}^0$ is an upper bound on the class of admissible estimators. The following corollaries follow immediately from Theorem 4.2 and Corollary 4.3.1.

Corollary 4.3.2. Suppose that the assumptions in Theorem 4.2 hold. Under the loss function $L_{\underset{\sim}{m}}$, $\underset{\sim}{\delta}^0$ is an upper bound on the class of admissible estimators, if $p > \lambda \max_{1\le j\le p} \beta_j$.

Corollary 4.3.3. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_{-1}, $\underset{\sim}{\delta}^C$ (given in (4.1.46)), with $\ell_0 < p-1$ is an upper bound on the class of admissible estimators.

Corollary 4.3.4. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_0, $\underset{\sim}{\delta}^P$(given in (4.1.47)), with $\ell < p-2$, is an upper bound on the class of admissible estimators.

Corollary 4.3.5. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_{-n}, $n = 1,2,\ldots$, $\underset{\sim}{\delta}^T$ (given in (4.1.48)), with $\ell < n(p-1)$, is an upper bound on the class of admissible estimators.

Corollary 4.3.6. Assume $X_i \overset{\text{indep.}}{\sim} NB(r_i,\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_0, $\underset{\sim}{\delta}^{NB}$(given in (4.1.49)), with $\ell < p-2$, is an upper bound on the class of admissible estimators.

There are many possible applications of these corollaries. Only two examples will be given here.

Example 4.1. Assume $X_i \overset{\text{indep.}}{\sim} Po(\theta_i)$, $i = 1,\ldots,p$. Under the loss function L_{-1}, the estimator $\underset{\sim}{\delta}^*$, with

$$\delta_i^*(\underset{\sim}{X}) = \left(\frac{\sum_{j=1}^p X_j}{\beta+p-1+\sum_{j=1}^p X_j} \right) X_i ,$$

was conjectured to be inadmissible for $\beta < 0$ in Brown (1974). To prove this, let ℓ_0 be any number such that $\beta+p-1 < \ell_0 < p-1$. For such ℓ_0, compare $\delta_i^C(\underset{\sim}{X})$(given in (4.1.46)) to $\delta_i^*(\underset{\sim}{X})$. Clearly, for sufficiently large $\underset{\sim}{x}$

$$\delta_i^*(\underset{\sim}{x}) \geq \delta_i^C(\underset{\sim}{x}) \tag{4.2.4}$$

$$\Leftrightarrow \left(1 - \frac{\beta+p-1}{\beta+p-1+\sum_{j=1}^p x_j}\right) x_i \geq \left(1 - \frac{\ell_0}{p-1+\sum_{j=1}^p x_j}\right) x_i$$

$$\Leftrightarrow \frac{\beta+p-1}{\beta+p-1+\sum_{j=1}^p x_j} \leq \frac{\ell_0}{p-1+\sum_{j=1}^p x_j} .$$

Since the last inequality holds for sufficiently large $\underset{\sim}{x}$, Corollary 4.3.3 and (4.2.4) imply that $\underset{\sim}{\delta}^*$ is inadmissible. This proves Brown's conjecture.

Example 4.2. Assume $X_i \overset{\text{indep.}}{\sim} NB(r,\theta_i)$, $i = 1,\ldots,p$. Recall from (1.3.9) that Hudson's estimator $\underset{\sim}{\delta}^H$ is

$$\delta_i^H(\underset{\sim}{X}) = \frac{X_i}{r-1+X_i} - \frac{(\#(\underset{\sim}{X})-3)^+ h(X_i)}{\sum_{j=1}^{p} h^2(X_j)}, \quad i = 1,\ldots,p,$$

where $\#(\underset{\sim}{x})$ is the number of indices j for which $x_j \geq 1$, and $h(x_i) = \sum_{k=1}^{x_i} (r-1+k)/k$ if $x_i > 0$, while $h(x_i) = 0$ otherwise. We claim that for $p \geq 3$, $\underset{\sim}{\delta}^H$ is inadmissible. To see this, let $\underset{\sim}{\delta}^{NB}$ be as in Corollary 4.3.6 with $\ell = p-2.5$. For sufficiently large $\underset{\sim}{x}$,

