LARGE DEVIATION INDICES AND BAHADUR EXACT SLOPES

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SUMMARY

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When it exists, the Bahadur exact slope $c(\theta_1)$ of a sequence $\{T_n: n \geq 1\}$ of test statistics can be used to compare this sequence of test statistics with other test sequences as tests of $H_0: \theta \in \Theta_0$ v.s. $H_1: \theta \in \Theta_1$. Since it is often fairly easy to show that $\lim_{n\to\infty} n^{-\frac{1}{2}} T_n = b(\theta_1)$ exists almost surely for $\theta_1 \in \Theta_1$, most attempts to calculate $c(\theta_1)$ in the literature try to compute $c(\theta_1)$ as a function $h(b(\theta_1))$ of $b(\theta_1)$. A result due to R. R. Bahadur states that for $c(\theta_1)$ to exist as a function of $b(\theta_1)$ it is sufficient that the index of large deviation

$$g(t) = \lim_{n\to\infty} \inf_{\theta \in \Theta_0} \left[- \frac{1}{n} \log P_\theta\{T_n \geq n^{\frac{1}{2}} t\}\right]$$

exists in a neighborhood of $b(\theta_1)$ and is continuous at $t = b(\theta_1)$; in which case, $c(\theta_1) = 2g(h(\theta_1))$. The present paper studies the consequences for the existence and properties of $g(t)$ implied by assuming that both $b(\theta_1)$ and $c(\theta_1)$ exist, all $\theta_1 \in \Theta_1$, and that a functional relationship $c(\theta_1) = h(b(\theta_1))$ holds between $c(\theta_1)$ and $b(\theta_1)$. Results obtained include inequalities for large deviation probabilities based on the function $h(t)$, and a converse of Bahadur's result. It is shown that existence of the relationship $c(\theta_1) = h(b(\theta_1))$ implies that the index of large deviation $g(t)$ must exist and be continuous at all but a countable number of points t in the interior of the image set $\{b(\theta_1): \theta_1 \in \Theta_1\}$.

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1. Introduction. Let (S,β) be a sample space of sequences $s = (x_1,x_2,\ldots)$, $x_i \in \mathcal{X}$, $i = 1,2,\ldots$, where $\mathcal{X}$ is an arbitrary space. Let $\{P_\theta: \theta \in \Theta_0 \cup \Theta_1\}$, $\Theta_0 \cap \Theta_1 = \phi$, be a collection of probability measures defined on (S,β). Let $\{T_n: n \geq 1\}$, $T_n = T_n(s)$, be a sequence of β-measurable functions from S to the real line R. In statistical applications, $T_n(s)$ is a function of s only through $x_1,x_2,\ldots,x_n$, and is thought of as a test statistic for testing $H_0: \theta \in \Theta_0$ v.s. $H_1: \theta \in \Theta_1$, with large values of T_n giving evidence against H_0.

For $t \in R$, define

$$F_n(t|\theta) = P_\theta\{T_n < t\}, \quad F_n(t) = \inf_{\theta \in \Theta_0} F_n(t|\theta), \tag{1.1}$$

$$L_n(t) = -\frac{1}{n}\log\{1 - F_n(t)\} = \inf_{\theta \in \Theta_0}\left[-\frac{1}{n}\log P_\theta\{T_n \geq t\}\right]. \tag{1.2}$$

The random quantity $1 - F_n(T_n(s))$ is the P-level of H_0 based on the test statistic T_n. Since large values of T_n offer evidence against H_0, an α-level test of H_0 v.s. H_1 based on T_n rejects H_0 whenever $1 - F_n(T_n(s)) \leq \alpha$, as equivalently whenever

$$-\frac{1}{n}\log\alpha \leq L_n(T_n(s)).$$

If for $\theta_1 \in \Theta_1$,

$$\text{a.s. } [P_{\theta_1}] \lim_{n\to\infty} L_n(T_n(s)) = \tfrac{1}{2}c(\theta_1), \tag{1.3}$$

then $c(\theta_1)$ is called the Bahadur exact slope of $\{T_n: n \geq 1\}$ at θ_1. Bahadur (1971) shows that exact slopes can be used to compare the asymptotic performances of test sequences. For this reason, Bahadur exact slopes have been computed for a wide variety of test sequences.

