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COMPARISON OF TWO-WAY CONTINGENCY TABLES

by

Jan F. Bjørnstad
Purdue University

Department of Statistics
Division of Mathematical Sciences
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SUMMARY

Comparison of two-way contingency tables using measures of association is considered. Multiple comparison procedures for both independent and dependent tables formed from larger multi-dimensional tables are proposed.

Key Words and Phrases: Contingency table, measure of association, multiple comparisons, asymptotic theory.

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1. Introduction

This paper deals with comparison of measures of association from two-way contingency tables that may be dependent. This allows for two-way tables that are faces of a multi-dimensional table or a generalized multi-dimensional table. A generalized multi-dimensional table is such that given any pair of two-way tables formed from the table, there is a set of observations for which there are responses for both tables, but there can also exist some observations giving results in only one of the two-way tables. The situation will be presented precisely in Section 4. We are given a set of R (generalized) multi-dimensional tables from independent samples. Typically the R tables consist of similar responses taken at different time periods. We are interested in the association between pairs of variables from the multi-dimensional tables. Let L denote the set of two-way tables corresponding to the pairs we are interested in. A two-way table from a larger multi-dimensional table is said to be permissible if the two-factor interaction terms in the two-way table's log-linear model is equal to the corresponding two-factor terms in the log-linear model of the whole multi-dimensional table. Let L_r^0 denote the set of permissible two-way tables from the multi-dimensional table r; r = 1,...,R. We will assume that

$$L \subset L^0 = \bigcup_{r=1}^{R} L_r^0 \qquad (1.1)$$

in order to insure that the interaction in the relevant two-way tables measures the true interaction between the corresponding variables. Let K be the number of two-way tables in L. We present a general theory for multiple comparisons of measures of association from these K two-way tables. The condition $L \subset L^0$ implies that the comparisons made are meaningful, but the mathematical theory itself is independent of (1.1). If all the multi-dimensional tables are two-way tables, then R = K, all the two-way tables are independent and (1.1) is of course satisfied.

For a presentation of measures of association, we refer to Goodman and Kruskal (1954) and Bjørnstad (1975). Also Altham (1970) proposes meaningful measures, based on cross-product ratios. In addition to measures presented in these papers, Srikantan (1970) proposes some measures based on canonical correlations. However, these do not seem to be appropriate for comparing tables. Bishop et al. (1975), Chapter 2, give necessary and sufficient conditions for a two-way table to be permissible. This will be discussed later.

In Section 2 we present the asymptotic theory for a given two-way table. In Sections 3 and 4 we consider the problem of ordering the tables according to the measures of association, assuming (1.1) is true. Section 3 describes a multinomial model that covers the case where a generalized four-way or three-way table is given, and presents the main asymptotic result. In Section 4 we first show how the general situation with K permissible two-way tables formed from a set of generalized multi-dimensional tables can be described, and then pairwise comparisons of the tables are considered. Section 5 deals with multiple comparison of independent two-way tables.

2. Asymptotic Theory for Measures of Association in a Two-way Table

Let n be the total number of observations in the table, X_{ij} the frequency in cell (i,j), $i = 1,\dots,I$; $j = 1,\dots,J$, and $q_{ij} = X_{ij}/n$. Let p_{ij} (assumed positive) denote the cell-probabilities. Let $q_i = (q_{i1},\dots,q_{iJ})$, $p_i = (p_{i1},\dots,p_{iJ})$ $q = (q_1,\dots,q_I)$, $p = (p_1,\dots,p_I)$. Let $d(p)$ be a measure of association with continuous partial derivatives. (Three measures suggested by Goodman and Kruskal, λ, λ_b, λ_r do not satisfy this. A similar theory for these measures has been developed in Goodman and Kruskal (1963, 1972)). Let now $n_{i.} = \sum_{j=1}^{J} X_{ij}$, $i = 1,\dots,I$.

The following two sampling methods will be considered:

(i) Multinomial sampling over the entire two-way cross classification.

(ii) Independent multinomial sampling in the rows so that the rows marginals $n_{1.},\dots,n_{I.}$, are fixed. (This includes, of course, the case where the column marginals are fixed instead.)

For comparison of possibly dependent tables we will only consider sampling method (i), i.e. multinomial sampling over the whole table for each table. (This seems to be the most frequent case when the tables are dependent.) For comparison of independent tables the theory covers both sampling methods.

2.1 Multinomial Sampling over the Whole Two-way Table

Let

$$\sigma_d^2 = \sigma_d^2(p) = \sum_{i,j} p_{ij}(d_{ij}-d^*)^2 \qquad (2.1)$$

where $d_{ij} = \frac{\partial d}{\partial p_{ij}}$, $d^* = \sum_{i,j} p_{ij} d_{ij}$. Then from Bjørnstad (1975) and Goodman and Kruskal (1972) we have:

If $\sigma_d^2 > 0$ then

$$\sqrt{n}(d(q)-d(p)) \xrightarrow{\mathcal{D}} N(0,\sigma_d^2)$$

and

$$\frac{\sqrt{n}(d(q)-d(p))}{\hat{\sigma}_d} \xrightarrow{\mathcal{D}} N(0,1) \tag{2.2}$$

where $\hat{\sigma}_d^2 = \sigma_d^2(q)$. Here $X_n \xrightarrow{\mathcal{D}} N(0,1)$ denotes that the distribution of X_n converges to $N(0,1)$.

2.2 Independent Multinomial Sampling in the Rows

Let $X_i = (X_{i1},\ldots,X_{iJ})$, $\tilde{p}_{ij} = p_{ij}/p_{i\cdot}$, $\tilde{p}_i = (\tilde{p}_{i1},\ldots,\tilde{p}_{iJ})$. Here $p_{i\cdot} = \sum_j p_{ij}$. Then $X_1,\ldots X_I$ are independent. $L(X_i)$ is multinomial $(n_{i\cdot},\tilde{p}_i)$. Let

$$\Sigma_i = D(\tilde{p}_i) - \tilde{p}_i'\tilde{p}_i, \qquad D(\tilde{p}_i) = \begin{bmatrix} \tilde{p}_{i1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \tilde{p}_{iJ} \end{bmatrix}$$

Then $\sqrt{n_{i\cdot}}(q_i/m_i-p_i) \xrightarrow{\mathcal{D}} N(0,\Sigma_i)$, and we get the following asymptotic distribution theory for the multinomial estimates.

