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ON THE CONVERGENCE OF THE EMPIRIC AGE DISTRIBUTION FOR ONE DIMENSIONAL SUPERCRITICAL AGE DEPENDENT BRANCHING PROCESSES

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"AGE DISTRIBUTION"

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ABSTRACT

The age distribution for the supercritical age dependent branching processes is shown to converge on the set of nonextinction to a particular distribution function if the offspring distribution $\{p_j\}$ satisfies $1 < \Sigma j p_j < \infty$.

Key Words: Age Distribution, Branching Process, Supercritical

1. INTRODUCTION

Consider an age-dependent branching process (See Harris [3] for definitions) governed by $\{p_j\}$, the common probability distribution of the number of progeny born to an individual at death, and $G(\cdot)$, the common distribution function (d.f.) of the length of life of an individual. For a realization ω of the process, let $Z(t,\omega)$ denote the total number of individuals alive at time t, $Z(x,t,\omega)$, the number among these that have ages no more than x, and $A(\cdot,t,\omega)$ defined by $A(x,t,\omega) = Z(x,t,\omega)/Z(t,\omega)$ denote the empiric age d.f. of those alive at time t.

There has been considerable interest shown in the past in the limiting behavior of the age distribution $A(\cdot,t,\omega)$, as $t \to \infty$. In [3], Harris showed the almost sure (a.s.) convergence of $A(\cdot,t,\omega)$ if $\{p_j\}$ has a second moment and $G(\cdot)$ is sufficiently regular. Later Jagers [4] obtained the same result assuming only that $\{p_j\}$ has a second moment. More recently, Athreya and Kaplan [1] showed the validity of the above result assuming that $\Sigma(j \cdot \log j) \cdot p_j$ is finite. After the present results were obtained, a later paper of Athreya and Kaplan [2] was brought to the attention of the author, where they have shown that the result holds if $1 < m = \Sigma j p_j < \infty$ and $G(\cdot)$ satisfies a certain tail condition. In contrast to these, the present paper assumes only $1 < m < \infty$, with no conditions imposed on $G(\cdot)$.

The approach adopted here follows in part the basic steps of [1], namely that we decompose $A(\cdot,t,\omega)$ into three terms, and then tackle each term separately. In so doing we use a rather interesting embedding technique leading to the final proof.

In Section 2, we give notation adopted from [1] and our basic assumptions. In Section 3, we state the main theorem and the three lemmas necessary to prove it. Section 4 deals with a law of large numbers, which may be of independent interest. Finally, to complete the proof of the basic theorem of Section 3, in Section 5, a theorem is given which provides certain lower bounds achieved through a special embedding. Of the three lemmas of Section 3, one is already proved in [1], while the other two are proven in Section 5, using also the results of Section 4.

2. NOTATION AND BASIC ASSUMPTIONS

We always assume, whether stated or not, that a) $p_0 = 0$ (rather than conditioning on the set of nonextinction), b) $1 < m = \Sigma j p_j < \infty$, and c) $G(0+) = 0$. For any realization ω and $x \geq 0$ we define,

$Z(t,\omega)$ = number of particles living at time t,

$Z(x,t,\omega)$ = number of particles of age $\leq x$ living at time t,

$A(x,t,\omega) = Z(x,t,\omega)/Z(t,\omega)$,

$\{x_i(t,\omega);\ i=1,2,\ldots,\ Z(t,\omega)\}$ = the age chart at time t,

$Z_{x_i(t,\omega)}(x,s,\omega)$ = number of particles alive at time t+s with ages $\leq x$, in a line of descent initiated by a particle of age $x_i(t,\omega)$ living at time t,

and

$$Z_{x_i(t,\omega)}(s,\omega) = \lim_{x\to\infty} Z_{x_i(t,\omega)}(x,s,\omega).$$