The most commonly used method for finding Bahadur exact slopes proceeds indirectly, rather than by a direct calculation of the limit (1.3). Usually the limit

$$b(\theta_1) = \text{a.s. } [P_{\theta_1}] \lim_{n\to\infty} n^{-\frac{1}{2}} T_n(s) \tag{1.4}$$

exists for all $\theta_1 \in \Theta_1$, and is easy to determine. Let

$$J = \{t: t = b(\theta_1), \text{ some } \theta_1 \in \Theta_1\} . \tag{1.5}$$

The indirect method for calculating $c(\theta_1)$ attempts to find $c(\theta_1)$ as a function

$$c(\theta_1) = h(b(\theta_1)) \tag{1.6}$$

of $b(\theta_1)$, where $h(t)$ maps J onto $[0,\infty)$.

A sufficient condition for (1.6) to hold has been given by Bahadur:

Theorem 1.1. [Bahadur (1971; Theorem 7.1)] Suppose that the limit (1.4) exists for all $\theta_1 \in \Theta_1$. A sufficient condition for $c(\theta_1)$ to exist, and for $c(\theta_1)$ to be a continuous function $h(b(\theta_1))$ of $b(\theta_1)$, all $\theta_1 \in \Theta_1$, is that for every $t \in J$,

(a) $g(u) = \lim_{n\to\infty} L_n(n^{\frac{1}{2}}u)$ exists in a neighborhood M_t of t,

(b) $g(t)$ is continuous at t.

In this case, $c(\theta_1) = 2g(b(\theta_1))$, all $\theta_1 \in \Theta_1$.

For $t \in R$, the quantity

$$g(t) = \lim_{n\to\infty} L_n(n^{\frac{1}{2}} t), \tag{1.7}$$

if it exists, is called the index of large deviation of $\{T_n: n \geq 1\}$ under H_0. The justification for this terminology is apparent from (1.2) and (1.7). Even if $g(t)$ does not exist at t, the lower index of large deviation, $\underline{g}(t)$, and the upper index of large deviation, $\bar{g}(t)$, defined by

$$(1.8) \qquad \underline{g}(t) = \varliminf_{n\to\infty} L_n(n^{\frac{1}{2}} t), \quad \bar{g}(t) = \varlimsup_{n\to\infty} L_n(n^{\frac{1}{2}} t),$$

always exist for all $t \in R$. The index of large deviation $g(t)$ is of interest to probabilists [Bahadur (1971)], independently of its use in calculating Bahadur exact slopes.

The present paper attempts to answer the following question: "Suppose that the goals of the indirect method for calculating Bahadur exact slopes (described in the paragraph before Theorem 1.1) are met. That is, suppose that the limits (1.3) and (1.4) exist for all $\theta_1 \in \Theta_1$, and that there exists a function h: $J \to [0,\infty)$ such that (1.6) holds. What consequences do these assumptions have for the existence and properties of the index of large deviation $g(t)$ defined by (1.7), or for the properties of $\underline{g}(t)$ and $\bar{g}(t)$?"

For the existence and properties of $g(t)$, the answer to this question is provided by Theorem 1.2 and Corollary 1.1, which are proven in Section 2.

Theorem 1.2. Assume that the limits (1.3) and (1.4) exist for all $\theta_1 \in \Theta_1$, and that J defined by (1.5) has a nonempty interior, int(J). If there exists a function h: $J \to [0,\infty)$ such that (1.6) holds, then

(i) The function $h(t)$ is nondecreasing for $t \in J$, and thus is continuous at all but a countable number of points t in J;

(ii) If $h(t)$ is continuous at $t \in$ int(J), then $g(t)$ defined by (1.7) both exists at t and is continuous at t, and $g(t) = \frac{1}{2} h(t)$;

(iii) In consequence, $g(t)$ exists and is continuous at all but a countable number of points $t \in$ int(J).

Corollary 1.1. Suppose that the limit (1.4) exists for all $\theta_1 \in \Theta_1$. For $c(\theta_1)$ to exist and to be a continuous function $h(b(\theta_1))$ of $b(\theta_1)$, all $b(\theta_1) \in \text{int}(J)$, it is necessary that $g(t)$ defined by (1.7) exist and that $g(t)$ be continuous at all $t \in \text{int}(J)$. If such is the case, then $h(t) = 2g(t)$, $t \in \text{int}(J)$.