LEMMA 1. *Let* $m_i = n_{i\cdot}/n$. *Assume there exists* $w_i > 0$ *such that* $\sqrt{n}(m_i-w_i) \to 0$ for $i = 1,\ldots I$. *Then*

a) $$\sqrt{n}(q-(w_1\tilde{p}_1, w_2\tilde{p}_2, \ldots, w_I\tilde{p}_I)) \xrightarrow{\mathcal{D}} N(0,\Sigma)$$

b) $$\sqrt{n}(\tilde{q}-(w_1\tilde{p}_1, \ldots w_I\tilde{p}_I)) \xrightarrow{\mathcal{D}} N(0,\Sigma)$$

where

$$\tilde{q} = (w_1 m_1^{-1} q_1, \ldots, w_I m_I^{-1} q_I)$$

and

$$\Sigma = \begin{bmatrix} w_1\Sigma_1 & & \bigcirc \\ & w_2\Sigma_2 & \\ & & \ddots \\ \bigcirc & & & w_I\Sigma_I \end{bmatrix}$$

From Lemma 1 we get the following result which generalizes somewhat the results for this case by Goodman and Kruskal (1972).

<u>THEOREM 1</u>. *Assume there exists* $w_i > 0$ *such that* $\sqrt{n}(m_i - w_i) \to 0$ *for* $i = 1,\ldots,I$. *Then if* $\tau_d^2 > 0$

a) $$\sqrt{n}[d(p_{1.}w_1^{-1}q_1, \ldots, p_{I.}w_I^{-1}q_I) - d(p)] \xrightarrow{\mathcal{D}} N(0,\tau_d^2)$$

b) $$\sqrt{n}[d(p_{1.}m_1^{-1}q_1, \ldots, p_{I.}m_I^{-1}q_I) - d(p)] \xrightarrow{\mathcal{D}} N(0,\tau_d^2)$$

where

$$\tau_d^2 = \sum_{i=1}^{I} \frac{p_{i.}^2}{w_i} \sum_{j=1}^{J} \tilde{p}_{ij}(d_{ij}-d_i^*)^2 = \sum_{i=1}^{I} \frac{p_{i.}}{w_i} \sum_{j=1}^{J} p_{ij}(d_{ij}-d_i^*)^2$$

and

$$d_i^* = \sum_{j=1}^{J} \tilde{p}_{ij} d_{ij}.$$

Proof. The results are proved by applying the δ-method to Lemma 1 (see for example Rao (1965) or Bishop et al. (1975)). Let

$$f(x_1,\ldots,x_I) = d(\frac{p_{1.}}{w_1}x_1,\ldots,\frac{p_{I.}}{w_I}x_I).$$

Let $\theta = (w_1\tilde{p}_1,\ldots,w_I\tilde{p}_I)$. Now f has total differential at θ given by $\overset{\circ}{f}(\theta) = [\frac{\partial f}{\partial x_{ij}}\Big|_{x=\theta}]^{1\times IJ}$, $\frac{\partial f}{\partial x_{ij}}\Big|_{x=\theta} = \frac{\partial d}{\partial y_{ij}}\Big|_{y=p} \cdot \frac{p_{i.}}{w_i} = \frac{p_{i.}}{w_i}d_{ij}(p)$. It follows that

$$\sqrt{n}[f(q)-f(\theta)] \xrightarrow{\mathcal{D}} Y\overset{\circ}{f}'(\theta) \quad ; \quad Y \sim N(0,\Sigma)$$

$$\sqrt{n}[f(\tilde{q})-f(\theta)] \xrightarrow{\mathcal{D}} Y\overset{\circ}{f}'(\theta).$$

It is readily seen that $\tau_d^2 = \overset{\circ}{f}(\theta)\Sigma\overset{\circ}{f}'(\theta)$, $f(\theta) = d(p)$, $f(q) = d(p_{1.}w_1^{-1}q_1,\ldots,p_{I.}w_I^{-1}q_I)$, $f(\tilde{q}) = d(p_1m_1^{-1}q_1,\ldots,p_Im_I^{-1}q_I)$, and the results follow. Q.E.D.

Remarks

1) Goodman and Kruskal (1972) assume that w_i, $p_{i.}$ are known and that $n_i = n.i.(nw_i)$ (nearest integer to nw_i). This implies that $\sqrt{n}(m_i-w_i) \to 0$, and Theorem 1 can be applied.

2) τ_d^2 is the same asymptotic variance as given in Goodman and Kruskal (1972), Section 3.

3) The theorem holds even if we cannot write d as a function of $\tilde{p}$ only.

To use this result to estimate or to test on $d(p)$ we only have to assume that the $p_{i.}$ are known. (Usually this means that $p_{i.} = N_i/N$, where N_i is the size of population i and N is the size of the whole population.) As is seen in the next result, we do not have to assume that the w_i are known as Goodman and Kruskal (1972) do.

If $p_{i\cdot}$ is known and $\sqrt{n}(m_i - w_i) \to 0$, $\forall i$, then a consistent estimator of τ_d^2 is given by

$$\hat{\tau}_d^2 = \sum_{i=1}^{I} p_{i\cdot}^2 m_i^{-2} \sum_{j=1}^{J} q_{ij}(\tilde{d}_{ij} - \tilde{d}_i^*)^2$$

where

$$\tilde{d}_{ij} = d_{ij}(p_{1\cdot}m_1^{-1}q_1, \ldots, p_{I\cdot}m_I^{-1}q_I); \quad \tilde{d}_i^* = \sum_{j=1}^{J} q_{ij} m_i^{-1} \tilde{d}_{ij}.$$

If w_i is known, different from $p_{i\cdot}$, then it is better to use w_i instead of m_i in $\hat{\tau}_d$.