$$\delta_i^{NB}(\underset{\sim}{x}) \leq \delta_i^H(\underset{\sim}{x}) \tag{4.2.5}$$

$$\Leftrightarrow \frac{x_i}{r-1+x_i} - \frac{\ell h(x_i)}{\sum_{j=1}^{p} h^2(x_j)} \leq \frac{x_i}{r-1+x_i} - \frac{(p-3)h(x_i)}{\sum_{j=1}^{p} h^2(x_j)}$$

$$\Leftrightarrow \frac{\ell h(x_i)}{\sum_{j=1}^{p} h^2(x_j)} \geq \frac{(p-3)h(x_i)}{\sum_{j=1}^{p} h^2(x_j)} .$$

Clearly the last inequality holds. Hence by (4.2.5) and Corollary 4.3.6, $\underset{\sim}{\delta}^H$ is inadmissible if $p \geq 3$.

CHAPTER V

OTHER RELATED PROBLEMS

In this chapter, some miscellaneous problems which relate to improving upon estimators will be considered. In Section 5.1, the problems of improving upon standard estimators for the parameters of Poisson and Chi-square distributions are compared. The comparison reveals the role played by discreteness.

In Section 5.2, an admissibility problem is discussed. An example is given in Section 5.3, which deals with the simultaneous estimation problem based on three observations having distributions of completely different forms. In Section 5.4, three generalizations are discussed.

Section 5.1 Comparison of the Poisson Case and the Chi-square Case.

Assume that $X_i \overset{\text{indep.}}{\sim} P_0(\theta_i)$, i=1,...,p. Estimators better than $\underset{\sim}{\delta}^0(\underset{\sim}{X})=\underset{\sim}{X}$ under the loss function, $L_{\underset{\sim}{m}}$, $m_i \leq 0$, i=1,...,p, were presented in Section 3.1. But no results have been given for the loss function, $L_{\underset{\sim}{m}}$, with positive m_i's.

To discuss this problem in detail, let us consider the loss function L_m, where m is an integer. Again, in trying to improve upon $\underset{\sim}{\delta}^o$, write a competitor as $\underset{\sim}{\delta}^* = \underset{\sim}{\delta}^o + \underset{\sim}{\phi}$ with $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$. From Theorem 2.1 and (3.1.2), we have $R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*)-R(\underset{\sim}{\theta},\underset{\sim}{\delta}^o) = E_{\underset{\sim}{\theta}} \mathcal{D}'_m(\underset{\sim}{\phi})$ where

$$\mathcal{D}'_m(\underset{\sim}{\phi}) = \begin{cases} \sum_{i=1}^{p} \{2(x_i-m)\ldots x_i\ \Delta_i\psi_i(\underset{\sim}{x})+(x_i-m+1)\ldots x_i\psi_i^2(\underset{\sim}{x})\} & \text{if } m>0 \quad (5.1.1) \\ \sum_{i=1}^{p} \{2x_i\Delta_i\psi_i(\underset{\sim}{x})+\psi_i^2(\underset{\sim}{x}) & \text{if } m=0 \quad (5.1.2) \\ \sum_{i=1}^{p} \{2\ \dfrac{\Delta_i\psi_i(\underset{\sim}{x})}{(x_i+1)\ldots(x_i-m-1)} + \dfrac{\psi_i^2(\underset{\sim}{x})}{(x_i+1)\ldots(x_i-m)}\} & \text{if } m<0 \quad (5.1.3) \end{cases}$$

and $\psi_i(\underset{\sim}{x}) = \phi_i(\underset{\sim}{x}-m\underset{\sim}{e}_i)$. For the case m > 0, the theorems in Section 2.2 do not yield any nontrivial solutions to $\mathcal{D}'_m(\underset{\sim}{\phi}) \leq 0$.