If attention is restricted to alternatives $\theta_1 \in \Theta_1$ such that $b(\theta_1) \in \text{int}(J)$, then Corollary 1.1 is the converse to Theorem 1.1. More important, Theorem 1.2 shows that the indirect method for calculating Bahadur exact slopes $c(\theta_1)$ as a function $c(\theta_1) = h(b(\theta_1))$ of $b(\theta_1)$ can hope to succeed only in cases where the index of large deviation $g(t)$ exists for all but at most a countable number of points t in $\text{int}(J)$.

In addition to a proof of Theorem 1.2, Section 2 provides some additional results relating functions $h(t)$ defined by (1.6) to the upper and lower indices of large deviation $\bar{g}(t)$ and $\underline{g}(t)$. Section 3 sketches an example of interest in connection with Theorems 1.1 and 1.2 and in its own right, illustrating that when $b(\theta_1)$ exists for all $\theta_1 \in \Theta_1$ and $g(t)$ exists, all $t \in J$, it is possible for $c(\theta_1)$ to exist and equal $2g(b(\theta_1))$ for all $\theta_1 \in \Theta_1$, even when $g(t)$ is not continuous at all points $t \in R$.

2. Proof of Theorem 1.2 and Related Results. Throughout this section, reference is made to the following assumptions:

Assumption A. For all $\theta_1 \in \Theta_1$,

(2.1) $$\text{a.s. } [P_{\theta_1}] \lim_{n\to\infty} n^{-\frac{1}{2}} T_n(s) = b(\theta_1),$$

(2.2) $$\text{a.s. } [P_{\theta_1}] \lim_{n\to\infty} L_n(T_n(s)) = \tfrac{1}{2} c(\theta_1).$$

Assumption B. $\text{Int}(J) \neq \phi$, where J is defined by (1.5).

Assumption C. There exists a function $h: J \to [0,\infty)$ such that

$$c(\theta_1) = h(b(\theta_1)), \text{ all } \theta_1 \in \Theta_1.$$

Let $\underline{g}(t)$ and $\bar{g}(t)$ be defined by (1.8).

Lemma 2.1. Each of the functions $F_n(t)$, $L_n(t)$, $\underline{g}(t)$, $\bar{g}(t)$ is nondecreasing in t.

Proof. Since $F_n(t|\theta)$ is a c.d.f., $F_n(t|\theta)$ is nondecreasing in t, all $\theta \in \Theta_0$. The stated assertions now follow directly from this fact and the definitions of $F_n(t)$, $L_n(t)$, $\underline{g}(t)$, and $\bar{g}(t)$. □

Lemma 2.2. For all $\varepsilon > 0$, $\theta_1 \in \Theta_1$,

$$(2.3) \quad \underline{g}(b(\theta_1)-\varepsilon) \le \bar{g}(b(\theta_1)-\varepsilon) \le \tfrac{1}{2}c(\theta_1) \le \underline{g}(b(\theta_1)+\varepsilon) \le \bar{g}(b(\theta_1)+\varepsilon).$$

Proof. From Assumption A, there exists $S_{\theta_1} \subset S$ such that $P_{\theta_1}(S_{\theta_1}) = 1$ and

$$(2.4) \quad \lim_{n\to\infty} n^{-\frac{1}{2}} T_n(s) = b(\theta_1), \quad \lim_{n\to\infty} L_n(T_n(s)) = \tfrac{1}{2} c(\theta_1), \quad \text{all } s \in S_{\theta_1}.$$

Fix $s \in S_{\theta_1}$. By (2.4), for each $\varepsilon > 0$, there exists $N = N(s,\theta_1,\varepsilon)$ such that $n \ge N$ implies

$$(2.5) \quad n^{\frac{1}{2}}(b(\theta_1) - \varepsilon) \le T_n(s) \le n^{\frac{1}{2}}(b(\theta_1) + \varepsilon).$$

