ON THE EXACT NON-NULL DISTRIBUTION OF WILKS' L_{vc} CRITERION IN THE COMPLEX CASE

by

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1. INTRODUCTION AND SUMMARY

Let $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$ be independent complex normal random p-vectors with mean vector $\underset{\sim}{\xi}$ and covariance matrix $\underset{\sim}{\Sigma}$, i.e., $\underset{\sim}{Z}_i \sim CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$. Let $\underset{\sim}{Z} = (\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N)$. Then $\underset{\sim}{Z} \sim CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, (see Goodman [4]) where the complex multivariate normal distribution is defined by

$$CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma}) = (\pi)^{-pN}|\underset{\sim}{\Sigma}|^{-N}\exp(-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{Z} - \underset{\sim}{\mu})(\overline{\underset{\sim}{Z} - \underset{\sim}{\mu}})') \tag{1.1}$$

and $\underset{\sim}{\mu} = (\underset{\sim}{\xi}, \underset{\sim}{\xi}, \ldots, \underset{\sim}{\xi})$ is a $p \times N$ complex matrix. Let us define

$$\underset{\sim}{Z}_0 = N^{-1}\sum_{i=1}^{N}\underset{\sim}{Z}_i \quad \text{and} \quad \underset{\sim}{S} = \sum_{i=1}(\underset{\sim}{Z}_i - \underset{\sim}{Z}_0)(\overline{\underset{\sim}{Z}_i - \underset{\sim}{Z}_0})' . \tag{1.2}$$

Then $N^{-1/2}(\underset{\sim}{Z}_0 - \underset{\sim}{\xi}) \sim CN(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ and S has an independent complex Wishart distribution which is defined by

$$CW(\underset{\sim}{S}; \underset{\sim}{p}, \underset{\sim}{N}, \underset{\sim}{\Sigma}) = [\Gamma_p(n)]^{-1}|\underset{\sim}{\Sigma}|^{-n}|\underset{\sim}{S}|^{n-p}\exp(-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S}) \tag{1.3}$$

with $n = N - 1$ and $\tilde{\Gamma}_p(n)$ is defined in the next section. $\underset{\sim}{\Sigma}$ and $\underset{\sim}{S}$ are Hermitian positive definite matrices of order p. In this paper, in order to study the structure of the covariance matrices of the complex multivariate normal populations, we derive the exact non-null moments and distribution of the Wilks' [12] L_{vc} criterion for testing

$H: \Sigma = \sigma^2[(1-\rho)I + \rho e e']$, σ and ρ unknown against the alternative $A \neq H$; μ unknown and $e' = (1, 1, \ldots, 1)$. We derive the distribution of L_{vc} in three series forms and compute powers for $p = 2$ for various values of N and the parameters involved for 5% significance level based on the null distribution and the percentage points of L_{vc} obtained in Singh [II]. In Section 2, we give some definitions and lemmas which are needed in our derivation. In Section 3, we obtain the non-null density of L_{vc} as a series of Meijer's [7] G-functions using Mellin [19] integral transform. Some special cases have also been discussed which are used to compute powers for the case $p = 2$. In Section 4, we obtain the density in an alternative series form using the method of contour integration [8] and in Section 5, the non-null moments of the criterion are used to obtain the distribution as a chi-square series employing methods similar to those of Box[1]. In Section 6, we tabulate the powers for various values of N and ρ for the case $p = 2$.

2. SOME DEFINITIONS AND RESULTS

We now give some definitions and lemmas of interest for the following derivation.

Definitions: Let k be a non-negative integer and let $\kappa = (k_1, k_2, \ldots, k_p)$ be a partition of k such that $k_1 \geq k_2 \geq \cdots \geq k_p \geq 0$, $\sum_{i=1}^{p} k_i = k$ and let

$$[a]_\kappa = \prod_{i=1}^{p} (a - i + 1)_{k_i} = \tilde{\Gamma}_p(a, \kappa)/\tilde{\Gamma}_p(a) \tag{2.1}$$

$$(a)_k = (a)(a+1) \cdots (a+k-1) \quad \text{and} \tag{2.2}$$

$$\tilde{\Gamma}_p(a) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(a - i + 1) = \int_{\bar{S}'=S>0} |\underset{\sim}{S}|^{a-p} \exp(-\mathrm{tr}\underset{\sim}{S})d\underset{\sim}{S} \tag{2.3}$$

Also the hypergeometric function of a matrix variate is defined by (see James [5]),

$$ {}_p\tilde{F}_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; \underset{\sim}{Z}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa} \frac{\tilde{C}_\kappa(\underset{\sim}{Z})}{k!} \tag{2.4}$$

Where $\tilde{C}_\kappa(\underset{\sim}{Z})$ denotes the zonal polynomial, a symmetric function in the characteristic roots of the hermitian matrix $\underset{\sim}{Z}$ (see James [5]) of degree k corresponding to the partition κ. In particular we have

$$ {}_0\tilde{F}_0(\underset{\sim}{Z}) = \exp(\mathrm{tr}\underset{\sim}{Z}) \quad \text{and} \quad {}_1\tilde{F}_0(a; \underset{\sim}{Z}) = |\underset{\sim}{I} - \underset{\sim}{Z}|^{-a} \tag{2.5}$$

Lemmas: We now give some lemmas which will be used in the sequel.

Lemma 2.1. Let $\underset{\sim}{R}$ be a complex symmetric matrix whose real part is positive definite and let $\underset{\sim}{T}$ be an arbitrary complex symmetric matrix. Then

$$\int_{\underset{\sim}{S}=\bar{\underset{\sim}{S}}'>\underset{\sim}{0}} \exp(-\mathrm{tr}\underset{\sim}{R}\underset{\sim}{S})|\underset{\sim}{S}|^{t-m}\tilde{C}_\kappa(\underset{\sim}{S}\underset{\sim}{T})d\underset{\sim}{S} = \tilde{\Gamma}_m(t, \kappa)|\underset{\sim}{R}|^{-t}\, \tilde{C}_\kappa(\underset{\sim}{T}\underset{\sim}{R}^{-1})$$

the integration being taken over the space of positive definite Hermitian (p.d.h.) $m \times m$ matrices.

We now define the Laplace transform of a function $f(\underset{\sim}{S})$ of the p.d.h. $m \times m$ matrix $\underset{\sim}{S}$

$$g(\underset{\sim}{Z}) = \int_{\underset{\sim}{S}=\bar{\underset{\sim}{S}}'>0} \exp(-\mathrm{tr}\underset{\sim}{S}\underset{\sim}{Z})f(\underset{\sim}{S})d\underset{\sim}{S} \quad \text{where} \quad \underset{\sim}{Z} = \underset{\sim}{X} + i\underset{\sim}{Y} \tag{2.6}$$

is a complex symmetric matrix; X and Y are real and it is assumed that the integral converges in the "half-plane" $R(Z) = X > X_0$ for some positive definite X_0. (See Constantine [2]). The following theorem will also be needed.