For the case of proportional sampling, i.e. $p_{i\cdot} = w_i$ is known for all i, it follows from Theorem 1 that if $\tau^2 > 0$ then

$\sqrt{n}(d(q) - d(p)) \xrightarrow{\mathcal{D}} N(0, \tau^2)$, where

$$\tau^2 = \tau^2(p) = \sum_{i,j} p_{ij}(d_{ij} - d_i^*)^2 \qquad (2.3)$$

A consistent estimator of τ^2 is $\hat{\tau}^2 = \tau^2(q)$.

3. A Multinomial Model for Two Two-way Contingency Tables Under Sampling Method (i) for Each Table

The situation can generally be described as a multinomial model with two possibly dependent sequences of trials as follows (where a sequence of trials corresponds to a two-way table). In sequence j, r_j events (the events cannot be the same in the two sequences) can occur with probabilities p_{ij} ; $i = 1,\ldots,r_j$; $j = 1, 2$. $\sum_i p_{ij} = 1$. Let $r = r_1 + r_2$. All p_{ij} are assumed to be positive. Let k_n be the total number of independent trials, and let n_j be the total number of trials in sequence j. Let $n = n_1 + n_2$. We always have $n \geq k_n$. If $n > k_n$ then some of the trials give observations in both sequences.

If each trial gives observations in both sequences then $n_1 = n_2 = k_n$, $n = 2k_n$ and the situation corresponds to a usual four- or three-way table. The other extreme is when each trial gives observations only in one sequence. Then $k_n = n$ and the situation corresponds to two two-way tables from independent samples. Let now M denote the set of trials that gives observations in both sequences and let $m = \#(M)$. Then $k_n = n - m$. For the trials in M, μ_{ij} is the probability of event i in sequence 1 and event j in sequence 2, $i = 1,\ldots,r_1$; $j = 1,\ldots,r_2$.

Let now N_{ij} be the total number of observations in cell i of sequence j. Then $n_j = \sum_{i=1}^{r_j} N_{ij}$. Denote the relative frequencies by $q_{ij} = N_{ij}/n_j$. Let $t_j = n_j/n$, $t = m/n$. We will assume that there exists $\pi_j > 0$, $\pi \geq 0$ such that

$$\sqrt{n}(t_j - \pi_j) \to 0, \quad j = 1, 2 \quad \text{and} \quad t \to \pi \quad \text{as} \quad n \to \infty \tag{3.1}$$

$(\pi_1 + \pi_2 = 1)$. Let

$$p_j = (p_{1j},\ldots,p_{r_j,j}) \;, \quad j = 1,2$$

$$q_j = (q_{1j},\ldots,q_{r_j,j}) \;, \quad j = 1,2$$

$$q = (q_1,q_2) \;, \qquad p = (p_1,p_2).$$

Σ_j is the covariance matrix of $\sqrt{n_j}q_j$, that is,

$$\Sigma_j = D(p_j) - p_j'p_j \quad \text{where} \quad D(p_j) = \begin{bmatrix} p_{1j} & & \\ & \ddots & \\ & & p_{r_j,j} \end{bmatrix}$$

Further let $\Lambda = \{\beta_{ij}\}^{r_1 \times r_2}$ where $\beta_{ij} = \mu_{ij} - p_{i1}p_{j2}$. We see that $n \operatorname{cov}(q_{i1}, q_{j2}) = \beta_{ij} t/(t_1 t_2)$.

The asymptotic distribution of $\sqrt{n}[(\pi_1 q_1, \pi_2 q_2) - (\pi_1 p_1, \pi_2 p_2)]$ can now be stated:

<u>LEMMA 2.</u> *Assume* (3.1) *holds. Then*

$$\sqrt{n}[(\pi_1 q_1, \pi_2 q_2) - (\pi_1 p_1, \pi_2 p_2)] \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

where

$$\Sigma = \begin{bmatrix} \pi_1 \Sigma_1 & \pi\Lambda \\ \pi\Lambda' & \pi_2 \Sigma_2 \end{bmatrix}^{r \times r} .$$

<u>Proof</u>. Assume first $\pi > 0$. Consider first the trials from M. Define $M_j = (M_{1j}, \ldots, M_{r_j,j})$, $j = 1,2$, $M = (M_1, M_2)$. Here M_{ij} is the frequency cell i of sequence j from the set M. Using the multivariate central limit theorem we find that

$$\sqrt{m}(M/m - p) \xrightarrow{\mathcal{D}} N(0, \Gamma)$$

where

$$\Gamma^{r \times r} = \begin{bmatrix} \Sigma_1 & \Lambda \\ \Lambda' & \Sigma_2 \end{bmatrix} .$$

Let now $N_j = (N_{1j}, \ldots, N_{r_j,j})$ and $L_j = N_j - M_j$, $j = 1,2$.

Assume now $\pi_j > \pi$ for $j = 1,2$. Let $n_j' = n_j - m$. Then

$$X_j^n = \sqrt{n_j'/n}\, \sqrt{n_j'}(L_j/n_j' - p_j) \xrightarrow{\mathcal{D}} N(0, (\pi_j - \pi)\Sigma_j).$$

Let $Y_j^n = \sqrt{n} t_j m n_j^{-1}(M_j/m - p_j)$. X_1^n and X_2^n are independent and (X_1^n, X_2^n) and (Y_1^n, Y_2^n) are independent. Now

$$\sqrt{n}[(t_1q_1,t_2q_2) - (t_1p_1,t_2p_2)] = Z_1^n + Z_2^n$$

where $Z_1^n = (X_1^n,X_2^n)$, $Z_2^n = (Y_1^n,Y_2^n)$. The result now follows from (3.1).

If one or both of the π_j are equal to π, then one or both of $X_j^n \xrightarrow{\mathcal{D}} 0$ and the result follows. If $\pi = 0$ then $Z_2^n \xrightarrow{\mathcal{D}} 0$ and the result follows. Q.E.D.