The problems of the existence of a nontrivial solution to (5.1.1) and of the admissibility of $\underset{\sim}{\delta}^o$ are not yet answered. To gain some insight, however, we compare the difference inequality

$$\mathcal{D}'_m(\underset{\sim}{\phi}) \leq 0 \qquad (5.1.4)$$

($\mathcal{D}'_m$ was given in (5.1.1) through (5.1.3)) to the differential inequality (1.3.10). For convenience, (1.3.10) is restated here:

$$\mathcal{D}_m(\underset{\sim}{\phi}) = \sum_{i=1}^{p} x_i^{m+1} \frac{\partial}{\partial x_i} \phi_i(\underset{\sim}{x})+b_i x_i^m\phi_i^2(\underset{\sim}{x}) \leq 0. \qquad (1.3.10)$$

The differential inequality was encountered in trying to improve upon the estimator $\underset{\sim}{x}/n+2$ under the loss function L_{m-2}, where $x_i/\theta_i \overset{\text{indep.}}{\sim} \chi^2_n$, i = 1,...,p. (cf. Section 1.3.3). Inspecting $\mathcal{D}_m$ and $\mathcal{D}'_m$, we see that, the difference inequality (5.1.4) for the Poisson case is analogous to the differential inequality (1.3.10) for the chi-square case. (The constants $b_1,\ldots,b_p$ are not significant in determining the form of a solution to (1.3.10).) For $m \leq 0$, as seen in (3.1.15), (3.1.18), (3.1.20) and (1.3.14), the solutions to $\mathcal{D}'_m \leq 0$ and $\mathcal{D}_m \leq 0$ are very similar. The improved estimators given by these solutions all correct the standard estimator by shrinking toward a point. For $m > 0$, the solutions to $\mathcal{D}_m \leq 0$ are all nonnegative and therefore indicate that the new estimator corrects the standard one by pulling away from $(0,\ldots,0)$. Due to the similarity between $\mathcal{D}_m$ and $\mathcal{D}'_m$, the solution to $\mathcal{D}_m \leq 0$ seems to suggest that the solutions to $\mathcal{D}'_m \leq 0$ (if any) are also nonnegative, and hence that better estimators pull $\underset{\sim}{\delta}^0$ away from $(0,\ldots,0)$. However, a theorem can be established in discrete cases which asserts that if $\underset{\sim}{\delta}^0 + \underset{\sim}{\phi}$ is as good as $\underset{\sim}{\delta}^0$, and $\underset{\sim}{\phi} = (\phi_1,\ldots,\phi_p)$ is such that $\phi_i(\underset{\sim}{x}) \geq 0$ all $\underset{\sim}{x}$, then all the ϕ_i's must be zero. In other words, $\underset{\sim}{\delta}^0$ can not be improved by pulling away from the origin. Therefore, these facts seem to suggest the admissibility of $\underset{\sim}{\delta}^0$ under the loss function L_m, $m > 0$.

Note that this also explains why the first aspect of Berger's phenomena (i.e. the correction terms changes sign according to the loss function) does not occur in the discrete case. But, clearly, the second aspect is certainly observed (i.e. the dimension needed for inadmissibility of $\underset{\sim}{\delta}^0$ depends on the loss function as shown in Peng (1975), Clevenson and Zidek (1975) and Tsui and Press (1977).). In fact a general theorem asserts that any estimator of $\underset{\sim}{\theta}$ can not be improved by positive correction terms if $\underset{\sim}{x}$ is distributed as in Lemma 2.2. This will be reported elsewhere.

Section 5.2. An Admissibility Problem

The main idea used in Chapters II, III, IV to prove inadmissibility has been solving an appropriate difference inequality.

Once a nontrivial solution, which satisfies the regularity conditions, is obtained, the estimator is known to be inadmissible.

It is therefore natural to ask whether the lack of a solution (except the zero solution) to the difference inequality corresponding to a particular $\underset{\sim}{\delta}^o$, implies that $\underset{\sim}{\delta}^o$ is admissible. The following example indicates that the conjecture is false for p = 1.