Convolution Theorem. If $g_1(Z)$, $g_2(Z)$ are the Laplace transforms of $f_1(S)$ and $f_2(S)$, then $g_1(Z)g_2(Z)$ is the Laplace transform of

$$f(R) = \int_{S=\bar{S}'>0}^{R} f_1(S)f_2(R-S)dS,$$

the integration being over the space of all S for which $0 < S < R$

Lemma 2.2. If R and S are $m \times m$ p.d.h. matrices, then

$$\int_{S=S'>0}^{I} |S|^{t-m}|I-S|^{u-m}\, \tilde{C}_\kappa(RS)dS = \tilde{\Gamma}_m(t,\kappa)\tilde{\Gamma}_m(u)\tilde{C}_\kappa(R)/\tilde{\Gamma}_m(t+u,\kappa)$$

Proof: Let

$$F(R) = \int_{S=\bar{S}'>0}^{I} |S|^{t-m}|I-S|^{u-m}\tilde{C}_\kappa(RS)dS \tag{2.7}$$

then $F(R)$ is a symmetric function of R, i.e., $F(R) = F(\bar{U}'RU)$ for all U s.t. $U\bar{U}' = I$. Therefore, we have

$$F(R) = F(I)\tilde{C}_\kappa(R)/\tilde{C}_\kappa(I) \tag{2.8}$$

In order to complete the proof, we need to show that

$$F(I)/\tilde{C}_\kappa(I) = \tilde{\Gamma}_m(t,\kappa)\tilde{\Gamma}_m(u)/\tilde{\Gamma}_m(t+u,\kappa)$$

Make the transformation $S \to R^{-1/2}TR^{-1/2}$. The Jacobian of the transformation is $|R|^m$. Under this transformation, we have from (2.7)

$$F(\underset{\sim}{R})|\underset{\sim}{R}|^{t+u-m} = \int_{\underset{\sim}{T}=\underset{\sim}{\bar{T}}'>\underset{\sim}{0}}^{\underset{\sim}{R}} |\underset{\sim}{\bar{T}}|^{t-m}|\underset{\sim}{R}-\underset{\sim}{T}|^{u-m}\,\tilde{C}_{\kappa}(\underset{\sim}{T})d\underset{\sim}{T} \tag{2.9}$$

Taking the Laplace transform on both sides of (2.9) we have

$$\int_{\underset{\sim}{R}=\underset{\sim}{\bar{R}}'>0} F(\underset{\sim}{R})|\underset{\sim}{R}|^{t+u-m}\exp(-\mathrm{tr}\underset{\sim}{R}\underset{\sim}{Z})d\underset{\sim}{R} =$$

$$\int_{\underset{\sim}{R}=\underset{\sim}{\bar{R}}'>\underset{\sim}{0}} \left[\int_{\underset{\sim}{T}=\underset{\sim}{\bar{T}}'>\underset{\sim}{0}}^{\underset{\sim}{R}} |\underset{\sim}{T}|^{t-m}|\underset{\sim}{R}-\underset{\sim}{T}|^{u-m}\tilde{C}_{\kappa}(\underset{\sim}{T})d\underset{\sim}{T}\right]\exp(-\mathrm{tr}\underset{\sim}{R}\underset{\sim}{Z})d\underset{\sim}{R} \tag{2.10}$$

After using (2.8) and lemmas (2.1), L.H.S. of (2.10) is given by

$$\text{L.H.S.} = F(\underset{\sim}{I})/C_{\kappa}(\underset{\sim}{I})\tilde{\Gamma}_p(t+u,\kappa)|\underset{\sim}{Z}|^{-(t+u)}\tilde{C}_{\kappa}(\underset{\sim}{Z}^{-1}) \tag{2.11}$$

Let $f_1(\underset{\sim}{T}) = |\underset{\sim}{T}|^{t-m}\tilde{C}_{\kappa}(\underset{\sim}{T})$ and $f_2(\underset{\sim}{T}) = |\underset{\sim}{T}|^{u-m}$ and $g_1(\underset{\sim}{Z})$, $g_2(\underset{\sim}{Z})$ be the Laplace transforms of $f_1(\underset{\sim}{T})$ and $f_2(\underset{\sim}{T})$ respectively, then using (2.3), lemma (2.1) and the covolution theorem, we have R.H.S. of (2.10) in the form

$$\text{R.H.S.} = g_1(\underset{\sim}{Z})g_2(\underset{\sim}{Z}) = \tilde{\Gamma}_m(t,\kappa)\tilde{\Gamma}_m(u)|\underset{\sim}{Z}|^{-(t+u)}\tilde{C}_{\kappa}(\underset{\sim}{Z}^{-1}) \tag{2.12}$$

which proves the lemma.

3. EXACT NON-NULL DISTRIBUTION OF L_{vc}

In this section we derive the non-null density of L_{vc} as a series of Miejer's G-functions [7] using Mellin-integral transform [9]. As in Pillai and Singh [10] using lemma (2.1) of [10], the test of $H: \underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$ reduces to that of $H: \underset{\sim}{\Sigma} = \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\underset{\sim}{I}_{p_2}\end{bmatrix}$,

$\sigma_1, \sigma_2 > 0$ and unknown, against the alternatives $A \neq H$; $p_2 = p - 1$. The likelihood ratio criterion is based on the statistic

$$L_{vc} = |\underset{\sim}{S}|/[s_{11}(\mathrm{tr}\underset{\sim}{S}_{22}/p_2)^{p_2}] \tag{3.1}$$

where

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}'_{12} & \underset{\sim}{S}_{22} \end{bmatrix}\begin{matrix} 1 \\ p_2 \end{matrix} \quad \text{with} \quad n = N - 1,$$

N being the size of the random sample from $CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}' > \underset{\sim}{0}$. Furthermore, we make use of the transformation $\begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{bmatrix} \to \begin{bmatrix} \underset{\sim}{x}_1/\sigma_1 \\ \underset{\sim}{x}_2/\sigma_2 \end{bmatrix}\begin{matrix} 1 \\ p_2 \end{matrix}$

Under this transformation the problem of testing H versus A reduces to the problem of testing H_1: $\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & \underset{\sim}{I}_{p_2} \end{bmatrix}$ versus $A_1 \neq H_1$, where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \underset{\sim}{\Sigma}_{12}/\sigma_1\sigma_2 \\ \underset{\sim}{\Sigma}'_{12}/\sigma_1\sigma_2 & \underset{\sim}{\Sigma}_{22} \end{bmatrix}\begin{matrix} 1 \\ p_2 \end{matrix} \qquad \sigma_1, \sigma_2 \text{ positive and unknown.}$$

From now on we assume that this has been done and we are testing H_1 versus A_1. We now define

$$T = s_{11}^{-1/2}\, \underset{\sim}{S}_{12}\, \underset{\sim}{S}_{22}^{-1}\, \underset{\sim}{S}'_{12}\, s_{11}^{-1/2} \tag{3.2}$$

Then the statistic L_{vc} can be written as

$$L_{vc} = |\underset{\sim}{S}_{22}|(1 - T)/(\mathrm{tr}\underset{\sim}{S}_{22}/p_2)^{p_2} \tag{3.3}$$

We now need the following lemma.

Lemma 3.1. The joint p.d.f. of $\underset{\sim}{T}$, $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$ is given by

$$f(\underset{\sim}{T}, \underset{\sim}{S}_{11}, \underset{\sim}{S}_{22}) = U(p_1, p_2, n, \underset{\sim}{\Sigma})|\underset{\sim}{S}_{11}|^{n-p_1}|\underset{\sim}{S}_{22}|^{n-p_2}\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{11})$$

$$\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{2.1}^{-1}\underset{\sim}{S}_{22})|\underset{\sim}{T}|^{p_2-p_1}|\underset{\sim}{I} - \underset{\sim}{T}|^{n-p_1-p_2}{}_0F_1(p_2, (\bar{\underset{\sim}{S}}_{11}^{1/2})'$$

$$\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'(\bar{\underset{\sim}{\Sigma}}_{1.2})^{-1}\underset{\sim}{S}_{11}^{1/2}\underset{\sim}{T}) \tag{3.4}$$

where

$$\underset{\sim}{\Sigma}_{1.2} = \underset{\sim}{\Sigma}_{11} - \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\bar{\underset{\sim}{\Sigma}}_{12}'$$

$$\underset{\sim}{\Sigma}_{2.1} = \underset{\sim}{\Sigma}_{22} - \bar{\underset{\sim}{\Sigma}}_{12}'\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12}$$

$$\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \bar{\underset{\sim}{\Sigma}}_{12}' & \Sigma_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix} \qquad \text{and} \qquad \underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \bar{\underset{\sim}{S}}_{12}' & \underset{\sim}{S}_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix}$$

and $p_1 + p_2 = p$, $p_2 \geq p_1 \geq 1$ without loss of generality.

$$\underset{\sim}{T} = \underset{\sim}{S}_{11}^{-1/2}\, S_{12}\, S_{22}^{-1}\, \bar{S}_{12}'(\bar{S}_{11}^{-1/2})'$$

$$U^{-1}(p_1, p_2, n, \underset{\sim}{\Sigma}) = \tilde{\Gamma}_{p_2}(n)\tilde{\Gamma}_{p_1}(n - p_2)|\underset{\sim}{\Sigma}_{1.2}|^{n}|\underset{\sim}{\Sigma}_{22}|^{n}\tilde{\Gamma}_{p_1}(p_2)$$

$\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$ and $\underset{\sim}{T}$ are p.d.h. and $\underset{\sim}{0} < \underset{\sim}{T} < \underset{\sim}{I}$.

