ON A GENERALIZED STORAGE MODEL WITH MOMENT ASSUMPTIONS

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ABSTRACT. This paper considers a semi-infinite storage model, of the type studied by Senturia and Puri [13] and Balagopal [2], defined on a Markov renewal process, $\{(X_n, T_n), n = 0, 1,\dots\}$, with $0 \equiv T_0 < T_1 <\dots$, a.s., where X_n takes values in the set $\{1, 2,\dots\}$. If at T_n, $X_n = j$, then there is a random 'input' $V_n(j)$ (a negative input implying a demand) of 'type' j, having distribution function $F_j(\cdot)$. We assume that $\{V_n(j)\}$ is an i.i.d. sequence of random variables, taken to be independent of $\{(X_n, T_n)\}$ and of $\{V_n(k)\}$, for $k \neq j$, and that $V_n(j)$ has first and second moments. Here the random variables $V_n(j)$ represent instantaneous 'inputs' (a negative value implying a demand) of type j for our storage model. Under these assumptions, we establish certain limit distributions for the joint process $(Z(t), L(t))$, where $Z(t)$ (defined in (2)) is the level of storage at time t and $L(t)$ (defined in (3)) is the demand lost due to shortage of supply during $[0,t]$. Different limit distributions are obtained for the cases when the 'average stationary input' ρ, as defined in (5), is positive, zero or negative.

KEY WORDS: MARKOV RENEWAL PROCESS; STORAGE MODELS; LIMIT DISTRIBUTIONS; TOTAL DEMAND LOST; AVERAGE STATIONARY INPUT; STORAGE LEVEL; SUPERCRITICAL, CRITICAL AND SUBCRITICAL CASES.

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1. INTRODUCTION. In literature, one finds a large number of models which have been proposed to approximate the mechanism associated with a semi-infinite storage facility. Typically, these models allow for a single type of random input and a deterministic output. Examples of such storage models can be found, along with further references, in papers by Ali Khan and Gani [1], Kendall [7], and Lloyd and Odoom [8] among others, as well as in books by Moran [9] and Prabhu [10]. In this paper, we consider a generalized storage model which was inspired by a model proposed by Senturia and Puri ([13], [14]). Their model, which allows both random inputs and random demands, arose in the context of certain biological situations. The interested reader may find a discussion of these specific situations as well as references to other areas of applicability in [12]. The model we propose to study here is described below.

Let $\{(X_n, T_n), n = 0, 1, \ldots\}$ be a Markov renewal process (M.R.P) with state space $J \times [0,\infty)$, $J \subset \{1, 2, \ldots\}$, defined on a complete probability space $(\Omega, \mathcal{G}, P)$, such that $T_0 = 0$, a.s., $P(X_0 = i) = a(i)$, for $i \in J$ with $\sum_{i \in J} a(i) = 1$ and

(1) $P(X_{n+1} = j, T_{n+1} - T_n \leq t | T_0, X_0, \ldots, T_n, X_n = i) = A(i,j,t)$,

for $i, j \in J$ and $t \in [0,\infty)$. We assume that $A(i,j,0) = 0$ and that $\sum_{j \in J} p(i,j) = 1$

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for each $i \in J$, where $0 \leq p(i,j) \equiv A(i,j,\infty)$. In addition, it is assumed that the embedded Markov chain (M.C) $\{X_n\}$ is aperiodic, irreducible, and positive recurrent with the associated stationary distribution given by $\underset{\sim}{\pi} = (\pi(1), \pi(2), \ldots)$.

For each $i \in J$, let $\{V_n(i), n = 0,1,\ldots\}$ be a sequence of i.i.d. random variables (r.v.'s), defined on $(\Omega, \mathcal{G}, P)$, which is assumed to be independent of $\{(X_n, T_n), n = 0, 1,\ldots\}$ and of $\{V_n(j), n = 0, 1,\ldots\}$, for $j \neq i$. Furthermore, we denote the common distribution function (d.f.) of $V_n(i)$ by $F_i(\cdot)$. Here, the r.v.'s $V_n(i)$ represent instantaneous 'inputs' of type i for our storage model. The reader may note our preference to call the $V_n(i)$'s as 'inputs' even though for negative values, strictly speaking, they are demands.

