ESTIMATING A QUANTILE OF AN EXPONENTIAL DISTRIBUTION

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SUMMARY

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The estimation of a quantile $\xi + b\sigma$ of an exponential distribution on the basis of a random sample of size $n \geq 2$ is considered. Here ξ and σ are unknown location and scale parameters and b is a given constant. For quadratic loss, it is established that the best equivariant estimator $\delta_o = x_{min} + (b-n^{-1})(\bar{x}-x_{min})$ is inadmissible if $0 \leq b < n^{-1}$ or $b > 1 + n^{-1}$. For $b > 1 + n^{-1}$ the estimator

$$\delta = \delta_o - 2(n+1)^{-1}[(b-1-n^{-1})(\bar{x}-x_{min})-(bn-1)x_{min}] ,$$

$0 < x_{min} < (b-1-n^{-1})(bn-1)^{-1}(\bar{x}-x_{min})$, $\delta = \delta_o$ elsewhere, provides a noticeable improvement over δ_o.

Key words: Quantile of the exponential distribution, location-scale parameter, best equivariant estimator, inadmissibility, minimaxness, quadratic loss.

1. INTRODUCTION

Let $\underset{\sim}{x} = (x_1,\ldots,x_n)(n \geq 2)$ be a random sample from a two-parameter exponential distribution. Thus all x's have a density $\sigma^{-1}p((u-\xi)\sigma^{-1})$, where $p(u) = e^{-u}$, $u \geq 0$; $p(u) = 0$, $u < 0$, and ξ and σ are unknown location and scale parameters. In this paper we consider the problem of estimating the parametric function $\theta = \xi + b\sigma$, where b is a given nonnegative constant. Clearly if $p = e^{-b}$, $0 < p < 1$, then θ is a p-quantile of the exponential distribution. We assume that the loss function is squared error scaled by σ^2, so as to make the loss invariant under location and scale transformations.

Quantile estimation, in particular for the exponential distribution, is important for reliability theory, life testing, etc. (cf. David (1970) p.121). Many papers have been dedicated to this subject (see, for instance, Epstein and Sobel (1954), Epstein (1962)). Also, for statistical decision theory it is of interest to find out if the best equivariant estimator or the maximum likelihood estimator of θ is admissible. In the case of a normal sample the problem of estimating θ was considered by Zidek (1971) who established inadmissibility of the best equivariant estimator for $b \neq 0$.

The usual (best equivariant) estimator of σ when ξ is a nuisance parameter is known to be inadmissible (see Arnold (1970), Zidek (1973), Brewster (1974)). This fact suggests inadmissiblity of the best equivariant estimator of $\xi + b\sigma$ and can be used to prove it if b is sufficiently large.

Indeed let $\delta_o(\underset{\sim}{x}) = \tilde{\xi}(\underset{\sim}{x}) + b\tilde{\sigma}(\underset{\sim}{x})$ be the best equivariant estimator of $\xi + b\sigma$, so that $\tilde{\xi}$ can be interpreted as an estimator of ξ and $\tilde{\sigma}$ as an estimator of σ. It is easy to see that $\tilde{\sigma}$ is the best equivariant estimator of σ for the quadratic loss. Also

$$E_{\xi\sigma}(\delta(\underset{\sim}{x})-\xi-b\sigma)^2 = E_{\xi\sigma}(\tilde{\xi}(x)-\xi)^2$$
$$+ 2bE_{\xi\sigma}(\tilde{\xi}(\underset{\sim}{x})-\xi)(\tilde{\sigma}(\underset{\sim}{x})-\sigma) + b^2E_{\xi\sigma}(\tilde{\sigma}(\underset{\sim}{x})-\sigma)^2.$$

Therefore for large values of b the main impact in the risk of δ is the risk corresponding to $\tilde{\sigma}$. Because of the mentioned inadmissibility result one can improve upon δ_o for large b by using an estimator $\tilde{\xi}(x) + b\hat{\sigma}(x)$ where $\hat{\sigma}$ is an improvement over $\tilde{\sigma}$.

In this paper we show that the region of "large values of b" for which δ_o is inadmissible is the interval $b > 1 + n^{-1}$. More precisely we prove inadmissibility of the traditional estimator δ_o for $0 \leq b < n^{-1}$ and $b > 1 + n^{-1}$, and in the latter case, for small sample sizes n, offer noticeable improvements upon δ_o. Thus, in particular, the natural estimator of the location parameter ξ is inadmissible. This estimator is also inadmissible when b is negative, but this case needs special methods and is not of much statistical interest.