Let for $x \geq 0$, $y \geq 0$, $M(x,t) = E\{Z(x,t)\}$, $M(t) = E\{Z(t)\}$, $M_y(t) = E\{Z_y(t)\}$, $M_y(x,t) = E\{Z_y(x,t)\}$, and $m = \Sigma j p_j$. Also for $x \geq 0$, $y \geq 0$, let $G_y(x) = (G(x+y)-G(y))/(1-G(y))$, $V(x) = m \int_0^\infty e^{-\alpha u} G_x(du)$,

$$n_1 = [\int_0^\infty e^{-\alpha t}(1-G(t))dt]/[m \int_0^\infty t\, e^{-\alpha t} G(dt)],$$

$$A(x) = [\int_0^x e^{-\alpha t}(1-G(t))dt]/[\int_0^\infty e^{-\alpha t}(1-G(t))dt],$$

and

$$V_t = \int_0^\infty V(x) Z(dx,t,\omega) = \sum_{i=1}^{Z(t,\omega)} V(x_i(t,\omega)),$$

where α is the Malthusian parameter as defined as the root of the equation $m \int_0^\infty e^{-\alpha t} G(dt) = 1$.

3. THE THEOREM AND THREE LEMMAS

The proof of the following theorem is based on a natural decomposition of $A(x,t)$ into three parts as in [1], and a separate lemma is proven for each part, the difference being that two of the three lemmas given below are stronger than those of [1] or [2].

THEOREM 3.1. *If* $1 < m < \infty$, *then*

(3.1) $$\lim_{t\to\infty} \sup_{x\geq 0} |A(x,t,\omega) - A(x)| = 0 \text{ a.s.}$$

Before indicating the proof of Theorem 3.1, we will first define the decomposition, and then give the three corresponding lemmas.

Clearly one may write (suppressing subscripts)

(3.2) $$Z(x,t+s) = \sum_{i=1}^{Z(t)} Z_{x_i}(x,s).$$

Also, as in [1], by defining

$$(3.3)\qquad a_t(x,s) = \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [Z_{x_i}(x,s) - M_{x_i}(x,s)]e^{-\alpha s},$$

$$(3.4)\qquad b_t(x,s) = \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [M_{x_i}(x,s)e^{-\alpha s} - n_1 V(x_i)A(x)],$$

and

$$(3.5)\qquad c_t = V_t/Z(t)$$

we have

$$A(x,t+s) = \frac{a_t(x,s)+b_t(x,s)+c_t A(x)}{a_t(\infty,s)+b_t(\infty,s)+c_t},$$

where $a_t(\infty,s)$ and $b_t(\infty,s)$ are the respective limits of $a_t(x,s)$ and $b_t(x,s)$ as $x \to \infty$. The following lemma and corollary are from [1].

<u>LEMMA 3.1.</u> <u>As</u> $s \to \infty$, <u>for fixed</u> x,

$$(3.6)\qquad \sup_{y\geq 0}\{|M_y(x,s)e^{-\alpha s} - n_1 V(y)A(x)|, |M_y(\infty,s)e^{-\alpha s} - n_1 V(y)|\} \to 0.$$

<u>COROLLARY 3.1</u>. <u>As</u> $s \to \infty$, <u>for fixed</u> x,

$$(3.7)\qquad \sup_{t,\omega}\{|b_t(x,s)|, |b_t(\infty,s)|\} \to 0.$$

The next two lemmas are proved at the end of Section 5 as the corollaries of results in Sections 4 and 5.

<u>LEMMA 3.2</u>. <u>If</u> $1 < m < \infty$ <u>and</u> $\delta > 0$

<u>then</u>,

$$(3.8)\qquad Y_{n\delta} = \frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} [Z_{x_i}(x,s) - M_{x_i}(x,s)] \to 0 \qquad \text{a.s.}$$

In [1] it was shown that $Y_t \to 0$ in probability, if $1 < m < \infty$, and that $Y_{n\delta} \to 0$ a.s., if $\Sigma(j \log j)p_j < \infty$. In [2] it was shown that

$Y_t \to 0$ a.s., if $1 < m < \infty$ and a certain tail condition on $G(\cdot)$ holds.