We consider a semi-infinite storage facility and let $Z(t)$ and $L(t)$ represent the level of storage at time t and the total demand lost or not met due to nonavailability of supply during $[0,t]$, respectively. More exactly, the process $\{Z(t), L(t)\}$ is defined constructively by

$$(2) \qquad Z(t) = \begin{cases} \max[0, V_0(X_0)], & 0 \leq t < T_1 \\ \max[0, Z(T_{n-1}) + V_n(X_n)], & T_n \leq t < T_{n+1}, \quad n \geq 1, \end{cases}$$

and

$$(3) \qquad L(t) = \begin{cases} \max[0, -V_0(X_0)], & 0 \leq t < T_1 \\ L(T_{n-1}) + \max[0, -(Z(T_{n-1}) + V_n(X_n))], & T_n \leq t < T_{n+1}, \ n \geq 1, \end{cases}$$

where $V_n(X_n) = V_n(j)$, whenever $X_n = j$. Let $M(t) = \sup\{n \geq 0 : T_n \leq t\}$. We assume that the sample paths of $X_{M(t)}$ are almost surely right continuous. Consequently, the process $\{Z(t), L(t)\}$, as defined by (2) and (3), is almost surely right continuous and separable.

Throughout the following, in order to eliminate certain trivial cases, we assume that there exist states i and j in J, not necessarily distinct, such that $F_i(0) < 1$ and $F_j(0-) > 0$. We also assume that

(4) $$\sum_{j \in J} \pi(j)\ E(|V_0(j)|) < \infty,$$

so that

(5) $$\rho \equiv \sum_{j \in J} \pi(j)\ E(V_0(j))$$

exists, which evidently represents here the 'average stationary input' to the storage facility. We follow the terminology of Senturia and Puri [13] in identifying the three distinct cases labelled as supercritical ($\rho>0$), critical ($\rho=0$), and subcritical ($\rho<0$), and study each case separately.

In [13] and [14], Senturia and Puri have considered a special case of the above model with $J = \{1,2\}$, where State 1 represents an input with $V_0(1) \geq 0$, a.s., and State 2 represents a demand with $V_0(2) \leq 0$, a.s. For their special case, they obtained limit distribution for $Z(t)$, as $t \to \infty$, in the supercritical and critical cases. More recently, with the same restrictions on J, $V_0(1)$, and $V_0(2)$, Balagopal [2] introduced and studied the r.v. $L(t)$ and obtained its limit distribution only for the subcritical case ($\rho<0$). The purpose of the present paper is to study the joint asymptotic behavior of $\{Z(t), L(t)\}$ in all three cases, namely the supercritical case, the critical case, and the subcritical case. This is accomplished for our generalized model, which have two distinct features. First, it allows a countable number of possibly different 'types' of 'inputs'. Second, it generalizes the old model further by ignoring the specific distinction between an input or a demand; instead anytime an input takes a negative value it is treated as a demand. Sections 3, 4, and 5 deal with the supercritical, critical, and subcritical cases respectively, but first we need the following few preliminaries.

2. PRELIMINARIES. Following Balagopal [2], let $Z_n = Z(T_n)$ and $L_n = L(T_n)$, for $n \geq 0$, so that $\{(Z_n, L_n), n = 0, 1, \ldots\}$ is an embedded discrete time process. By defining $S_n \equiv \sum_{i=0}^{n} V_i(X_i)$, we can expand Z_n recursively to yield the relationships

(6) $$Z_n = S_n + L_n$$

and

(7) $$L_n = \max\ (0, -S_0, -S_1, \ldots, -S_n).$$

The following definitions will be needed in the sequel. For $i \in J$, arbitrary but fixed, let $\ell(0) \equiv \ell(0,i) = 0$, a.s.,

(8) $$\ell(n) \equiv \ell(n,i) \equiv \inf \{m : m > \ell(n-1),\ X_m = i\},\ n \geq 1,$$

(9) $$Y(n) \equiv Y(n,i) \equiv S_{\ell(n)-1} - S_{\ell(n-1)-1},\ n \geq 1,$$

where $S_{-1} = 0$, and $N(t) = \sup \{n \geq 0 : T_{\ell(n)} \leq t\}$.

