

A BAYESIAN APPROACH TO THE SYMMETRIC  
MULTIPLE COMPARISONS PROBLEM  
IN THE TWO-WAY BALANCED DESIGN

by

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ABSTRACT

A class of Bayes rules is constructed for the comparison of all pairs of means resulting from combinations of treatment levels in a balanced two-factor design. For each pair of means  $u_{ij}$ ,  $u_{i'j'}$ , it is decided whether  $\eta = u_{ij} - u_{i'j'}$  is positive, negative, or zero; losses for incorrect decision are linear in  $|\eta|$ . When considered jointly, decisions made in the component pairwise comparison problems must be consistent with one another in the sense that they produce non-circular rankings of the cell means. The loss for incorrect decision in the overall problem is taken to be the sum of the component losses. Finally, the usual assumptions of independent normally distributed observations  $X_{ijk}$  with means  $u_{ij} = E(X_{ijk})$  and common error variance  $\sigma_e^2$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq c$ ,  $1 \leq k \leq K$ , are made. The prior distributions selected for this problem have the useful conjugate property that prior information about the magnitudes of contrasts for main effects and interactions in the  $u_{ij}$ 's, and for the magnitude of the variance  $\sigma_e^2$ , are pooled with similar information obtained from the data when forming the joint posterior density of the means  $u_{ij}$  and  $\sigma_e^2$ .

Each Bayes rule for the overall multiple comparisons problem is shown to result from the simultaneous applications of the corresponding

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Bayes rules for all component pairwise comparison problems. A computer program for simultaneously implementing the component Bayes rules is constructed. An important feature of this program is the use of bounds on the posterior Bayes risk function which in many cases enable the action with minimum posterior risk to be chosen without having to explicitly calculate these posterior risks. Since exact calculation of the posterior risks requires numerical evaluation of complicated double and/or triple integrals, computer time is drastically reduced by use of these bounds. Nevertheless, the exact Bayes rules are expensive in terms of computer time. For this reason, a large-sample approximation to the Bayes rule is proposed which is much easier (and cheaper) to apply, and which performs similarly to the exact Bayes rule on a diversity of examples of data of moderate sample size (error d.f.=15,30) taken from standard statistical textbooks.

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CHAPTER I  
INTRODUCTION

The purpose of this research is to develop and present a Bayes rule for the symmetric multiple comparisons problem in balanced two-factor designs. The model and solution developed here are a generalization of the model and solution of Duncan and Waller (1974) for one-factor designs. The proposed Bayesian procedure is presented in five chapters.

The three objectives of the present chapter are the following:

- (i) to give a summary of previous approaches to the multiple comparisons problem in one-factor designs;
- (ii) to discuss the one available approach to the multiple comparisons problem in two-factor designs, and to indicate the weaknesses of that approach;
- (iii) to indicate the potential usefulness of the proposed Bayes rule.

### 1.1. Pairwise Multiple Comparisons in One-Factor Designs

The data for a multiple comparisons problem in balanced one-factor designs consists of independent observations  $X_{ij}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , with

$$X_{ij} = u_i + e_{ij}, \quad 1 \leq i \leq n; 1 \leq j \leq m. \quad (1.1.1)$$

Assuming that the errors  $e_{ij}$  are independent, identically distributed

$N(0, \sigma_e^2)$  random variables, a reduction of the data by sufficiency yields the sufficient statistic

$$(\bar{X}, S^2) = (\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n, S^2), \quad (1.1.2)$$

where the sample means  $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$  from the  $n$  "cells" of the design are independent, normally distributed variables with common variance  $\sigma_e^2/m$ , and with means

$$E(\bar{X}_i) = u_i, \quad 1 \leq i \leq n \quad (1.1.3)$$

equal to the true cell means. Also, the sample variance  $S^2$  is independent of  $(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)$  and has distribution

$$S^2 \sim \frac{\sigma_e^2}{n(m-1)} \chi_{n(m-1)}^2, \quad (1.1.4)$$

where  $\chi_v^2$  has the chi-square distribution with  $v$  degrees of freedom.

Given this data, the multiple comparisons problem is to make all  $N = \frac{1}{2} n(n-1)$  pairwise comparisons between the true cell means  $u_1, u_2, \dots, u_n$ , where for each pair  $(u_i, u_j)$  of means,  $i < j$ , one of the following three decisions is to be made:

$$\begin{aligned} d_{ij}^+ &: \text{ say } "u_i > u_j" , \\ d_{ij}^0 &: \text{ say } "u_i = u_j" \\ &\text{ or } "u_i \text{ and } u_j \text{ are not significantly different",} \\ d_{ij}^- &: \text{ say } "u_i < u_j" . \end{aligned} \quad (1.1.5)$$

The multiple comparisons problem is symmetric if all component comparisons between means  $u_i$  and  $u_j$ , leading to one of the decisions (1.1.5), are regarded as being equally important.

Using concepts borrowed from hypothesis testing, the classical approaches to the multiple comparisons problem derive and compare procedures on the basis of their comparisonwise Type I and Type II error rates:

$$\begin{aligned}\alpha_{ij} &= \text{comparisonwise Type I error rate for } u_i \text{ v.s. } u_j \\ &= P\{\text{taking } d_{ij}^+ \text{ or } d_{ij}^- | u_i = u_j\} \quad (1.1.6)\end{aligned}$$

and

$$\begin{aligned}\beta_{ij} &= \text{comparisonwise Type II error rate for } u_i \text{ v.s. } u_j \\ &= P\{\text{taking } d_{ij}^0 | u_i \neq u_j\} . \quad (1.1.7)\end{aligned}$$

However, since the individual comparisons are part of an overall summary of the given experiment, experimentwise error rates are also considered. For example,

$$\begin{aligned}\alpha_E &= \text{experimentwise Type I error rate} \\ &= P\{\text{taking } d_{ij}^+ \text{ or } d_{ij}^- \text{ for some pair } (u_i, u_j) | u_1 = u_2 = \dots = u_n\} . \quad (1.1.8)\end{aligned}$$

If the experiment has only  $n=2$  treatments (cells), the comparisonwise and experimentwise Type I error rates are equal (i.e.,  $\alpha_{12} = \alpha_E$ ). In general, it follows from the Bonferroni inequality that

$$\alpha_E \leq \sum_{i < j} \alpha_{ij} , \quad (1.1.9)$$

although the right-hand side of (1.1.9) is typically a very crude bound for  $\alpha_E$ .

If  $n > 2$ , some true cell means can be equal and others can be unequal, leading to the need to evaluate experimentwise "Type I" error rates which give the probability of falsely saying  $d_{ij}^+$  or  $d_{ij}^-$  for at least one pair  $(u_i, u_j)$ ,  $i \neq j$ , for which  $u_i = u_j$ . Such "Type I" error

rates are midway, both in value and in the restrictiveness of their assumptions, between the comparisonwise (1.1.6) and experimentwise (1.1.8) extremes.

Classical multiple comparisons procedures are designed to control one (and sometimes more than one) of the above Type I error rates. Choosing a procedure to control the experimentwise Type I error rate  $\alpha_E$  results in a procedure which has low (conservative) comparisonwise Type I error rates  $\alpha_{ij}$ , and which also tends to have high comparisonwise Type II error rates whenever the true cell means  $\mu_i$  and  $\mu_j$  are not widely separated. On the other hand, controlling the comparisonwise Type I error rates  $\alpha_{ij}$  at a fixed value (say .05) results in an experimentwise Type I error rate  $\alpha_E$  which increases rapidly with the number  $n$  of means to be compared. However, procedures which control the  $\alpha_{ij}$ 's typically have comparatively lower comparisonwise Type II error rates  $\beta_{ij}$ . The decision as to which type of Type I error rate to control, and the comparison of resulting procedures in terms of the other Type I (or "Type I") and Type II error rates, requires that one have fairly detailed knowledge of the types of patterns of the true cell means  $\mu_1, \mu_2, \dots, \mu_n$  anticipated to occur for the given problem. Since such detailed information is usually not available, some procedures have been suggested which attempt to use the data to arrive at an insight into the pattern of true cell means  $\mu_1, \dots, \mu_n$  which has actually occurred. One such approach will be discussed in Section 1.2.

Examples of the various types of classical multiple-comparison procedures are given in the following.



The Least Significant Difference (LSD) Procedure. The LSD procedure is a purely comparisonwise procedure, in that only the comparisonwise Type I error rates are controlled at a fixed value  $\alpha$  (typically .01, .05, or .10). For each pair of sample means, the LSD procedure decides:

$$\begin{aligned} d_{ij}^+ & \text{ if } \frac{\sqrt{m}(\bar{X}_i - \bar{X}_j)}{S} > \sqrt{2} t(\alpha, n(m-1)) , \\ d_{ij}^0 & \text{ if } \frac{\sqrt{m}|\bar{X}_i - \bar{X}_j|}{S} \leq \sqrt{2} t(\alpha, n(m-1)) , \\ d_{ij}^- & \text{ if } \frac{\sqrt{m}(\bar{X}_i - \bar{X}_j)}{S} < -\sqrt{2} t(\alpha, n(m-1)) , \end{aligned} \quad (1.1.10)$$

where  $t(\alpha, \gamma)$  is the  $100(1 - \frac{\alpha}{2})^{\text{th}}$  percentile of the  $t$ -distribution with  $\gamma$  degrees of freedom.

Tukey's Honestly Significant Difference (HSD) Procedure. The HSD procedure controls the experimentwise Type I error  $\alpha_E$  at  $\alpha$ . The procedure has the form (1.1.10) except that in place of  $\sqrt{2} t(\alpha, n(m-1))$ , the quantity  $q(\alpha, n, n(m-1))$  is used, where  $q(\alpha, p, \gamma)$  is the  $100(1 - \alpha)^{\text{th}}$  percentile of the studentized range for  $p$  means and  $\gamma$  degrees of freedom.

Scheffé Method (SSD). The Scheffé procedure also controls the experimentwise Type I error rate  $\alpha_E$ , but is typically more conservative than the HSD procedure since the SSD method is intended to apply to all contrasts (not just pairwise contrasts) of means. The SSD procedure has the form (1.1.10) except that  $[2(n-1)F(\alpha, n-1, n(m-1))]^{\frac{1}{2}}$  is used in place of  $\sqrt{2} t(\alpha, n(m-1))$ , where  $F(\alpha, \nu_1, \nu_2)$  is the  $100(1 - \alpha)^{\text{th}}$  percentile of the  $F$ -distribution with  $\nu_1$  and  $\nu_2$  degrees of freedom. The SSD method also can be applied to unbalanced designs, correlated sample means, and general linear contrasts.

Fisher's Protected Least Significant Difference Procedure. In this procedure, the usual F test for  $H_0: u_1 = u_2 = \dots = u_n$  at a specified level of significance  $\alpha'$  is used as a preliminary screening device. If the F test fails to reject  $H_0$ , the decision  $d_{ij}^0$  is made for all pairs of means  $(u_i, u_j)$ ,  $i < j$ . Otherwise, the LSD procedure is used at a controlled comparisonwise Type I error rate of  $\alpha_{ij} = \alpha$ , all  $i \neq j$ . As a consequence of the preliminary F-test, it is guaranteed that  $\alpha_E \leq \alpha'$ , thus  $\alpha'$  bounds the experimentwise Type I error rate  $\alpha_E$ . Fisher's protected LSD is an example of an adaptive procedure.

Newman-Keuls Multiple Range Test (NKT). The NKT procedure has the form of the HSD procedure, with  $q(\alpha, n_{ij}(\bar{X}), n(m-1))$  replacing  $q(\alpha, n, n(m-1))$ , where  $n_{ij}(\bar{X})$  is the number of means between (and including)  $\bar{X}_i$  and  $\bar{X}_j$ , when the sample means  $\bar{X}_k$ ,  $1 \leq k \leq n$ , are arranged in ascending order. In addition, whenever the difference between  $\bar{X}_i$  and  $\bar{X}_j$  for some  $i, j$  is declared insignificant, all cell pairs of means  $\bar{X}_k$  and  $\bar{X}_l$  which appear between  $\bar{X}_i$  and  $\bar{X}_j$  in the ascending order of sample means are also declared to have insignificant differences. The NKT procedure, like the Fisher protected LSD procedure, is an adaptive procedure having the property that  $\alpha_E \leq \alpha$ . It differs from the Fisher procedure in attempting to control the experimentwise "Type I" error rates (where  $u_1 = u_2 = \dots = u_n$  is possibly false, but yet groups of means are equal), rather than merely trying to bound  $\alpha_E$ .

Another adaptive procedure similar to the NKT, and with similar goals, is Duncan's Multiple Range Test (Duncan 1947, 1951, 1955). Other multiple comparisons procedures, and the classical theory for comparing such procedures, can be found in the textbooks of Federer (1955),

Scheffé (1959), Miller (1966, 1977) and Lindman (1974), and also in review papers of Gill (1973), O'Neill and Wetherill (1971) and Thomas (1973). The last two papers provide extensive bibliographies.

### 1.2. Duncan and Waller's Bayesian Decision-Theoretic Approach.

The classical procedures described in Section 1.1 have in common the attempt to apply hypothesis testing (2 decision) concepts to what is clearly a multiple decision problem. It is not surprising, therefore, that in attempting to compare such procedures, one gets lost in a multitude of types (and rates) of error. Indeed, because the various procedures do not even fix the same kind of Type I error rate, comparing them on the basis of an assumed common rate of error is misleading, and has resulted in confusion in the literature.