LEMMA 3.3. *If* $1 < m < \infty$, *then for some* $\eta > 0$

$$\liminf_{t\to\infty} [V_t/Z(t)] > \eta \quad \text{a.s.} \tag{3.9.}$$

It was shown in [1] that (3.9) holds if $\inf_{x \in \text{supp } G} V(x) > 0$ or instead if $\Sigma(j \log j)p_j < \infty$.

PROOF: (of Theorem 3.1) The above lemmas show that if $\delta > 0$ then $A_{n\delta}(x) \to A(x)$ a.s. This fact and the continuity of $A(x)$ along with technical arguments in [1], give us Theorem 3.1.

4. A LAW OF LARGE NUMBERS

The following Proposition 4.1 was originally used by the author in obtaining results applicable in proving lemma 3.2. It was pointed out to the author by Professor Prem Puri that [2] contained a very similar result which in fact implied our Proposition 4.1. Consequently we state Proposition 4.1 and the appropriate lemma from [2], and then remark how the lemma implies the proposition.

PROPOSITION 4.1. *Let* T_{ij} *be a measureable array of non-negative random variables and* K_i *a sequence of non-negative random variables such that:*

a) For some $s < \infty$, $P(T_{ij} \in [0,s]) = 1$, for $j=1,2,\ldots,K_i$, $i=1,2,\ldots$

b) There exists a constant $C > 1$, such that $\liminf_{i\to\infty} [K_{i+1}/K_i] > C$ a.s.

Also let $X_{ij}(t)$ *be row independent, identically distributed realizations of a process such that for any* $s > 0$, *there is an integrable random variable* X *where*

a) $P(|X_{ij}(t)| > \lambda) \leq P(|X| > \lambda)$, *if* $t \in [0,s]$,

b) $E\, X_{ij}(t) = EX = 0$ *for all* $t \geq 0$, *and*

c) $\{X_{ij}\}$ are independent of $\{K_i, T_{i1}, \ldots, T_{iK_i}\}$ for $j=1,2,\ldots,K_i$ and $i=1,2,\ldots$

Then

$$\lim_{i\to\infty}\left[\sum_{j=1}^{K_i} X_{ij}(T_{ij})/K_i\right] = 0 \quad \text{a.s.} \tag{4.1}$$

LEMMA (Athreya and Kaplan). Let $\{X_{i1}, \ldots, X_{iK_i}\}$, $i=1,2,\ldots,$ be an array of random variables such that,

a) for each i, $X_{i1}, \ldots, X_{iK_i}$ are independent,

b) $E\, X_{ij} = 0 \quad j=1,2,\ldots,K_i, \quad i=1,2,\ldots,$

c) $\sup_{i,j} P(|X_{ij}| > x) < C[1-Q(x)]$, for all large x, where C is an absolute constant and Q is an integrable cumulative distribution function. Assume further that $\liminf_{i\to\infty}[K_{i+1}/K_i] > 1$. Then for every $\mu > 0$

$$\sum_{i=1}^{\infty} P\left(\left|\frac{1}{K_i}\cdot\sum_{j=1}^{K_i} X_{ij}\right| > \mu\right) < \infty. \tag{4.2}$$

PROOF (Proposition 4.1). Without loss of generality assume

$$P(K_{i+1}/K_i > C) = 1.$$

In this case the upper bound for the sum in equation (4.2) found in [2] is a function only of C. By a conditioning argument, the same upper bound holds for $\sum_{i=1}^{\infty} P\left(\left|\sum_{j=1}^{K_i} X_{ij}(T_{ij})/K_i\right| > \mu\right)$ where $X_{ij}(T_{ij})$ are as in equation (4.1). Convergence is now obvious. □

5. A THEOREM BASED ON AN IMBEDDING AND PROOFS OF LEMMAS 3.2 AND 3.3.

By considering a certain type of process imbedded within the Bellman-Harris process, the following theorem is proved, which in turn implies lemma 3.3. A corollary combined with the results of Section 4 proves lemma 3.2.