<u>Remark</u>. The condition in Lemma 2 is always satisfied if we consider t_j as constants as $n \to \infty$, or more precisely $n_j = n.i.(n\pi_j)$ for all n.

Suppose now that f is a function in r variables with continuous partial derivatives. We are interested in the asymptotic distribution of $f(q)$. Let M_{ij}° be the number of observations from M that falls in cell i of sequence 1 and cell j of sequence 2 and let $m_{ij} = M_{ij}^\circ/m$.

We need the following quantities.

$$f_{i1}(p) = \left.\frac{\partial f}{\partial x_i}\right|_{x=p} \text{ for } i \le r_1,\ f_{i2}(p) = \left.\frac{\partial f}{\partial x_{i+r_1}}\right|_{x=p} \text{ for } i \le r_2,\ \hat{f}_{ij} = f_{ij}(q),$$

$$\bar{f}p_1 = \sum_{i=1}^{r_1} p_{i1}f_{i1}(p),\ \bar{f}p_2 = \sum_{i=1}^{r_2} p_{i2}f_{i2}(p),\ \bar{f}q_1 = \sum_{i=1}^{r_1} q_{i1}\hat{f}_{i1},\ \bar{f}q_2 = \sum_{i=1}^{r_2} q_{i2}\hat{f}_{i2}.$$

The main result is as follows.

<u>THEOREM 2</u>. *Assume (3.1) holds. Let*

$$\sigma_f^2 = \pi_1^{-1}\sum_{i=1}^{r_1} p_{i1}(f_{i1}(p)-\bar{f}p_1)^2 + \pi_2^{-1}\sum_{i=1}^{r_2} p_{i2}(\bar{f}_{i2}(p)-fp_2)^2$$

$$+ (2\pi/\pi_1\pi_2)\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\beta_{ij}f_{i1}(p)f_{j2}(p).$$

Let further $\hat{\sigma}_f^2$ *be the following estimator of* σ_f^2.

$$\hat{\sigma}_f^2 = t_1^{-1}\sum_{i=1}^{r_1} q_{i1}(\hat{f}_{i1}-\bar{f}q_1)^2 + t_2^{-1}\sum_{i=1}^{r_2} q_{i2}(\hat{f}_{i2}-\bar{f}q_2)^2$$

$$+ (2t/t_1t_2)\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}(m_{ij}-q_{i1}q_{j2})\hat{f}_{i1}\hat{f}_{j2}.$$

If $\sigma_f^2 > 0$ *then*

(a) $$\sqrt{n}(f(q)-f(p)) \xrightarrow{\mathcal{D}} N(0,\sigma_f^2)$$

(b) $$\sqrt{n}(f(q)-f(p))/\hat{\sigma}_f \xrightarrow{\mathcal{D}} N(0,1)$$

<u>Proof</u>. The results are proved by applying the δ-method to Lemma 2. Let $g(x_1,\ldots,x_r) = f(x_1/\pi_1,\ldots,x_{r_1}/\pi_1, x_{r_1+1}/\pi_2,\ldots,x_r/\pi_2)$. Let $\mathring{g}(\theta) = (\frac{\partial g}{\partial x_i}\Big|_{x=\theta})^{1\times r}$; $\theta = (\pi_1 p_1, \pi_2 p_2)$. Then

$$\sqrt{n}(g(\pi_1 q_1,\pi_2 p_2) - g(\theta)) \xrightarrow{\mathcal{D}} N(0,\sigma^2) \quad \text{if} \quad \sigma^2 > 0$$

where $\sigma^2 = \mathring{g}(\theta)\Sigma\mathring{g}'(\theta)$. Now $g(\pi_1 q_1, \pi_2 q_2) = f(q)$, $g(\theta) = f(p)$ and it is readily seen that $\sigma^2 = \sigma_f^2$. (b) follows from the fact that $\hat{\sigma}_f^2$ is a consistent estimator of σ_f^2. Q.E.D.

Let $\mu^* = \sum_i \sum_j \mu_{ij}(f_{i1} + f_{j2})$. When the two sequences of trials correspond to faces from a three-or four-way table then

$$\sigma_f^2 = 2 \sum_i \sum_j \mu_{ij}(f_{i1} + f_{j2} - \mu^*)^2$$

4. Pairwise Multiple Comparison of Measures of Association

4.1 The General Situation

The situation with K two-way tables from a set of generalized multi-dimensional tables can be formulated as follows. Let the number of row- and column-classes in table number k be respectively I_k and J_k for $k = 1,\ldots,K$. Let $r_k = I_k \cdot J_k$. Let p_{ijk} denote the cell-probabilities in table k. We assume $p_{ijk} > 0$ and we have

$$\sum_{i=1}^{I_k} \sum_{j=1}^{J_k} p_{ijk} = 1 \ ; \quad k = 1,\ldots,K.$$

Let n_k be the number of observations in table k and let q_{ijk} denote the relative frequency in cell (i,j) of table k. Put $n = \sum_{k=1}^{K} n_k$, $t_k = n_k/n$. For each pair (k,ℓ) of tables we let $M_{k\ell}$ be the set of trials that give observations in both table k and ℓ. Further $n_{k\ell} = \#(M_{k\ell})$, $t_{k\ell} = n_{k\ell}/n$. For the trials in M_{rt}, $\mu_{ijh\ell}^{rt}$ is the probability of falling in cell (i,j) of table r and cell (h,ℓ) of table t. Throughout we will assume there exists constants $\pi_k > 0$, $\pi_{k\ell} \geq 0$ such that

$$\sqrt{n}(t_k - \pi_k) \to 0, \quad k = 1,\ldots,K \quad \text{and} \quad t_{k\ell} \to \pi_{k\ell} \quad \text{as} \quad n \to \infty \qquad (4.1)$$