Duncan (1961, 1965), building upon earlier theoretical work of Lehmann (1957) concerning multiple decision problems, attempts to formulate the multiple comparisons problem as a multiple decision problem. In a multiple decision formulation, all component decisions for pairs of means  $(u_i, u_j)$  are viewed as a whole, and losses are assigned which evaluate how well the decision vector:

$$\underline{d} = (d_{12}, d_{13}, \dots, d_{1n}, d_{23}, \dots, d_{2n}, \dots, d_{n-1,n}) , \quad (1.2.1)$$

composed of decisions  $d_{ij}$ ,  $i < j$  of (1.1.5) conforms to the true state of affairs as described by the vector

$$\underline{\theta} = [(u_1, \dots, u_n), \sigma_e^2] = (\underline{u}, \sigma_e^2) \quad (1.2.2)$$

of true cell means and random error  $\sigma_e^2$ . To keep the flavor of the multiple comparisons context, in which individual comparison decisions  $d_{ij}$  have importance in their own right, the formulation of the multiple

comparisons problem must clearly indicate how the overall loss due to the combined decision vector  $\underline{d}$  depends upon the component decisions  $d_{ij}$ . To this end, Duncan uses an additive-loss model:

$$L(\underline{\theta}, \underline{d}) = \sum_{i < j} L_{ij}(\underline{\theta}, d_{ij}), \quad (1.2.3)$$

in which the overall loss for the decision  $\underline{d}$  is expressed as the sum of losses for the  $n(n-1)/2$  component decisions  $d_{ij}$  which compare  $u_i$  to  $u_j$ . Note that in (1.2.3), the component loss  $L_{ij}(\underline{\theta}, d_{ij})$  only depends upon the decision vector  $\underline{d}$  through the decision  $d_{ij}$  actually made for that component problem. These properties (additivity and dependence of  $L_{ij}(\cdot, \cdot)$  only on  $d_{ij}$ ) provide the clear indication of the contribution of the component decisions to the overall performance of  $\underline{d}$  that is desired.

Further, note that each of the classical multiple comparisons procedures discussed in Section 1.1 has the property that the form of the component decision rule is independent of the indices  $i$  and  $j$  of the means being compared. This symmetry, or exchangeability, with respect to the indices implies that the component problems themselves are viewed as being symmetric with respect to the indices. Without any prior opinion concerning the nature (cost, value, importance) of the treatment levels being compared in the design, such an assumption of symmetry is reasonable.

Indeed, the likelihood function  $p(\bar{X}, S^2 | \underline{u}, \sigma_e^2)$  is symmetric in the cell indices. Let

$$N_p(\underline{z} | \underline{\eta}, \Sigma) = [(2\pi)^p |\Sigma|]^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\underline{z} - \underline{\eta})' \Sigma^{-1} (\underline{z} - \underline{\eta})\} \quad (1.2.4)$$

be the density function of a  $p$ -variate normally distributed random vector  $\underline{z}$  with mean vector  $\underline{\eta}$  and covariance matrix  $\Sigma$ . Let

$$f(w|\nu, \tau) = \frac{w^{\frac{1}{2}\nu-1} \exp\{-w/2\tau\}}{\Gamma(\frac{1}{2}\nu)(2\tau)^{\frac{1}{2}\nu}}, \quad w > 0, \quad (1.2.5)$$

be the density function of a random variable  $w \sim \tau\chi_{\nu}^2$ . Then, from the Section 1.1,

$$p(\bar{\underline{X}}, S^2 | \underline{u}, \sigma_e^2) = N_n(\bar{\underline{X}} | \underline{u}, \frac{\sigma_e^2}{m} I_n) f(S^2 | n(m-1), \frac{\sigma_e^2}{n(m-1)}). \quad (1.2.6)$$

Interchanging the indices is equivalent to transforming  $\bar{\underline{X}}, \underline{u}$  to  $Q\bar{\underline{X}}, Q\underline{u}$ , where  $Q$  is an  $n \times n$  permutation matrix. It now follows from (1.2.4) and (1.2.6), and the fact that  $Q^{-1}Q = I_n$ , that

$$p(Q\bar{\underline{X}}, S^2 | Q\underline{u}, \sigma_e^2) = p(\bar{\underline{X}}, S^2 | \underline{u}, \sigma_e^2)$$

for all permutation matrices  $Q$ .

If the decision problem is to be symmetric in the indices, then the loss function  $L(\underline{\theta}, \underline{d})$  must be symmetric in the indices. The loss function (1.2.3) with

$$L_{ij}((\underline{u}, \sigma_e^2), d_{ij}^+) = \begin{cases} 0 & , \quad u_i > u_j, \\ \kappa_1 |u_i - u_j|, & u_i \leq u_j, \end{cases}$$

$$L_{ij}((\underline{u}, \sigma_e^2), d_{ij}^0) = \kappa_0 |u_i - u_j|, \quad (1.2.7)$$

$$L_{ij}((\underline{u}, \sigma_e^2), d_{ij}^-) = \begin{cases} \kappa_1 |u_i - u_j|, & u_i > u_j, \\ 0 & , \quad u_i \leq u_j, \end{cases}$$

with  $\kappa_1 > \kappa_0$ , is Duncan's choice for such a symmetric loss function.

Finally, in Section 1.1, it was also mentioned that evaluation of competing multiple comparisons procedures would require some knowledge of the configuration of the means  $u_i$  in  $n$ -dimensional space, and that an investigator might hope to obtain such knowledge from the data. A Bayesian formulation of the decision problem, expressing prior information on  $\underline{u}$  in the form of a prior distribution  $p(\underline{u})$ , and incorporating information about  $\underline{u}$  from the data in terms of the posterior distribution  $p(\underline{u}|\bar{X}, S^2, \sigma_e^2)$ , is one intuitively appealing way to quantitatively provide the desired input. Duncan chooses the (conjugate) prior density

$$p(\underline{u}|m_u, \sigma_u^2) = N_n(\underline{u}|m_u \underline{1}_n, \sigma_u^2 \underline{I}_n), \quad (1.2.8)$$

where  $m_u$  is the "grand mean" of the population of  $u_i$ 's and  $\sigma_u^2$  is the variance of that population. Here,  $\underline{1}_t$  represents the  $t$ -dimensional column vector, all of whose elements are equal to 1. It is easy to see that for any permutation matrix  $Q$ ,

$$p(Q\underline{u}|m_u, \sigma_u^2) = p(\underline{u}|m_u, \sigma_u^2),$$

so that  $p(\underline{u}|m_u, \sigma_u^2)$  defined by (1.2.8) has the necessary property of being symmetric in the indices. Also note that the marginal distribution of  $\bar{X}$  is

$$p(\bar{X}|m_u, \sigma_u^2, \sigma_e^2) = N_n(\bar{X}|m_u \underline{1}_n, (\frac{1}{m} \sigma_e^2 + \sigma_u^2) \underline{I}_n), \quad (1.2.9)$$

which is the familiar Model II ANOVA distribution for  $\bar{X}$ .

Having formulated the multiple comparisons problem as a symmetric, multiple decision problem with additive loss, and having chosen a symmetric conjugate prior for  $\underline{u}$ , Duncan uses the additive loss theorem of Lehmann (1957), specialized to his particular problem, to show that the Bayes rule

$$\underline{\phi}(\bar{X}, S^2) = (\phi_{12}(\bar{X}, S^2), \dots, \phi_{n-1, n}(\bar{X}, S^2)) \quad (1.2.10)$$

for the problem is that rule whose components  $\phi_{ij}(\bar{X}, S^2)$  are the Bayes rules for the component decision problems having data  $(\bar{X}, S^2)$ , actions  $\{d_{ij}^+, d_{ij}^0, d_{ij}^-\}$ , loss  $L_{ij}((\underline{u}, \sigma_e^2), d_{ij})$  defined by (1.2.7), and prior  $p(\underline{u} | m_u, \sigma_u^2)$ . In so doing, Duncan assumes that  $\sigma_u^2$ ,  $\sigma_e^2$  are known. If these parameters are unknown, Duncan suggests estimating them in the usual way from an ANOVA table - thus defining an Empirical Bayes procedure. Waller (1967) and Waller and Duncan (1969, 1974) generalize this result by defining the prior  $p(\underline{u}, \sigma_e^2)$  as a mixture

$$p(\underline{u}, \sigma_e^2) = \iint p(\underline{u} | m_u, \sigma_u^2) p(\sigma_u^2, \sigma_e^2) p(m_u) d\sigma_u^2 dm_u \quad (1.2.11)$$

of priors of the form (1.2.8), with the mixing distribution being the product  $p(\sigma_u^2, \sigma_e^2) p(m_u)$  of a certain truncated conjugate joint inverse chi-squared distribution  $p(\sigma_u^2, \sigma_e^2)$  for  $\sigma_u^2$ ,  $\sigma_e^2$ , and an arbitrary (not necessarily proper) distribution  $p(m_u)$  for  $m_u$ .

All of the Bayes (or Empirical Bayes) rules resulting from this approach have the form of (1.1.10), with  $S^2$  replaced by a posterior estimate  $S_e^2$  of  $\sigma_e^2$ , and with  $\sqrt{2} t(\alpha, n(m-1))$  replaced by the critical value

$$\sqrt{2} t(\kappa, \hat{F}^*, \nu_1, \nu_2) . \quad (1.2.12)$$

This critical value depends on four arguments. The first is

$$\kappa = \frac{\kappa_1}{\kappa_0} ,$$

which can be interpreted as reflecting the relative seriousness of comparisonwise Type I to comparisonwise Type II errors. The second is a posterior estimate

$$\hat{F}^*(\bar{X}, S^2)$$

of the ratio

$$F^* = \frac{m\sigma_u^2 + \sigma_e^2}{\sigma_e^2} \quad (1.2.13)$$

of the expected mean squares (EMS) of the between mean square and within mean square in the ANOVA table. The last two are the numerator and denominator degrees of freedom  $\nu_1$  and  $\nu_2$  which are, respectively, pooled from the degrees of freedom appropriate for the data and from prior degrees of freedom for estimates of  $m\sigma_u^2 + \sigma_e^2$  and  $\sigma_e^2$  given by the prior. If  $\sigma_u^2$  and  $\sigma_e^2$  are known,

$$\hat{F}^*(\bar{X}, S^2) = F^*, \quad \nu_1 = \nu_2 = \infty. \quad (1.2.14)$$

In the case of the prior of indifference on  $\sigma_u^2$  and  $\sigma_e^2$ ,

$$\hat{F}^*(\bar{X}, S^2) = F, \quad \nu_1 = n - 1, \nu_2 = n(m - 1), \quad (1.2.15)$$

where  $F$  is the usual  $F$ -statistic for testing  $H_0: u_1 = u_2 = \dots = u_n$ . In the former case (1.2.14),

$$\sqrt{2} t(\chi, \hat{F}^*, \nu_1, \nu_2) = t_\infty(\chi) [F^*/(F^* - 1)]^{\frac{1}{2}} \quad (1.2.16)$$

where  $t_\infty(\chi)$  is taken from tables in Duncan (1965). In general, the critical value (1.2.12) is found from tables in Waller and Duncan (1969).

As was anticipated, the Waller-Duncan Bayes rules are (except when  $\sigma_u^2$  and  $\sigma_e^2$  are known) adaptive, in that the critical value (1.2.12) depends upon the data. Indeed, when  $\hat{F}^*(\bar{X}, S^2)$  is large (indicating heterogeneous treatment means  $u_1, u_2, \dots, u_n$ ), the critical value is reduced, converging, as  $\hat{F}^* \rightarrow \infty$ , to the critical value of the LSD procedure for some  $\alpha$ . In other words, a large value of  $\hat{F}^*$  allows the investigator to concentrate on controlling the comparisonwise Type I error rates, thereby



decreasing comparisonwise Type II error rates and increasing the power of the individual comparisons. When  $\hat{F}^*$  is small (indicating homogeneous treatment means), the critical value is increased, thereby allowing the procedure to control the experimentwise Type I error rate. The fact that the data is used to select the type of Type I error rate to be controlled, and the fact that the critical value is a function of all the data rather than only the number  $n$  of treatments compared (in contrast with the HSD and SSD procedures) makes the Waller-Duncan Bayesian procedure intuitively attractive. Indeed, of the procedures discussed in Section 1.1, only the Fisher protected LSD procedure has the comparable property of incorporating the observed  $F$ -value into the decision process. However, in contrast to the Fisher protected LSD, which uses the observed  $F$  value only in a 0-1 "go, no go" fashion, the Waller-Duncan procedure uses the  $F$  value in a smooth fashion to choose the critical level.

There have been many simulation studies comparing the properties of multiple comparison procedures; for example, Chen (1960), Balaam (1963), Broadman and Moffitt (1971), Thomas (1974), Ury and Wiggins (1976), Keselman and Rogan (1978, 1979). However, only the simulation studies of Carmer and Swanson (1971, 1973), have included the Waller-Duncan procedures in the list of procedures to be compared. Although these studies are done from a classical perspective, using rates of error, and suffer from a failure to put all procedures on a common ground (e.g., fix a common kind of Type I error rate for all), they do have the merit of attempting to compare procedures in terms of their overall (over component pairwise comparisons) ability to make correct

decisions. Consequently, it is of interest to note that in the Carmer-Swanson (1973) study, the two types of adaptive procedure based on the sample F-statistic (namely, Fisher's protected LSD and the Waller-Duncan procedures) are said to exhibit the best overall performance. Relative to Fisher's LSD, the Waller-Duncan procedures are slightly more sensitive when the F statistic is large, and somewhat more conservative when the F-statistic is small. Although Carmer and Swanson suggest that the Fisher LSD procedure would be preferred because of its simplicity, they fail to note one disadvantage of this procedure. This disadvantage is that it is difficult to generalize the Fisher procedure in a meaningful way to designs more complex than single-factor designs. In contrast, it will be seen in the present study that the Waller-Duncan approach can be usefully generalized to two-factor designs.