Since $L_n(t)$ is nondecreasing in t (Lemma 2.1),

$$(2.6) \quad L_n(n^{\frac{1}{2}}(b(\theta_1) - \varepsilon)) \le L_n(T_n(s)) \le L_n(n^{\frac{1}{2}}(b(\theta_1) + \varepsilon)),$$

all $n \ge N$. The result (2.3) now follows from (2.6), (1.8) and (2.4) by first taking $\underline{\lim}_{n\to\infty}$ on all sides of (2.6), then taking $\overline{\lim}_{n\to\infty}$ on all sides of (2.6), and finally recalling that $\underline{g}(t) \le \bar{g}(t)$, all $t \in R$. □

The following lemma summarizes known facts about nondecreasing functions of a real variable [Apostol (1958)].

Lemma 2.3. Let $r(t)$ be nondecreasing in t for $t \in Q$, $Q \subset R$.

(i) The limits $r(t-)$, $r(t+)$ from the left and right, respectively, exist at all $t \in \text{int}(Q)$. Indeed, for all $t \in \text{int}(Q)$,

$$(2.7)\quad \begin{aligned} r(t-) &= \lim_{\substack{u \to t \\ u \in Q \cap (-\infty,t)}} r(u) = \sup\{u: u < t\}\ , \\ r(t+) &= \lim_{\substack{u \to t \\ u \in Q \cap (t,\infty)}} r(u) = \inf\{u: u > t\}\ , \end{aligned}$$

and

$$(2.8)\quad r(t-) \leq r(t) \leq r(t+)\ .$$

(ii) $r(t)$ has only a countable number of points of discontinuity in Q;

(iii) For $t \in \text{int}(Q)$, $r(t-)$ is left continuous at t, $r(t+)$ is right continuous at t, and $r(t)$ is continuous at t if and only if $r(t-) = r(t+) = r(t)$.

Remark. If $Q \cap (t,\infty) \neq \phi$, all facts stated about $r(t+)$ in Lemma 2.3 are valid, while all facts stated about $r(t-)$ in Lemma 2.3 are valid if $Q \cap (-\infty,t) \neq \phi$.

The next lemma, proven as Theorem 7.2 in Bahadur (1971), requires only Assumption A, and provides what Bahadur calls a "partial converse" to Theorem 1.1.

Lemma 2.4. If only Assumption A holds, let

$$g_1(t) = \begin{cases} \inf\{\frac{1}{2}c(\theta_1): \theta_1 \in \Theta_1,\ b(\theta_1) > t\}\ , & \text{if} \quad J \cap (t,\infty) \neq \phi\ , \\ \infty\ , & \text{otherwise}\ . \end{cases}$$

$$g_2(t) = \begin{cases} \sup\{\frac{1}{2}c(\theta_1): \theta_1 \in \Theta_1,\ b(\theta_1) < t\}\ , & \text{if} \quad J \cap (-\infty,t) \neq \phi\ , \\ 0\ , & \text{otherwise}\ . \end{cases}$$

Then for all $t \in R$,

$$(2.9)\quad g_2(t) \leq \underline{g}(t) \leq \bar{g}(t) \leq g_1(t)\ .$$

One final lemma is needed before the proof of Theorem 1.2 can be given.

Lemma 2.5. Under Assumptions A, B, C,

(i) The function $h(t)$ is nondecreasing in t for all $t \in J$;

(ii) For all $t \in \mathrm{int}(J)$, the limits $h(t-)$ and $h(t+)$ exist, and

$$h(t-) = 2g_2(t), \quad h(t+) = 2g_1(t) . \tag{2.10}$$

(iii) Consequently, for all $t \in \mathrm{int}(J)$,

$$\tfrac{1}{2}h(t-) \le \underline{g}(t) \le \bar{g}(t) \le \tfrac{1}{2}h(t+) . \tag{2.11}$$

(iv) Further, for all $t \in \mathrm{int}(J)$,

$$\tfrac{1}{2}h(t-) \le \underline{g}(t-) \le \bar{g}(t-) \le \tfrac{1}{2}h(t) \le \underline{g}(t+) \le \bar{g}(t+) \le \tfrac{1}{2}h(t+) . \tag{2.12}$$