THEOREM 5.1. *If* $1 < m < \infty$, *then for some* C_1, C_2, *both positive and finite*,

(5.1) $$\lim_{t\to\infty} \inf[Z(C_1,t)/Z(t)] > C_2 \quad \text{a.s.}$$

Before proving the above theorem, we shall need the concept of what we call a short term branching process. Without loss of generality assume $G(t) < 1$ for $t < \infty$. (The theorem is trivially true if not.) As usual $p_0 = 0$.

Fix $K > 0$ such that

(5.2) $$G(K)\cdot m > 1, \text{ and}$$

(5.3) $$G(2K)-G(K) > 0.$$

With this particular K, for a particle born at time 0, define

(5.4) $\bar{Z}(\tau) = \{$number of particles alive at time τ, descended from the original particle, such that 1) each has a lifespan $\leq K$ in length, and 2) each of its ancestors, up to and including the original particle, had lifespan $\leq K\}$.

The above definition implies that $\bar{Z}(\tau)$ is itself a Bellman-Harris process (on set ancestor lives $\leq K$) with lifetime distribution

(5.5) $$\bar{G}(x) = G(x)/G(K), \quad 0 \leq x \leq K,$$

and offspring distribution

(5.6) $$\bar{p}_n = \sum_{m=n}^{\infty} \binom{m}{n}(G(K))^n(1=G(K))^{m-n}p_m, \quad n=0,1,\ldots$$

Evidently

(5.7) $$\Sigma n\bar{p}_n = G(K)\Sigma np_n = G(K)m > 1,$$

so that $0 < \beta < 1$, where

(5.8) $$\beta \equiv p(\bar{Z}(\tau) > 0, \text{ for all } \tau > 0).$$

Another concept we need is that of particles of order n at time t. Recursively, define them as follows.

The particles of order one at time t are those particles ever born up to time t such that 1) their life-length is >K, 2) no ancestor (born at or after t=0) has life-length >K.

We also define

(5.9) $Z_1(t)$ = {number of particles of order one born by time t},

(5.10) $$S_{1i}(\tau) = \sum_{j=1}^{J_i} \bar{Z}_{1i}^j(\tau)$$

where $\bar{Z}_{1i}^j(\tau)$ is the short term branching process at time τ after the birth of the jth of J_i progeny of the ith of $Z_1(t)$ particles of order one born by t, as well as,

(5.11) Y_{1i} = life-length of ith particle of order one.

We add that t is suppressed in some expressions for notational convenience, and $i=1,2,\ldots,Z_1(t)$.

Assume that particles of order n at time t have been defined and that this set is not null. Define particles of order n+1 at time t to be those particles born up to time t such that 1) each has lifespan >K, and 2) each has an ancestor with lifespan >K and the nearest such ancestor is a particle of order n.

Clearly one may define $Z_{n+1}(t)$, $S_{n+1,i}(\tau)$, $\bar{Z}^j_{n+1,i}(\tau)$, and $Y_{n+1,i}$ for particles of order n+1 just as they were defined for particles of order one. Also note that if there are no particles of order n at time t there are none of higher order at time t.

Also define

(5.12) $$X_{ni} = I_{[Y_{ni} \in (K,2K]]} \cdot I_{[\underline{\lim}\, S_{ni}(\tau) > 0]}$$

for $i=1,2,\ldots, Z_n(t)$, $n=1,2,\ldots, [t/K]+1$. We note that $Z_n(t)$ is void if $n > [t/K]+1$. For notational simplicity let

(5.13) $$n_t = [t/K]+1 \quad \text{and}$$

(5.14) $$N_t = \sum_{n=1}^{n_t} Z_n(t).$$

The purpose of the previous definitions was to define the random variables $\{X_{ni}\}$ and $\{Z_n(t)\}$. The following lemmas concern distributional aspects of $\Sigma\Sigma X_{ni}$. These results will be crucial to the proof of theorem 5.1. Before proceeding to the lemmas, more notation is needed. Let