### 1.3. Multiple Comparisons in Balanced Two-Factor Designs.

Suppose that data  $X_{ijk}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq c$ ,  $1 \leq k \leq K$ , are obtained from a balanced two-factor ANOVA design. Here, the index  $i$ ,  $1 \leq i \leq r$ , indicates the level of the row factor (A); the index  $j$ ,  $1 \leq j \leq c$ , indicates the level of the column factor (B), and  $k$  is the index of replication. It is assumed that

$$X_{ijk} = u_{ij} + e_{ijk} . \quad (1.3.1)$$

Here, the  $u_{ij}$  are the true cell means resulting from the treatment combination of level  $i$  of factor A and level  $j$  of factor B. The errors  $e_{ijk}$  are assumed to be i.i.d.  $N(0, \sigma_e^2)$  random variables.

By sufficiency, the data can be reduced to the sufficient statistic

$$(X, S_e^2) = [(\bar{X}_{ij}), S_e^2] \quad (1.3.2)$$

where the  $\bar{x}_{ij}$ 's are the sample cell means, and  $S_e^2$  is the pooled sample variance defined by

$$rc(K-1)S_e^2 = \sum_{i=1}^r \sum_{j=1}^c \sum_{k=1}^K (x_{ijk} - \bar{x}_{ij})^2 .$$

Let

$$U = ((U_{ij})) : r \times c \quad (1.3.3)$$

be the matrix of true cell means. For any  $p \times q$  matrix  $A$  with columns  $a^{(1)}, a^{(2)}, \dots, a^{(q)} : p \times 1$ , define

$$\text{vec}(A) = \begin{pmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(q)} \end{pmatrix} : pq \times 1 . \quad (1.3.4)$$

Then it is well known that the likelihood of  $(X, S_e^2)$  is given by

$$\begin{aligned} p(X, S_e^2 | U, \sigma_e^2) \\ = N_{rc}(\text{vec}(X) | \text{vec}(U), K^{-1} \sigma_e^2 I_{rc}) \cdot f(S_e^2 | rc(K-1), \frac{\sigma_e^2}{rc(K-1)}) . \end{aligned} \quad (1.3.5)$$

In the form (1.3.1), the multiple comparisons problem is identical to that for the one-factor problem discussed in Sections 1.1 and 1.2. For each pair of cell means  $u_{ij}$  and  $u_{i'j'}$ , one of the following three decisions is to be made:

$$\begin{aligned} d_{ij, i'j'}^+ : & \text{ say } "u_{ij} > u_{i'j'}" \\ d_{ij, i'j'}^0 : & \text{ say } "u_{ij} = u_{i'j'}" \text{ or } "u_{ij} \text{ and } u_{i'j'} \\ & \text{ are not significantly different"} \\ d_{ij, i'j'}^- : & \text{ say } "u_{ij} < u_{i'j'}" . \end{aligned} \quad (1.3.6)$$

The choice of one of these three decisions defines the action space for

any component problem of the multiple decision problem.

However the two-factor nature of the design introduces the possibility of a simpler model than (1.3.1) - namely, the additive or no-interaction model

$$X_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk}, \quad 1 \leq i \leq r, \quad 1 \leq j \leq c, \quad 1 \leq k \leq K. \quad (1.3.7)$$

Comparing (1.3.1) and (1.3.7), it is seen that

$$u_{ij} - u_{i'j'} = \alpha_i - \alpha_{i'} + \beta_j - \beta_{j'}. \quad (1.3.8)$$

Thus, under the model (1.3.7), comparing true cell means  $u_{ij}$  and  $u_{i'j'}$  for  $j=j'$  is the same as comparing the main effects  $\alpha_i$  and  $\alpha_{i'}$  for factor A. Similarly, when  $i=i'$ , comparing  $u_{ij}$  and  $u_{i'j'}$  is the same as comparing main effects  $\beta_j$  and  $\beta_{j'}$  for factor B. In general, when  $i \neq i'$ ,  $j \neq j'$ , comparison of  $u_{ij}$  and  $u_{i'j'}$  involves comparison of both types of main effects.

In the model (1.3.7), the differences  $\alpha_i - \alpha_{i'}$  are estimated by  $\bar{X}_{i.} - \bar{X}_{i'.$ , and the differences  $\beta_j - \beta_{j'}$  are estimated by  $\bar{X}_{.j} - \bar{X}_{.j'}$ , where  $\bar{X}_{i.}$  and  $\bar{X}_{.j}$  are, respectively, the  $i^{\text{th}}$  row mean and  $j^{\text{th}}$  column mean of  $X$ . Hence under the model (1.3.7), more accurate estimates of  $u_{ij} - u_{i'j'}$  can be made, thereby lessening the possibility of error when deciding among the actions (1.3.6).

In the literature, the following two-stage process for making multiple comparisons among the means  $u_{ij}$  has been recommended. See Anderson and McLean (1974), Hicks (1973) :

Stage 1. Test the model (1.3.7) versus the model (1.3.1) using the usual ANOVA F-test for no interaction.

Stage 2. If the model (1.3.7) cannot be rejected by the data, perform separate (and independent) multiple comparisons among

the  $\alpha_i$ 's (equivalently among the means  $\bar{u}_{i\cdot}$ ) and among the  $\beta_j$ 's (equivalently among the means  $\bar{u}_{\cdot j}$ ). Otherwise, compare the means  $u_{ij}$  directly by means of the usual one-factor approaches.

Anderson and McLean, and Hicks, specifically advocate use of the Newman-Keuls (NKT) approach in Stage 2, but actually each of the procedures in Section 1.1 (and also the Waller-Duncan procedures in Section 1.2) is applicable. If the  $\alpha_i$ 's and  $\beta_j$ 's are to be compared [Model (1.3.7) is not rejected by the data], these procedures apply since  $\bar{X}_{i\cdot}$ ,  $1 \leq i \leq r$ , and  $\bar{X}_{\cdot j}$ ,  $1 \leq j \leq c$ , satisfy the assumptions of Section 1.1, where in the former case  $n=r$ ,  $m=cK$ , and in the latter case  $n=c$ ,  $m=rK$ .

Unfortunately, this recommended approach has two major flaws. The first, and most obvious, flaw is that comparison of  $u_{ij}$  and  $u_{i'j'}$  involves both the difference  $\alpha_i - \alpha_{i'}$  and the difference  $\beta_j - \beta_{j'}$  when  $i \neq i'$ ,  $j \neq j'$ , and conclusions about the relative values of  $\alpha_i$  vs.  $\alpha_{i'}$  and  $\beta_j$  vs.  $\beta_{j'}$  do not clearly combine to yield conclusions about the relative values of  $u_{ij}$  and  $u_{i'j'}$  [See (1.3.8)]. For example, if one decides that  $\alpha_i > \alpha_{i'}$  and  $\beta_j < \beta_{j'}$ , any of the decisions  $d_{ij, i'j'}^+$ ,  $d_{ij, i'j'}^0$ ,  $d_{ij, i'j'}^-$  is still possible for  $u_{ij}$  vs.  $u_{i'j'}$ :

$$\alpha_i - \alpha_{i'} = 5, \beta_j - \beta_{j'} = -3 \Rightarrow u_{ij} > u_{i'j'},$$

$$\alpha_i - \alpha_{i'} = 5, \beta_j - \beta_{j'} = -5 \Rightarrow u_{ij} = u_{i'j'},$$

$$\alpha_i - \alpha_{i'} = 5, \beta_j - \beta_{j'} = -7 \Rightarrow u_{ij} < u_{i'j'}.$$

It is possible to use the method to compare  $u_{ij}$  to  $u_{i'j}$  or  $u_{ij}$  to  $u_{ij'}$ , but if the cell means to be compared do not lie in the same column, or in the same row, no conclusions about  $u_{ij}$  vs.  $u_{i'j'}$  can be reached by this method.

It should be noted that the two-stage procedure described above usually is used to choose the best combination of factor levels. However, if the best combination is  $(i^*, j^*)$  and  $i \neq i^*, j \neq j^*$ , it may still not be possible to decide whether to take the action  $d_{i^*j^*,ij}^+$  or  $d_{i^*j^*,ij}^0$  for comparing  $u_{i^*j^*}$  to  $u_{ij}$ . Thus a minimal subclass of treatment combinations  $(i,j)$  not significantly different from  $(i^*,j^*)$  may not be identifiable.

The second objection to the above proposed method is that the initial test of no interaction can reach the wrong conclusion. This is particularly serious if the F-test makes a Type II error [fails to reject model (1.3.7) when model (1.3.1) is correct]. If interactions among the factors levels are present, the  $u_{ij}$ 's cannot possibly be compared by comparing row means  $\bar{u}_{i.}$  and column means  $\bar{u}_{.j}$ , as the following 2x2 example indicates:

		B	
		1	2
A	1	2	3
	2	5	2

Here,  $u_{12} > u_{11}$ , but  $\bar{u}_{.2} < \bar{u}_{.1}$ . If, contrary to the interaction among the factors exhibited in this table, the F-test falsely declared no interaction, and if the multiple comparison procedure on the column means of the  $u_{ij}$ 's correctly decided that  $\bar{u}_{.2} < \bar{u}_{.1}$ , the false conclusion  $d_{11,12}^+$  would result. As a consequence, the component comparison of  $u_{11}$  and  $u_{12}$  would result in a very serious Type III error (deciding  $d^+$  when  $d^-$  is correct, or vice versa).

The comparison of cell means under the additive model (1.3.7) has been relatively unexplored in the literature. It can be shown that this problem is equivalent to a problem in which the cell means  $\bar{X}_{ij}$  are replaced by the dependent mean-like quantities

$$\mu + \alpha_i + \beta_j = \bar{X}_{i.} + \bar{X}_{.j} - \bar{X}_{..} , \quad 1 \leq i \leq r, 1 \leq j \leq c , \quad (1.3.9)$$

where

$$\bar{X}_{..} = \frac{1}{r} \sum_{i=1}^r \bar{X}_{i.} = \frac{1}{c} \sum_{j=1}^c \bar{X}_{.j} .$$

For this problem, the Scheffé (SSD) procedure is directly applicable. However, the Scheffé procedure is very conservative, and it should be possible to improve upon it. One could also use a LSD procedure, since the test for [see (1.3.8)]

$$H_0: \alpha_i - \alpha_{i'} + \beta_j - \beta_{j'} = 0$$

is still a t-test. This LSD procedure could be protected by a preliminary simultaneous F-test of

$$H_0: \alpha_1 = \alpha_2 = \dots = \alpha_r = 0; \beta_1 = \beta_2 = \dots = \beta_c = 0 .$$

However, because of the dependence among the quantities (1.3.9), the properties of the LSD and protected LSD procedures would not be easy to derive, and would likely be qualitatively different than the corresponding properties of the LSD and protected LSD procedures in the one-factor model.

Even if a satisfactory classical multiple comparisons procedure for the additive model (1.3.7) is available, the objections raised against the use of the preliminary F-test in Stage I would still give compelling reasons against using the two-stage procedure. In

addition, it is doubtful whether the model (1.3.7) ever precisely describes real data. The real question that an investigator should be asking is whether the true configuration of the means  $u_{ij}$  is close enough to the model (1.3.7) that information from the row means  $\bar{X}_{i\cdot}$ ,  $\bar{X}_{i\cdot}$ , and column means  $\bar{X}_{\cdot j}$ ,  $\bar{X}_{\cdot j}$  will provide useful information to help in comparing  $u_{ij}$  and  $u_{i'j'}$ . This last point strongly argues for a Bayesian approach (see Section 1.2), and motivates the extension of Waller and Duncan's Bayesian approach to the two-factor case. This extension will be the subject of the remaining four chapters of this study.



CHAPTER II  
GENERALIZATION OF THE WALLER-DUNCAN FORMULATION  
TO TWO-FACTOR DESIGNS

In this chapter, Waller and Duncan's (1969, 1974) Bayesian decision theoretic formulation of the multiple comparisons problem in one-factor designs is generalized to two-factor designs. Two basic changes are made in the structure of the decision problem to take account of the multi-factorial nature of the design. First, symmetry is now defined by invariance of the decision problem over permutations of row indices and column indices. Second, the prior distribution is generalized to include prior opinion about the magnitudes of possible interactions between the row factor (A) and column factor (B). The new Bayesian decision structure is described in Section 2.1. In Section 2.2, it is shown that the Bayes decision rules can be given by simultaneous application of Bayes rules for the component pairwise comparison problems, and that such component Bayes rules have one of three basic forms, depending upon whether the means compared come from the same row (Type I), the same column (Type II) or from different rows and columns (Type III) of the design. An initial representation for these Bayes rules is also given in Section 2.2.

### 2.1. The Decision Structure and Prior Distribution

As remarked in Section 1.3, the data of a balanced two-factor design with normally distributed errors can be reduced by sufficiency to  $(X, S_e^2)$ , or equivalently to  $(X, SSE)$ , where  $X = ((\bar{X}_{ij}))$  is the matrix of sample means  $\bar{X}_{ij}$  for the cells,  $1 \leq i \leq r$ ,  $1 \leq j \leq c$ , and where

$$SSE = rc(K-1)S_e^2$$

is the error sum of squares obtained from the usual ANOVA table. Here,  $K$  = the number of replications per cell.

The likelihood function for  $(U, \sigma_e^2)$  obtained from  $(X, SSE)$  is

$$p(X, SSE | U, \sigma_e^2) = N_{rc}(\text{vec}(X) | \text{vec}(U), K^{-1} \sigma_e^2 I_{rc}) f(SSE | rc(K-1), \sigma_e^2), \quad (2.1.1)$$

where  $U = ((u_{ij}))$  is the matrix of true cell means, and the functions  $N_n(\cdot | \cdot, \cdot)$ ,  $f(\cdot | \cdot, \cdot)$  are defined by (1.2.4) and (1.2.5) respectively.