Let $M^{rt}_{ijh\ell}$ be the total frequency from set M_{rt} that falls in cell (i,j) of table r and cell (h,ℓ) of table t. The relative frequencies are denoted by

$$m^{rt}_{ijh\ell} = M^{rt}_{ijh\ell}/n_{rt}.$$

We use the following notation:

$$p_k = (p_{11k},\ldots,p_{I_kJ_k,k}) \;;\quad k = 1,\ldots,K$$

$$q_k = (q_{11k},\ldots,q_{I_kJ_k,k}) \;;\quad k = 1,\ldots,K$$

$$p = (p_1,\ldots,p_K)$$

$$q = (q_1,\ldots,q_K)$$

$$m^{rt} = (m^{rt}_{11,11},\ldots,m^{rt}_{I_rJ_r,I_tJ_t})$$

$$m = \{m^{rt} | r=1,\ldots,K,\ t = 1,\ldots,K;\ r < t\}$$

$$\mu^{rt} = (\mu^{rt}_{11,11},\ldots,\mu^{rt}_{I_rJ_r,I_tJ_t})$$

$$\mu = \{\mu^{rt} | r=1,\ldots,K,\ t = 1,\ldots,K;\ r < t\}$$

Let d be the chosen measure of association with continuous partial derivatives as function of the cell-probabilities; d_k is the measure d in table k. Then d_k is a function of r_k variables with continuous partial derivatives, i.e. $d_k = d_k(p_k)$. Let $\hat{d}_k = d_k(q_k)$. The asymptotic variance of $\sqrt{n_k}\,\hat{d}_k$ is from (2.1).

$$\sigma^2_k = \sum_{i=1}^{I_k}\sum_{j=1}^{J_k} p_{ijk}(d_{ijk}-d^*_k)^2 \;;\quad k = 1,\ldots,K$$

where $d_{ijk} = \dfrac{\partial d_k}{\partial p_{ijk}}$, $d^*_k = \sum_{i,j} p_{ijk}d_{ijk}$. A consistent estimator of $\hat{\sigma}^2_k$ is $\hat{\sigma}^2_k = \hat{\sigma}^2_k(q_k)$. Let

$$\rho_{rt} = \sum_{i=1}^{I_r} \sum_{j=1}^{J_r} \sum_{h=1}^{I_t} \sum_{\ell=1}^{J_t} \mu_{ijh\ell}^{rt} d_{ijr} d_{h\ell t} - d_r^* \cdot d_t^*.$$

It will be seen that $(\pi_{rt}/\sqrt{\pi_r \cdot \pi_t})\rho_{rt}$ can be considered as the asymptotic covariance of $(\sqrt{n_r}\,\hat{d}_r, \sqrt{n_t}\,\hat{d}_t)$. A consistent estimator of ρ_{rt} is

$$\hat{\rho}_{rt} = \sum_{i=1}^{I_r} \sum_{j=1}^{J_r} \sum_{h=1}^{I_t} \sum_{\ell=1}^{J_t} m_{ijh\ell}^{rt} \hat{d}_{ijr} \hat{d}_{h\ell t} - \hat{d}_r^* \hat{d}_t^*.$$

We will first consider the case $K = 2$; i.e. comparison of the two measures d_1 and d_2, under assumption (1.1), so that the interaction terms in the two tables are meaningful.

4.2 Comparison of Two Tables

To simplify notation, let $\rho = \rho_{12}$, $\hat{\rho} = \hat{\rho}_{12}$, $m_{ijh\ell} = m_{ijh\ell}^{12}$, $m = m^{12}$, $\mu_{ijh\ell} = \mu_{ijh\ell}^{12}$, $\mu = \mu^{12}$, $M = M_{12}$, $\pi = \pi_{12}$, $t = t_{12}$. We see that the situation is exactly as described in Section 3. From Theorem 2, letting $f = d_1 - d_2$ we immediately get the following result, assuming (4.1) holds.

THEOREM 3. $\sqrt{n}(\hat{d}_1 - \hat{d}_2 - (d_1 - d_2))$ *is asymptotically normal with asymptotic variance*

$$\sigma^2 = \sigma_1^2/\pi_1 + \sigma_2^2/\pi_2 - \rho 2\pi/(\pi_1\pi_2).$$

A consistent estimator of σ^2 *is*

$$\hat{\sigma}^2 = \hat{\sigma}_1^2/t_1 + \hat{\sigma}_2^2/t_2 - \hat{\rho} 2t/(t_1 t_2).$$

We define the two tables to be independent if q_1 and q_2 are stochastically independent. Hence, if $n_{12} = 0$ then obviously the two tables are independent. For $n_{12} > 0$ we have the following result.

LEMMA 3. *If* $n_{12} > 0$, *then tables* 1 *and* 2 *are independent if and only if*

$$\mu_{ijh\ell} = p_{ij1} \cdot p_{h\ell 2} \quad \textit{for all} \quad (i,j),\ (h\ell).$$

Proof. We have that

$$\mathrm{cov}(q_{ij1}, q_{h\ell 2}) = (n_{12}/n_1 n_2)\ (\mu_{ijh\ell} - p_{ij1} \cdot p_{h\ell 2}).$$

Assume q_1, q_2 are independent. Then $\mathrm{cov}(q_{ij1}, q_{h\ell 2}) = 0$ for all (i,j), (h,ℓ), and the "only if" -part follows.

The observations in M can be described as follows. Let

$$X_{ij} = \begin{cases} 1 & \text{if observation falls in cell } (i,j) \text{ of table 1} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_h = \begin{cases} 1 & \text{if observation falls in cell } (h,\ell) \text{ of table 2} \\ 0 & \text{otherwise.} \end{cases}$$

$X = (X_{11}, \ldots, X_{I_1 J_1})$, $Y = (Y_{11}, \ldots, Y_{I_2 J_2})$. In M we have n_{12} independent observations (X^t, Y^t) of (X,Y). q_1, q_2 are independent if X, Y are independent. Assume now $\mu_{ijh\ell} = p_{ij1} \cdot p_{h\ell 2}$, for all (i,j), (h,ℓ). This implies that $(X_{ij}, Y_{h\ell})$ are independent for all (i,j), (h,ℓ). Hence X, Y are independent, and the result follows.