We conclude with the following proposition which is a direct consequence of the fact (see Çinlar [5]) that

(10) $$\lim_{t\to\infty} M(t)/t = \beta^{-1} \equiv \left[\sum_{j\in J} \pi(j) \int_0^\infty t\ d(\Sigma_{\ell\in J} A(j,\ell,t))\right]^{-1}.$$

PROPOSITION 1. *If* (4) *holds*, *then*, *as* $t\to\infty$,

(11) $$L(t)/t \to \beta^{-1} \max\ (0,-\rho),\ \text{a.s.},$$

(12) $$Z(t)/t \to \beta^{-1} \max\ (0,\rho),\ \text{a.s.}$$

PROOF. From (6) and (7), it follows that (see Puri [11] and Chung [4]) $L_n/n \to \max\ (0,-\rho)$ and $Z_n/n \to \max\ (0,\rho)$ almost surely as $n \to \infty$. The result follows from the law of large numbers and (10). □

3. THE SUPERCRITICAL CASE. We consider here the case with $\rho > 0$. For this since $L(t)$ is a nondecreasing function of t, the law of large numbers implies that $L(t)$ converges to a proper r.v. L almost surely, as $t \to \infty$. However, in order to study the limit behavior of $\{Z(t), L(t)\}$, we first establish the following limit result for the marginal distribution of $Z(t)$.

LEMMA 1. *If* $\rho > 0$, $\beta < \infty$, *and*

$$\sigma^2 \equiv E\{[Y(1) - T_{\ell(1)} \cdot \rho\beta^{-1}]^2 \,|X_0 = i\} < \infty, \tag{13}$$

then

$$\lim_{t\to\infty} P(Z(t) - t\rho\beta^{-1} \le x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2}) = \Phi(x), \tag{14}$$

where $\phi(\cdot)$ *is the standard normal d.f.*

PROOF. Using (6), we write

$$Z(t) = [S_{M(t)} - S_{\ell(N(t))-1}] + S_{\ell(N(t))-1} + L_{M(t)}. \tag{15}$$

The finiteness of L implies that $L_{M(t)}\, t^{-1/2} \overset{P}{\to} 0$, while the key renewal theorem can be used to show that $(S_{M(t)} - S_{\ell(N(t))-1})\, t^{-1/2} \overset{P}{\to} 0$. In addition, we note that

$$S_{\ell(N(t))-1} - t\rho\beta^{-1} \equiv \sum_{m=2}^{N(t)} [Y(m) - (T_{\ell(m)} - T_{\ell(m-1)})\rho\beta^{-1}] + R, \tag{16}$$

where $Rt^{-1/2} \overset{P}{\to} 0$, as $t\to\infty$. Furthermore $\{Y(m) - (T_{\ell(m)} - T_{\ell(m-1)})\, \rho\beta^{-1}\}$ is a sequence of i.i.d. r.v.'s having mean zero and variance σ^2. Consequently, following Chung ([4], page 100), we can show that

(17) $$\lim_{t\to\infty} P(Z(t) - t\rho\beta^{-1} \leq x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2})$$

$$= \lim_{t\to\infty} P(\sum_{m=2}^{N(t)} [Y(m) - (T_{\ell(m)} - T_{\ell(m-1)})\rho\beta^{-1}] \leq x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2})$$

$$= \Phi(x)$$ □

Using the above lemma, we now establish a limit distribution for the process $\{Z(t), L(t)\}$.

THEOREM 1. If y is a continuity point for $P(L \leq y)$, $\rho>0$, $\beta<\infty$, and $\sigma^2<\infty$, then

(18) $$\lim_{t\to\infty} P(Z(t) - t\rho\beta^{-1} \leq x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2}, L(t) \leq y) = \Phi(x) \cdot P(L \leq y).$$

PROOF. As in the proof of Lemma 1, it suffices to prove that

(19) $$\lim_{t\to\infty} P(\sum_{m=2}^{N(t)} D(m) \leq x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2}, L(t) \leq y)$$

$$= \Phi(x) \cdot P(L \leq y),$$

where $D(m) = Y(m) - (T_{\ell(m)} - T_{\ell(m-1)})\rho\beta^{-1}$. However, $\{(\sigma n^{1/2})^{-1} \sum_{m=2}^{n} D(m)\}$ is a strongly mixing sequence with limit d.f. $\Phi(\cdot)$ (see Puri [11]). Since $N(t)t^{-1} \to \pi(i)\beta^{-1}$, a.s., as $t \to \infty$, there exists a sequence of positive real numbers $\{\varepsilon(t)\}$, with $\varepsilon(t)$ decreasing to zero as $t \to \infty$, such that