Example 5.1. Let X be a one-dimensional random variable having logarithmic distribution, i.e.

$$P(X=x) = \frac{1}{-\log(1-\theta)} \frac{\theta^x}{x}, \quad x = 1,2,\ldots,$$

for some unknown parameter θ, $0 < \theta < 1$. It is clear that the unbiased estimator $\delta^o(X)$,

$$\begin{aligned} \delta^o(X) &= \frac{X}{X-1} && \text{if } X \geq 2 \\ &= 0 && \text{if } X = 1, \end{aligned}$$

is inadmissible, since it estimates θ by some number greater than 1. Thus $\delta^o(X)$ can certainly be improved if X/X-1 is replaced by 1 when $X \geq 2$. However we will consider the problem of improving upon δ^o . by using Theorem 2.1. Under square error loss, the difference inequality (See (2.1.12) and (2.1.14) with $q_i = 1$) has the form

$$\mathcal{D}'(\underset{\sim}{\phi}) = 2v(x)\Delta\phi(x)+\phi^2(x) \leq 0 \qquad (5.2.1)$$

where

$$\begin{aligned} v(x) &= \frac{x}{x-1} && x \geq 2 \\ &= 0 && x = 1. \end{aligned}$$

The following lemma will show that the only solution to (5.2.1) is $\phi(x) \equiv 0$. Therefore the lack of a nontrivial solution to the difference inequality does not necessarily imply admissibility.

Lemma 5.1. If $\phi(x)$ satisfies (5.2.1), then $\phi(x) = 0$, $x = 1,2,\ldots$.

Proof: First, we will show that $\phi(\cdot)$ is bounded. Clearly $\phi(\cdot)$ is nonincreasing function, since $\Delta\phi(x)$ must be nonpositive. Also for $x = 1$, (5.2.1) becomes $\phi^2(1) \leq 0$, implying $\phi(1) = 0$. Thus $\phi(x) \leq 0$, for $x = 1,2,\ldots$. Since $v(x) \leq 2$ for all nonnegative integer x, it follows that

$$\begin{aligned} 0 &\geq \mathscr{L}'(\phi) \\ &\geq 4\Delta\phi(x) + \phi^2(x) \\ &\geq 4\phi(x) + \phi^2(x) \end{aligned} \tag{5.2.2}$$

which, since $\phi(x) \leq 0$, implies that

$$-4 \leq \phi(x) \leq 0.$$

Next, let ℓ be the limit of $\phi(x)$ as $x \to \infty$. (This limit exists, since ϕ is bounded and nonincreasing.) We then complete the proof by showing that $\ell = 0$. Now clearly

$$-4 \leq \ell \leq 0.$$

Suppose that $\ell < 0$, then there exists some $N > 0$ such that $\phi(x) < 0$ for all $x > N$. By (5.2.2),

$$\frac{4(\phi(x)-\phi(x-1))}{\phi(x)} + \phi(x) \geq 0 \tag{5.2.3}$$

Letting x go to infinity, (5.2.3) implies that $\ell \geq 0$, which is a contradiction.

Q.E.D.

Section 5.3. An Example of an Improved Simultaneous Estimator Based on Discrete and Continuous Observations

The theorems in Chapter II were designed for the case when the observations $X_1,\ldots,X_p$ are independently from discrete exponential families. Of course, the most common situations occur when the distributions of $X_1,\ldots,X_p$ have the same form, as considered in Chatper III. However, it is also interesting to observe the following example which deals with an estimation problem based on three independent random variables, X_1, X_2 and X_3 having distributions of completely different forms.

Example 5.1. Assume that X_1, X_2 and X_3 are independent random variables; $X_1 \sim Po(\theta_1)$, $X_2 \sim N(\theta_2,1)$ and $X_3/\theta_3 \sim \chi^2_n$. It is desired to estimate $(\theta_1,\theta_2,\theta_3)$ under the loss function $L_{\underset{\sim}{m}}$, $\underset{\sim}{m} = (0,0,-1)$. i.e.