(5.15) $$p(x_{11},\ldots,x_{n_t m_{n_t}}, m_1,\ldots,m_{n_t})$$

$$= P(X_{11}=x_{11},\ldots,X_{n_t m_{n_t}}=x_{n_t m_{n_t}}, Z_1(t)=m_1,\ldots,Z_{n_t}(t)=m_{n_t})$$

We note that some x are necesarily 0 if $m_i = 0$ for some $i \leq n_t$. We also use p(,...,) for marginals also. The following is trivial.

$$(5.16)\quad p(x_{11},\ldots,x_{n_t m_{n_t}},m_1,\ldots,m_{n_t}) = p(x_{11},\ldots,x_{1m_1}|m_1)$$

$$\cdot p(x_{21},\ldots,x_{2m_2}|m_2,x_{11},\ldots,x_{1m_1},m_1)\cdot\ldots$$

$$\cdot p(x_{n_t 1},\ldots,x_{n_t m_{n_t}}|m_{n_t},x_{n_t-1,1},\ldots,x_{1m1},m_1)$$

$$\cdot p(m_1,\ldots,m_{n_t}).$$

<u>LEMMA 5.1</u>. <u>On</u> $\{Z_i = m_i\}$ <u>for</u> $m_i \geq 1$,

(5.17)

$$p(x_{i1},\ldots,x_{im_i}|m_i,x_{i-1,1},\ldots,x_{11},m_1) = p^{\sum_{j=1}^{m_i} x_{ij}}(1-p)^{m_i-\sum_{j=1}^{m_i} x_{ij}}.$$

In (5.17), p has the value

$$(5.18)\qquad p = [(G(2K)-G(K))/(1-G(K))]\cdot[1-E(1-\beta)^J]$$

where J is the random number of offspring at split and β is as in (5.8).

<u>PROOF</u> (lemma 5.1) Clear $X_{i1},\ldots,X_{im_i}$ are Bernoulli (p) random variables for p as in (5.18). Using definition 2.3 from [3], one may defind particles by sequences, $<i_1,\ldots,i_k>$, for example, is a particle of generation k+1. Now if we condition on event that $\{\alpha_1,\ldots,\alpha_{m_i}$ (α's represent sequences) are all and only particles of order i at time t, and $x_{i-1,1},\ldots,x_{11},m_1$ also occur$\}$ then conditioned on this set

$$p(x_{i1},\ldots,x_{im_i}) = p^{\sum_{j=1}^{m_i} x_{ij}}(1-p)^{m_i-\sum_{j=1}^{m_i} x_{ij}},$$

since the future is conditionally independent of the past. Unconditioning on the particular α's gives (5.17). □

LEMMA 5.2.

(5.19)

$$p(x_{11},\ldots,x_{n_t,m_{n_t}}|m_1,\ldots,m_{n_t}) = p^{\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} x_{ni}}(1-p)^{N_t-\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} x_{ni}} .$$

PROOF. lemma 5.1 and equation (5.16). By Chebyshev, the following holds:

$$(5.20)\quad p(|[(\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni})/N_t]-p| > \varepsilon|m_1,m_2,\ldots,m_{n_t}) \le \Sigma p(1-p)]/\varepsilon^2 N_t.$$

Since N_t is really an overestimate of the number of particles of age >K at time t+K, one may argue, by means of results of Jagers [4], that for any $\delta > 0$, there exists M_δ finite and $C > 1$ such that

$$(5.21)\qquad N_{m\cdot\delta} > C^m \quad \text{a.s.}$$

if $m \ge M_\delta$. Consequently, we obtain

LEMMA 5.3. For $\delta > 0$,

$$(5.22)\qquad P([\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni}]/N_t < p/2 \quad \text{i.o.}, t=m\delta) = 0.$$

PROOF. Let $\varepsilon = p/2$. Since $M_\delta < \infty$ a.s. using (5.20)

(5.23)

$$\sum_{m=M_\delta}^{\infty} P([\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni}]/N_t < p/2,\ t=m\delta) \le \sum_{m=1}^{\infty}[p(1-p)]/[(\varepsilon/2)^2C^m] < \infty.$$

Borel-Contelli gives the rest. □

Now the tools are available to prove Theorem 5.1.