Proof. Let $\underset{\sim}{S}_{1.2} = \underset{\sim}{S}_{11} - \underset{\sim}{S}_{12}\,\underset{\sim}{S}_{22}^{-1}\,\bar{\underset{\sim}{S}}_{12}'$. It is easy to prove that $\underset{\sim}{S}_{1.2}$ and $(\underset{\sim}{S}_{12}, \underset{\sim}{S}_{22})$ are independently distributed and

$\underset{\sim}{S}_{1.2} \sim CW(\underset{\sim}{S}_{1.2}; p_1, n - p_2, \underset{\sim}{\Sigma}_{1.2})$. Also

$\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}_{12}' \sim CW(\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{S}_{12}'; p_1, p_2, \underset{\sim}{\Sigma}_{1.2}, \underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}')$ given $\underset{\sim}{S}_{22}$, i.e.,

$\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}\bar{\underset{\sim}{S}}'_{12}$ has noncentral complex Wishart distribution with mean matrix $\underset{\sim}{\beta}\underset{\sim}{F}^{-1}$, where $\underset{\sim}{F}$ is s.t. $\underset{\sim}{F}^{-1}(\bar{\underset{\sim}{F}}')^{-1} = \underset{\sim}{S}_{22}$, given $\underset{\sim}{S}_{22}$, where non-central Wishart density is given by (see James [5])

$$CW(\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12};\ p_1,\ p_2,\ \underset{\sim}{\Sigma}_{1.2},\ \underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}') = \exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}')$$

$${}_0\tilde{F}_1(p_2;\ \underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12})\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12})$$

$$|\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}|^{p_2-p_1}/[|\underset{\sim}{\Sigma}_{1.2}|^n\tilde{\Gamma}_{p_1}(p_2)] \tag{3.5}$$

Now, the joint conditional distribution of $\underset{\sim}{S}_{1.2}$ and $\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}$ given $\underset{\sim}{S}_{22}$ is given by

$$dH|\underset{\sim}{S}_{22} = U_1(p_1,\ p_2,\ n,\ \underset{\sim}{\Sigma})\,{}_0\tilde{F}_1(p_2;\ \underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}'\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12})$$

$$|\underset{\sim}{S}_{1.2}|^{n-p_1-p_2}\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{1.2})\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{\beta}\underset{\sim}{S}_{22}\bar{\underset{\sim}{\beta}}')|\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}|^{p_2-p_1}$$

$$\exp(-\mathrm{tr}\underset{\sim}{\Sigma}_{1.2}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12})d(\underset{\sim}{S}_{1.2})d(\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}) \tag{3.6}$$

$$U_1^{-1}(p_1,\ p_2,\ n;\ \underset{\sim}{\Sigma}) = |\underset{\sim}{\Sigma}_{1.2}|^n\tilde{\Gamma}_{p_1}(p_2)\tilde{\Gamma}_{p_1}(n-p_2) \tag{3.7}$$

We now make the following transformation

$$\underset{\sim}{S}_{11} = \underset{\sim}{S}_{1.2} + \underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}$$

$$\underset{\sim}{T} = \underset{\sim}{S}_{11}^{-1/2}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\bar{\underset{\sim}{S}}'_{12}(\bar{\underset{\sim}{S}}'_{11})^{-1/2} \tag{3.8}$$

The Jacobian of the transformation is $|\underset{\sim}{S}_{11}|^{p_1}$ (see Khatri [6]). Hence, the joint conditional density of $\underset{\sim}{T}$, and $\underset{\sim}{S}_{11}$ given $\underset{\sim}{S}_{22}$ is given by

$$h(S_{11}, T|S_{22}) = U_1(p_1, p_2, n, \Sigma)\,{}_0\tilde{F}_1(p_2;\ (\bar{S}_{11}')^{1/2}\Sigma_{1.2}^{-1}\beta S_{22}\bar{\beta}'\Sigma_{1.2}^{-1}S_{11}^{1/2}T)$$

$$\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}S_{11})|S_{11}|^{n-p_1}|I - T|^{n-p_1-p_2}|T|^{p_2-p_1}\exp(-\mathrm{tr}\bar{\beta}'\Sigma_{1.2}^{-1}\beta S_{22})$$

Also $S_{22} \sim CW(n, p_2, \Sigma_{22})$. If $g(S_{22})$ denotes the density of S_{22}, then the joint density of S_{11}, S_{22}, and T is $h(S_{11}, T|S_{22})g(S_{22})$ which will be the same as (3.4) after using the identity
$\Sigma_{22}^{-1} + \bar{\beta}'\Sigma_{1.2}^{-1}\beta = \Sigma_{2.1}^{-1}$.

Now we need the following theorem in order to derive $E(L_{vc})^h$.

Theorem 3.1.

$$E[\exp(-t\ \mathrm{tr}S_{22})|S_{22}|^h(1-T)^h] = U_3(p_2, n, \Sigma, h)\sum_{j=0}^{\infty}\sum_{J}\sum_{k=0}^{\infty}\sum_{\kappa}$$

$$(t+1)^{-p_2(h+n)+k+j}[h]_\kappa[n]_J\,\tilde{C}_\kappa(I - \Sigma_{2.1}^{-1})\,\tilde{C}_J(I - \Sigma_{2.1}^{-1} + (\bar{\Sigma}_{1.2})^{-1}\bar{\beta}'\beta)/k!j! \tag{3.9}$$

where

$$U_3(p_2, n, \Sigma, h) = \tilde{\Gamma}_{p_2}(n-1+h)/[\tilde{\Gamma}_{p_2}(n-1)|\Sigma_{22}|^n] \quad . \tag{3.10}$$

Proof. Let $V = \exp(-t\,\mathrm{tr}S_{22})|S_{22}|^h(1-T)^h$. Now using lemma (3.1) with $p_1 = 1$, we obtain

$$E[V] = U(1, p_2, n, \Sigma)\int_{s_{11}=\bar{s}_{11}>0}\int_{\bar{S}_{22}'=S_{22}>0}\int_{\bar{T}'=T>0}^{I}(s_{11})^{n-1}|S_{22}|^{n+h-p_2}$$

$$\exp(-\mathrm{tr}\Sigma_{1.2}^{-1}s_{11})\exp(-\mathrm{tr}(\Sigma_{2.1}^{-1} + t\ I)S_{22})|T|^{p_2-p_1}|I - T|^{n-p_1-p_2+h}$$

$$\sum_{k=0}\sum_{\kappa}\tilde{C}_\kappa(\bar{s}_{11}^{-1/2}\ \Sigma_{1.2}^{-1}\ \beta S_{22}\bar{\beta}'\ \Sigma_{1.2}^{-1}\ s_{11}^{1/2}T)/([p_2]_\kappa k!)ds_{11}dS_{22} \tag{3.11}$$