$$L_{\underset{\sim}{m}}(\underset{\sim}{\theta},\underset{\sim}{a}) = L_1(\theta_1,a_1) + L_2(\theta_2,a_2) + L_3(\theta_3,a_3)$$

where $L_1(\theta_1,a_1) = (\theta_1-a_1)^2$, $L_2(\theta_2,a_2) = (\theta_2-a_2)^2$ and $L_3(\theta_3,a_3) = \theta_3^{-1}(\theta_3-a_3)^2$. A standard estimator is

$$\delta^o(\underset{\sim}{X}) = (\delta_1^o(X_1),\delta_2^o(X_2),\delta_3^o(X_3))$$

where

$$\delta_i^o(X_i) = X_i, \quad i = 1,2,$$

and

$$\delta_3^o(X_3) = X_3/n+2.$$

It is known (see Hodges and Lehmann (1951)) that for each coordinate treated separately, $\delta_i^o(x_i)$ is an admissible estimator for θ_i under the loss function L_i. However, $\underset{\sim}{\delta}^o$ is inadmissible under $L_{\underset{\sim}{m}}$ and can be improved by an argument similar to Stein's technique. (Stein (1973).)

Write a competitor as $\underset{\sim}{\delta}^* = (\delta_1^*, \delta_2^*, \delta_3^*)$ with

$$\delta_i^*(\underset{\sim}{x}) = x_i + \phi_i(\underset{\sim}{x}) \quad i = 1,2, \tag{5.3.1}$$

and

$$\delta_3^*(\underset{\sim}{x}) = \frac{x_3}{n+3}\,(1+\phi_3(\underset{\sim}{x})). \tag{5.3.2}$$

Under certain regularity conditions on ϕ_3, the identity

$$\theta_3^{-1}\{E_{\underset{\sim}{\theta}}(\delta_3^*(\underset{\sim}{X}) - \theta_3)^2 - E_{\underset{\sim}{\theta}}(\frac{X_3}{n+2} - \theta_3)^2\}$$

$$= E_{\underset{\sim}{\theta}}\{\frac{4X_3^2}{(n+2)^2}\,\frac{\partial\phi_3(\underset{\sim}{X})}{\partial X_3} + \frac{X_3}{n+2}\,\phi_3^2(\underset{\sim}{X}) + \frac{4X_3^2}{(n+2)^2}\,\phi_3(\underset{\sim}{X})\frac{\partial\phi_3(\underset{\sim}{X})}{\partial X_3}\} \tag{5.3.3}$$

is derived in Berger (1978). If $\phi_3(\underset{\sim}{x}) \geq 0$ and $\frac{\partial\phi_3(\underset{\sim}{x})}{\partial x_3} < 0$, the expression on the right hand side of (5.3.3) is bounded above by

$$E_{\underset{\sim}{\theta}}\,\{\frac{4X_3^2}{(n+2)^2}\,\frac{\partial\phi_3(\underset{\sim}{X})}{\partial X_3} + \frac{X_3}{n+2}\,\phi_3^2(\underset{\sim}{X})\} \tag{5.3.4}$$

Then together with (1.3.2) and (1.3.5), this implies

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) - R(\underset{\sim}{\theta},\underset{\sim}{\delta}^0) \leq E_{\underset{\sim}{\theta}}\{2X_1\Delta_1\phi_1(\underset{\sim}{X}) + \phi_1^2(\underset{\sim}{X}) + 2\frac{\partial\phi_2(\underset{\sim}{X})}{\partial_2} + \phi_2^2(\underset{\sim}{X})$$

$$+ 4\frac{X_3^2}{(n+2)^2}\,\frac{\partial\phi_3(\underset{\sim}{X})}{\partial X_3} + \frac{X_3}{n+2}\,\phi_3^2(\underset{\sim}{X})\}$$

under certain regularity conditions on ϕ_i. Let

$$\mathcal{L}'(\underset{\sim}{\phi}(x)) = x_1\Delta_1\phi_1(\underset{\sim}{x}) + \frac{\partial\phi_2(\underset{\sim}{x})}{\partial x_2} + \frac{2x_3^2}{(n+2)^2}\,\frac{\partial\phi_3(\underset{\sim}{x})}{\partial x_3}$$

$$+ \phi_1^2(\underset{\sim}{x}) + \phi_2^2(\underset{\sim}{x}) + \frac{x_3}{n+2}\,\phi_3^2(\underset{\sim}{x})$$

A solution to $\mathcal{L}'(\underset{\sim}{\phi}) \leq 0$ can be found by an argument similar to those used in Chapter II. Indeed it can be described as following:

