NOTE ON A CLARIFICATION OF RESTRICTION ERRORS

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1. BACKGROUND

In a paper, Anderson (1970), the term "restriction" error was introduced. Later the ideas were extended to many designs in a few papers, but mainly in Anderson and McLean (1974). During this time many statisticians remarked that they agreed with the objective, mainly to make practitioners aware that when investigators use treatments as a blocking factor they can not validly test for treatments using the "within" error. These same practitioners found it difficult to understand the meaning of the term restriction error. The purpose of this note is to try to clarify this concept.

2. TREATMENTS AS BLOCKS

For experiments in which the investigator has used only one leval of a given treatment over an entire block of experimental units, we call this treatment the primary treatment. One may liken the primary treatment to the usual whole plot treatment except that in a split plot type design (Anderson and McLean, Chapter 7, 1974), the whole plot treatment, by definition, is replicated; however, here the primary treatment is not replicated. Consequently, there is no unbiased estimate of the effect of the primary treatment nor a valid test of the primary treatment effect (shown later). Of course we assume that this primary treatment is randomly assigned to the block of experimental units.

Continuing the analogy with split plot designs, we call the treatment at the next stage (in which the different levels of this treatment are assigned, at random, to the experimental units within the block having been given only one of the levels of the primary treatment) the secondary treatment. Of course the secondary treatment is similar, in concept, to the split plot treatment.

Let us next use an example to bring these ideas into focus. For experiments in which the investigator has used primary treatment levels as blocks (the philosophy of doing this will be discussed later) we have insisted that the "restriction" error ($\delta_{(i)}$) be included in the model. The resulting model is

$$y_{ij} = \mu + B_i + \delta_{(i)} + T_j + \epsilon_{ij} \qquad (1)$$

$$i=1,\ldots,b \quad j=1,\ldots,t$$

where

y_{ij} = the response from the j^{th} treatment (secondary) assigned randomly to an experimental unit in the i^{th} block (primary treatment)

B_i = effect of the i^{th} block which is actually the effect of the i^{th} level of the primary treatment possibly used to create the block,

$\delta_{(i)}$ = restriction error associated and confounded with the i^{th} block (primary treatment),

T_j = effect of the j^{th} secondary treatment level.

ϵ_{ij} = the random error associated with the experimental unit in the i^{th} block (primary treatment) to which the j^{th} secondary treatment is assigned.

The work of Wilk (1955) and Wilk and Kempthorne (1955) indicates that part of this $\delta_{(i)}$ may be due to technical error associated with the primary treatment. Another possible part is due to treatment error associated with primary treatment as described by Zyskind and Kempthorne (1960). There may be other causes of "restriction" error, but all of these take place when there is a restriction on randomization. This is to say that the term "restriction" error makes a broad brush coverage of peculiarities that influence the size of the additional variation appearing when a restriction on randomization is made on the primary treatments by the experimenter. The restriction on the randomization is not necessarily the cause of this error but merely an indicator that one or more of these errors may exist and be included as part of the observation for all responses within this confined group of experimental units. The boundaries of this confinement are indicated in the randomization procedure.

(Figure 1 here)

One can think of the restriction on the randomization scheme as illustrated in Figure 1 in two equivalent ways. The first is to say that all experimental units within B_4, say, have been restricted such that a randomly selected primary treatment has been applied to all of these units. The other interpretation is that the randomization of all possible levels of the secondary treatment, T, is confined or restricted within each of the blocks. Thus, we think of this concept as the application of the i^{th} level of the primary treatment to all the experimental units in the i^{th} block that gives rise to the " restriction" error, $\delta_{(i)}$.

It follows from equation (1) that $\delta_{(i)}$ is the error that would form the basis for testing the B_i effect (primary treatment); however, there are no degrees of freedom for estimating $\delta_{(i)}$ because that error is completely confounded with B_i. Hence there is no test for B_i and the experimenter knows he must redesign his experiment before taking the first observation if he wants to test the primary treatment effect.

Next, we must emphasize the assumption made to allow one to write ϵ_{ij} in equation (1) without including the interaction term, BT_{ij}, is that this interaction is zero. Of course, an interaction of primary and secondary treatments almost always exists. Hence one needs to redesign the experiment to allow for a better test for T_j and even BT_{ij} if it appeared in the equation. This assumption, that $BT_{ij} = 0$, allows experimenters, in many cases, to falsely think that they can take fewer observations and still get reliable results. The expected mean squares that are developed using equation (1) warns the experimenter of this false economy.