Using the monotone convergence theorem, the interchange of the integral and summation signs is valid. Now using lemma (2.2) in order to integrate with respect to $\underset{\sim}{T}$, we get from (3.11)

$$E[V] = U_2 \sum_{k=0}^{\infty} \sum_{\kappa} \int_{s_{11}>0} \int_{\underset{\sim}{S}'_{22}=\underset{\sim}{S}_{22}} s_{11}^{n-1} |S_{22}|^{n+h-p_2} \exp(-\mathrm{tr}\Sigma_{1.2}^{-1} s_{11})$$

$$\exp(-\mathrm{tr}(t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1})\underset{\sim}{S}_{22})\tilde{C}_{\kappa}(s_{11}\underset{\sim}{\bar{\beta}}'\underset{\sim}{\beta}\Sigma_{1.2}^{-2}\underset{\sim}{S}_{22})/(k![n+h]_{\kappa})ds_{11}d\underset{\sim}{S}_{22} \qquad (3.12)$$

where

$$U_2 = U\,(1, p_2, n, \underset{\sim}{\Sigma})\tilde{\Gamma}(p_2)\tilde{\Gamma}(n - p_2 + h)/\tilde{\Gamma}(h + n) \qquad (3.13)$$

Now using lemma (2.1) to integral with respect to S_{22} and then in turn using monotone covergence theorem and the relation (2.5), we get

$$E[V] = U_4\ |t\ I + \underset{\sim}{\Sigma}_{2.1}^{-1}|^{-(n+h)} \int_{s_{11}>0} s_{11}^{n-1} \exp(-(\Sigma_{1.2}^{-1} - \underset{\sim}{\beta}(t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1})^{-1}$$

$$\underset{\sim}{\bar{\beta}}'\Sigma_{1.2}^{-2})s_{11})ds_{11} \qquad (3.14)$$

where $U_4 = U_2\tilde{\Gamma}_{p_2}(n+h)$. Now integrating with respect to s_{11} and using relation (2.5), we get

$$E[V] = U_3(p_2, n, \underset{\sim}{\Sigma}, h)\,|t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1}|^{-h}|t\,\underset{\sim}{I} + \underset{\sim}{\Sigma}_{2.1}^{-1} - \underset{\sim}{\bar{\beta}}'\underset{\sim}{\beta}\underset{\sim}{\Sigma}_{1.2}^{-1}|^{-n} \qquad (3.15)$$

where $U_3(p_2, n, \underset{\sim}{\Sigma}, h)$ is given by (3.10).

Now adding and substracting $\underset{\sim}{I}$ inside each of the two determinants and using (2.5), we have

$$E[V] = U_3(p_2, n, \underset{\sim}{\Sigma}, h)(t+1)^{-p_2(h+n)}\ {}_1\tilde{F}_0(h;\ (t+1)^{-1}(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1}))$$

$${}_1\tilde{F}_0(n;\ (t+1)^{-1}(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1} + \underset{\sim}{\Sigma}_{1.2}^{-1}\ \underset{\sim}{\bar{\beta}}'\underset{\sim}{\beta})). \qquad (3.16)$$

which can be expressed as (3.9) after using (2.4).

Theorem 3.2. For any finite p the p.d.f. of L_{vc} is given by

$$p(L_{vc}) = D_1(p_2, n, \Sigma)(L_{vc})^{-(p_2+1)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} p_2^{-(k+j)}$$

$$B(J, \kappa, p_2, n, \Sigma) G_{2p_2\ \ 2p_2}^{2p_2\ \ 0}\left[L_{vc}\middle| \begin{matrix} c_1, c_2, \ldots, c_{p_2}; d_1, d_2, \ldots, d_{p_2} \\ a_1, a_2, \ldots, a_{p_2}; b_1, b_2, \ldots, b_{p_2} \end{matrix}\right] \quad (3.17)$$

where

$$D_1(p_2, n, \Sigma) = (2\pi)^{(p_2-1)/2} p_2^{\frac{1}{2}-np_2} / \left(\prod_{i=1}^{p_2} \Gamma(n-i)|\Sigma_{22}|^n\right)$$

$$B(J, \kappa, p_2, n, \Sigma) = [n]_J \Gamma(np_2 + k + j)\tilde{C}_\kappa(I - \Sigma_{2.1}^{-1})\tilde{C}_J(I - \Sigma_{2.1}^{-1} + \Sigma_{1.2}^{-1}\bar{\beta}'\beta)/k!j! \quad (3.18)$$

$$a_i = p_2 + n - i, \quad b_i = p_2 - i + 1 + k_i \quad (3.19)$$

$$c_i = p_2 + 1 - i, \quad d_i = p_2 + n + (k + j + i - 1)p_2^{-1}; \quad i = 1, 2, \ldots, p_2 .$$

Proof: First, we evaluate the h-th moment of L_{vc} as the method of derivation of the density of L_{vc} depends on lemma (2.4) of Pillai and Singh [10], concerning the Mellin transform. Integrating both sides of (3.9) with respect to t, p_2h times under the integral sign and putting $t = 0$ in the final result, we get

$$E[L_{vc}]^h = U_3(p_2, n, \Sigma, h)\, p_2^{p_2h} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} [n]_J[h]_\kappa \tilde{C}_\kappa(I - \Sigma_{2.1}^{-1})\tilde{C}_J(I - \Sigma_{2.1}^{-1}$$

$$+ \Sigma_{1.2}^{-1}\bar{\beta}'\beta)/((np_2 + k + j)_{hp_2} k!j!) \quad (3.20)$$

Let

$$D(p_2, n, \Sigma) = 1/\left(|\Sigma_{22}|^n \prod_{i=1}^{p_2} \Gamma(n-i)\right), \quad (3.21)$$

then

$$E[L_{vc}]^h = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J,\kappa, p_2, n, \underset{\sim}{\Sigma}) p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma(h+n-i)$$

$$\prod_{i=1}^{p_2} (h-i+1)_{k_i} / \Gamma(p_2(h+n)+k+j) \qquad (3.22)$$

where $B(J,\kappa, p_2, n, \underset{\sim}{\Sigma})$ is defined by (3.18). Now using Mellin integral transform on both sides of (3.22) we get the density of L_{vc} in the form

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J,\kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$(2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_{vc})^{-(h+1)} p_2^{p_2 h} \prod_{i=1}^{p_2} \frac{\Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i}}{\Gamma(p_2(h+n)+k+j)} \, dh. \qquad (3.23)$$

Now applying the transformation $h \to h + p_2$ and using Gauss - Legendre's multiplication theorem (see (3.22) of [10]) on $\Gamma(p_2(h+n)+k+j))$ we get

$$p(L_{vc}) = D_1(p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{-(p_2+1)} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J,\kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} (2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} (L_{vc})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h+p_2-i+1+k_i) \prod_{i=1}^{p_2} \Gamma(h+n+p_2-i)}{\prod_{i=1}^{p_2} \Gamma(h+p_2-i+1) \prod_{i=1}^{p_2} \Gamma(h+p_2+n+(k+j+i-1)/p_2)} \, dh$$

$$(3.24)$$

where $C_1 = C + p_2$ and $D_1(p_2, n, \underset{\sim}{\Sigma})$ is given by (3.18). (3.24) can also be written as

$$p(L_{vc}) = D_1(p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{-(p_2+1)} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)}(2\pi i)^{-1} \int_{C_i - i\infty}^{C_1 + i\infty} (L_{vc})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h+a_i) \prod_{i=1}^{p_2} \Gamma(h+b_i)}{\prod_{i=1}^{p_2} \Gamma(h+c_i) \prod_{i=1}^{p_2} \Gamma(h+d_i)} dh \qquad (3.25)$$

$a_i^{s'}$, $b_i^{s'}$, $c_i^{s'}$, and $d_i^{s'}$ being defined in (3.19). Noticing that the integrals in (3.25) are in the form of Meijer's G-function (see (2.4) of [10], we can write (3.25) in the form (3.17).