(20) $$P(|N(t) - \pi(i)\beta^{-1}t| \geq \pi(i)\beta^{-1}\varepsilon(t)t) \leq \varepsilon(t)\pi(i)\beta^{-1},$$

for all $t > 0$. For $t > 0$, let

(21) $\Lambda(t) = \{\omega\in\Omega : |N(t) - \pi(i)\beta^{-1}t| \leq \pi(i)\beta^{-1}\varepsilon(t)t\}$,

(22) $a \equiv a(t) = [(1-\varepsilon(t))\ \pi(i)\beta^{-1}t]$

and

(23) $b \equiv b(t) = [(1+\varepsilon(t))\ \pi(i)\beta^{-1}t] + 1$,

where $[\alpha]$ is the largest integer $\leq \alpha$. Equations (21) - (23) lead us to the decomposition

(24) $$\sum_{m=2}^{N(t)} D(m) = \sum_{m=2}^{a} D(m)\ I(\Lambda(t)) + \sum_{m=a+1}^{N(t)} D(m)\ I(\Lambda(t)) + \sum_{m=2}^{N(t)} D(m)\ I(\Lambda(t)^c),$$

where $I(A)$ is the indicator function of the set A and A^c is the complement of the set A. An application of Kolmogorov inequality shows that $t^{-1/2} \sum_{m=a+1}^{N(t)} D(m)\ I(\Lambda(t)) \overset{P}{\to} 0$, while (20) implies that $t^{-1/2} \sum_{m=2}^{N(t)} D(m)\ I(\Lambda(t)^c) \overset{P}{\to} 0$. Hence, using the strongly mixing property of $\{(\sigma n^{1/2})^{-1} \sum_{m=2}^{n} D(m)\}$ and the nondecreasing nature of $L(t)$, we can find a $K = K(\varepsilon, x, y) > 0$, where $\varepsilon>0$ is arbitrary but fixed, $x \in (-\infty, \infty)$, and y is a continuity point for $P(L \leq y)$, such that $t > K$ implies that

(25) $$\left|P\left(\sum_{m=2}^{a} D(m) \leq x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2},\ L(t) \leq y\right) - \Phi(x) \cdot P(L \leq y)\right| \leq \varepsilon.$$

But ε was arbitrary and so we obtain (19). □

4. THE CRITICAL CASE. In order to find a limit distribution for $\{Z(t), L(t)\}$ when $\rho=0$, we first investigate the structure of the discrete time embedded process, $\{Z_n, L_n\}$. Again we take $i \in J$ to be arbitrary but fixed. Using (7), we can write the demand lost up to, but not including, the epoch of the nth return to state i, as

$$L_{\ell(n)-1} = \max(0, -S_0, -S_1, \dots, -S_{\ell(n)-1}) \tag{26}$$

$$= \max(0, U_1, \dots, U_n),$$

where

$$U_1 = -S_{\ell(1)-1} + \max(0, S_{\ell(1)-1} - S_{\ell(1)-2}, \dots, S_{\ell(1)-1} - S_0, S_{\ell(1)-1}) \tag{27}$$

and, for $2 \le m \le n$,

$$U_m = -S_{\ell(m)-1} + \max(0, S_{\ell(m)-1} - S_{\ell(m)-2}, \dots, S_{\ell(m)-1} - S_{\ell(m-1)}). \tag{28}$$

Similarly, the level of storage at the instant before the nth return to state i is

$$Z_{\ell(n)-1} = \max(0, U_1^*, U_2^*, \dots, U_n^*), \tag{29}$$

where

$$U_1^* = S_{\ell(n)-1} - S_{\ell(1)-1} + \max(S_{\ell(1)-1}, S_{\ell(1)-1} - S_0, \dots, S_{\ell(1)-1} - S_{\ell(1)-2}) \tag{30}$$

while, for $2 \le m \le n$,

$$U_m^* = S_{\ell(n)-1} - S_{\ell(m-1)-1} + \max(0, -(S_{\ell(m-1)} - S_{\ell(m-1)-1}), \dots, \tag{31}$$

$$-(S_{\ell(m)-2} - S_{\ell(m-1)-1})).$$

Thus, if we define $\alpha(n) = \sum_{m=2}^{n} Y(m)$, for $n \geq 2$ and zero for $n = 0$ and 1, and let $r(m) = \sum_{j=\ell(m-1)}^{\ell(m)-1} |V_j(X_j)|$, for $1 \leq m \leq n$, then, after some manipulation, we obtain