It sometimes will be notationally convenient to index the cell means  $\bar{X}_{ij}$ ,  $u_{ij}$  by the index  $t$  of the location of these means in  $\text{vec}(X)$ ,  $\text{vec}(U)$ , respectively. Thus, for example,  $u_{ij}$  will be alternatively denoted by  $u_t$ , where

$$t = (j-1)r + i, \quad 1 \leq i \leq r, \quad 1 \leq j \leq c. \quad (2.1.2)$$

Conversely, the  $t$ -th coordinate  $u_t$  of  $\text{vec}(U)$  is the  $(i, j)$ th coordinate of  $U$ , where

$$i = t - r \left[ \frac{t-1}{r} \right], \quad j = \left[ \frac{t-1}{r} \right] + 1, \quad t=1, 2, \dots, n, \quad (2.1.3)$$

$n = rc$ , and  $[v]$  is the greatest integer  $\leq v$ . Note from (2.1.3) that  $u_{t_1}$  and  $u_{t_2}$  correspond to cell means  $u_{i_1 j_1}$ ,  $u_{i_2 j_2}$  in the same row of  $U$  (i.e.  $i_1 = i_2$ ) if and only if  $t_1 \pmod{r} = t_2 \pmod{r}$ , while  $u_{t_1}$  and  $u_{t_2}$

correspond to cell means in the same column of  $U$  if and only if

$$\left[ \frac{t_1 - 1}{r} \right] = \left[ \frac{t_2 - 1}{r} \right].$$

Under the above notational conventions, the action space for the problem of comparing all pairs of means  $(u_{t_1}, u_{t_2})$ ,  $t_1 \neq t_2$  from  $\text{vec}(U)$  consists of vectors

$$\underline{d} = (d_{12}, d_{13}, \dots, d_{t_1 t_2}, \dots, d_{n-1, n}), \quad (2.1.4)$$

whose components  $d_{t_1 t_2}$ ,  $1 \leq t_1 < t_2 \leq n$ , can be any one of the following component actions:

$$\begin{aligned} d_{t_1 t_2}^+ &: \text{ say } "u_{t_1} > u_{t_2}" , \\ d_{t_1 t_2}^0 &: \text{ say } "u_{t_1} = u_{t_2}" \text{ or } "u_{t_1} \text{ and } u_{t_2} \\ &\quad \text{are not significantly different,"} \\ d_{t_1 t_2}^- &: \text{ say } "u_{t_1} < u_{t_2}" . \end{aligned} \quad (2.1.5)$$

The possible component actions are identical to those considered in Equation (1.3.6) in Section 1.3.

One aspect of the action space not emphasized in Section 1.2 is that action vectors  $\underline{d}$  with inconsistent component actions are to be excluded from the action space. An action vector  $\underline{d}$  has inconsistent components if there exist indices  $t_1, t_2, t_3$  for which any one of the following assertions is true:

$$\begin{aligned} \text{(i)} \quad & d_{t_1 t_2} = d_{t_1 t_2}^+, d_{t_2 t_3} = d_{t_2 t_3}^+, d_{t_1 t_3} \neq d_{t_1 t_3}^+, \\ \text{(ii)} \quad & d_{t_1 t_2} = d_{t_1 t_2}^-, d_{t_2 t_3} = d_{t_2 t_3}^-, d_{t_1 t_3} \neq d_{t_1 t_3}^-, \end{aligned}$$

$$(iii) \quad d_{t_1 t_2} = d_{t_1 t_2}^+, \quad d_{t_2 t_3} = d_{t_2 t_3}^0, \quad d_{t_1 t_3} = d_{t_1 t_3}^-,$$

$$(iv) \quad d_{t_1 t_2} = d_{t_1 t_2}^-, \quad d_{t_2 t_3} = d_{t_2 t_3}^0, \quad d_{t_1 t_3} = d_{t_1 t_3}^+.$$

The above definition of inconsistency is motivated by the transitivity property of the orderings  $<$  and  $\leq$  for real numbers.

Paralleling Section 1.2, the loss for taking action vector  $\tilde{d}$  when the true state of nature is  $\theta = (U, \sigma_e^2)$  is defined to be the sum

$$L(\theta, \tilde{d}) = \sum_{1 \leq t_1 < t_2 \leq n} L_{t_1 t_2}(\theta, d_{t_1 t_2}) \quad (2.1.6)$$

of the losses for the component actions  $d_{t_1 t_2}$  defined in (2.1.5), where

$$L_{t_1 t_2}(\theta, d_{t_1 t_2}^+) = \begin{cases} 0 & , \quad \text{if } u_{t_1} > u_{t_2}, \\ \chi_1 |u_{t_1} - u_{t_2}| & , \quad \text{if } u_{t_1} \leq u_{t_2}, \end{cases}$$

$$L_{t_1 t_2}(\theta, d_{t_1 t_2}^0) = \chi_0 |u_{t_1} - u_{t_2}|, \quad (2.1.7)$$

$$L_{t_1 t_2}(\theta, d_{t_1 t_2}^-) = \begin{cases} \chi_1 |u_{t_1} - u_{t_2}| & , \quad \text{if } u_{t_1} > u_{t_2}, \\ 0 & , \quad \text{if } u_{t_1} \leq u_{t_2}. \end{cases}$$

The decision structure for the multiple comparisons problem for the two-factor design is now complete. Because all component loss functions (2.1.7) have identical form, it is apparent from (2.1.6) that the loss function  $L(\theta, \tilde{d})$  is invariant (symmetric) under permutation of either the row indices  $i$  or column indices  $j$  of the design. To demonstrate that the likelihood function (2.1.1) is similarly symmetric under permutation of row and column indices, it is sufficient to show that

$$p(Q_1 X Q_2, S_e^2 | Q_1 U Q_2, \sigma_e^2) = p(X, S_e^2 | U, \sigma_e^2) \quad (2.1.8)$$

for all  $X$ ,  $U$ , and all permutation matrices  $Q_1: r \times r$ ,  $Q_2: c \times c$ . It is well known that

$$\text{vec}(Q_1 A Q_2) = (Q_2' \otimes Q_1) \text{vec}(A) \quad (2.1.9)$$

for any matrix  $A: r \times c$ , where " $\otimes$ " is the Kronecker product. Using the fact that  $Q_2' \otimes Q_1$  is an  $n \times n$  orthogonal matrix, the result (2.1.8) now follows directly from (2.1.1), (2.1.9) and Theorem 1.3.2 of Bickel and Doksum (1977).

Attention can now be turned to finding a prior distribution for  $\theta = (U, \sigma_e^2)$  which is symmetric under row and column permutations, and which incorporates prior opinions about the magnitudes of the interactions (if any) between the row and column factors.

We will use prior distributions for  $(U, \sigma_e^2)$  that are conjugate to the model (2.1.1). This will allow prior opinion about the magnitudes of main effects, interactions and error to be pooled with sample information (taken from the usual ANOVA table) in forming a posterior distribution for  $(U, \sigma_e^2)$ . Although the model (2.1.1) is intended to be a Model I ANOVA representation, where the  $u_{ij}$ 's are regarded as fixed unknown constants, the fact that the  $u_{ij}$ 's in a Bayesian approach are given a joint (prior) distribution allows us to utilize Model II theory in constructing and analyzing such a prior model. In Model II ANOVA, the elements  $u_{ij}$  of  $U$  are assumed to have the joint density defined by

$$\text{vec}(U) \sim N_n(\mu \mathbf{1}_n, \Sigma_U),$$

where  $\mu$  is any constant,

$$\Sigma_u = \sigma_A^2 [I_c \otimes I_r] + \sigma_B^2 [I_c \otimes I_r] + \sigma_{AB}^2 [I_c \otimes I_r], \quad (2.1.10)$$

and  $\sigma_A^2, \sigma_B^2, \sigma_{AB}^2$  are nonnegative constants. In such a case, it is known that the marginal distribution of  $X$  is given by

$$\text{vec}(X) \sim N(\mu_n, \Sigma_X), \quad (2.1.11)$$

where

$$\Sigma_X = \Sigma_u + \frac{\sigma_e^2}{K} I_n.$$

Define  $H_\ell$  to be any  $\ell \times \ell$  orthogonal matrix whose first row is

$$\frac{1}{\sqrt{2}} (1, -1, 0, \dots, 0)$$

and whose  $\ell$ -th row is  $\ell^{-\frac{1}{2}} \mathbf{1}_\ell$ . Using (2.1.10), the definition of  $H_\ell$ , and known facts about Kronecker products, it is straightforward to show that the matrix  $(H_c' \otimes H_r')$  is an  $n \times n$  orthogonal matrix whose columns are eigenvectors of  $\Sigma_X$ .

Also, the distinct eigenvalues of  $K\Sigma_X$  are

$$\begin{aligned} E_A &= cK\sigma_A^2 + K\sigma_{AB}^2 + \sigma_e^2, \\ E_B &= rK\sigma_B^2 + K\sigma_{AB}^2 + \sigma_e^2, \\ E_{AB} &= K\sigma_{AB}^2 + \sigma_e^2, \end{aligned} \quad (2.1.12)$$

having respective multiplicities  $r-1$ ,  $c-1$ , and  $(r-1)(c-1)$ ; plus one additional eigenvalue  $E_A + E_B - E_{AB}$  of multiplicity 1. The quantities  $E_A$ ,  $E_B$ ,  $E_{AB}$  are recognizable as the expected values of the mean squares MSA, MSB, MSAB for A, B and AB, respectively, that would appear in the usual ANOVA table for the two-factor design under model II assumptions.

The columns of  $H_c' \otimes H_r'$  are also eigenvectors for  $\Sigma_u$ . The distinct eigenvalues of  $K\Sigma_u$  are  $E_A - \sigma_e^2$ ,  $E_B - \sigma_e^2$ ,  $E_{AB} - \sigma_e^2$ , and  $E_A + E_B - E_{AB} - \sigma_e^2$ .

Since  $H_c' \otimes H_r'$  is a constant (known) matrix,  $\Sigma_u$  depends functionally only upon  $E_A$ ,  $E_B$ ,  $E_{AB}$  and  $\sigma_e^2$ ; that is,

$$\Sigma_u = \Sigma_u(E_A, E_B, E_{AB}, \sigma_e^2).$$

For Model II ANOVA, Box and Tiao (1973) define a class of conjugate prior distributions for  $E_A$ ,  $E_B$ ,  $E_{AB}$  and  $\sigma_e^2$

$$g(E|S, f) = \frac{(S)^{\frac{1}{2}f} \exp\{-\frac{1}{2} \frac{S}{E}\}}{E^{\frac{1}{2}f+1} \Gamma(\frac{f}{2}) 2^{\frac{1}{2}f}}, \quad E > 0. \quad (2.1.13)$$

The functions (2.1.13) define density functions when  $S > 0$ ,  $f > 0$ . Indeed,  $g(E|S, f)$  is a member of the family of inverse chi-squared densities. These densities form a class of conjugate prior densities for the family of chi-squared densities (1.2.5). The constant  $S$  may be thought of as being a sample sum of squares from a previous experiment which yielded an estimate of  $E$ . The constant  $f$  can be regarded as the degrees of freedom for  $S$  in this prior experiment - that is, as a measure of the strength of our prior belief. If a prior estimate  $S$  of  $E$  is not available, then the Jeffreys' prior of indifference

$$g(E|0, 0) = \frac{1}{E}, \quad E > 0 \quad (2.1.14)$$

for scale parameters can be used. The function  $g(E|0, 0)$  is not a density function, but can be regarded as the limiting function of (2.1.13) as  $S$  and  $f$  approach 0.

Define

$$\begin{aligned}
 &g(E_A, E_B, E_{AB}, \sigma_e^2 | \underline{SS}_p, \underline{f}) \\
 &= g(E_A | SSA_p, f_p) g(E_B | SSB_p, f_B) g(E_{AB} | SSAB_p, f_{AB}) g(\sigma_e^2 | SSE_p, f_p)
 \end{aligned} \tag{2.1.15}$$

where

$$\underline{SS}_p = (SSA_p, SSB_p, SSAB_p, SSE_p)', \quad \underline{f} = (f_A, f_B, f_{AB}, f_e)'.$$

Let

$$\underline{E} = (E_A, E_B, E_{AB}, \sigma_e^2)', \quad \mathcal{E} = \{\underline{E}: 0 < \sigma_e^2 < E_{AB} < E_A, E_B\}. \tag{2.1.16}$$

By (2.1.12), any density for  $\underline{E}$  can be positive only when  $\underline{E} \in \mathcal{E}$ . Thus, the prior density for  $\underline{E}$  is defined to be the truncation of the joint density  $g(\underline{E} | \underline{SS}_p, \underline{f})$  of (2.1.15) to the region  $\mathcal{E}$ . That is,

$$\pi(\underline{E} | \underline{SS}_p, \underline{f}) = \begin{cases} Dg(\underline{E} | \underline{SS}_p, \underline{f}), & \text{if } \underline{E} \in \mathcal{E}, \\ 0 & \text{, otherwise,} \end{cases} \tag{2.1.17}$$

where

$$D = D(\underline{SS}_p, \underline{f}) = \begin{cases} \left[ \int_{\mathcal{E}} g(\underline{E} | \underline{SS}_p, \underline{f}) d\underline{E} \right]^{-1}, & \text{if } f_A, f_B, f_{AB}, f_e > 0, \\ 1 & \text{, if } f_A \text{ or } f_B \text{ or } f_{AB} \text{ or } f_e = 0. \end{cases}$$

If  $f_A, f_B, f_{AB}, f_e > 0$ , the function (2.1.17) is a prior joint density function for  $\underline{E}$ . Otherwise, (2.1.17) is a formal (diffuse, improper) prior. Box and Tiao (1973) note that under Model II ANOVA assumptions, the family of distributions (2.1.17) is conjugate to the family of joint distributions of the sums of squares SSA, SSB, SSAB, SSE in the usual



ANOVA table obtained from  $(X, SSE)$ . Thus, the posterior density of  $\underline{E}$  given  $(X, SSE)$  is

$$p(\underline{E}|X, SSE) = \pi(\underline{E}|\underline{SS}^*, \underline{f}^*) \quad (2.1.18)$$

where

$$\begin{aligned} \underline{SS}^* &= (SSA^*, SSB^*, SSAB^*, SSE^*)' = (SSA_p + SSA, SSB_p + SSB, SSAB_p + SSAB, SSE_p + SSE)', \\ \underline{f}^* &= (f_A^*, f_B^*, f_{AB}^*, f_e^*)' \\ &= (f_A^{r-1}, f_B^{c-1}, f_{AB}^{(r-1)(c-1)}, f_e^{rc(K-1)})'. \end{aligned} \quad (2.1.19)$$

It is obvious that when  $K=1$ , the prior density of  $\sigma_e^2$  should be of the form (2.1.13). In this problem,  $\theta = (U, \sigma_e^2)$  is the parameter of interest, and  $E_A, E_B, E_{AB}$  (and also  $\mu$ ) serve as variables of integration (hyper-parameters) by which to define the family of prior distributions for  $\theta$  as

$$\begin{aligned} \lambda(\theta|\underline{SS}_p, \underline{f}) &= \lambda(U, \sigma_e^2|\underline{SS}_p, \underline{f}) \\ &= \int_{\sigma_e^2}^{\infty} \int_{E_{AB}}^{\infty} \int_{E_{AB}}^{\infty} \int_{-\infty}^{\infty} N_n(\text{vec}(U)|\underline{\mu}_n, \Sigma_u(\underline{E})) \\ &\quad \cdot \pi(\underline{E}|\underline{SS}_p, \underline{f}) h(\mu) d\mu dE_A dE_B dE_{AB} \end{aligned} \quad (2.1.20)$$

for  $(U, \sigma_e^2)$ . In (2.1.20),  $h(\mu)$  is permitted to be any probability density function on  $(-\infty, \infty)$ , or even the indifference prior  $h(\mu) \equiv 1$ ,  $-\infty < \mu < \infty$ . The matrix  $\Sigma_u(\underline{E})$  is defined by (2.1.10) and (2.1.12).