Our technique is a slight modification of the method originally proposed by Stein (1964), which was later formalized by Brewster and Zidek (1973) and used by Sharma (1977) in the estimation problem of σ^{-1}.

Note that the quantile estimation problem is also closely related to the estimation of the location parameter ξ with unknown scale parameter σ. Indeed if $p_1(u) = p(u+b)$ is the density of the shifted exponential distribution, then the best equivariant estimator of the location parameter ξ in the family $\sigma^{-1}p_1((u-\xi)\sigma^{-1})$ coincides with the best equivariant estimator of a quantile $\xi + b\sigma$ in the family $\sigma^{-1}p((u-\xi)\sigma^{-1})$.

2. A CLASS OF MINIMAX ESTIMATORS

Let $x = x_{\min} = \min_{1\le j\le n} x_j$, $y = \bar{x} - x = n^{-1}\sum_1^n x_j - x$. Then (x,y) is a version of the minimal sufficient statistic and has density

$$(n\sigma^{-1})^n[(n-2)!]^{-1}\exp\{-n(x+y-\xi)\sigma^{-1}\}y^{n-2} \tag{2.1}$$

if $y > 0$, $x > \xi$, and 0 otherwise.

The problem of estimating $\theta = \xi + b\sigma$ is invariant under the affine group, and if $\delta(x,y)$ is an equivariant estimator then

$$\delta(cx+d, cy) = c\delta(x,y) + d$$

for all $c > 0$ and d. This implies that δ has the form $\delta(x,y) = x + \lambda y$ for some constant λ. If the loss is measured by $(\delta-\theta)^2\sigma^{-2}$, an examination of the risk of δ reveals that the best choice of λ is

$$\lambda = E_{01}yE_{01}(x-b)[E_{01}y^2]^{-1} = b - n^{-1} .$$

This (best equivariant) estimator $\delta_o(x,y) = x + \lambda y$ has a constant risk and is known to be minimax. ("E_{01}" = $E_{\xi=0,\sigma=1}$).

For $\lambda > 1$ or $\lambda < 0$ we seek an improvement upon $\delta_o(x,y)$ in the class of estimators

$$\delta(x,y) = x + \lambda y - \lambda y f((x+y)/x) , \tag{2.2}$$

where f is a (measurable) function of $(x+y)/x$.

THEOREM 1. The estimator δ_o is inadmissible if $0 \le b < n^{-1}$ or $b > 1 + n^{-1}$. For $b > 1 + n^{-1}$ the estimator

$$\delta(x,y) = x + (b-n^{-1})y - 2[(b-1-n^{-1})y - (bn-1)x](n+1)^{-1}$$

for $0 < x < (bn-1)^{-1}(b-1-n^{-1})y$, and $\delta(x,y) = \delta_o(x,y)$ otherwise, improves upon δ_o.

Proof. Since the risk of any estimator of the form (2.2) depends only on ξ/σ we can take $\sigma = 1$. One has

$$\begin{aligned}\Delta(\xi) &= E_{\xi 1}(\delta_o(x,y)-\xi-b)^2-E_{\xi 1}(\delta(x,y)-\xi-b)^2\\ &= E_{\xi 1}[\delta_o(x,y)-\delta(x,y)][\delta_o(x,y)+\delta(x,y)-2\xi-2b]\\ &= 2\lambda E_{\xi 1}\{f(1+y/x)(xy+\lambda y^2-\lambda y^2 f(1+y/x)/2-(\xi+b)y)\,|x>0\}P_{\xi 1}(x>0)\\ &\quad + 2\lambda E_{\xi 1}\{f(1+y/x)(xy+\lambda y^2-\lambda y^2 f(1+y/x)/2-(\xi+b)y)\,|x<0\}P_{\xi 1}(x<0)\\ &= \Delta_o(\xi)+\Delta_1(\xi). \end{aligned} \tag{2.4}$$

For $\lambda > 1$ we construct an estimator δ of the form (2.2) with $f(1+y/x) = 0$ for $x < 0$ and $f(1+y/x) \geq 0$ for $x > 0$, such that $\Delta_o(\xi) \geq 0$ for $\xi \geq 0$. Clearly for all ξ $\Delta_1(\xi) = 0$ and if $\xi < 0$, then

$$\Delta(\xi) = \Delta_o(\xi) = \Delta_o(0)-2\lambda\xi E_{\xi 1}\{f(1+y/x)y/x>0\}P_{\xi 1}\{x>0\} .$$

Thus inequalities $\Delta_o(0) \geq 0$ and $f \geq 0$ imply that $\Delta_o(\xi) \geq 0$ for $\xi < 0$, and the estimator δ with these properties will improve over δ_o. We construct the function f not being identically zero, so that δ is different from δ_o with positive probability and is minimax.