(32) $\quad |L_{\ell(n)-1} - \max(0, -\alpha(2), \ldots, -\alpha(n))| \leq r(1) + R(n)$, a.s.,

and

(33) $\quad |Z_{\ell(n)-1} - \max(0, \alpha(n), \alpha(n) - \alpha(n-1), \ldots, \alpha(n) - \alpha(2))| \leq R(n)$,

a.s., where $R(n) = \max\,(r(1), \ldots, r(n))$. After defining $\sigma^2 = E[Y(1)^2 | X_0 = i]$ and $\tau^2 = E\{[\sum_{m=0}^{\ell(1)-1} |V_m(X_m)|]^2 | X_0 = i\}$, we obtain the following limit result for $\{Z_n, L_n\}$.

LEMMA 2. *If* $\rho=0$ *and* $\tau^2<\infty$, *then*

(34) $$\lim_{n\to\infty} P(Z_n \leq x\sigma \cdot (\pi(i)n)^{1/2},\ L_n \leq y\sigma \cdot (\pi(i)n)^{1/2})$$

$$= \int_0^x \int_0^y g(s,t)\,ds\,dt,$$

where

(35) $$g(s,t) = \begin{cases} (2/\pi)^{1/2} \cdot (s+t) \exp\{-(s+t)^2/2\}, & s,\ t \geq 0 \\ , & \text{otherwise.} \end{cases}$$

PROOF. For $i \in J$ as above, let $M(i,n) = \sup\{m \geq 0 : \ell(m) \leq n\}$. In ([4], Section 14), Chung shows that

(36) $$n^{-1/2}|L_n - L_{\ell(M(i,n))-1}| \overset{P}{\to} 0,\quad n^{-1/2}|Z_n - Z_{\ell(M(i,n))-1}| \overset{P}{\to} 0,$$

as $n \to \infty$. Consider now the space $D[0,1]$ of right continuous functions having left hand limits, defined on $[0,1]$, and define $G_n(\cdot) \in D[0,1]$ by

(37) $$G_n(t) \equiv G_n(t,\omega) \equiv (\sigma \cdot (\pi(i)n)^{1/2})^{-1} \cdot \alpha([M(i,n)t]),$$

where $[a]$ is the integer part of a. Since $M(i,n)n^{-1} \to \pi(i)$, a.s., as $n \to \infty$, for $W(t)$, the standard Brownian motion defined on $D[0,1]$, $G_n(\cdot)$ converges in law to $W(\cdot)$ (see Billingsley ([3], page 146)). We also note that $r(1)n^{-1/2}$ and $R(n)n^{-1/2}$ both converge to zero in probability, as $n \to \infty$. This, together with (32), (33) and (36), implies that

(38) $$\lim_{n\to\infty} P(Z_n \le x\sigma \cdot (\pi(i)n)^{1/2},\ L_n \le y\sigma \cdot (\pi(i)n)^{1/2})$$

$$= P(W(1) + \sup_{0\le t\le 1} [-W(t)] \le x,\ \sup_{0\le t\le 1} [-W(t)] \le y)$$

$$= \int_0^x \int_0^y g(s,t)dsdt,$$

where $g(s,t)$ is as given in (35). □

Using Lemma 2, we can now establish the following limit theorem for $\{Z(t), L(t)\}$.

THEOREM 2. If $\rho=0$, $\beta<\infty$ and $\tau^2<\infty$, then

(39) $$\lim_{t\to\infty} P(Z(t) \le x\sigma \cdot (t\pi(i)\beta^{-1})^{1/2},\ L(t) \le y\sigma \cdot (t\pi(i)\beta^{-1})^{1/2})$$

$$= \int_0^x \int_0^y g(s,t)dsdt,$$

where $g(s,t)$ is given by (35).

