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INSTANTANEOUS FEEDBACK IN POINT PROCESS DIFFERENTIALS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

by

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1. Introduction and Statement of the Problem

It is desirable in applications to express a stochastic process as a solution of a stochastic differential equation where the equation has a point process differential whose jump intensity depends on the paths of the solution. This can be described symbolically as:

$$X_t = K_t + \int_0^t F(X)_s dM_s + \int_0^t G(X)_s dN(\lambda(X))_s \tag{1.1}$$

where $N(\lambda(X))_t$ is a point process whose jump intensity at time t is $\lambda(X_u, u \le t)$. In this paper we slightly extend the theory of stochastic differential equations to include an equation which allows us to create such a model.

In section two we prove an existence and uniqueness theorem which is not covered in the existing literature. We show solutions exist up to explosion times. We then use this equation to show a solution exists to an equation such as (1.1). In section three we discuss ways to assure that the explosion time is a.s. infinite. In section four we indicate how the model might be used in economics. For all inexplained terms we refer the reader to [2] or [3].

We wish to thank the economist B. Wernerfelt for suggesting this investigation to one of us and for his advice.

2. Theorems and Proofs

Rather than seek ultimate generality in proving existence and uniqueness of solutions, we content ourselves with penultimate, which suffices for applications. On a fixed filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ let Z be an adapted process with cadlag (i.e., right continuous and left limited) paths.

Let $h_t = |\Delta Z_t|$, where $\Delta Z_t = Z_t - Z_{t-}$. Define the random measure

(2.1) $$\beta(dt,dx) = \sum_{s>0} \varepsilon_{(s,h_s)}(ds,dx)$$

where ε_a is point mass at a. Let K be another adapted, cadlag process, and let M be a semimartingale. Under appropriate conditions on F, G, and H we will show a unique solution exists up to an explosion time of:

(2.2) $$X_t = K_t + \int_0^t F(X)_s dM_s + \int_0^t \int_{\mathbb{R}} G(X)_s 1_{\{H(X)_{s-}<x\}} \beta(ds,dx).$$

(2.3) DEFINITION. An operator F mapping adapted, cadlag processes to predictable processes is M-acceptable for a semimartingale M if a unique solution exists to the equation $X_t = K_t + \int_0^t F(X)_s dM_s$ for every cadlag adapted process K.

There has been much recent progress in finding sufficient conditions for F to be acceptable (cf., e.g., [2] or [3]). One simple condition given by M. Emery [1] is that if X, Y are adapted, cadlag and $X^{T-} = X_t 1_{\{t<T\}} + X_{T-} 1_{\{t\geq T\}}$, then $X^{T-} = Y^{T-}$ implies $(FX)^T = (FY)^T$ for any stopping time T; that F(0) is dM integrable; and that there exists a constant K such that $(FX-FY)^* \leq K(X-Y)^*$, where $X_t^* = \sup_{s\leq t}|X_s|$.

(2.4) DEFINITION. Let D[0,t] be the space of cadlag functions on [0,t]. Let j be a functional such that $j(f,\omega,t)$ is an adapted process with paths in D[0,t] for any $f \in D[0,t]$. An operator H is said to be cadlag postive if there exists such a j such that $H(X)_t(\omega) = j(X_u(\omega), u\leq t; \omega, t)$ and $H(X)_{t-}(\omega) > 0$ for each adapted process X with paths in D[0,t].

(2.5) THEOREM. *Let* F *be* M-*acceptable*, H *be cadlag positive, and let* G *map adapted cadlag processes to predictable processes such that* $Y^{T-} = Z^{T-}$ *implies* $G(Y)^T = G(Z)^T$. *Then there exists a unique solution of* (2.2) *determined up to a strictly postive explosion time* T.

Proof. Let X^1 be the unique solution of

$$X_t^1 = K_t + \int_0^t F(X^1)_s dM_s .$$

Let $T^1 = \inf\{t>0: h_t > H(X^1)_{t-}\}$. Then $T^1 > 0$ a.s. Let X^2 be the unique solution of

$$X_t^2 = K_t + \int_0^t F(X^2)_s dM_s + GX_{T^1}^2 1_{[\![T^1,\infty[\![} .$$

Then $X^1 = X^2$ on $[\![0,T^1[\![$. Proceeding inductively let

$$T^n = \inf\{t>T^{n-1}: h_t \; H(X^n)_{t-}\}$$

and let X^{n+1} satisfy the equation

$$X_t^{n+1} = K_t + \int_0^t F(X^{n+1})_s dM_s + \sum_{i=1}^n GX_{T^i}^{n+1} 1_{[\![T^i,\infty[\![} .$$

Note that $X^n = X^{n+k}$ on $[\![0,T^n[\![$. Let $T = \sup_n T^n$, and let $X = X^n$ on $[\![0,T^n[\![$. Then X is well defined on $[\![0,T[\![$ and is a solution of (2.2) up to T. □

We call T an explosion time because if $T < \infty$ then either $|X_{T-}| = \infty$ or X_{T-} does not exist as a limit.