Combining (2.1) and 2.4) one obtains with $\eta = n\xi$, $n > 0$, $z = 1+y/x$

$$\begin{aligned}\Delta_o(\eta) &= 2\lambda e^{\eta}[(n-2)!]^{-1}n^{-2}[\int_1^\infty f(z)[1+\lambda(z-1)-\lambda(z-1)f(z)/2](z-1)^{n-1}z^{-n-2}dz\\ &\quad \times \int_{\eta z}^\infty u^{n+1}e^{-u}du-(\eta+\lambda n-1)\int_1^\infty f(z)(z-1)^{n-1}z^{-n-1}dz\int_{\eta z}^\infty u^n e^{-u}du]. \end{aligned} \tag{2.5}$$

By using formulas

$$\int_1^\infty h(z)\int_{\eta z}^\infty u^n e^{-u}dudz = \eta^{n+1}\int_1^\infty z^n e^{-z}\int_1^z h(t)dtdz$$

and

$$\eta\int_1^\infty k(z)e^{-\eta z}dz = \int_1^\infty k'(z)e^{-\eta z}dz$$

(the latter assumes k is differentiable and $k(1) = 0$) we deduce that

$$\Delta_o(\eta) = 2\lambda e^{\eta}\eta^{n+1}n^{-2}[(n-2)!]^{-1}\int_1^{\infty} g(z)e^{-\eta z}dz ,$$

where

$$g(z) = \lambda(1-f(z)/2)f(z)(z-1)^n z^{-1}$$
$$+ z^n \int_1^z f(t)(t-1)^{n-1}t^{-n-2}[(\lambda-1)(t-n-1)-\lambda(n+1)(t-1)f(t)/2-nt/z]dt. \quad (2.6)$$

Thus if $\lambda > 1$ and for $z > 1$

$$f(z) = 2\max\{0,(\lambda-1)\lambda^{-1} - n(z-1)^{-1}\}(n+1)^{-1},$$

then $g(z)$ is positive and the corresponding estimator improves upon δ_o.

In the case when $0 \leq b < n^{-1}$, i.e. when $-n^{-1} \leq \lambda < 0$ we construct an improvement over δ_o such that $f(1+y/x) = 0$ for $x \geq 0$ and $f(1+y/x) \geq 0$ for $x < 0$. Then $\Delta_o(\xi) = 0$ for all ξ, so that to prove inadmissibility of δ_o it suffices to consider negative values of ξ and find f such that $\Delta_1(\xi) \geq 0$. We take $f(1+y/x) = 0$ for $x+y > 0$. Then with $\eta = n|\xi|$, $z = 1-y|x|^{-1}$, $0 < z < 1$ one obtains

$$\Delta_1(\eta) = 2|\lambda|e^{-\eta}n^{-2}[(n-2)!]^{-1}$$
$$\times \{\sum_{k=0}^{\infty} \eta^{n+k+2}[\int_0^1 z^k(1-z)^n f(z)[(1-z)^{-1}+|\lambda|-|\lambda|f(z)/2]dz[k!(n+k+2)]^{-1}$$
$$- (\eta-nb)\sum_{k=0}^{\infty} \eta^{n+k+1}\int_0^1 z^k(1-z)^{n-1}f(z)dz[k!(n+k+1)]^{-1}\}$$
$$= 2|\lambda|e^{-\eta}\eta^{n+1}n^{-2}[(n-2)!]^{-1}$$
$$\times \{(1-|\lambda|n)(n+1)^{-1}\int_0^1 (1-z)^{n-1}f(z)dz$$
$$+ \sum_{k=0}^{\infty} \eta^{k+1}(k!)^{-1}\int_0^1 z^k(1-z)^{n-1}f(z)$$
$$\times [(1-z)[(1-z)^{-1}+|\lambda||\lambda|f(z)/2](n+k+2)^{-1}-(n+k+1)^{-1}$$
$$+ (1-|\lambda|n)z(k+1)^{-1}(n+k+2)^{-1}]dz\} . \quad (2.8)$$