Q.E.D.

Remarks. 1) If tables 1 and 2 are independent then the estimated asymptotic variance of $\sqrt{n}(\hat{d}_1 - \hat{d}_2)$ is $\hat{\sigma}_1^2/t_1 + \hat{\sigma}_2^2/t_2$.

2) If $n_1 = n_{12} \leq n_2$, that is all the observations in one table come from the set M, then the estimated asymptotic variance is $\hat{\sigma}_1^2/t_1 + \hat{\sigma}_2^2/t_2 - 2\hat{\rho}/t_2$, so in particular if M consists of all the trials then the estimated asymptotic variance of $\sqrt{n_{12}}\,(\hat{d}_1-\hat{d}_2)$ is equal to $\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 2\hat{\rho}$.

3) From Theorem 3 we can now construct confidence intervals and test hypotheses about d_1-d_2.

Let us consider the case $n_1 = n_2 = n_{12}$ and suppose the two tables are from a 3-way table. For example, let table 1 consist of variables 1 and 2 and let table 2 consist of variables 1 and 3. Usually this means that variable 1 is the primary factor and variables 2 and 3 are considered as explaining factors. Since in this case $\mu_{ijh\ell} = 0$ and $p_{ij1}p_{h\ell 2} > 0$ for $i \neq h$, the tables cannot be independent, but they can still be permissible as we shall see. Let $\bar{p}_{ij\ell} = \mu_{iji\ell}$. The saturated log-linear model is:

$$\log \bar{p}_{ij\ell} = u + u_{1(i)} + u_{2(j)} + u_{3(\ell)} + u_{12(ij)} + u_{13(i\ell)} + u_{23(j\ell)} + u_{123(ij\ell)},$$

where

$$\sum_i u_{1(i)} = \sum_j u_{2(j)} = \sum u_{3(\ell)} = \sum_j u_{12(ij)} = \text{etc.} = 0.$$

From Bishop et al. (1975, Theorem 2.4-1) and Goodman (1972, formulas (55)-(56)) we find that the two tables are permissible if $u_{23} = u_{123} = 0$. That is, if $u_{23} = u_{123} = 0$ then u_{12} remains unchanged when we sum over variable 3 to get table 1 and u_{13} remains unchanged when we sum over variable 2 to get table 2.

As an example, consider the $2 \times 2 \times 2$ table. If we take as the measure of association a 1-1 function of the cross-product ratio, as suggested by Edwards (1963), then it is readily seen that $\rho = 0$. However if we use other types of measures ρ will not necessarily vanish. This is quite natural since the log-linear terms are functions of cross-product-ratios.

If the two tables are from a regular 4-way table and the tables consist of different variables, then the tables may be independent. If so, it is seen by applying Theorem 2.5-1 in Bishop et al. (1975) or results from Goodman (1972) that the tables also are permissible. However, also in this case, independence is not necessary for assumption (1.1) to be true.

4.3 Multiple Comparisons of K Tables

The asymptotic variance of $\sqrt{n}(d_i - d_j)$ is given by $\sigma_{ij}^2 = \sigma_i^2/\pi_i + \sigma_j^2/\pi_j - 2\rho_{ij}\pi_{ij}/\pi_i\pi_j$ with estimator $\hat{\sigma}_{ij}^2 = \hat{\sigma}_i^2/t_i + \hat{\sigma}_j^2/t_j - 2\hat{\rho}_{ij}t_{ij}/t_i t_j$. From now on we assume $\sigma_{ij}^2 > 0$ for all $i < j$. Let $\alpha_K = \alpha/K(K-1)$. Using the Bonferroni inequality we see that conservative simultaneous confidence intervals for all differences $d_i - d_j$ are given by

$$\hat{d}_i - \hat{d}_j \pm x(\alpha_k)\hat{\sigma}_{ij}/\sqrt{n} \tag{4.2}$$

Here $x(\varepsilon) = (1-\varepsilon)100$-percentile of the $N(0,1)$ - distribution. Therefore the multiple comparison procedure consists in stating $d_i > d_j$ when $T_{ij} = \sqrt{n}(\hat{d}_i - \hat{d}_j)/\hat{\sigma}_{ij} > x(\alpha_k)$. Let $\underline{d} = (d_1, \ldots, d_K)$. For a given $\underline{d}$ of values of the measures we define $\alpha(\underline{d})$ to be the probability of at least one false statement "$d_i > d_j$". We shall consider $\alpha(\underline{d})$ generally. For this purpose we use the same approach as Spjøtvoll (1971) and let V_i for $i = 1,\ldots,t$ be disjoint index sets with $\bigcup_i V_i = \{1,\ldots,K\}$. Let $v_i = \#(V_i)$. $\omega(V_1,\ldots,V_t)$ is the set of all $\underline{d}$ such that $d_i = d_j$ if $i,j \in V_h$ and $d_i \neq d_j$ if (i,j) belongs to different V_h's. The following result about $\alpha(\underline{d})$ holds.

THEOREM 4. *If* $\underline{d} \in \omega(V_1,\ldots,V_t)$ *then*

a) *for* $t < K$: $$\limsup_n \alpha(\underline{d}) = \limsup_n P\Big(\bigcup_{h=1}^{t} \max_{i,j\in V_h} T_{ij} > x(\alpha_K)\Big) \qquad (4.3)$$

b) $$\limsup_n \alpha(\underline{d}) \leq \Big(1 - \frac{t-1}{K}\Big)\Big(1 - \frac{t-1}{K-1}\Big)\alpha. \qquad (4.4)$$

Proof.

$$\lim P\Big(\bigcup_{g\neq h} \bigcup_{i\in V_g} \bigcup_{j\in V_h} (\text{false statement "}d_i > d_j\text{"})\Big) = 0. \qquad (4.5)$$

(4.4) is therefore proved for $t = K$.