We now assume $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ supports a Lévy process Z (i.e., Z has stationary, independent increments) with Lévy measure $\nu(dx) = 1_{\{x\neq 0\}} \frac{1}{x^2} dx$. (Thus Z has a lot of small jumps.) Set $h_t = |\Delta Z_t|$, where h is the process used to define $\beta(dt,dx)$ in (2.2). Let λ be a cadlag positive operator and define a new cadlag positive operator H by $H(Y) = 1/\lambda(Y)$.

Recall that if N is a point process, its (stochastic) jump intensity is a nonnegative, adapted, measurable process λ such that $N_t - \int_0^t \lambda_s ds$ is a local martingale. Such a λ need not exist for an arbitrary point process.

(2.6) THEOREM. With h and H as given above, and with X the unique solution of (2.2), then up to an explosion time T we may write X as the solution of

$$X_t = K_t + \int_0^t F(X)_s dM_s + \int_0^t G(X)_s dN_s$$

where N is a point process with jump intensity $\lambda(X)_t$.

Proof. Define $N_t = \int_0^t \int_{\mathbb{R}} 1_{\{H(X)_{s-}<x\}}\beta(ds,dx)$ where H and β are as described in the preceding paragraphs. In view of Theorem (2.5), one need check only that the intensity of N is indeed $\lambda(X_u;u\leq t)_{0\leq t<T}$. Let C be a bounded predictable process and define the process:

$$J_t = \int_0^t \int_0^\infty C_s 1_{\{H(X)_{s-}>x\}}\beta(ds,dx).$$

We then have that

$$J_t = \sum_{s\leq t} C_s 1_{\{H(X)_{s-}<|\Delta Z_s|\}} 1_{\{\Delta Z_s\neq 0\}}.$$

The dual predictable projection of J (cf [2, pp. 90-92]), $\tilde{J}$, is (for $t<T$):

$$\tilde{J}_t = \int_0^t \int_{-\infty}^\infty C_s 1_{\{H(X)_{s-}<|x|\}} \tfrac{1}{2}\,\nu(dx)ds$$

$$= \int_0^t \int_{-\infty}^\infty C_s 1_{\{H(X_{s-})<|x|\}} \frac{1}{2x^2}\, dxds$$

$$= \int_0^t C_s \int_{H(X)_{s-}}^{\infty} (1/x^2)dxds$$

$$= \int_0^t C_s(1/H(X)_{s-})ds$$

$$= \int_0^t C_s\lambda(X_u;u<s)ds,$$

and the result follows. □

3. Infinite Explosion Times

We discuss here conditions one can place on G and H in order to ensure that the explosion time in Theorem 2.5 is a.s. infinite. Which conditions one might want to use will depend on the application.

In order to have an "explosion" at a finite time T one easily sees from the proof of Theorem 2.5 that one must have $\liminf_{t\to T} H(X)_t = 0$. By the definition of cadlag positive this implies that X cannot be extended from $[\![0,T[\![$ to $[\![0,T]\!]$ and be cadlag.

Since X must either explode or have an oscillatory discontinuity, we see that necessary conditions to have an "explosion" at $T < \infty$ are:

(3.1) $$\liminf_{t\to T} H(X)_t = 0$$

(3.2) $$\sum_{n=1,\infty} |GX_{T^n}| = \infty \quad \text{a.s.}$$

where the stopping times (T^n) are as constructed in the proof of Theorem 2.5.

One easily sees from (3.1) and (3.2) above that the following conditions are each sufficient to assure that there is no explosion at a finite time.

(3.3) Take the cadlag positive operator H to have the additional property that for some $\varepsilon > 0$, Y cadlag on $[\![0,T]\!]$ implies $H(Y) \geqq \varepsilon$ on $[\![0,T]\!]$.

(3.4) Take the operator G in (2.2) to be nonnegative and have compact support in the sense that there exists an $N > 0$ such that $\sup_{s<T} |Y_s| \geqq N$ implies $GY_T = 0$.

4. An Economics Example

Suppose we wish to model the purchase of an inexpensive convenience good, where a consumer scouts for new offers of sales of the goods. If prices have been relatively low the consumer may reduce his efforts to find new bargains, hence the consumer scouts with varying intensity and therefore finds new offers at varying rates, with the rate intensity depending on the price. All the while, the consumer is continually consuming at the lowest price yet seen.

Mathematically, let:

$X_t(\omega)$ = last price (or offer) seen before or at time t

$Y_t(\omega)$ = last price actually paid by the consumer before or at time t

$g(t,\omega,x)$ = change at time t in size of offer if last offer was x

$\lambda(t,\omega,y)$ = instantaneous arrival rate of offers at time t if last price paid by the consumer was y.

The model described here implies that

$$Y_t(\omega) = \inf_{0 \leqq s \leqq t} X_s(\omega)$$

and X will satisfy the equation:

$$X_t = X_0 + \int_0^t g(s,\omega,X_{s-}(\omega))dN(\lambda(s,\omega,Y_{s-}(\omega)))$$

where the mathematical interpretation of this equation is given in Theorems 2.5 and 2.6.

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