PROOF. We first note the decompositions

$$Z(t) = (Z_{M(t)} - Z_{\ell(N(t))-1}) + Z_{\ell(a)-1} I(\Lambda(t)) + (Z_{\ell(N(t))-1} - Z_{\ell(a)-1}) I(\Lambda(t)) + Z_{\ell(N(t))-1} I(\Lambda(t)^c) \tag{40}$$

and

$$L(t) = (L_{M(t)} - L_{\ell(N(t))-1}) + L_{\ell(a)-1} I(\Lambda(t)) + (L_{\ell(N(t))-1} - L_{\ell(a)-1}) I(\Lambda(t)) + L_{\ell(N(t))-1} I(\Lambda(t)^c), \tag{41}$$

where $\Lambda(t)$, a and b are given by (21), (22) and (23) respectively. Again, the key renewal theorem can be used to show that, as $t\to\infty$,

$$t^{-1/2}|Z_{M(t)} - Z_{\ell(N(t))-1}| \overset{P}{\to} 0, \quad t^{-1/2}|L_{M(t)} - L_{\ell(N(t))-1}| \overset{P}{\to} 0. \tag{42}$$

The increasing nature of L_n allows us to write, for each $\omega \in \Lambda(t)$,

$$t^{-1/2}|L_{\ell(N(t))-1} - L_{\ell(a)-1}| \le t^{-1/2}|L_{\ell(b)-1} - \gamma(b)| + t^{-1/2}|\gamma(b) - \gamma(a)| + t^{-1/2}|L_{\ell(a)-1} - \gamma(a)|, \tag{43}$$

where $\gamma(n) = \max(0, -\alpha(2), \ldots, -\alpha(n))$. However, $R(n)n^{-1/2} \overset{P}{\to} 0$, and so, from (32), the first and last terms on the right side of (43) converge to zero in probability on $\Lambda(t)$. In addition, since $\gamma(b) - \gamma(a) \le \sup_{a<m\le b} |\sum_{n=a+1}^{m} Y(n)|$, Kolmogorov inequality can be used to show that the second term on the right side of (43) also converges to zero in probability on $\Lambda(t)$. In a similar fashion, for $\omega \in \Lambda(t)$, we use the inequality

$$(44)\qquad |Z_{\ell(N(t))-1} - Z_{\ell(a)-1}| \le R(b) + \sup_{a<m\le b} |\alpha(m) - \alpha(a)| + |\gamma(b) - \gamma(a)| + R(a)$$

to show that $t^{-1/2}|Z_{\ell(N(t))-1} - Z_{\ell(a)-1}|I(\Lambda(t)) \overset{P}{\to} 0$.

Consequently, making use of the fact that $I(\Lambda(t)^c) \overset{P}{\to} 0$, Lemma 2 and the above, we obtain that

$$(45)\qquad \lim_{t\to\infty} P(Z(t) \le x\sigma \cdot (t\,\pi(i)\beta^{-1})^{1/2},\ L(t) \le y\sigma \cdot (t\,\pi(i)\beta^{-1})^{1/2})$$

$$= \lim_{t\to\infty} P(Z_{\ell(a)-1} \le x\sigma \cdot (t\,\pi(i)\beta^{-1})^{1/2},\ L_{\ell(a)-1} \le y\sigma \cdot (t\,\pi(i)\beta^{-1})^{1/2})$$

$$= \int_0^x \int_0^y g(s,t)\,ds\,dt. \qquad \square$$

5. THE SUBCRITICAL CASE. Before proving the existence of a limit distribution for the process $\{Z_n, L_n\}$ when $\rho<0$, we require the following definition.

DEFINITION. Let $\{\hat{X}_n,\ n = 0,1\ldots\}$ be a M.C with state space J, defined on $(\Omega, \mathcal{A}, P)$ and independent of all other variables previously defined on $(\Omega, \mathcal{A}, P)$. In addition, let the initial distribution be $\pi = (\pi(1), \pi(2),\ldots)$ and let entries of the associated transition matrix be defined by

$$(46)\qquad \hat{p}_{ij} = (\pi(j)/\pi(i))p_{ji},\quad i, j \in J.$$

We refer to $\{\hat{X}_n\}$ as the dual M.C for $\{X_n\}$, also sometimes called as the 'reversed' M.C.

The next lemma follows immediately from the above definition and equations (6) and (7).

LEMMA 3. *If* $a(i) = \pi(i)$ *for all* $i \in J$, *then, for* $n \geq 0$,

(47) $$P(Z_n \leq x,\ L_n \leq y) = P(\max(0, \hat{S}_0, \ldots, \hat{S}_n) \leq x,$$

$$- \hat{S}_n + \max(0, \hat{S}_0, \ldots, \hat{S}_n) \leq y),$$

where $\hat{S}_n = \sum_{j=0}^{n} V_j(\hat{X}_j)$.

Lemma 4, below, gives a limit distribution for $\{Z_n, L_n\}$ when $a(i) = \pi(i)$, for all $i \in J$. This restriction will be subsequently removed in Theorem 3.