The analysis of variance (ANOVA) table for equation (1) is given in Table 1.

(Table 1 here)

The major item of interest in Table 1 is that the expected mean squares indicate that blocks which resulted from a classification of a primary treatment has no test available. This result holds whether blocks are considered either fixed or random.

3. BLOCK PECULIARITIES

Kempthorne (1952) in section 8.2 shows that the expected mean square for blocks contains no error variance component if the number of experimental units is equal to the number of treatments in a block. The general concept for this result comes from a general model[1] that the block expectation of the sum of square is

$$\left(1 - \frac{(\tau-1)}{(m-1)} \right)\sigma^2 + t\Sigma b_i^2$$

where τ = number of treatments and m = number of experimental units per block. Now, all of this is predicated upon blocks being a factor of classification and, by definition, has no restriction error associated with individual blocks. It has been our experience that most experimenters want to infer to a large number of experimental units within each block. In these cases $m >> \tau$ and infinite theory applies. It follows that the expected mean square for blocks, under the assumption that blocking is a factor of classification, is

$$\sigma^2 + t\Sigma b_i^2.$$

In addition, when one uses equation (1) with the above conditions it necessarily follows that σ_δ^2 must be equal to zero.

In our previous publications concerning block designs, we allowed that there could be a "restriction" error similar to that found in equation (1) for all block designs including those cases where blocking is a factor of classification. The main reason we do this is that it has been our experience in many industrial experiments that the operators have changed some settings from one replicate to another and introduced a treatment and/

[1]Personal communication with Dr. O. Kempthorne June, 1980.

or technical error which adds to the block effect. The "restriction" error with blocks then warns the investigator to be careful about making a test on blocks.

If, after careful examination of the experiment, the research worker is willing to say he definitely has replicates or blocking is a factor of classification rather than a treatment factor (primary treatment) and there are many possible experimental units for the treatments in each block, he can assume $\delta_{(i)}$ is zero and test blocks using the error in Table 1.

The overall conclusion, then, for handling blocks is to place the $\delta_{(i)}$ in the model and after the experiment is run, the investigator must decide whether or not the experiment was conducted with true replicates or blocks before analyzing for block effects. In most cases where blocking is a factor of classification, experimenters are not interested in blocks, per se, but rather the interactions of blocks with secondary treatments to form a correct error for the test on the secondary treatment. For the case in which the experimenter has a primary treatment of interest that was used as a blocking factor, he needs to replicate the experiment in order to obtain a valid test on the primary treatment.

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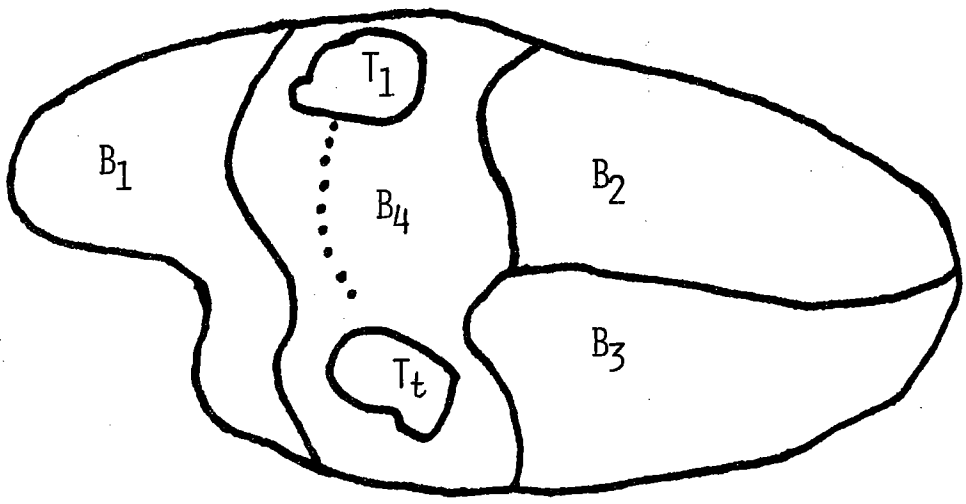

Figure 1. Randomization scheme

Table 1

ANOVA for Equation (1)

Source	df	MS
Blocks (primary treatment)	b-1	$\sigma^2 + \tau\sigma^2_\delta + t\phi(B)$
Treatments (secondary treatment)	t-1	$\sigma^2 + b\phi(T)$
Error	(b-1)(t-1)	σ^2
Total	bt-1	