Special Cases. We now discuss the cases $p_2 = 1$ and $p_2 = 2$.

$p_2 = 1$. Putting $p_2 = 1$ in (3.17), we obtain

$$p(L_{vc}) = \frac{(L_{vc})^{-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2)^k G_{2\ 2}^{2\ 0}\left[L_{vc}\middle|\begin{matrix} 1 & n+k+1 \\ n & k+1 \end{matrix}\right] \qquad (3.26)$$

where $\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \rho \\ \bar{\rho} & 1 \end{bmatrix}$, $|\rho|^2 = \rho\bar{\rho}$.

Now using (2.5) of [10], (3.26) can be put in the form

$$p(L_{vc}) = \frac{(L_{vc})^{n-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2))^k$$

$$_2F_1(n, -k, 1; 1 - L_{vc}), \quad 0 < L_{vc} < 1. \qquad (3.27)$$

In particular, under the null hypothesis, H_1: $\rho = 0$, the null density of L_{vc} is given by

$$p_1(L_{vc}) = (L_{vc})^{n-2}\Gamma(n)/\Gamma(n-1), \quad 0 < L_{vc} < 1. \qquad (3.28)$$

$p_2 = 2$. In this case $\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & C\rho_{13} \\ \bar{\rho}_{12} & 1 & C\rho_{23} \\ C\bar{\rho}_{13} & C\bar{\rho}_{23} & C^2 \end{bmatrix}$, $C = \sigma_3/\sigma_2$

Now putting $p_2 = 2$ in (3.17), we obtain

$$p(L_{VC}) = \frac{\Gamma(n)|\underset{\sim}{\Sigma}_{22}|^{-n}}{\Gamma(n-2)\tilde{\Gamma}_2(n)} 2^{1-2n}(\pi)^{3/2}(L_{VC})^{-3} \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} 2^{-(k+j)}$$

$$[n]_J \Gamma(2n+k+j)\tilde{C}_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1}_{2.1})\tilde{C}_J(\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-2}_{2.1} + \underset{\sim}{\Sigma}^{-1}_{1.2}\, \underset{\sim}{\bar{\beta}}'\underset{\sim}{\beta})/k!j!$$

$$G^{4\;0}_{4\;4}\left[L_{VC}\middle|\begin{matrix} c_1,\ c_2;\ d_1,\ d_2 \\ a_1,\ a_2;\ b_1,\ b_2 \end{matrix}\right] \tag{3.29}$$

where

$$a_1 = n+1, \quad a_2 = n; \quad b_1 = 2+k_1, \quad b_2 = 1+k_2$$

$$c_1 = 2, \quad c_2 = 1; \quad d_1 = 2+n+(k+j)/2, \quad d_2 = 2+n+(k+j+1)/2$$

Also under the null hypothesis we have

$$p_1(L_{VC}) = \pi^{3/2}\, 2^{1-2n}\Gamma(2n)/[\Gamma(n-2)\tilde{\Gamma}_2(n)](L_{VC})^{-3} G^{2\;0}_{2\;2}\left[L_{VC}\middle|\begin{matrix} 2+n, & n+3/2 \\ n\ , & n+1 \end{matrix}\right] \tag{3.30}$$

which after using the duplication formula of gamma functions and (2.5) of [10], can be written as

$$p_1(L_{VC}) = \frac{\Gamma(n)\Gamma(n+1/2)}{\Gamma(n-1)\Gamma(n-2)\Gamma(7/2)} (L_{VC})^{n-3}(1-L_{VC})^{5/2}\, {}_2F_1(3/2, 1, 7/2; 1-L_{VC}) \tag{3.31}$$

$$0 < L_{VC} < 1$$

Using the relation $_2F_1(a, b, C; 1) = \Gamma(C)\Gamma(C-a-b)/\Gamma(C-a)\Gamma(C-b)$ (see Erdelyi (3)), it can be checked that

$$\int_0^1 p_1(L_{vc})dL_{vc} = 1.$$

4. THE EXACT NON-NULL DISTRIBUTION OF L_{vc} CRITERION THROUGH CONTOUR INTEGRATION

Starting from (3.23) of Section 3, we have

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{J=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$(2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_{vc})^{-(h+1)} p_2^{p_2 h} \frac{\prod_{i=1}^{p_2} \Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i}}{\Gamma(p_2(h+n)+k+j)} dh \tag{4.1}$$

For simplifications, make the transformation $h+n \to h$. Then (4.1) can be written as

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{n-1}$$

$$p_2^{-np_2}(2\pi i)^{-1} \int_{C_1-i\infty}^{C_1+i\infty} (L_{vc}/p_2^{p_2 h})^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h-i) \prod_{i=1}^{p_2} (h-n-i+1)_{k_i}}{\Gamma(p_2 h+k+j)} dh \tag{4.2}$$

where $C_1 = C+n$ and

$$D^{-1}(p_2, n, \underset{\sim}{\Sigma}) = |\Sigma_{22}|^n \prod_{i=1}^{p_2} \Gamma(n-i)$$

$$B(J, \kappa, p_2, n, \underset{\sim}{\Sigma}) = [n]_J \Gamma(np_2+k+j) \tilde{C}_\kappa(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1}) \tag{4.3}$$

$$\tilde{C}_J(\underset{\sim}{I} - \underset{\sim}{\Sigma}_{2.1}^{-1} + \underset{\sim}{\Sigma}_{1.2}^{-1}\bar{\beta}'\beta)/k!j!$$

Let

$$L_1 = L_{vc}/p_2^{p_2}, \tag{4.4}$$

then (4.2) can be written as

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})(L_{vc})^{n-1} p_2^{-np_2} f(L_{vc}). \tag{4.5}$$

where

$$f_{J,\kappa}(L_{vc}) = (2\pi i)^{-1} \int_{C_1 - i\infty}^{C_1 + i\infty} G(h)dh \tag{4.6}$$

and

$$G_{J,\kappa}(h) = (L_1)^{-h} \prod_{i=1}^{p_2} \Gamma(h-i) \prod_{i=1}^{p_2} (h-n-i+1)_{k_i} / \Gamma(p_2 h + k + j) \tag{4.7}$$

For ease in typing, the functions $f_{J,\kappa}$, $G_{J,\kappa}$, $R_{J,\kappa}$ will be written as f, G, R respectively throughout this Chapter.

We now consider a special case.

$\underline{p_2 = 1}$. We have from (4.2)

$$p(L_{vc}) = \frac{1}{\Gamma(n-1)} (L_{vc})^{n-1} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} (-|\rho|^2/(1-|\rho|^2))^k$$

$$(2\pi i)^{-1} \int_{C_1 - i\infty}^{C_1 + i\infty} (L_{vc})^{-h} \Gamma(h-1)(h-n)_k / \Gamma(h+k)dh \tag{4.8}$$

The integral in (4.8) will be evaluated by contour integration. The poles of the integrand are at points

$$h = -\ell, \quad \ell = -1, 0, 1, 2, \ldots \tag{4.9}$$

The residue at these points can be found by putting $h = t - \ell$ in (4.8) and taking the residue of the integrand at $t = 0$. The integrand is given by

$$G(t-\ell) = (L_{vc})^{-t+\ell}\Gamma(t-\ell-1)(t-\ell-n)_k/\Gamma(t-\ell+k). \qquad (4.10)$$

To evaluate the integral in (4.8), we need to consider separately, the cases (A) $\ell < k$ (B) $\ell \geq k$.