If we put for $0 < z < 1$

$$f(z) = \max\{0, 2-2|\lambda|^{-1}(1-z)^{-1} \sup_{k\geq 0}[(n+k+1)^{-1}-(1-|\lambda|n)z(k+1)^{-1}]\},$$

then every term of the series in (2.8) will be nonnegative. Straight forward calculation shows that if $(n+i+1)^{-1} < |\lambda| < (n+i)^{-1}$, then for z taking values in the interval (d_i, e_i), where

$$d_i = (1-|\lambda|n)i(i+1)[(n+i+1)(n+i)]^{-1}, \quad e_i = (i+1)(n+i+1)^{-1}$$

the following inequality holds

$$\sup_{k\geq 0}[(n+k+1)^{-1}-(1-|\lambda|n)z(k+1)^{-1}] = (n+i+1)^{-1}-(1-|\lambda|n)z(i+1)^{-1}<|\lambda|(1-z).$$

Therefore if the nonnegative integer i is defined by inequalities $(n+i+1)^{-1} < |\lambda| < (n+i)^{-1}$ and

$$f(z) = 2-2|\lambda|^{-1}(1-z)^{-1}[(n+i+1)^{-1} - (1-|\lambda|n)z(i+1)^{-1}]$$

for $z \in (d_i, e_i)$, $f(z) = 0$ elsewhere, then the corresponding estimator δ improves upon δ_o. If $|\lambda| = (n+i)^{-1}$ then the function f can be defined so that the only negative term in (2.8), say, a_i corresponds to η^i. However, $a_i^2 < 4a_{i-1}a_{i+1}$ so that the whole sum is positive.

Remark 1. For $b > 1+n^{-1}$ the improvement over δ_o can be achieved by replacing y by its improvement suggested, for instance, by Zidek (1973). This estimator has the form

$$\delta_1(x,y) = x + y - b\max\{0, (y-nx)(n+1)^{-1}\}, \quad x > 0$$

$\delta_1(x,y) = \delta_o(x,y)$, $x < 0$. This estimator also improves upon δ_o, but only for $b > 2(1+n^{-1})$. Moreover the numerical evaluation of its risk shows that δ_1 exhibits poorer performance than the estimator obtained in Theorem 1.

Remark 2. For $b = n^{-1}$ the estimator δ_o is admissible. Indeed if $b = n^{-1}$ then the Pitman's estimator of the location parameter ξ for a fixed value of σ is x_{min}, which is independent of σ and coincides with

$\delta_o(x,y)$. Thus inadmissibility of δ_o in this case would imply inadmissibility of the Pitman's estimator x_{min} for fixed σ. The latter is known to be false (Stein (1959)).

The same method of admissibility proof (for one-dimensional case) can be used to show that for $b = 1 + n^{-1}$ the estimator δ_o is admissible within the class (2.2). In fact if $\lambda = 1$ δ_o is a generalized Bayes estimator within this class with the generalized prior density proportional to η^{-1} for $\eta > 0$. It is not known, however, if it is absolutely admissible.

Analysis of the proof of the Theorem 1 shows that the following result is true.

THEOREM 2. For $\lambda > 0$ given any estimator δ of the form (2.2), the estimator

$$\tilde{\delta}(x,y) = \begin{cases} \delta(x,y) - 2\lambda y \max\{0, 1 - f(z) - bnz(n+1)^{-1}\lambda^{-1}(z-1)^{-1}\} & x > 0 \\ \delta(x,y) & x < 0 , \end{cases}$$

where $z = (x+y)/x$, has a risk function which is at least as small as that of $\delta(x,y)$ and which is strictly smaller if the two estimators are not equal almost everywhere.

For $-n^{-1} \le \lambda < 0$ the result remains true if

$$\tilde{\delta}(x,y) = \delta(x,y) - 2\lambda y \max\{0, 1 - f(z) - |\lambda|^{-1}(1+z)^{-1}[(n+i+1)^{-1} - (1-|\lambda| n) z (i+1)^{-1}],$$

if $x + y < 0$, $z = 1 - y|x|^{-1}$; $\tilde{\delta}(x,y) = \delta(x,y)$ otherwise. Here i is the non-negative integer defined by inequalities

$$(n+i+1)^{-1} < |\lambda| \le (n+i)^{-1}.$$

3. DISCUSSION AND NUMERICAL RESULTS

To compare the performance of δ_o and the estimator obtained in the Theorem 1 we consider the relative improvement