To show that the family of prior distributions (2.1.20) is symmetric under row and column permutations of  $U$ , it is enough [see (2.1.1)] to demonstrate that

$$N_n(\text{vec}(U)|\underline{\mu}_n, \Sigma_u) = N_n(\text{vec}(Q_1 U Q_2)|\underline{\mu}_n, \Sigma_u). \quad (2.1.21)$$

Note that for any integer  $l > 0$ , and any  $l \times l$  permutation matrix  $Q$ ,

$$QI_{\sim l} = I_{\sim l}, QI_{\sim l}Q' = I_{\sim l}.$$

The equality (2.1.21) now follows from (2.1.9), (2.1.10) and (1.2.4).

Having defined the data  $(X, SSE)$ , parameter  $\theta = (U, \sigma_e^2)$ , likelihood function  $p(X, SSE|U, \sigma_e^2)$ , actions  $d$ , loss function  $L(\theta, d)$ , and prior distribution  $\lambda(\theta|\underline{SS}_p, \underline{f})$ , we have completed our formulation of the multiple comparisons problem for the means in a balanced two-factor design as a Bayesian multiple decision problem. Further, this Bayesian decision problem has been shown to be symmetric under permutation of row and column indices of  $X$  and  $U$ , a fact that is exploited in Section 2.2.

## 2.2. Component Bayes Rules

Any component problem of the multiple decision problem described in Section 2.1 can be formally defined as follows:

Data:  $(X, SSE)$ ;

Parameter Space:  $\Theta = \{\theta = (U, \sigma_e^2): U \text{ a real } r \times c \text{ matrix, } \sigma_e^2 > 0\}$ ;

Likelihood Function:  $p(X, SSE|\theta) = p(X, SSE|U, \sigma_e^2)$  defined by (2.1.1);

Actions:  $d_{t_1 t_2}^+$ ,  $d_{t_1 t_2}^0$ ,  $d_{t_1 t_2}^-$  defined by (2.1.5);

Loss Function:  $L_{t_1 t_2}(\theta, d_{t_1 t_2})$  defined by (2.1.7);

Prior Density:  $\lambda(\theta) = \lambda(\theta|\underline{SS}_p, \underline{f})$  defined by (2.1.20).

Except where it is necessary to identify a particular component problem, we will omit the subscripts  $t_1, t_2$  on the actions and loss function.

Thus, respectively,

$$d^+, d^0, d^- = d_{t_1 t_2}^+, d_{t_1 t_2}^0, d_{t_1 t_2}^-; L(\theta, d) = L_{t_1 t_2}(\theta, d_{t_1 t_2}).$$

The component Bayes Rule  $\varphi^{(\lambda)}_{t_1, t_2}(X, SSE) = \varphi^{(\lambda)}_{t_1, t_2}(X, SSE)$  can be described as a 3-dimensional vector of probabilities,

$$(\varphi^{(\lambda)}(+), \varphi^{(\lambda)}(0), \varphi^{(\lambda)}(-)),$$

for taking actions  $d^+$ ,  $d^0$ ,  $d^-$  when  $(X, SSE)$  is observed. It follows from the theory of Bayes decision procedures that

$$\varphi^{(\lambda)}(X, SSE) = \begin{cases} (1, 0, 0), & \text{if } r^{(\lambda)}(d^+) < r^{(\lambda)}(d^0), r^{(\lambda)}(d^-), \\ (0, 1, 0), & \text{if } r^{(\lambda)}(d^0) < r^{(\lambda)}(d^+), r^{(\lambda)}(d^-), \\ (0, 0, 1), & \text{if } r^{(\lambda)}(d^-) < r^{(\lambda)}(d^+), r^{(\lambda)}(d^0), \end{cases} \quad (2.2.1)$$

where

$$r^{(\lambda)}(d) = r^{(\lambda)}(d|X, SSE) = \int_{\Theta} L(\theta, d) \lambda(\theta|X, SSE) d\theta \quad (2.2.2)$$

is the posterior Bayes risk for the component problem. Here

$$\lambda(\theta|X, SSE) = \frac{p(X, SSE|\theta)\lambda(\theta)}{\int_{\Theta} p(X, SSE|\theta)\lambda(\theta)d\theta} = \frac{p(X, SSE, \theta)}{p(X, SSE)} \quad (2.2.3)$$

is the posterior density of  $\theta$ . If two or more actions yield the same minimum posterior Bayes risk,  $\varphi^{(\lambda)}(X, SSE)$  randomizes among these actions. However, it is shown in Chapter 3 that such ties have probability zero [over the distribution of  $(X, SSE)$  given  $\theta$ ] for all  $\theta$ . Hence,  $\varphi^{(\lambda)}(X, SSE)$  can be described as a nonrandomized decision rule.

Our motivation for deriving the component Bayes rules comes from the following theorem.

**Theorem 2.2.1.** Simultaneous application of the component Bayes rules  $\varphi^{(\lambda)}_{t_1, t_2}(X, SSE)$  in all component problems,  $1 \leq t_1 < t_2 \leq n$ , yields the Bayes rule for the multiple comparisons decision problem of Section 2.1.

Proof In Section 3.3 of Chapter 3, it will be shown that simultaneous application of the component Bayes rules yields consistent action vectors  $d$  (see Section 2.1 for the definition of consistent action vectors) with probability one, for all  $\theta \in \Theta$ . [Here, the probability referred to is taken over the conditional distribution of  $(X, SSE)$  given  $\theta$ .] The assertion of the theorem is now an immediate consequence of the additive structure (2.1.6) of the experimentwise loss function  $L(\theta, d)$  and Duncan's (1965) Additive Loss Theorem.  $\square$

We now seek to simplify the form of the posterior Bayes risk function (2.2.2). Recall that

$$\lambda(\theta) = \int_{\sigma_e^2}^{\infty} \int_{E_{AB}}^{\infty} \int_{E_{AB}}^{\infty} \int_{-\infty}^{\infty} p(U|\mu, \underline{E}) h(\mu) \pi(\underline{E}) d\mu dE_A dE_B dE_{AB},$$

where

$$\pi(\underline{E}) = \pi(\underline{E} | \underline{SS}_p, \underline{f}), \quad p(U|\mu, \underline{E}) = N_n(\text{vec}(U) | \mu \mathbf{1}_n, \Sigma_U(\underline{E})). \quad (2.2.4)$$

Thus, in a formal sense (since neither  $\pi(\underline{E})$  nor  $h(\mu)$  need to be proper densities),  $p(X, SSE, \theta)$  in Equation (2.2.3) is the marginal density of  $X, SSE, \theta$  obtained from the joint density

$$p(X, SSE, U, \mu, \underline{E}) = p(X, SSE | U, \sigma_e^2) p(U | \mu, \underline{E}) h(\mu) \pi(\underline{E}).$$

However, using an obvious notation, it is also true that

$$p(X, SSE, U, \mu, \underline{E}) = p(U | X, SSE, \mu, \underline{E}) p(\mu | X, SSE, \underline{E}) p(\underline{E} | X, SSE) p(X, SSE). \quad (2.2.5)$$

Recall from Section 2.1, Equation (2.1.18), that

$$p(\underline{E} | X, SSE) = \pi(\underline{E} | \underline{SS}^*, \underline{f}^*),$$

where  $\underline{SS}^*$  and  $\underline{f}^*$  are defined by (2.1.19). Also, from (2.1.1) and (2.2.4),

$$p(X, U | SSE, \mu, \underline{E}) = N_n(\text{vec}(X) | \text{vec}(U), K^{-1} \sigma_e^2 \mathbf{I}_n) N_n(\text{vec}(U) | \mu \mathbf{1}_n, \Sigma_U(\underline{E})),$$

from which standard results about the multivariate normal distribution yield

$$p(U|X, SSE, \mu, \underline{E}) = N_n(\text{vec}(U) | \mu\psi(\underline{E})\mathbf{1}_n + (\mathbf{I}_n - \psi(\underline{E}))\text{vec}(X), \psi(\underline{E})\Sigma_u(\underline{E})), \quad (2.2.6)$$

where

$$\psi(\underline{E}) = \frac{\sigma_e^2}{K} (\Sigma_u(\underline{E}) + \frac{\sigma_e^2}{K} \mathbf{I}_n)^{-1}. \quad (2.2.7)$$

From (2.1.5), the component loss function  $L_{t_1 t_2}(\theta, d_{t_1 t_2})$  for any component problem depends upon  $\theta$  only through

$$\eta = u_{t_1} - u_{t_2}. \quad (2.2.8)$$

Further, it is apparent from (2.2.6) that the conditional distribution of  $\eta$  given  $X, SSE, \mu, \underline{E}$  is a normal distribution, independent of SSE, determined by its conditional mean  $m(X, \mu, \underline{E})$  and conditional variance  $\sigma^2(\underline{E})$ . [That the conditional variance of  $\eta$  is independent of  $X$  and  $\mu$  is obvious from (2.2.6).] Thus, from (2.2.2), (2.2.3), (2.2.5), and (2.2.6),

$$r^{(\lambda)}(d) = \int_{-\infty}^{\infty} L(\eta, d) \lambda(\eta | X, SSE) d\eta, \quad (2.2.9)$$

$\lambda(\eta | X, SSE)$

$$= \int_{\underline{E}} \int_{-\infty}^{\infty} N_1(\eta | m(X, \mu, \underline{E}), \sigma^2(\underline{E})) p(\mu | X, SSE, \underline{E}) \pi(\underline{E} | SS^*, f^*) d\eta d\underline{E}, \quad (2.2.10)$$

In (2.2.9), we have now modified our notation for the component loss function to reflect the fact that it depends on  $\theta$  only through  $\eta$ .

One last reduction is possible. In Section 2.1, it was noted that the columns of  $H_c^* \otimes H_r^*$  are eigenvectors of

$$\Sigma_X(\underline{E}) = \Sigma_U(\underline{E}) + \frac{\sigma_e^2}{K} I_n.$$

Since  $\underline{1}_n$  is proportional to the last column of  $H_C' \otimes H_r'$ , it follows that

$$\left( \Sigma_U(\underline{E}) + \frac{\sigma_e^2}{K} I_n \right) \underline{1}_n = \gamma(\underline{E}) \underline{1}_n,$$

where  $\gamma(\underline{E})$  is the eigenvalue of  $\Sigma_X(\underline{E})$  corresponding to  $\underline{1}_n$ . Thus, from (2.2.7),

$$\mu\psi(\underline{E}) \underline{1}_n = \frac{\mu\sigma_e^2}{K\gamma(\underline{E})} \underline{1}_n. \quad (2.2.11)$$

Since

$$\eta = u_{t_1} - u_{t_2} = \left( u_{t_1} - \frac{\mu\sigma_e^2}{K\gamma(\underline{E})} \right) - \left( u_{t_2} - \frac{\mu\sigma_e^2}{K\gamma(\underline{E})} \right),$$

it follows from (2.2.6) and (2.2.11) that the conditional mean  $m(X, \mu, \underline{E})$  of  $\eta$  does not depend upon  $\mu$ ; that is,

$$m(X, \mu, \underline{E}) = m(X, \underline{E}).$$

Consequently, the integral over  $\mu$  can be taken in (2.2.10), yielding

$$\lambda(\eta | X, SSE) = \int_{\underline{E}} N_1(\eta | m(X, \underline{E}), \sigma^2(\underline{E})) \pi(\underline{E} | SS^*, f^*) d\underline{E}. \quad (2.2.12)$$

The component Bayes rule can now be determined from (2.2.1), (2.2.9) and (2.2.12), once we have identified the functional form of  $m(X, \underline{E})$  and  $\sigma^2(\underline{E})$ . This is not an easy exercise, so that it is fortunate that we need only make this calculation for three component problems, as shown by the following argument.