LEMMA 4. *If* $a(i) = \pi(i)$ *for all* $i \in J$, $\rho<0$, *and*

(48) $$\hat{\sigma}^2 \equiv E\{[Y(1) - \ell(1)\rho]^2\ X_0 = i\} < \infty$$

then

(49) $$\lim_{n\to\infty} P(Z_n \leq x,\ L_n + n\rho \leq y\hat{\sigma} \cdot [n\pi(i)]^{1/2}) = P(\hat{Z} \leq x) \cdot \Phi(y),$$

where $\hat{Z} = \sup(0, \hat{S}_0, \hat{S}_1, \ldots) < \infty$, a.s. *and* x *is a continuity point for* $P(\hat{Z} \leq x)$.

PROOF. Let $i \in J$ be arbitrary but fixed and define $\hat{\ell}(n) = \inf\{m : m > \hat{\ell}(n-1),\ X_m = i\}$, $n \geq 1$, with $\hat{\ell}(0) \equiv 0$. It is easy to show that $E(Y(1)|X_0 = i) = E(\hat{S}_{\hat{\ell}(1)-1}|\hat{X}_0 = i)$ and that $\hat{\sigma}^2 = E(\hat{S}_{\hat{\ell}(1)-1} - \hat{\ell}(1)\rho)^2|\hat{X}_0 = i\}$. Furthermore, since $\rho<0$, $\hat{Z}_n \equiv \max(0, \hat{S}_0, \hat{S}_1, \ldots, \hat{S}_n)$ increases almost surely to a proper random variable $\hat{Z}$, as $n \to \infty$, so that $\hat{Z}_n/n^{1/2} \overset{P}{\to} 0$. Consequently, because of Lemma 3, it suffices to consider the limit distribution of $\{\hat{Z}_n, -(\hat{S}_n - n\rho)[\hat{\sigma}(n\pi(i))^{1/2}]^{-1}\}$. Using the central limit theorem for functionals defined on a M.C (see Chung ([4], Section 16)), we can show that $\{-(\hat{S}_n - n\rho)[\hat{\sigma}\cdot(n\pi(i))^{1/2}]^{-1}\}$ is a strongly mixing sequence with limit d.f. $\Phi(\cdot)$. Hence, for each $\varepsilon>0$, $\exists K = K(\epsilon,x,y) > 0$ such that for $n > K$

(50) $$P(\hat{Z}_n \le x) - P(\hat{Z} \le x) \le \varepsilon$$

(51) $$|P(\hat{Z} \le x, -\hat{S}_n + n\rho \le y\hat{\sigma}\cdot(n\pi(i))^{1/2}) - P(\hat{Z} \le x)\cdot\Phi(y)| \le \varepsilon.$$

But this implies that for $n > K$,

(52) $$|P(\hat{Z}_K \le x, -\hat{S}_n + n\rho \le y\hat{\sigma}\cdot(n\pi(i))^{1/2}) - P(\hat{Z} \le x)\cdot\Phi(y)| \le 2\varepsilon.$$

Since ε was arbitrary, the monotonicity of $\hat{Z}_n$ implies the result. □

In order to remove the restriction that $a(i) = \pi(i)$ for all $i \in J$, following Hoel, et al [6] we define another irreducible, aperiodic, and positive recurrent M.C, $\{\bar{X}_n, n = 0, 1,\dots\}$, on $(\Omega, \mathcal{G}, P)$, which is independent of any variables previously defined on $(\Omega, \mathcal{G}, P)$. Furthermore, we take its state space to be J, the transition matrix to be $\underset{\sim}{P}$, and the initial distribution to be $\underset{\sim}{\pi}$. Define $T = \min\{n > 0 : X_n = \bar{X}_n\}$.