Let

$$h_1(x_1) = \frac{1}{k} \qquad \text{if } x_1 = 1, 2, \ldots$$
$$= 0 \qquad \text{if } x_1 \le 0$$

$$h_2(x_2) = x_2 \qquad x_2 \in R$$

and

$$h_3 = \frac{-(n+2)^2}{2} x_3^{-1} \qquad x_3 \in R$$

and, for some $b > 0$, define

$$D = b + h_1(x_1)h_1(x_1+1) + h_2^2(x_2) + h_3(x_3) \quad .$$

Furthermore, let

$$c(k) = 0 \qquad k = 0$$
$$= 2/n+2 \qquad k = 1, 2, \ldots$$

Then

$$\phi_i(x) = \frac{-c(x_1)h_i(x_i)}{D} \quad , \ i = 1,2,3, \qquad (5.3.6)$$

is a solution to $\mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) \le 0$ with $E_{\underset{\sim}{\theta}} \mathscr{D}'(\underset{\sim}{\phi}(\underset{\sim}{x})) < 0$, for all $\underset{\sim}{\theta}$ This solution also satisfies the required regularity conditions, and hence under the loss function $L_{\underset{\sim}{m}}$, $\underset{\sim}{\delta}^0$ is dominated by $\underset{\sim}{\delta}^*$ defined (componentwise) as in (5.3.2) where ϕ_1, ϕ_2 and ϕ_3 are given in (5.3.6).

The implications of the above example are intersting . First, although the distributions of X_1, X_2 and X_3 are very different and each δ_i is an admissible estimator of θ_i based on X_i, δ^0 is inadmissible under $L_{\underset{\sim}{m}}$. Therefore Stein's phenomena seems to be very general. Second, the improved estimator corrects $\underset{\sim}{\delta}^o$ very differently in each coordinate. For i = 1,2,$\underset{\sim}{\delta}^*$ corrects $\underset{\sim}{\delta}^o$ by shrinking toward zero while, for i = 3, by pulling away from zero. Finally, in the correction terms, given by (5.3.6), h_1, h_2, and h_3 (so are d_1,d_2, and d_3) are determined independently of each other. (i.e. h_i depends only on the distribution of X_i and L_i.) However, the ϕ_i's are obtained by combining these h_i's and d_i's in a definite way.

Section 5.4. Other Generalizations

There are many other possible generalizations of the results of this research. Only three of them are discussed here.

(a) All the results can be easily extended to the loss functions of the form $L(\underset{\sim}{\theta},\underset{\sim}{a})=\sum_{i=1}^{p} z_i\theta_i^{m_i}(\theta_i - a_i)^2$, where $z_1,\ldots,z_p$ are some positive constants. There are two ways to deal with such a loss function.

(i) Include these constants $z_1,\ldots,z_p$ in the difference inequality and solve it. Clearly, the difference inequality can be solved by using theorems in Section 2.2, if and only if the difference inequality corresponding to the loss function $L_{\underset{\sim}{m}}$(i.e. z_i = 1, i = 1,...,p) can be solved.

(ii) Apply the results in Berger (1977a), in which the problem is decomposed into p subproblems under the loss function $\sum_{i=1}^{j} \theta_i^{m_i}(\theta_i - a_i)^2$, j = 1,...,p. Improved estimators can be found for the original problems, once improved estimators are found under at least one of the subproblems.

(b) The idea of an "upper bound" on the admissible class (cf. Section 4.2) has certainly an analog in the continuous case. However, the concept is more complicated than in the discrete case, since the correction terms, frequently encountered, are not necessarily of the same sign.

(c) All the distributions of $X_1,\ldots,X_p$ considered in this work were assumed to be as in (1.2.2) with $t_i(x_i) > 0$ if and only if $X_i = 0,1,\ldots$. For the case $t_i(x_i) > 0$ if and only if $x_i = a_i, a_i + 1,\ldots,$ for some integer a_i, a simple transformation

$$X_i' = X_i - a_i$$

will make our results applicable to the estimation problem based on $X_1',\ldots,X_p'$.

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