Recall that the decision problem in Section 2.1 was shown to be symmetric (invariant) in the row and column indices. If the cells  $(i_1, j_1)$ ,  $(i_2, j_2)$  whose means are to be compared come from the same row ( $i_1 = i_2 = i$ ), choose a permutation  $\rho$  of the row indices such that  $\rho(i) = 1$ , and a

permutation  $\tau$  of the column indices such that  $\tau(j_1) = 1$ ,  $\tau(j_2) = 2$ . Choose the  $r \times r$  permutation matrix  $Q_1$  and the  $c \times c$  permutation matrix  $Q_2$  such that these matrices act to permute the rows and columns of  $X$  (and  $U$ ) in the manner specified by  $\rho$  and  $\tau$  respectively. Then, by symmetry of the problem,

$$\varphi_{(i,j_1),(i,j_2)}^{(\lambda)}(X,SSE) = \varphi_{(1,1),(1,2)}^{(\lambda)}(Q_1 X Q_2, SSE).$$

Similarly, if the cells  $(i_1, j_1)$ ,  $(i_2, j_2)$  come from the same column ( $j_1 = j_2 = j$ ), appropriate choice of  $\rho$ ,  $\tau$  and the corresponding permutation matrices  $Q_1$ ,  $Q_2$  shows that

$$\varphi_{(i_1,j),(i_2,j)}^{(\lambda)}(X,SSE) = \varphi_{(1,1),(2,1)}^{(\lambda)}(Q_1 X Q_2, SSE).$$

Finally, if the cells come neither from the same row nor from the same column ( $i_1 \neq i_2$ ,  $j_1 \neq j_2$ ),

$$\varphi_{(i_1,j_1),(i_2,j_2)}^{(\lambda)}(X,SSE) = \varphi_{(1,1),(2,2)}^{(\lambda)}(Q_1 X Q_2, SSE).$$

Thus, all Type I component Bayes rules  $\varphi^{(\lambda)}(X,SSE)$ , corresponding to cells  $(i_1, j_1)$ ,  $(i_2, j_2)$  in the same row ( $i_1 = i_2 = i$ ) can be obtained from  $\varphi_{(1,1),(1,2)}^{(\lambda)}(X,SSE)$ . Similarly, all Type II component Bayes rules, corresponding to cells in the same column ( $j_1 = j_2 = j$ ) can be obtained from  $\varphi_{(1,1),(2,1)}^{(\lambda)}(X,SSE)$ . Finally, all Type III component Bayes rules, corresponding to cells neither in the same column nor in the same row, can be obtained from  $\varphi_{(1,1),(2,2)}^{(\lambda)}(X,SSE)$ . Since the above three classes of component Bayes rules partition the class of all component Bayes rules, we can confine our derivation of component Bayes rules to the three prototype cases:

$$\eta_I = u_{11} - u_{12}, \quad \eta_{II} = u_{11} - u_{21}, \quad \eta_{III} = u_{11} - u_{22}. \quad (2.2.13)$$

For the comparisons specified by the  $\eta$ 's in (2.2.13), we now proceed to find the conditional mean  $m(X, \underline{E})$  and conditional variance  $\sigma^2(\underline{E})$ . In finding  $m(X, \underline{E})$  and  $\sigma^2(\underline{E})$ , we can assume without loss of generality (for reasons given above) that  $\mu=0$  in (2.2.6). Thus, conditional upon  $X, \underline{E}$ ,  $\text{vec}(U)$  has mean vector  $(I_n - \psi(\underline{E}))\text{vec}(X)$  and covariance matrix

$$\begin{aligned} \psi(\underline{E})\Sigma_U(\underline{E}) &= \frac{\sigma_e^2}{K} (\Sigma_U(\underline{E}) + \frac{\sigma_e^2}{K} I_n)^{-1} \Sigma_U(\underline{E}) \\ &= \frac{\sigma_e^2}{K} (I_n - \psi(\underline{E})). \end{aligned}$$

Define

$$a_1(\underline{E}) = 1 - \frac{\sigma_e^2}{E_{AB}}, \quad a_2(\underline{E}) = \frac{\sigma_e^2}{E_{AB}} - \frac{\sigma_e^2}{E_B}, \quad a_3(\underline{E}) = \frac{\sigma_e^2}{E_{AB}} - \frac{\sigma_e^2}{E_A}. \quad (2.2.14)$$

Theorem 2.2.2. If the random matrix  $U$ :  $r \times c$  in vector form as  $\text{vec}(U)$  has mean vector  $(I_n - \psi(\underline{E}))\text{vec}(X)$  and covariance matrix  $K^{-1} \sigma_e^2 (I_n - \psi(\underline{E}))$ , where  $\psi(\underline{E})$  is given by (2.2.7), then:

Type I:  $\eta_I = u_{11} - u_{12}$  has mean and variance:

$$m_I(X, \underline{E}) = a_1(\underline{E})(\bar{X}_{11} - \bar{X}_{12}) + a_2(\underline{E})(\bar{X}_{.1} - \bar{X}_{.2}),$$

$$\sigma_I^2(\underline{E}) = \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + r^{-1} a_2(\underline{E})];$$

Type II:  $\eta_{II} = u_{11} - u_{21}$  has mean and variance:

$$m_{II}(X, \underline{E}) = a_1(\underline{E})(\bar{X}_{11} - \bar{X}_{21}) + a_3(\underline{E})(\bar{X}_{1.} - \bar{X}_{2.}),$$

$$\sigma_{II}^2(X, \underline{E}) = \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + c^{-1} a_3(\underline{E})];$$



Type III:  $\eta_{III} = u_{11} - u_{22}$  has mean and variance:

$$m_{III}(X, \underline{E}) = a_1(\underline{E})(\bar{X}_{11} - \bar{X}_{22}) + a_2(\underline{E})(\bar{X}_{.1} - \bar{X}_{.2}) + a_3(\underline{E})(\bar{X}_{1.} - \bar{X}_{2.})$$

$$\sigma_{III}^2(X, \underline{E}) = \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + r^{-1}a_2(\underline{E}) + c^{-1}a_3(\underline{E})].$$

Proof. We have previously noted that the columns of the  $n \times n$  orthogonal matrix  $H = H_c' \otimes H_r'$  are eigenvectors of

$$\Sigma_X(\underline{E}) = \Sigma_U(\underline{E}) + \frac{\sigma_e^2}{K} I_n = \frac{\sigma_e^2}{K} \psi^{-1}(\underline{E}).$$

Indeed,

$$\Sigma_X(\underline{E})H = HD(\underline{E})$$

where  $D(\underline{E})$  is a diagonal matrix with diagonal elements  $d_{tt}(\underline{E})$  defined by

$$Kd_{tt}(\underline{E}) = \begin{cases} (E_A + E_B - E_{AB}), & \text{if } t = n = rc, \\ E_A & , \text{ if } n - r + 1 \leq t \leq n - 1, \\ E_B & , \text{ if } t = r, 2r, \dots, (c-1)r, \\ E_{AB} & , \text{ otherwise.} \end{cases}$$

Thus

$$I_n - \psi(\underline{E}) = HD^*(\underline{E})H', \quad (2.2.15)$$

where  $D^*(\underline{E})$  is a diagonal matrix with diagonal elements

$$d_{tt}^*(\underline{E}) = \frac{Kd_{tt}(\underline{E}) - \sigma_e^2}{Kd_{tt}(\underline{E})}, \quad 1 \leq t \leq n.$$

Let  $\underline{s}_\ell = (0, 0, \dots, 0, 1)'$ :  $\ell \times 1$ . Then it can easily be shown that

$$D^*(\underline{E}) = a_1(\underline{E})(I_c \otimes I_r) + a_2(\underline{E})(I_c \otimes \underline{s}_r \underline{s}_r') + a_3(\underline{E})(\underline{s}_c \underline{s}_c' \otimes I_r). \quad (2.2.16)$$

Let  $\underline{v}_\ell = (1, 0, 0, \dots, 0)^\top: \ell \times 1$  and  $\underline{h}_\ell = (1, -1, 0, 0, \dots, 0)^\top: \ell \times 1$ .

Notice that since  $\underline{h}_\ell^\top$  is  $\sqrt{2}$  times the first row of  $H_\ell$ ,

$$H_\ell \underline{h}_\ell = \sqrt{2} \underline{v}_\ell, \quad \underline{v}_\ell^\top H_\ell = \frac{1}{\sqrt{2}} \underline{h}_\ell^\top.$$

Also

$$H_{\ell\ell}^\top \underline{s}_\ell = \frac{1}{\sqrt{\ell}} \underline{1}_\ell, \quad \underline{v}_\ell^\top H_{\ell\ell}^\top \underline{s}_\ell = \frac{1}{\sqrt{\ell}}, \quad \underline{v}_\ell^\top \underline{s}_\ell = 0.$$

The following results now follow by direct multiplication with Kronecker products; using the representations (2.2.15) and (2.2.16),

$$\begin{aligned} (\underline{h}_c^\top \otimes \underline{v}_r^\top)(I_{n-\psi}(\underline{E})) &= a_1(\underline{E})(\underline{h}_c^\top \otimes \underline{v}_r^\top) + \frac{1}{r} a_2(\underline{E})(\underline{h}_c^\top \otimes \underline{1}_r^\top), \\ (\underline{v}_c^\top \otimes \underline{h}_r^\top)(I_{n-\psi}(\underline{E})) &= a_1(\underline{E})(\underline{v}_c^\top \otimes \underline{h}_r^\top) + \frac{1}{c} a_3(\underline{E})(\underline{1}_c^\top \otimes \underline{h}_r^\top), \\ (\underline{h}_c^\top \otimes \underline{h}_r^\top)(I_{n-\psi}(\underline{E})) &= a_1(\underline{E})(\underline{h}_c^\top \otimes \underline{h}_r^\top). \end{aligned} \quad (2.2.17)$$

Now note that

$$u_{11} - u_{12} = \underline{v}_r^\top U \underline{h}_c = (\underline{h}_c^\top \otimes \underline{v}_r^\top) \text{vec}(U). \quad (2.2.18)$$

so that, using (2.2.17),

$$\begin{aligned} m_I(X, \underline{E}) &= (\underline{h}_c^\top \otimes \underline{v}_r^\top)(I_{n-\psi}(\underline{E})) \text{vec}(X) \\ &= a_1(\underline{E})(\underline{h}_c^\top \otimes \underline{v}_r^\top) \text{vec}(X) + \frac{1}{r} a_2(\underline{E})(\underline{h}_c^\top \otimes \underline{1}_r^\top) \text{vec}(X) \\ &= a_1(\underline{E})(\bar{X}_{11} - \bar{X}_{12}) + a_2(\underline{E})(\bar{X}_{.1} - \bar{X}_{.2}). \end{aligned}$$

Similarly, since

$$u_{11} - u_{21} = \underline{h}_r^\top U \underline{v}_c, \quad u_{11} - u_{22} = \underline{v}_r^\top U \underline{h}_c + \underline{h}_r^\top U \underline{v}_c - \underline{h}_r^\top U \underline{h}_c,$$

the asserted results for  $m_{II}(X, \underline{E})$  and  $m_{III}(X, \underline{E})$  follow from (2.2.17).

From (2.2.17) and (2.2.18), the variance of  $u_{11} - u_{12}$  is

$$\sigma_I^2(\underline{E}) = \frac{\sigma_e^2}{K} (\underline{h}_c^\top \otimes \underline{v}_r^\top)(I_{n-\psi}(\underline{E}))(\underline{h}_c^\top \otimes \underline{v}_r^\top)$$

$$\begin{aligned}
&= \frac{\sigma_e^2}{K} [a_1(\underline{E})(\underline{h}'_c \otimes \underline{v}'_r) + \frac{1}{r} a_2(\underline{E})(\underline{h}'_c \otimes \underline{1}'_r)](\underline{h}_c \otimes \underline{v}_r) \\
&= \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + \frac{1}{r} a_2(\underline{E})].
\end{aligned}$$

The formula for  $\sigma_{II}^2(\underline{E})$  is established in a corresponding fashion.

Finally, since

$$\begin{aligned}
u_{11} - u_{22} &= [(\underline{h}'_c \otimes \underline{v}'_r) + (\underline{v}'_c \otimes \underline{h}'_r) - (\underline{h}'_c \otimes \underline{h}'_r)]\text{vec}(U), \\
&\equiv \underline{w}'\text{vec}(U),
\end{aligned}$$

it follows from (2.2.17) that

$$\begin{aligned}
\sigma_{III}^2(\underline{E}) &= \underline{w}' \frac{\sigma_e^2}{K} (I_n - \psi(\underline{E}))\underline{w} \\
&= \frac{\sigma_e^2}{K} [a_1(\underline{E})\underline{w}'\underline{w} + \frac{1}{r} a_2(\underline{E})(\underline{h}'_c \otimes \underline{1}'_r)\underline{w} + \frac{1}{c} a_3(\underline{E})(\underline{1}'_c \otimes \underline{h}'_r)\underline{w}] \\
&= \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + \frac{1}{r} a_2(\underline{E}) + \frac{1}{c} a_3(\underline{E})],
\end{aligned}$$

as asserted.  $\square$

Up to this point, we have not used the explicit form of the component loss function  $L(\theta, d)$ . Only the fact that  $L(\theta, d)$  is a function of  $\theta$  only through  $\eta$  has been used. However, using the explicit form (2.1.7) of  $L(\theta, d)$  as a function of  $\eta$ , together with (2.2.9) and (2.2.12),

$$r^{(\lambda)}(d^+) = \chi_1 \int_{\mathcal{E}} \left[ \int_{-\infty}^0 |n| N_1(n | m(X, \underline{E}), \sigma^2(\underline{E})) dn \right] \pi(\underline{E} | \underline{S}\underline{S}^*, \underline{f}^*) d\underline{E},$$

$$r^{(\lambda)}(d^0) = \chi_0 \int_{\mathcal{E}} \left[ \int_{-\infty}^{\infty} |n| N_1(n | m(X, \underline{E}), \sigma^2(\underline{E})) dn \right] \pi(\underline{E} | \underline{S}\underline{S}^*, \underline{f}^*) d\underline{E},$$

and

$$r^{(\lambda)}(d^-) = \chi_1 \int_{\mathcal{E}} \left[ \int_0^{\infty} n N_1(n | m(X, \underline{E}), \sigma^2(\underline{E})) dn \right] \pi(\underline{E} | \underline{S}\underline{S}^*, \underline{f}^*) d\underline{E}.$$