$$\rho(\xi) = \frac{R(\delta_o,\xi) - R(\delta,\xi)}{R(\delta_o,\xi)} ,$$

where $R(\delta,\xi) = E_\xi(\delta - \xi - b)^2$ is the risk function. Note that

$$R(\delta_o,\xi) = n^{-2} + (b - n^{-1})^2(n+1)^{-1} ,$$

so that for the estimator (2.3) one obtains (in percents)

$$\rho(\eta) = \frac{400}{(n-2)!(n+1)(n+1+(n\lambda)^2)} [\eta(\frac{\lambda n\eta^2}{n-1} - \frac{(\lambda+n-1)\eta}{n-1} + (\lambda n+1)(\lambda-1)]\Gamma(n,t)$$

$$-\eta^2(\eta+\lambda n+1)\lambda n\Gamma(n-1,t) - (\frac{\lambda n+\lambda+n-1}{n^2-1} - \frac{(\lambda+n-1)\eta}{n(n-1)})\Gamma(n+2,t)$$

$$-(\frac{\lambda n+\lambda+n-1}{n-1} - \frac{(\lambda n^2+\lambda+n-1)\eta}{n(n-1)})\eta\Gamma(n+1,t)$$

$$+(\frac{\lambda n}{\lambda n+\lambda-1})^n e^\eta(\frac{\lambda n+\lambda-1}{n-1} - \frac{\lambda n}{n+1} - \frac{(\lambda n+\lambda-1)\eta}{n(n-1)})\Gamma(n+2,t+\eta)], \qquad (3.1)$$

where $t = n\lambda\eta/(\lambda-1)$, $\eta = \xi n$ and $\Gamma(n,t) = \int_t^\infty e^{-x}x^{n-1}dx = e^{-t}(n-1)!\sum_{m=0}^{n-1} t^m/m!$ is the incomplete gamma function. The formula (3.1) was used for numerical evaluation of the maximal improvement over δ_o. Instead of λ or b the probability $p = e^{-b}$ was employed for p = 0.2,0.1,0.05,0.01,0.001, 0.00001, so that p-percentile estimation is considered.

Note that as $\lambda\to\infty$ or $p\to 0$,

$$\rho(\eta) \to \frac{400\eta}{(n-2)!n(n+1)} \int_{\eta n}^\infty u^{n-2}(u-\eta n)e^{-u}du.$$

The maximal improvement for large λ occurs when $n = 3$, $\eta = \sqrt{2}/e = .47$

and equals $100(2\sqrt{2} + 2)e^{-\sqrt{2}}/3 = 13.04\%$. When $n = 8$ the maximal improvement upon the traditional estimator is about 6% if p is smaller than 0.01. Since in application expenses due to tests and lengthy testing procedures often dictate small sample sizes for quantile estimation, these results seem to be relevant to practical use in the fatigue life determination. "In fact, the Federal Aviation Administration (FAA) allows for the qualification of components based on samples of size six or less for fatigue tests" (Dyer, Keating and Hensley (1977) p. 270).

A peculiar feature of the Table is that the maximal improvement is the largest when $n = 3$. Typical values of the parameter $\eta = \xi n$ where the maximal improvement is achieved, belong to the interval (0.05, 0.5). Thus the maximal improvement of the estimator $\delta(x - c, y) + c$ (which is minimax if δ is) occurs when η is in the interval $(c + 0.05,\ c + 0.5)$. Thus if prior information about ξ/σ is available, one should choose a constant c such that the prior mean of $n\xi/\sigma$ is in the interval specified above and use the corresponding estimator.

Maximal improvement (in percents) over the estimator of the p-procentile

n	p 0.2	0.1	0.05	0.01	0.001	0.00001
2	0.109	2.591	4.825	7.644	9.332	10.587
3	0.621	3.407	5.532	8.205	9.878	11.183
4	0.879	3.442	5.327	7.702	9.223	10.443
5	0.959	3.265	4.238	7.048	8.423	9.536
6	0.961	3.040	4.539	6.429	7.673	8.690
7	0.931	2.816	4.176	5.883	7.014	7.947
8	0.889	2.661	3.853	5.410	6.446	7.304
9	0.848	2.424	3.571	4.999	5.955	6.749
10	0.817	2.261	3.323	4.643	5.529	6.268
20	0.552	1.345	1.935	2.681	3.189	3.620

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