<u>CASE A: $\ell < k$; $\ell = -1, 0, 1, \ldots, k-1$.</u> In this case, after expanding the gamma functions (4.10) can be written as

$$G(t-\ell) = (L_{vc})^{-t+\ell}\Gamma(t+1)(t-\ell-n)_k/(t\prod_{\delta=1}^{\ell+1}(t-\delta)\Gamma(t+k-\ell)). \qquad (4.11)$$

The integrand $G(t-\ell)$ in (4.11) has a simple pole of first order at $t=0$, and the residue at this point is given by

$$R_\ell = \lim_{t\to 0} t\, G(t-\ell) \quad ,$$

and

$$R_\ell = (L_{vc})^{\ell}(-\ell-n)_k{}^{(-1)^{\ell+1}}/((\ell+1)!\Gamma(k-\ell)). \qquad (4.12)$$

<u>CASE B: $\ell \geq k$; $\ell = k, k+1, \ldots$.</u> After expanding the gamma functions in (4.10), we get

$$G(t-\ell) = (L_{vc})^{-t+\ell}(t-\ell-n)_k \prod_{\delta=1}^{\ell-k}(t-\delta)/\prod_{\delta=1}^{\ell+1}(t-\delta). \qquad (4.13)$$

The integrand in (4.13) does not have any pole at $t=0$.

Thus from (4.12) and (4.13) and using Cauchy's residue theorem, the integral in (4.8) for this case is given by

$$f(L_{vc}) = \sum_{\ell=1}^{k-1} R_\ell \;, \qquad (4.14)$$

and the density (4.8) is given by

$$p(L_{vc}) = \frac{(L_{vc})^{n-2}}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!} \left\{\frac{-|\rho|^2}{1-|\rho|^2}\right\}^k \sum_{v=0}^{k} \frac{(-L_{vc})^v(-v-n+1)_k}{v!\Gamma(k+1-v)} ,$$

$$0 < L_{vc} < 1 , \qquad (4.15)$$

which after using Vandermonde's theorem (see Erdély; [3])

$$_2F_1(-n, b; c; 1) = (c-b)_n/(c)_n \quad c \neq 0, -1, -2, \ldots \qquad (4.16)$$

and for other b and c, reduces to (3.27) of Section 3. This form of the density has been used for power computations, which are presented in Table (2.1).

Now for finding the density of L_{vc} for $p_2 \geq 2$, we still use the method of contour integration but the density now will involve psi functions and their derivative. We will make use of lemma (4.1) of Pillai and Singh [10] in this connection. Throughout the rest of this paper all empty products $\prod_{i=m}^{n}(\cdot)$ and empty sums $\sum_{i=m}^{n}(\cdot)$ for $m > n$ will be treated as 1 and 0 respectively.

Now from (4.7), the poles of the integrand $G(h)$ are at points

$$h = -\ell, \ \ell = -p_2, -p_2+1, \ldots, -1, 0, 1, 2, \ldots \qquad (4.17)$$

To compute the residue at these poles, we put $h = t - \ell$ in (4.7) and find the residue at $t = 0 \ \forall \ \ell$. Now, (4.7) can be written as

$$G(t-\ell) = (L_1)^{-t+\ell} \prod_{i=1}^{p_2} (t-\ell-n-i+1)_{\kappa_i} \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)+k+j) \qquad (4.18)$$

Let $c = k+j-p_2\ell$. Two cases arise: (A) $\ell \geq 0$ (B) $\ell < 0$.

Let

$$GP(t) = \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)+k+j) \tag{4.19}$$

The poles of the integrand in (4.18) are the poles of (4.19)

<u>CASE A: $\ell \geq 0$</u>. Two subcases: (A1) $c \leq 0$ and (A2) $c > 0$.

<u>SUBCASE A1: $\ell \geq 0$ and $c \leq 0$</u>. Expanding the gamma functions in (4.19) we obtain

$$GP(t) = p_2(\Gamma(t+1))^{p_2} t^{-(p_2-1)} \prod_{\delta=1}^{-c} (tp_2-\delta)/(\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (t-\delta)) \tag{4.20}$$

Thus for $\ell \geq 0$ and $k+j \leq p_2\ell$, the pole of the integrand $G(t-\ell)$ is of order $p_2 - 1$.

In the following the functions A, GP, B, C, G, R depend upon j and k, but for the ease of typing the subscripts j, k will be supressed. Now using (4.20), (4.18) can be written as

$$G(t-\ell) = (L_1)^{\ell} a_0' t^{-(p_2-1)} A_{j,k}(t) \tag{4.21}$$

where

$$a_0' = (-1)^{k+j-p_2(p_2+1)/2} \cdot p_2(-c)! \prod_{i=1}^{p_2} (-\ell-n-i+1)_{k_i} / \prod_{i=1}^{p_2} (\ell+1)! \tag{4.22}$$

$$A(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=0}^{k_i-1} (1+t/(\delta-\ell-n-i+1))(\Gamma(t+1))^{p_2} \prod_{\delta=1}^{-c} (1-tp_2/\delta)/ \\ (\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)) \tag{4.23}$$

The residue of order $p_2 - 1$ at $t = 0$ is given by,

$$R_{\ell} = (L_1)^{\ell} a_0'/\Gamma(p_2-1) \left(\frac{d}{dt}\right)_{t=0}^{p_2-2} \exp(\log A(t)) . \tag{4.24}$$

Using (4.36), (4.37), (4.38) of [10], we can write $\log A(t)$ as

$$\log A(t) = a_1 t + a_2 t^2/2! + a_3 t^3/3! + \cdots , \tag{4.25}$$

where

$$a_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} 1/(\delta - n - \ell - i + 1) - \sum_{\delta=1}^{-c} (p_2/\delta) + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \tag{4.26}$$

and for $q \geq 2$, we have

$$a_q = (p_2 - p_2^q)\psi_{q-1}(1) + (q-1)! \left[\sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} (-1)^{q+1}/(\delta - n - \ell - i + 1)^q - \sum_{i=1}^{-c} (p_2/\delta)^q + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \right] .$$

Using (4.25) in (4.24) and lemma (4.1) of [10], we get

$$R_\ell = (L_1)^\ell a_0' D_{p_2-2}(L_1;\, a)/\Gamma(p_2 - 1) \tag{4.27}$$

where

$$D_{p_2-2}(L_1;\, a) = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & a_1 & -1 & \cdots & \\ a_3 & 2a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{p_2-2} & \binom{p_2-3}{1} a_{p_2-3} & \binom{p_2-4}{2} a_{p_2-4} & \cdots & a_1 \end{vmatrix} \tag{4.28}$$

where a_q's are defined in (4.26).