Define

$$\Delta = \Delta(X, \underline{SS}^*, \underline{f}^*) = \int_{\mathcal{E}} m(X, \underline{E}) \pi(\underline{E} | \underline{SS}^*, \underline{f}^*) d\underline{E}, \quad (2.2.19)$$

$$\Gamma = \Gamma(X, \underline{SS}^*, \underline{f}^*) = \int_{\mathcal{E}} \left[ \int_{-\infty}^{\infty} |\eta| N_1(\eta | m(X, \underline{E}), \sigma^2(\underline{E})) d\eta \right] \pi(\underline{E} | \underline{SS}^*, \underline{f}^*) d\underline{E}. \quad (2.2.20)$$

Note that

$$\begin{aligned} r^{(\lambda)}(d^-) - r^{(\lambda)}(d^+) &= \kappa_1 \int_{\mathcal{E}} \left[ \int_{-\infty}^{\infty} \eta N_1(\eta | m(X, \underline{E}), \sigma^2(\underline{E})) d\eta \right] \pi(\underline{E} | \underline{SS}^*, \underline{f}^*) d\underline{E} \\ &= \kappa_1 \Delta, \end{aligned}$$

while

$$\begin{aligned} r^{(\lambda)}(d^-) - r^{(\lambda)}(d^0) &= \frac{1}{2}(r^{(\lambda)}(d^-) - r^{(\lambda)}(d^+)) + \frac{1}{2}(r^{(\lambda)}(d^-) + r^{(\lambda)}(d^+)) - r^{(\lambda)}(d^0) \\ &= \frac{\kappa_1}{2} \left[ \Delta + \left( \frac{\kappa_1 - 2\kappa_0}{\kappa_1} \right) \Gamma \right], \end{aligned}$$

$$r^{(\lambda)}(d^0) - r^{(\lambda)}(d^+) = \frac{\kappa_1}{2} \left[ \Delta - \left( \frac{\kappa_1 - 2\kappa_0}{\kappa_1} \right) \Gamma \right].$$

It now follows from (2.2.1) that

$$\varphi^{(\lambda)}(X, SSE) = \begin{cases} (1, 0, 0), & \text{if } \Delta > \left( \frac{\kappa_1 - 2\kappa_0}{\kappa_1} \right) \Gamma, \\ (0, 1, 0), & \text{if } |\Delta| < \left( \frac{\kappa_1 - 2\kappa_0}{\kappa_1} \right) \Gamma, \\ (0, 0, 1), & \text{if } \Delta < - \left( \frac{\kappa_1 - 2\kappa_0}{\kappa_1} \right) \Gamma. \end{cases} \quad (2.2.21)$$

Remark 1. The notation in (2.2.19) and (2.2.20) is somewhat imprecise, since we know from Theorem 2.2.2 that in the component problem where  $u_{i_1 j_1}$  is compared to  $u_{i_2 j_2}$ ,  $\Delta$  and  $\Gamma$  depend upon  $X$  only through  $\underline{SS}^*$  and the quantities:

$$T_1(X) = \bar{x}_{i_1 j_1} - \bar{x}_{i_2 j_2}, \quad T_2(X) = \bar{x}_{\cdot j_1} - \bar{x}_{\cdot j_2}, \quad T_3(X) = \bar{x}_{i_1 \cdot} - \bar{x}_{i_2 \cdot}. \quad (2.2.22)$$

However, in different component problems,  $T_1(X)$ ,  $T_2(X)$ , and  $T_3(X)$  depend upon different elements of  $X$ . Thus, at the level of generality here, where a general form for the Bayes rule is being described, the notation (2.2.19) and (2.2.20) suffices.

Remark 2. Since  $\tau \geq 0$ , as is apparent from (2.2.20), we see that  $\varphi^{(\lambda)}(X, SSE)$  can never take action  $d^0$  if  $\kappa_1 \leq 2\kappa_0$ . Hence, in the following chapters it is assumed that

$$0 < 2\kappa_0 < \kappa_1,$$

or, equivalently, that

$$0 < \rho = \frac{\kappa_1 - 2\kappa_0}{\kappa_1} < 1. \quad (2.2.23)$$

Remark 3.  $\Delta = \Delta(X, SS^*, f^*)$  is the posterior mean  $E(\eta | X, SSE)$  of  $\eta$ . Note from (2.2.21), (2.2.23) and the fact that  $\tau \geq 0$ , that when  $\Delta > 0$ , our choice of action lies between  $d^+$  and  $d^0$ , while when  $\Delta < 0$ , the choice lies between  $d^-$  and  $d^0$ . Thus, the value of  $\Delta$  can be used to eliminate one action ( $d^+$  or  $d^-$ ) from consideration. It follows from Theorem 2.2.2 that

$$\Delta(X, SS^*, f^*) = \sum_{j=1}^3 T_j(X) \left[ \int_{\mathcal{E}} a_j(\underline{E}) \pi(\underline{E} | SS^*, f^*) d\underline{E} \right], \quad (2.2.24)$$

where the  $a_j(\underline{E})$ 's are defined by (2.2.14) and the  $T_j(X)$ 's are defined by (2.2.22). An algorithm for computing  $\Delta = \Delta(X, SS^*, f^*)$  is given in Chapter 4.

Having simplified the form of component Bayes rules to (2.2.21), our basic goal in this section has been achieved. In Chapter 3, Section 3.1, we briefly indicate, in the special context where  $\underline{\Xi} = (E_A, E_B, E_{AB}, \sigma_e^2)$  is assumed known, how the component Bayes rules (and thus the experimentwise Bayes rule defined in Theorem 2.2.1) make use of information about main effects and interactions.

CHAPTER III  
PROPERTIES OF THE COMPONENT BAYES RULES

This chapter is divided into three sections. Section 3.1 shows how the component Bayes rules explicitly depend upon information about expected patterns of cell means  $u_{ij}$  in the simplest case where this information is expressed in terms of known measures of interaction ( $\sigma_{AB}^2$ ), main effects for row and column ( $\sigma_A^2, \sigma_B^2$ ) and error ( $\sigma_e^2$ ). This special case can be interpreted as the limiting case of the general model of Chapter II, when all pooled degrees of freedom  $f_A^*, f_B^*, f_{AB}^*, f_e^*$  [see (2.1.19)] become infinitely large. Study of this special case thus gives insight into the adaptive characteristic of the overall Bayes rule of Theorem 2.2.1 viewed as a multiple comparisons procedure.

Returning to the general model of Chapter II, Section 3.2 establishes some useful properties of the functions  $\Delta(X, \underline{SS}^*, \underline{f}^*)$  and  $\Gamma(X, \underline{SS}^*, \underline{f}^*)$  which define the component Bayes rules. Bounds are found for  $\Gamma(X, \underline{SS}^*, \underline{f}^*)$ , leading to a screening algorithm which serves to reduce the number of component problems in which this function must be calculated. Further, it is shown that with probability one under  $p(X, SSE|\theta)$  for all  $\theta \in \Theta$ , the component Bayes rules are nonrandomized, verifying an assertion made earlier in Chapter II. Finally, in Section 3.3, a proof of the fact that simultaneous application of the component Bayes rules yields consistent action vectors  $\underline{d}$  with probability one [under

$p(X, SSE|\theta)$ , all  $\theta \in \Theta$  ] is given. This proof thus completes the verification of Theorem 2.2.1.

### 3.1. The Nature of the Component Bayes Rules when $\sigma_A^2, \sigma_B^2, \sigma_{AB}^2$ and $\sigma_e^2$ are Known

Suppose that in place of the prior distribution  $\lambda(\theta)$  defined in Equation (2.1.20) of Chapter II, the prior density

$$p(U|\underline{E}) = N_n(\text{vec}(U) | \underline{\mu}_n, \Sigma_U(\underline{E}))$$

is assigned to  $U$ , and it is assumed that  $\underline{E} = (E_A, E_B, E_{AB}, \sigma_e^2)'$  is known. The resulting prior distribution  $\lambda_0(\theta) = \lambda_0(\theta|\underline{E})$  is then the product of  $p(U|\underline{E})$  and a mass function placing probability one on the known value of  $\sigma_e^2$ . Such a prior  $\lambda_0(\theta)$  can be viewed as the limit of the prior distribution  $\lambda(\theta|\underline{SS}_p, \underline{f})$  when each element of  $\underline{f} = (f_A, f_B, f_{AB}, f_e)'$  tends to infinity, and

$$\frac{SSA_p}{f_A} \rightarrow E_A, \quad \frac{SSB_p}{f_B} \rightarrow E_B, \quad \frac{SSAB_p}{f_{AB}} \rightarrow E_{AB}, \quad \frac{SSE_p}{f_e} \rightarrow \sigma_e^2.$$

Alternatively, the posterior distribution obtained from  $\lambda_0(\theta)$  can be viewed as the limit of the posterior distribution derived from  $\lambda(\theta|\underline{SS}_p, \underline{f})$  when each element of the "pooled" degrees of freedom vector  $\underline{f}^*$  tends to infinity, and

$$\frac{SSA^*}{f_A^*} \rightarrow E_A, \quad \frac{SSB^*}{f_B^*} \rightarrow E_B, \quad \frac{SSAB^*}{f_{AB}^*} \rightarrow E_{AB}, \quad \frac{SSE^*}{f_e^*} \rightarrow \sigma_e^2.$$

In either case, the known quantities  $E_A, E_B, E_{AB}, \sigma_e^2$  correspond to the "mean squares" that would appear in an ANOVA table derived from "data" pooled from prior information and  $(X, SSE)$ . Assuming that  $\underline{E} \in \mathcal{E}$ , so that



$$0 < \sigma_e^2 \leq E_{AB} \leq E_A, E_B < \infty,$$

the variance components

$$\sigma_A^2 = \frac{E_A - E_{AB}}{cK}, \quad \sigma_B^2 = \frac{E_B - E_{AB}}{rK}, \quad \sigma_{AB}^2 = \frac{E_{AB} - \sigma_e^2}{K}, \quad \sigma_e^2$$

are known constants.

Repeating the steps that led to the component Bayes rules  $\varphi^{(\lambda)}(X, SSE)$ , in Section 2.2 (except that now no integration over  $\underline{E} \in \mathcal{E}$  is necessary) it can be shown that the component Bayes rule

$\varphi^{(\lambda)}(i_1, j_1), (i_2, j_2)(X)$  for comparing  $u_{i_1 j_1}$  to  $u_{i_2 j_2}$  is the following:

$$\varphi^{(\lambda)}(i_1, j_1), (i_2, j_2)(X) = \begin{cases} (1, 0, 0) & \text{if } \Delta(\underline{T}(X), \underline{a}(\underline{E})) > \rho \Gamma(\underline{T}(X), \underline{a}(\underline{E})), \\ (0, 1, 0) & \text{if } |\Delta(\underline{T}(X), \underline{a}(\underline{E}))| < \rho \Gamma(\underline{T}(X), \underline{a}(\underline{E})), \\ (0, 0, 1) & \text{if } \Delta(\underline{T}(X), \underline{a}(\underline{E})) < -\rho \Gamma(\underline{T}(X), \underline{a}(\underline{E})), \end{cases} \quad (3.1.1)$$

where  $1 - \rho = 2\chi_0 \chi_1^{-1}$ ,

$$\underline{T}(X) = (T_1(X), T_2(X), T_3(X))' = (\bar{X}_{i_1 j_1} - \bar{X}_{i_2 j_2}, \bar{X}_{\cdot j_1} - \bar{X}_{\cdot j_2}, \bar{X}_{i_1 \cdot} - \bar{X}_{i_2 \cdot})', \quad (3.1.2)$$

$$\underline{a}(\underline{E}) = (a_1(\underline{E}), a_2(\underline{E}), a_3(\underline{E}))' = \left(1 - \frac{\sigma_e^2}{E_{AB}}, \frac{\sigma_e^2}{E_{AB}} - \frac{\sigma_e^2}{E_B}, \frac{\sigma_e^2}{E_{AB}} - \frac{\sigma_e^2}{E_A}\right)', \quad (3.1.3)$$

$$\Delta(\underline{T}(X), \underline{a}(\underline{E})) = \underline{a}'(\underline{E})\underline{T}(X), \quad (3.1.4)$$

$$\Gamma(\underline{T}(X), \underline{a}(\underline{E})) = \int_{-\infty}^{\infty} |n| N_1(n | \Delta(\underline{T}(X), \underline{a}(\underline{E})), \sigma^2(\underline{E})) dn, \quad (3.1.5)$$

and

$$\sigma^2(\underline{E}) = \frac{2\sigma_e^2}{K} \left[ a_1(\underline{E}) + \frac{1}{r} (1 - \delta_{j_1 j_2}) a_2(\underline{E}) + \frac{1}{c} (1 - \delta_{i_1 i_2}) a_3(\underline{E}) \right]. \quad (3.1.6)$$

Here  $\delta_{st}$  is the Kronecker delta function:

$$\delta_{st} = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

Note that:

(i) In Type I problems ( $i_1=i_2$ ):  $1 - \delta_{i_1 i_2} = 0$ ,  $T_3(X) = 0$ .

(ii) In Type II problems ( $j_1=j_2$ ):  $1 - \delta_{j_1 j_2} = 0$ ,  $T_2(X) = 0$ .

When  $\Delta = \rho\Gamma, \varphi^{(\lambda)}(X)$  would randomize between  $d^+$  and  $d^0$ , while when  $\Delta = -\rho\Gamma, \varphi^{(\lambda)}(X)$  randomizes between  $d^-$  and  $d^0$ .