Assume now $t < K$. From (4.5):

$$\limsup_n \alpha(\underline{d}) = \limsup_n P\Big(\bigcup_{h=1}^{t} \max_{i,j\in V_h} T_{ij} > x(\alpha_K)\Big). \qquad (4.6)$$

Hence (4.3) is proved. From (4.6):

$$\limsup_n \alpha(\underline{d}) \leq \sum_{h=1}^{t} \sum_{\substack{i<j \\ i,j\in V_h}} \lim_n P(|T_{ij}| > x(\alpha_K))$$

$$= \sum_{h=1}^{t} \sum_{\substack{i<j \\ i,j\in V_h}} (2\alpha)/K(K-1) = (\alpha/K(K-1))\Big(\sum_{h=1}^{t} v_h^2 - K\Big).$$

$\sum_{h=1}^{t} v_h^2$ takes its maximum value when one v_h equals $K - t + 1$ and the rest is 1. Hence

$$\sum_{h=1}^{t} v_h^2 \leq (K-t+1)^2 + (t-1).$$

This implies:

$$\limsup_n \alpha(\underline{d}) \leq (\alpha/K(K-1))](K-t+1)^2 - (K-t+1)]$$

$$= (1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1})\alpha. \qquad \text{Q.E.D.}$$

The upper bound in (4.4) increases as t decreases and has maximum for $t = 1$, such that $\limsup_n \alpha(\underline{d}) \leq \alpha$ for all $\underline{d}$. Some values of $(1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1})$ are listed in the table below.

TABLE 4.1. Values of $(1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1})$

K \ t	1	2	3	4	5	9	19
2	1	0					
3	1	0.333	0				
4	1	0.5	0.167	0			
5	1	0.6	0.3	0.1			
10	1	0.8	0.622	0.467	0.333	0.022	
20	1	0.9	0.805	0.716	0.623	0.347	0.005

The result in (4.4) is not only of theoretical interest but can also be used in practice if we have some prior knowledge of the tables. For instance if we know the measures can be separated at least in two groups, we can use this to get a higher significance-level on each test and get more powerful tests. In this case $t \geq 2$ and $\lim\sup \alpha(d) \leq (1-1/K)(1-1/(K-1))\alpha$. Now we determine α' such that $(1-1/K)(1-1/(K-1))\alpha' = \alpha$, i.e. $\alpha' = \alpha/(1-1/K)(1-1/(K-1))$ and use level $\alpha'/K(K-1)$ on each comparison. For example if $K = 5$ and $\alpha = .05$ then $\alpha' = .083$.

5. Multiple Comparisons of Independent Two-way Tables

Typically the two-way tables are from independent samples, but also the slightly more general case that $q_1,\ldots,q_k$ are independent is included.

We shall now allow for both sampling method (i) and sampling method (ii). For those tables using sampling method (ii), independent multinomial sampling in the rows, we will assume for simplicity that $p_{i\cdot k} = w_{ik}$ is known for all i. Then, from (2.3) the asymptotic variance of $\sqrt{n}\,\hat{d}_k$ is given by

$$\tau_k^2(p_k) = \sum_{i=1}^{v_k} \sum_{j=1}^{w_k} p_{ijk}(d_{ijk}-d^*_{ik})^2,$$

estimated by $\hat{\tau}_k^2 = \tau_k^2(q_k)$. Here $d^*_{ik} = p_{i\cdot k}^{-1} \sum_{j=1}^{w_k} p_{ijk}d_{ijk}$; $p_{i\cdot k} = \sum_j p_{ijk}$.

The estimated asymptotic variance can then be written as

$$s_k^2 = \begin{cases} \hat{\sigma}_k^2 & \text{if sampling method (i) is used in table k} \\ \hat{\tau}_k^2 & \text{if sampling method (ii) is used in table k.} \end{cases}$$

Throughout this section we will assume that the asymptotic variances corresponding to s_k^2 are positive. Moreover (4.1) is assumed to hold and if sampling method (ii) is used in table k we will assume that $\sqrt{n}_k(n_{i\cdot k}/n_k - p_{i\cdot k}) \longrightarrow 0$. Here $n_{i\cdot k}$ are the row-marginals in table k.

Then $\sqrt{n}(\hat{d}_k - d_k)/s_k$ is asymptotically $N(0,1)$. Let $s_{ij}^2 = s_i^2/t_i + s_j^2/t_j$. Simultaneous confidence intervals are given by (4.2) replacing $\hat{\sigma}_{ij}$ with

s_{ij}. Theorem 4 is of course still valid for the corresponding pairwise comparisons procedure, and in addition, since $\max_{i,j \in V_h} T_{ij}$ for $h = 1,\ldots,t$ are independent, we have

$$\limsup_n \alpha(\underline{d}) = 1 - \prod_{h=1}^{t} \limsup_n P[\max_{i,j \in V_h} T_{ij} \leq x(\alpha_K)]$$

where now

$$T_{ij} = \sqrt{n}(\hat{d}_i - \hat{d}_j)/s_{ij}. \qquad (5.1)$$

Next we consider the problem of testing the homogenity-hypothesis

$$H: d_1 = \ldots = d_K \qquad (5.2)$$

The usual test-statistic is

$$U = \sum_{k=1}^{K} n_k(\hat{d}_k - \bar{d})^2/s_k^2$$

where $\bar{d} = (\sum_i n_i/s_i^2)^{-1} \sum_i n_i \hat{d}_i/s_i^2$, so that $U = \min_d \sum_{k=1}^{K} n_k(\hat{d}_k - d)^2/s_k^2$.

Under H, U is asymptotically chi-square distributed with K-1 degrees of freedom. (A proof can be found for example in Rao (1965, Ch. 6a 2(v))). It follows that H is rejected if

$$U > z(K-1,\alpha) \qquad (5.3)$$

where $z(\nu,\alpha)$ is the $100(1-\alpha)$ percentile in the chi-square distribution with ν degrees of freedom.