<u>SUBCASE A2: $\ell \geq 0$ and $c > 0$</u> i.e., $k + j > p_2\ell$. Expanding the gamma functions in (4.19), we get

$$G(P(t) = (\Gamma(t+1))^{p_2}\ t^{-p_2}/(\prod_{i=1}^{p_2}\ \prod_{\delta=1}^{\ell+i}\ (t-\delta)\Gamma(p_2t+c)) \qquad (4.29)$$

Thus in this case we have a pole of order p_2 at $t=0$. Using (4.29) in (4.18), we have

$$G(t-\ell) = (L_1)^{\ell}\ t^{-p_2}\ b_0'\ \exp(\log B(t)) \qquad (4.30)$$

where

$$b_0' = (-1)^{\ell p_2+p_2(p_2+1)/2} \prod_{i=1}^{p_2} (-\ell-n-i+1)_{k_i} / \prod_{i=1}^{p_2} (\ell+i)! \qquad (4.31)$$

and

$$B(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=1}^{k_i-1} (1+t/(\delta-\ell-n-i+1))(\Gamma(t+1))^{p_2}/(\Gamma(tp_2+c)$$
$$\prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)) \ .$$

Using (4.36), (4.37), and (4.38) Pillai and Singh [10], $\log B(t)$ can be written as

$$\log B(t) = -\log\Gamma(c) + b_1t + b_2t^2/2! + b_3t^3/3! + \cdots \ , \qquad (4.32)$$

where

$$b_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=1}^{k_i-1} 1/(\delta-n-\ell-i+1) + p_2(\psi(1)-\psi(c)) + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \qquad (4.33)$$

and for $q \geq 2$, we have

$$b_q = p_2\psi_{q-1}(1) - p_2^q\psi_{q-1}(c) + (q-1)!\left[\sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} (-1)^{q+1}/(\delta-n-\ell-i+1)^q + \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q\right]$$

using (4.32) and lemma (4.1) Pillai and Singh [10], the residue at t=0 is given by

$$R_{\ell} = (L_1)^{\ell} b_0 D_{p_2-1}(L_1;\ b)/\Gamma(p_2)\ , \tag{4.34}$$

where

$$b_0 = b_0'/\Gamma(c) \tag{4.35}$$

and b_0' is given by (4.31) and b_q's are given by (4.33). The determinant $D_{p_2-1}(L_1;\ b)$ is equal to the determinant on the right hand side of (4.28) with p_2-1 rows and a_q's replaced by b_q's; $q = 1, 2, \ldots, p_2-1$.

<u>CASE B:</u> $\ell < 0$ i.e., $\ell = -p_2, -p_2+1, \ldots, -2, -1$. For this case, (4.19) after the expansion of gamma functions can be written as

$$GP(t) = (t)^{-(p_2+\ell+1}(\Gamma(t+1))^{p_2+\ell+1} \prod_{i=1}^{-\ell-1} \Gamma(t-\ell-i)/(\Gamma(tp_2+c) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (t-\delta)) \tag{4.36}$$

Thus in this case, we have a pole of order $p_2+\ell+1$ at $t = 0$. Using (4.36) in (4.18), we have

$$G(t-\ell) = (L_1)^{\ell} C_0'(t)^{-(p_2+\ell+1)} C(t) \tag{4.37}$$

where

$$C_0' = (-1)^{(p_2+\ell)(p_2+\ell+1)/2} \prod_{i=1}^{p_2} (-\ell-n-i+1)_{k_i} / \prod_{i=-\ell}^{p_2} (\ell+i)! \tag{4.38}$$

$$C(t) = (L_1)^{-t} \prod_{i=1}^{p_2} \prod_{\delta=0}^{k_i-1} (1 + t/(\delta - \ell - n - i + 1)) \prod_{i=1}^{-\ell-1} \Gamma(t - \ell - i)$$

$$(\Gamma(t+1))^{p_2+\ell+1} / (\Gamma(tp_2 + c) \prod_{i=-\ell}^{p_2} \prod_{\delta=1}^{\ell+i} (1 - t/\delta)) \tag{4.39}$$

Thus the residue at $t = 0$ is given by

$$R_\ell = (L_1)^\ell C_0 \left(\frac{d}{dt}\right)_{t=0}^{p_2+\ell} \exp(\log C(t))/\Gamma(p_2 + \ell + 1) \tag{4.40}$$

where after using (4.36), (4.37), and (4.38) of [10], $C(t)$ can be written as

$$\log C(t) = C_0'' + C_1 t + C_2 t^2/2! + C_3 t^3/3! + \cdots \tag{4.41}$$

where

$$C_0'' = \log(\prod_{i=1}^{-(\ell+1)} \Gamma(-\ell - i)/\Gamma(c), \quad \text{where} \quad c = k + j - p_2\ell \tag{4.42}$$

$$C_1 = -\log L_1 + \sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} 1/(\delta - \ell - n - i + 1) + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} 1/\delta$$

$$+ \sum_{i=1}^{-(\ell+1)} \psi(-i - \ell) - p_2\psi(c) + (p_2 + \ell + 1)\psi(1)$$

and for $q \geq 2$, we have

$$C_q = \sum_{i=1}^{-(\ell+1)} \psi_{q-1}(-\ell - i) - p_2^q \psi_{q-1}(c) + (p_2 + \ell + 1)\psi_{q-1}(1) +$$

$$(q-1)! \left[\sum_{i=1}^{p_2} \sum_{\delta=0}^{k_i-1} (-1)^{q+1}/(\delta - \ell - n - i + 1)^q + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q \right]$$

and let

$$C_0 = C_0' \cdot \exp(C_0'')$$

Now appealing to lemma (4.1) of Pillai and Singh [10], and using (4.41) in (4.40), we have

$$R_\ell = (L_1)^\ell C_0 D_{p_2+\ell}(L_1;\, \mathbf{c})/\Gamma(p_2+\ell+1) \tag{4.43}$$

where the determinant $D_{p_2+\ell}(L_1; \mathbf{c})$ is equal to the determinant on the right hand side of (4.28) with a_q's replaced by C_q's, $q = 1, 2, \ldots, p_2+\ell$ and have $p_2+\ell$ rows. Hence, for any $p_2 \geq 1$, we have from (4.5), (4.6) and Cauchy's residue theorem, the non-null density of L_{vc} in the form

$$p(L_{vc}) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$(L_{vc})^{n-1} p_2^{-np_2} \left[\sum_{\substack{\ell \geq 0 \\ k+j \leq p_2\ell}} R_\ell + \sum_{\substack{\ell \geq 0 \\ k+j > p_2\ell}} R_\ell + \sum_{\ell=-1}^{-p_2} R_\ell \right] \tag{4.44}$$

where R_ℓ's are given in (4.27), (4.34), and (4.43). If we put $p_2 = 1$ in (4.44), we get (4.15).

5. DISTRIBUTION OF L_{vc} AS A CHI-SQUARE SERIES

In this section, we express the density of L_{vc} as a chi-square series using methods similar to those of Chapter I.

Let $\lambda = (L_{vc})^n$ and $\lambda^* = -2\rho\log\lambda$, where ρ is chosen so that the rate of convergence of the resulting series can be controlled, $\rho \geq 0$. Let $\phi(t)$ be the characteristic function of λ^*. Then

$$\phi(t) = E(L_{vc})^{-2it\rho n} \tag{5.1}$$

In Section 3, we obtained the non-null moments $E[L_{vc}]^h$ for integral values of h. But the result (3.22) can be extended to any complex number h by analytic continuation. So, we have for any complex number h

$$E[L_{vc}]^h = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma(h+n-i) \prod_{i=1}^{p_2} (h-i+1)_{k_i} / \Gamma(p_2(h+n)+k+j) \tag{5.2}$$

where $B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})$ is defined by (3.18). Using (5.2), (5.1) can be written as

$$\phi(t) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-2np_2\rho it} \prod_{\delta=1}^{p_2} (1-2it\rho n-\delta)_{k_\delta} \prod_{\delta=1}^{p_2} \Gamma(n(1-2it\rho)-\delta)/\Gamma(np_2(1-2\rho it)+k+j) \tag{5.3}$$

Note that $\phi(0) = 1$ (using $\underset{\sim}{\Sigma}_{22}^{-1} = \Sigma_{2.1}^{-1} - \underset{\sim}{\Sigma}_{1.2}^{-1} \underset{\sim}{\beta}'\underset{\sim}{\beta}$) and for $t \neq 0$, (5.3) can be written as

$$\phi(t) = D(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, k, p_2, n, \underset{\sim}{\Sigma}) \exp(\log G(t)) \tag{5.4}$$

where $G_{j,k}(t)$ is denoted by $G(t)$ and is given by

$$G(t) = \frac{p_2^{-2np_2 it} \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)-\delta+n(1-\rho)) \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)+k_\delta+1-\delta-n\rho)}{\Gamma(np_2\rho(1-2it)+k+j+p_2 n(1+\rho)) \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it)+1-\delta-n\rho)} \tag{5.5}$$