PROOF. (theorem 5.1) It is easy to see that equation (5.1) is equivalent to showing, for some C_1', C_2', that

$$(5.24)\qquad \lim_{t\to\infty} \inf[Z(C_1',t)/[Z(t)-Z(C_1',t)]] > C_2' > 0 \quad \text{a.s.}$$

Now note that by definition of X_{ni} (equation (5.12)),

(5.25) $$\sum_{n=1}^{n_t} \sum_{i=1}^{Z_n(t)} X_{ni} \leq Z(2K,t+2K)$$

and

(5.26) $$N_t \geq Z(t+2K)-Z(2K,t+2K).$$

This yields the lower bound

(5.27) $$[\sum_{n=1}^{n_t} \sum_{i=1}^{Z_n(t)} X_{ni}]/N_t \leq Z(2K,t+2K)/[Z(t+2K)-Z(2K,t+2K)].$$

Directly from lemma 5.3 we may infer that, if $\delta > 0$

(5.28) $$P(Z(2K,t+2K)/[Z(t+2K)-Z(2K,t+2K)] < p/2 \quad \text{i.o.} \quad t=m\cdot\delta) = 0.$$

It is not difficult to show that

(5.29)

$$\liminf_{t\to\infty} Z(4K,t)/[Z(t)-Z(4K,t)] \geq \liminf_{m\to\infty} Z(2K,m\cdot\delta)/[Z(m\delta)-Z(2K,m\delta)] \quad \text{a.s.}$$

Therefore with $C_1' = 4K$, $C_2' = p/2$ we have (5.24). □

PROOF (lemma 3.3). If we let

(5.30) $$a = \inf_{x\in[0,C_1]} V(x) > 0$$

then

(5.31) $$\liminf_{t\to\infty} V_t/Z(t) \geq a \liminf_{t\to\infty} Z(C_1,t)/Z(t)$$

(5.32) $$\geq a\cdot C_2 \quad \text{a.s.}$$ □

COROLLARY 5.1. For some $K' > 0$, there is a constant $C' > 1$, such that

(5.33) $$\liminf_{t\to\infty} Z(t+K')/Z(t) > C' \quad \text{a.s.}$$

PROOF. Rather than provide a tedious proof, we remark that if K' is chosen to satisfy

(5.34) $$\inf_{x \in [0,C_1]} G_x(K') > \eta$$

for some $\eta > 0$, then an asymptotic bound from below may be found for the proportion of $Z(t)$ particles which split in $(t,t+K']$. This, and the fact that $m > 1$, can be used to find a suitable C'.

PROOF (lemma 3.2). Let $\delta > 0$. Let $K' = \ell \cdot \delta$ for some ℓ (can always increase K' and still satisfy 5.33). We only need to show

(5.35) $$Y_{n(\ell \cdot \delta)+i\delta} \to 0 \quad \text{a.s.}$$

for $i=1,2,\ldots,\ell$, as $n \to \infty$. If we let $K_n = Z(n\ (\ell \cdot \delta)+i\delta)$, $n=1,2,\ldots,$

(5.36) T_{nj} = [s - (time from $(n\ell\delta+i\delta)$ to split of jth particle of $Z(n\ell\delta+i\delta)$] $\vee$ 0,

and

(5.37) $X_{nj}(s)$ = (number of particles of age $\leq x$, s time after the split of jth particle of $Z(n\ell\delta+i\delta)$),

then proposition 4.1 implies (5.35) for each $i=1,2,\ldots,\ell \cdot a$. This gives lemma 3.2. □

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