Goodman (1963) considers some hypotheses of the form (5.2) for comparison of 2×2-tables. The tests developed there are special cases of (5.3). Goodman (1964b) analyzes three-factor interaction in a three-way table. For the case of fixed layer marginals in the $2 \times 2 \times K$-table the hypothesis of zero three-factor interaction is a hypothesis of the form (5.2) for K 2×2-tables. The tests proposed in Goodman (1964b) for this situation also follow from (5.3). Zelen (1971) considers also comparison of several 2×2-tables, by cross-product ratios, and presents an exact conditional test.

We shall now consider simultaneous confidence intervals for all linear functions $\sum c_k d_k$. Let

$$\hat{\sigma}^2_{c'\hat{d}} = \sum_{k=1}^{K} c_k^2 s_k^2 / n_k.$$

By applying the well-known algebraic result that if y is a $(K \times 1)$-vector then $y'y \leq z$ if and only if $|h'y| \leq \sqrt{z}\,\sqrt{h'h}$ *for all* $h = (h_1, \ldots, h_K)'$ (see e.g. Miller (1966, Lemma 2)), we find that

$$\sum n_k(\hat{d}_k - d_k)^2 / s_k^2 \leq z(K,\alpha) \Leftrightarrow |\sum_k c_k(\hat{d}_k - d_k)| \leq \sqrt{z(K,\alpha)}\,\hat{\sigma}_{c'\hat{d}}; \quad \forall c\}$$

(as in the proof of Theorem 1 in Miller (1966)). Hence $[\sum_{k=1}^{K} c_k \hat{d}_k \pm \sqrt{z(K,\alpha)}\,\hat{\sigma}_{c'\hat{d}}]$ are simultaneous confidence intervals for all linear functions $\sum c_k d_k$.

We decide $\sum c_k d_k > 0$ if $\sum c_k \hat{d}_k > \sqrt{z(K,\alpha)}\,\hat{\sigma}_{c'\hat{d}}$. It is readily seen that

$$\limsup_n P(\text{at least one false statement: } \sum c_k d_k > 0) = \begin{cases} \alpha & \text{if } \underline{d} = 0 \\ \leq \alpha & \text{if } \underline{d} \neq 0. \end{cases}$$

Next, we consider linear contrasts defined as $\sum_k c_k d_k$ with $\sum_k c_k = 0$. Let $d^0 = (\sum_i n_i/s_i^2)^{-1} \sum_i n_i d_i / s_i^2$. Then it follows exactly as for Scheffé's simultaneous confidence intervals in analysis of variance (see Scheffé (1959)) that

$$U_1 = \sum n_k(\hat{d}_k - \bar{d} - (d_k - d^0))^2/s_k^2 \leq z(K-1,\alpha)$$

$$\Updownarrow$$

$$\sum c_k(\hat{d}_k - d_k) \leq \sqrt{Z(K-1,\alpha)}\hat{\sigma}_{c'\hat{d}} \quad \forall c, \sum c_k = 0 \qquad (5.4)$$

Since U_1 is asymptotically chi-square distributed with $K-1$ degrees of freedom, it follows that simultaneous confidence-intervals for all linear contrasts are given by

$$\sum c_k\hat{d}_k \pm \sqrt{z(K-1,\alpha)}\hat{\sigma}_{c'\hat{d}} \qquad (5.5)$$

Also, we decide $\sum c_i d_k > 0$ if $\sum c_k\hat{d}_k > \sqrt{z(K-1,\alpha)}\hat{\sigma}_{c'\hat{d}}$. Hence, from (5.4) we get as in analysis of variance that the hypothesis (5.2) is accepted if and only if no contrasts are found to be different from zero. It is also easily seen that

$$\limsup_n P(\text{at least one false statement: } \sum c_k d_k > 0) = \begin{cases} \alpha & \text{under } H \\ \leq \alpha & \text{otherwise.} \end{cases}$$

Marascuilo (1966) stated the above results for linear contrasts without proofs.

The simultaneous confidence intervals in Goodman (1964b), Chapter 3.1 for the 2 × 2 × K-table in the case of fixed layer marginals follow from (5.5). For $K > 2$, an alternative procedure for pairwise comparison is to use (5.5), i.e. state $d_i > d_j$ if $T_{ij} > \sqrt{z(K-1,\alpha)}$; where T_{ij} is given by (5.1). For the usual choices of $\alpha, x^2(\alpha_K) < z(K-1,\alpha)$ such that using the linear contrast procedure will typically be less powerful. Similar remarks have been made by Goodman (1964a, 1964b).

To illustrate the procedures in this section, let us consider the comparison of 2 × 2-tables. The natural measure of association is essentially the cross product ratio (see Edwards (1963)). Let $\Delta_k = p_{11k}p_{22k}/p_{12k}p_{21k}$. Δ_k is the cross product ratio in table k. Let $\hat{\Delta}_k = \Delta_k(q_k)$. To simplify the presentation we assume that sampling method (i) is used in every table. Let $u_k = q_{11k}^{-1} + q_{22k}^{-1} + q_{12k}^{-1} + q_{21k}^{-1}$. In this case $s_k^2 = \hat{\sigma}_k^2 = \hat{\Delta}_k^2 u_k$. Let further $\rho_k = \ln \Delta_k$, $\hat{\rho}_k = \ln \hat{\Delta}_k$. Then the estimated variance for $\sqrt{n}\,\hat{\rho}_k$ is u_k. The simultaneous confidence intervals for $\rho_i - \rho_j$ are given by:

$$\rho_i - \rho_j \pm x(\alpha_K)(u_i/n_i + u_j/n_j)^{1/2} \tag{5.6}$$

For $k = 2$, (5.6) is the same as in Goodman (1964a), p. 97. For general K, (5.6) also follows from Goodman (1964b), Chapter 3.1. Simultaneous confidence intervals for linear contrasts in Δ_k are given by

$$\sum c_k\hat{\Delta}_k \pm (z(K-1,\alpha))^{1/2}(\sum_k c_k^2\hat{\Delta}_k^2 u_k/n_k)^{1/2}.$$

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