THEOREM 3. If $\rho<0$, x is a continuity point for $P(\hat{Z} \le x)$ and $\hat{\sigma}^2<\infty$, where $\hat{\sigma}^2$ is defined in (48), then

(53) $$\lim_{n\to\infty} P(Z_n \le x, L_n + n\rho \le y\hat{\sigma}\cdot(n\pi(i))^{1/2}) = P(\hat{Z} \le x)\cdot\Phi(y).$$

PROOF. Hoel, et. al. [6] have shown that $T < \infty$, a.s. Hence, for any $\varepsilon > 0$, fixed, we can choose finite constants $B = B(\varepsilon)$, $C = C(B,\varepsilon)$ and $D = D(B,\varepsilon)$ such that

(54) $P(T \le B) > 1-\varepsilon,\ P(Z_B \le C) > 1-\varepsilon,\ P(L_B \le D) > 1-\varepsilon.$

For $n > B$, noting that

(55)
$$P(X_n = j_0, X_{n+1} = j_1, \ldots, X_{n+m} = j_m, T \le n)$$
$$= P(\bar{X}_n = j_0, \bar{X}_{n+1} = j_1, \ldots, \bar{X}_{n+m} = j_m, T \le n),$$

for any $m \ge 0$ and $j_\ell \in J$, $0 \le \ell \le m$, we obtain the inequality

(56)
$$P(Z_n \le x,\ L_n + n\rho \le y\hat{\sigma} \cdot (n\pi(i))^{1/2})$$
$$\ge P(\max(0, \bar{S}_n - \bar{S}_{n-1}, \ldots, \bar{S}_n - \bar{S}_{B+1}) \le x,$$
$$\max[0, -(\bar{S}_{B+1} - \bar{S}_B), \ldots, -(\bar{S}_n - \bar{S}_B)] + D + n\rho \le y\hat{\sigma} \cdot (n\pi(i))^{1/2})$$
$$- 3\varepsilon - P(\bar{S}_n - \bar{S}_B + C > x),$$

where $\bar{S}_n = \sum_{m=0}^{n} V_m(\bar{X}_m)$. Similarly,

(57)
$$P(Z_n \le x,\ L_n + n\rho \le y\hat{\sigma} \cdot (n\pi(i))^{1/2})$$
$$\le P(\max(0, \bar{S}_n - \bar{S}_{n-1}, \ldots, \bar{S}_n - \bar{S}_B) \le x,$$
$$\max[0, -(\bar{S}_{B+1} - \bar{S}_B), \ldots, -(\bar{S}_n - \bar{S}_B)] - D + n\rho \le y\hat{\sigma} \cdot (n\pi(i))^{1/2})$$
$$+ 2\varepsilon.$$

However, since $\{\bar{X}_n\}$ is a stationary M.C, Lemma 4 and the fact that ε is arbitrary, can be used to arrive at the desired result. □

6. A FEW CONCLUDING REMARKS (a) For the supercritical case (Section 3) it is also true that L_n converges a.s. to L, as $n \to \infty$. Moreover from Chung ([4], Section 16), we see that when $\rho > 0$,

(58) $$\lim_{n\to\infty} P(Z_n - n\rho \leq x\hat{\sigma} \cdot (n\,\pi(i))^{1/2}) = \Phi(x),$$

whenever

(59) $$\hat{\sigma}^2 \equiv E\{[Y(1) - \ell(1)\rho]^2 | X_0 = i\},$$

is finite. Consequently, an argument similar to the proof of theorem 1, implies that

(60) $$\lim_{n\to\infty} P(Z_n - n\rho \leq x\hat{\sigma} \cdot (n\,\pi(i))^{1/2}, L_n \leq y) = \Phi(x) \cdot P(L \leq y),$$

for all continuity points y of $P(L \leq Y)$, whenever $\hat{\sigma}^2 < \infty$, where L is as defined in theorem 1.

(b) For the subcritical case (Section 5), a limit distribution for L(t), similar to the one given in [2], can also be established for our model. In particular, we have the following proposition whose proof is left to the interested reader.

PROPOSITION. If $\rho < 0$, $\beta < \infty$, and $\tau^2 \equiv E\ [\{\sum_{m=0}^{\ell(1)-1} |V_m(X_m)|]^2 | X_0 = i\}$ is finite, then

(61) $$\lim_{t\to\infty} P(L(t) + \rho N(t)[\pi(i)]^{-1} \leq x\sigma \cdot [N(t)]^{1/2}) = \Phi(x),$$

where σ^2 is the variance of Y(m).

Again, in the continuous time case, the limit behavior of Z(t), for the subcritical case, was left out as it involves rather delicate analysis. The work pertaining to this is still in progress and will be reported elsewhere.

(c) Finally, the results reported in this paper are subject to the assumption of the finiteness of the first two moments of the random variables $V_0(j)$'s. Analagous results have been obtained without any such assumption (see Woolford [15]). These will appear in a forthcoming paper.

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