Throughout this section functions G, W, w, R all depend upon j and k, but for simplicity of notation the subscripts or the superscripts j, k will not be explicitly given unless necessary. From (5.5) taking logarithm on both sides, we get

$$\log G(t) = -2np_2 it \log p_2 + \sum_{\delta=1}^{p_2} \log\Gamma(n\rho(1-2it) - \delta + n(1-\rho))$$
$$+ \sum_{\delta=1}^{p_2} \log\Gamma(n\rho(1-2it) + k_\delta + 1 - \delta - n\rho) - \log\Gamma(np_2\rho(1-2it) + k + j$$
$$+ p_2 n(1-\rho)) - \sum_{\delta=1}^{p_2} \log\Gamma(n\rho(1-2it) + 1 - \delta - n\rho) \qquad (5.6)$$

Using the expansion (5.7) of Pillai and Singh [10], for each of the gamma functions in (5.6), we obtain

$$\log G(t) = (p_2 - 1)/2 \log 2\pi - (k + j + p_2 n - \tfrac{1}{2}) \log p_2$$
$$- (j + p_2 + (p_2^2 - 1)/2) \log(n\rho(1-2it)) + \sum_{r=1}^{m} (\rho n(1-2it))^r w_r$$
$$+ R_{m+1}^0(n, t) \quad , \qquad ((5.7)$$

where the coefficients w_r are given by

$$w_r = \left[\sum_{\delta=1}^{p_2} B_{r+1}(1-\delta-n\rho) - \sum_{\delta=1}^{p_2} B_{r+1}(1+k_\delta-\delta-n\rho) + B_{r+1}(k+j+p_2 n(1-\rho))/p_2^r\right.$$
$$\left. - \sum_{\delta=1}^{p_2} B_{r+1}(n(1-\rho)-\delta)\right](-1)^r/(r(r+1)) \qquad (5.8)$$

Thus $G(t)$ is given by

$$G(t) = (2\pi)^{(p_2-1)/2} (n\rho(1-2it))^{-(j+p_2+(p_2^2-1)/2)} p_2^{-(k+j+p_2n-\frac{1}{2})}$$

$$\sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-r} + R'_{m+1}(n, t) \tag{5.9}$$

where W_r is the coefficient of $((1-2it)\rho n)^{-r}$ in the expansion of $\exp(\sum_{r=1}^{m}((1-2it)\rho n)^{-r} w_r)$.

Let $u = p_2 + p_2^2/2 + j - \frac{1}{2}$. Then (5.9) can be written as

$$G(t) = (2\pi)^{(p_2-1)/2} p_2^{-(k+j+p_2n-\frac{1}{2})} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-(r+u)} + R'_{m+1}(n, t) \tag{5.10}$$

Hence the characteristic function of λ^* is given by

$$\phi(t) = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-(r+u)} + R''_{m+1}(n, t). \tag{5.11}$$

where

$$D_1(p_2, n, \underset{\sim}{\Sigma}) = D(p_2, n, \underset{\sim}{\Sigma})(2\pi)^{(p_2-1)/2} p_2^{(\frac{1}{2}-np_2)}$$

Since $(1-i\beta t)^{-\alpha}$ is the characteristic function of the gamma density $g_\alpha(\beta, x)$, where

$$g_\alpha(\beta, x) = [\beta^\alpha \Gamma(\alpha)]^{-1} x^{\alpha-1} e^{-x/\beta} \tag{5.12}$$

Thus the density of λ^* can be derived from (5.11) in the form

$$p(\lambda^*) = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma}) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r g_{r+u}(2, \lambda^*) + R'''_{m+1}(n) \tag{5.13}$$

Hence the probability that λ^* is larger than any value, say λ_0 is

$$p[\lambda^* > \lambda_0] = D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma}) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r G_{r+u}(2, \lambda_0) + R_{m+1}(n), \tag{5.14}$$

where

$$G_{r+u}(2, \lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2, x) dx \tag{5.15}$$

and

$$R_{m+1}(n) = (2\pi)^{-1} D_1(p_2, n, \underset{\sim}{\Sigma}) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} B(J, \kappa, p_2, n, \underset{\sim}{\Sigma})$$

$$p_2^{-(k+j)} \int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} \bar{e}^{it\lambda^*} \sum_{r=0}^{\infty} W_r (\rho n)^{-(r+u)} (1 - 2it)^{-(r+u)} [\exp(R_{m+1}'''(n)) - 1] dt d\lambda^* \tag{5.16}$$

From (5.14), we get the distribution of λ^* as a series of chi-square distributions. Now

$$P[\lambda^* > \lambda_0] = P[-2\rho \log(L_{vc})^n > \lambda_0] = P[L_{vc} < \exp(-\lambda_0/2n\rho)] \tag{5.17}$$

Therefore, once we know the distribution of λ^*, the distribution of L_{vc} can be obtained by using (5.17).

6. <u>POWER COMPUTATIONS OF L_{vc} CRITERION</u>

Powers have been computed for $p = 2$ using (3.27) and (4.15) which have been tabulated in Table (2.1). The computations were carried out on CDC 6500 computer at Purdue University Computing Center. Before computing the power for specific values of the parameter the total probability for that case has been computed and the number of decimals included in the tables were determined depending upon the number of places of accuracy obtained in the total probabilities. From Table (2.1), we observe that power increases with the sample size N as well as the parameter $|\rho|$.

Table 2.1

Power Computations For Wilks' L_{vc} Criterion

$p = 2$

N \ $\lvert\rho\rvert^2$	$.0^{4}1$	$.0^{3}1$	$.0^{2}1$	$.0^{2}5$	.01	.1	.15
3	.05000095	.0500095	.050095	.05048	.05096	.06047	.06655
4	.050002	.050023	.05023	.05117	.05236	.07670	.9301
5	.050004	.050038	.05038	.05191	.05385	.0947	.1228
6	.050005	.05005	.05053	.05266	.05537	.1136	.1542
7	.050007	.050068	.05068	.05342	.05691	.1331	.1866
8	.050008	.050083	.05083	.05418	.05846	.1531	.2195
9	.0500097	.050098	.05098	.05494	.06002	.1735	.2528
10	.050011	.05011	.05113	.05571	.06158	.1942	.2861
15	.050019	.05019	.05188	.05956	.06952	.2993	.4469
20	.050026	.05026	.05264	.06347	.07762	.4026	.5877
25	.050034	.05034	.05339	.06741	.08589	.4992	.7022
30	.050041	.05041	.05415	.07140	.09429	.5864	.7905
40	.050056	.05056	.05568	.07950	.1115	.7284	.9028
50	.05007	.05071	.05721	.08774	.1292	.8290	.9579
60	.05009	.05086	.05874	.09614	.1473	.8960	.9827

Table 2.1 (Continued)

N \ $\|\rho\|^2$	.2	.25	.3	.35	.4	.45
3	.07331	.08089	.08944	.09916	.1103	.1232
4	.1117	.1332	.1579	.1864	.2192	.2572
5	.1553	.1926	.2352	.2835	.3377	.3980
6	.2010	.2543	.3138	.3794	.4502	.525
7	.2477	.3160	.3905	.4695	.5512	.63
8	.2946	.3765	.4631	.5515	.6387	.73
9	.3409	.4347	.5306	.6245	.712	
10	.3863	.4901	.5924	.6883	.77	
15	.5891	.7141	.8147	.88		
20	.7409	.8524	.923	.99		
25	.8441	.9284	.99			
30	.9097	.967				
35	.9494					
40	.9724					

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