Proof. To prove Assertion (i), let $t < t^*$, where $t, t^* \in J$. Then by (1.5) there exist $\theta_1, \theta_1^* \in \Theta_1$ such that $t = b(\theta_1)$, $t^* = b(\theta_1^*)$. Let

$$\varepsilon^* = \tfrac{1}{2}(t^* - t) > 0.$$

Applying (2.3),

$$\tfrac{1}{2}h(t) = \tfrac{1}{2}c(\theta_1) \le \underline{g}(t+\varepsilon^*) = \underline{g}(t^*-\varepsilon^*) \le \tfrac{1}{2}c(\theta_1^*) = \tfrac{1}{2}h(t^*),$$

from which it follows that $h(t) \le h(t^*)$, proving Assertion (i). Assertion (ii) now follows from Assertion (i), Lemma 2.3(i), and the definitions of $g_1(t)$ and $g_2(t)$. Applying Assertion (ii) and (2.9), Assertion (iii) directly follows. Finally, to prove Assertion (iv), note that $t \in \mathrm{int}(J)$ implies that $(t-\varepsilon, t+\varepsilon) \in \mathrm{int}(J)$ for all small enough $\varepsilon > 0$. Applying (2.3) and (2.10),

$$\tfrac{1}{2}h((t-\varepsilon)-) \le \underline{g}(t-\varepsilon) \le \bar{g}(t-\varepsilon) \le \tfrac{1}{2}h(t) \le \underline{g}(t+\varepsilon) \le \bar{g}(t+\varepsilon) \le \tfrac{1}{2}h((t+\varepsilon)+) .$$

Taking limits as $\varepsilon \to 0$, using the fact that $h(t-)$ is left continuous and $h(t+)$ is right continuous at all $t \in \mathrm{int}(J)$ [Lemma 2.3(iii)], proves (2.12). □

Proof of Theorem 1.2 and Corollary 1.1. Assertion (i) of Theorem 1.2 is a direct consequence of Lemma 2.5(i) and Lemma 2.3(ii). Assertion (ii) of Theorem 1.2 follows directly from Lemma 2.3(iii) and (2.11). Assertion (iii) of Theorem 1.2 follows from Assertions (i) and (ii). Finally, Corollary 1.1 is an immediate consequence of Theorem 1.2(ii). □

3. An Interesting Example. The following example, which is of independent interest, provides an illustration of a situation where the Bahadur exact slope $c(\theta_1)$ can exist as a function $c(\theta_1) = 2g(b(\theta_1))$ of the limit $b(\theta_1)$, even when the index of large deviation $g(t)$ is not continuous at all points $t \in \text{int}(J)$.

Let $\Theta_0 = \{0\}$, $\Theta_1 = (0,\infty)$, and let $G(u) = P\{U < u\}$ be the c.d.f. of some nonnegative random variable U. [Thus, $G(0-) = 0$.] Because, $G(u)$ is a c.d.f., it must be right continuous at all $u \in [0,\infty)$, but $G(u)$ can be discontinuous at a countable number of points u. Under P_θ, $\theta \in \Theta_0 \cup \Theta_1$, let $X_1(s), X_2(s), \ldots,$ be i.i.d., with $X_i(s)$ having the same distribution as $\theta + U$, $i = 1,2,\ldots$. Let

$$T_n(s) = n^{\frac{1}{2}} \min_{1 \le i \le n} X_i(s) . \tag{3.1}$$

Then, it is easily shown that

$$L_n(n^{\frac{1}{2}} t) = \log[(1 - G(t))^{-1}] = g(t) = \lim_{n\to\infty} L_n(n^{\frac{1}{2}} t) \tag{3.2}$$

for all $n \geq 1$, $t \geq 0$, and

(3.3) $n^{-\frac{1}{2}} T_n(s)$ converges <u>from the right</u> to $b(\theta_1) = \theta_1$, a.s.$[P_{\theta_1}]$,

for all $\theta_1 \in \Theta_1$. Since $G(t)$ is right continuous, it follows from (3.2) and (3.3) that for all $\theta_1 \in \Theta_1$,

$$2c(\theta_1) = \text{a.s.}[P_{\theta_1}] \lim_{n\to\infty} L_n(T_n(s)) = g(\theta_1) = g(b(\theta_1)). \tag{3.4}$$

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FOOTNOTES

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