Observe that  $\varphi^{(\lambda)}(X) = \varphi^{(\lambda)}(\underline{I}(X))$ . That is, the component Bayes rule depends upon  $X$  only through the elements of  $\underline{I}(X)$ . The first element,  $T_1(X) = \bar{X}_{i_1 j_1} - \bar{X}_{i_2 j_2}$  of  $\underline{I}(X)$  is the contrast in the cell means that would be used to make comparisons between  $u_{i_1 j_1}$  and  $u_{i_2 j_2}$  if the two-factor design were treated as a one-factor design. Under the two-stage approach mentioned in Section 1.3, this approach would be used if it were believed that  $\sigma_{AB}^2 > 0$ , or equivalently if

$$\frac{E_{AB}}{\sigma_e^2} > 1.$$

The second and third elements of  $\underline{I}(X)$  are column main effect ( $T_2$ ) and row main effect ( $T_3$ ) contrasts which the two-stage approach would use if it were believed that  $\sigma_{AB}^2 = 0$ , or equivalently if

$$\frac{E_{AB}}{\sigma_e^2} \approx 1.$$

Note that with  $\Delta = \Delta(\underline{T}(X), \underline{a}(E))$ ,  $\sigma^2 = \sigma^2(E)$ ,

$$\begin{aligned} \Gamma(\underline{T}(X), \underline{a}(E)) &= \int_{-\infty}^{\infty} |\eta| N_1(\eta | \Delta, \sigma^2) d\eta \\ &= \sigma \int_{-\infty}^{\infty} |v| N_1\left(v \left| \frac{\Delta}{\sigma}, 1\right.\right) dv, \end{aligned} \quad (3.1.7)$$

where we have made the change of variables  $v = \sigma^{-1}\eta$ . Thus, letting

$$Q(\omega) = \int_{-\infty}^{\infty} |v| N_1(v | \omega, 1) dv, \quad (3.1.8)$$

it is easily seen from (3.1.1) and the fact that  $\sigma = \sigma(E) > 0$ , that

$$\varphi^{(\lambda)}(\underline{T}(X)) = \begin{cases} (1, 0, 0) & \text{if } \frac{\Delta}{\sigma} - \rho Q\left(\frac{\Delta}{\sigma}\right) > 0 \\ (0, 1, 0) & \text{if } \frac{\Delta}{\sigma} - \rho Q\left(\frac{\Delta}{\sigma}\right) < 0, \frac{\Delta}{\sigma} + \rho Q\left(\frac{\Delta}{\sigma}\right) > 0, \\ (0, 0, 1) & \text{if } \frac{\Delta}{\sigma} + \rho Q\left(\frac{\Delta}{\sigma}\right) < 0. \end{cases} \quad (3.1.9)$$

Lemma 3.1.1. If  $|c| < 1$ ,  $\omega + cQ(\omega)$  is a strictly increasing function of  $\omega$ .

Proof. It is easily seen that

$$\omega + cQ(\omega) = \int_{-\infty}^{\infty} g(v) N_1(v | \omega, 1) dv,$$

where

$$g(v) = \begin{cases} (1+c)v & \text{if } v \geq 0 \\ (1-c)v & \text{if } v < 0 \end{cases} = v + c|v|,$$

is (since  $|c| < 1$ ) a strictly increasing function of  $v$ . Since  $N_1(v | \omega, 1)$  has strict monotone likelihood in  $v$ , the assertion follows by modifying the proof of Lemma 2, page 74, of Lehmann (1959) to account for strict inequalities.  $\square$

Lemma 3.1.2. If  $|c| < 1$ ,  $\omega + cQ(\omega)$  is continuous in  $\omega$ , and

$$\lim_{\omega \rightarrow \pm\infty} [\omega + cQ(\omega)] = \pm\infty. \quad (3.1.10)$$

Thus, there exists a unique  $\omega_c$ ,  $-\infty < \omega_c < \infty$ , such that

$$\omega_c + cQ(\omega_c) = 0. \quad (3.1.11)$$

Since  $Q(\omega) \geq 0$ , it follows that  $c\omega_c \leq 0$ .

Proof. Continuity of  $\omega + cQ(\omega)$  will follow from continuity of  $Q(\omega)$ .

However, for any  $\omega_1, \omega_2$ ,  $-\infty < \omega_1, \omega_2 < \infty$ ,

$$\begin{aligned} |Q(\omega_1) - Q(\omega_2)| &= \left| \int_{-\infty}^{\infty} |v| N_1(v|\omega_1, 1) dv - \int_{-\infty}^{\infty} |v| N_1(v|\omega_2, 1) dv \right| \\ &\leq \int_{-\infty}^{\infty} \left| |z + \omega_1| - |z + \omega_2| \right| N_1(z|0, 1) dz \\ &\leq |\omega_1 - \omega_2|, \end{aligned} \quad (3.1.12)$$

from which the continuity of  $Q(\omega)$  follows. Letting  $\omega_1 = \omega$ ,  $\omega_2 = 0$  in (3.1.12),

$$0 \leq Q(\omega) \leq |\omega| + \int_{-\infty}^{\infty} |z| N_1(z|0, 1) dz = |\omega| + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

Using this result, it is straightforward to show that (3.1.10) holds for  $|c| < 1$ . The existence of  $\omega_c$  now follows from Lemma 3.1.1, the continuity of  $\omega + cQ(\omega)$ , and (3.1.10).  $\square$

Lemma 3.1.3. For all  $\omega$ ,  $-\infty < \omega < \infty$ , all  $c$ ,  $|c| < 1$ ,

$$-\omega + cQ(-\omega) = -(\omega - cQ(\omega)). \quad (3.1.13)$$

Proof. It is easily shown that  $Q(-\omega) = Q(\omega)$ . The result (3.1.13) now directly follows.  $\square$

From (3.1.13) it follows that  $\omega_\rho = -\omega_{-\rho}$ , where  $\omega_\rho$  and  $\omega_{-\rho}$  are defined by (3.1.11) with  $c = \rho$ ,  $c = -\rho$ , respectively. Thus, from Lemma 3.1.1 and (3.1.11), the component Bayes rule  $\varphi^{(\lambda)}(\underline{T}(X))$  has equivalent form:

$$\varphi^{(\lambda)}(\underline{T}(X)) = \begin{cases} (1,0,0), & \text{if } \Delta(\underline{T}(X), \underline{a}(\underline{E})) > t^*\sigma(\underline{E}), \\ (0,1,0), & \text{if } |\Delta(\underline{T}(X), \underline{a}(\underline{E}))| < t^*\sigma(\underline{E}), \\ (0,0,1), & \text{if } \Delta(\underline{T}(X), \underline{a}(\underline{E})) < -t^*\sigma(\underline{E}), \end{cases} \quad (3.1.14)$$

where  $t^*$  is the unique solution of the equation

$$t^* - \rho Q(t^*) = 0. \quad (3.1.15)$$

For a given  $\rho = 1 - 2(\chi_0/\chi_1)$ , a computer program has been written to find  $t^*$ . A listing of this program, together with a table of  $t^*$  for selected values of

$$\frac{\chi_1}{\chi_0} = \frac{2}{1-\rho},$$

is provided in Appendix A.

Note that if

$$\Delta(\underline{T}(X), \underline{a}(\underline{E})) = \pm t^*\sigma(\underline{E}), \quad (3.1.16)$$

the component Bayes rule should randomize over  $d^+$  and  $d^0$ , or over  $d^-$  and  $d^0$ , respectively. However, since  $\Delta(\underline{T}(X), \underline{a}(\underline{E}))$  is a linear combination of the continuous variables  $T_i(X)$ ,  $1 \leq i \leq 3$ , the event defined by (3.1.16) has probability 0 under the density  $p(X, SSE|U, \sigma_e^2)$  for all  $(U, \sigma_e^2)$ . Thus,  $\varphi^{(\lambda)}(\underline{T}(X))$  can be written as a nonrandomized rule as in (3.1.14).

The decision made by the component Bayes rule  $\varphi^{(\lambda)}(\underline{T}(X))$  depends entirely upon the value of the statistic

$$\frac{\Delta(\underline{T}(X), \underline{a}(\underline{E}))}{\sigma(\underline{E})} = \sum_{i=1}^3 T_i(X) \frac{a_i(\underline{E})}{\sigma(\underline{E})}.$$

The meaning of the functions  $T_1(X), T_2(X), T_3(X)$  has already been discussed. From (3.1.3) and (3.1.6) we see that as the ratio  $(\sigma_e^2)^{-1}E_{AB}$  increases to infinity

$$\frac{a_1(\bar{E})}{\sigma(\bar{E})} \rightarrow \left( \frac{K}{\sigma_e^2} \right)^{\frac{1}{2}}, \quad \frac{a_i(\bar{E})}{\sigma(\bar{E})} \rightarrow 0, \quad i=2,3,$$

and the decision made by the Bayes rule depends entirely upon the comparison statistic  $T_1(X) = \bar{X}_{i_1 j_1} - \bar{X}_{i_2 j_2}$  for the cell means. Hence, strong evidence of interaction places the burden of the comparison of  $u_{i_1 j_1}$  to  $u_{i_2 j_2}$  on cell data. On the other hand, as  $\sigma_{AB}^2 \rightarrow 0$ , so that  $(\sigma_e^2)^{-1}E_{AB} \rightarrow 1$ ,

$$\frac{a_1(\bar{E})}{\sigma(\bar{E})} \rightarrow 0,$$

and information from the cell means is ignored. In Type I problems ( $i_1 = i_2$ ), when  $(\sigma_e^2)^{-1}E_{AB} \rightarrow 1$ ,

$$\frac{a_2(\bar{E})}{\sigma(\bar{E})} \rightarrow \left[ \frac{rK}{\sigma_e^2} \left( 1 - \frac{E_{AB}}{E_B} \right) \right]^{\frac{1}{2}}$$

so that with strong evidence of column effects ( $E_{AB}^{-1}E_B \rightarrow \infty$ ), information from the column comparison  $T_2(X) = \bar{X}_{.j_1} - \bar{X}_{.j_2}$  determines the decision, while with no evidence of column effects ( $E_{AB}^{-1}E_B \rightarrow 1$ ),  $\sigma^{-1}\Delta \rightarrow 0$  and the decision  $d^0$  is taken. Type II problems act similarly, except that the values of  $E_{AB}^{-1}E_A$  and  $T_3(X) = \bar{X}_{i_1.} - \bar{X}_{i_2.}$  determine the decision. In Type III problems with  $(\sigma_e^2)^{-1}E_{AB} \rightarrow 1$ , the decision depends upon both row and column effects, with row effects  $T_3(X)$  being ignored when  $E_{AB}^{-1}E_A \rightarrow 1$ ,

column effects  $T_2(X)$  being ignored when  $E_{AB}^{-1}E_B \rightarrow 1$ , and  $d^0$  being taken always when main effects are not believed to exist. Remember that the specialized prior of this section yields a posterior distribution which approximates the posterior resulting from the more general model of Section 2.1 and a large amount of information pooled from the prior and the data:

$$\tilde{f}^* \rightarrow \infty, \tilde{E} = \lim \left( \frac{SSA^*}{\tilde{f}_A^*}, \frac{SSB^*}{\tilde{f}_B^*}, \frac{SSAB^*}{\tilde{f}_{AB}^*}, \frac{SSE^*}{\tilde{f}_e^*} \right).$$

We see that the component Bayes rules use information (from prior and data) to adaptively choose, in a continuous fashion, among the alternative procedures advocated in the two-stage classical multiple comparisons procedure described in Section 1.3.

### 3.2. Properties of the Component Bayes Rules in the General Model

In Section 2.2, it was noted that the component Bayes rule for comparing  $u_{i_1 j_1}$  to  $u_{i_2 j_2}$  depends upon the data  $(X, SSE)$  only through  $\tilde{SS}^*$ , as defined in (2.1.19), and

$$\tilde{T} = (T_1, T_2, T_3)^* = (\bar{X}_{i_1 j_1} - \bar{X}_{i_2 j_2}, \bar{X}_{\cdot j_1} - \bar{X}_{\cdot j_2}, \bar{X}_{i_1 \cdot} - \bar{X}_{i_2 \cdot})^*.$$

It will be notationally convenient not to continually exhibit the dependence of the component Bayes rule on  $\tilde{SS}^*$ , and its pooled degrees of freedom (vector)  $\tilde{f}^*$ . To this end, let

$$\pi^*(E) = \pi(E | \tilde{SS}^*, \tilde{f}^*).$$

Also define

$$m(\tilde{T}, E) = m(X, E) = \sum_{i=1}^3 a_i(E) T_i, \quad (3.2.1)$$

$$\sigma^2(\underline{E}) = \frac{2\sigma_e^2}{K} [a_1(\underline{E}) + \frac{1}{r} (1 - \delta_{j_1 j_2}) a_2(\underline{E}) + \frac{1}{c} (1 - \delta_{i_1 i_2}) a_3(\underline{E})]; \quad (3.2.2)$$

where  $a_i(\underline{E})$ ,  $1 \leq i \leq 3$ , are defined in (2.2.14) and  $\delta_{st}$  is the Kronecker delta function. From (2.2.21), the component Bayes rule is then defined by

$$\varphi^{(\lambda)}(\underline{T}) = \varphi^{(\lambda)}(X, SSE) = \begin{cases} (1, 0, 0), & \text{if } \Delta(\underline{T}) > \rho\Gamma(\underline{T}), \\ (0, 1, 0), & \text{if } |\Delta(\underline{T})| < \rho\Gamma(\underline{T}), \\ (0, 0, 1), & \text{if } \Delta(\underline{T}) < -\rho\Gamma(\underline{T}), \end{cases} \quad (3.2.3)$$

where

$$\Delta(\underline{T}) = \int_{\mathcal{E}} m(\underline{T}, \underline{E}) \pi^*(\underline{E}) d\underline{E},$$

$$\Gamma(\underline{T}) = \int_{\mathcal{E}} \left[ \int_{-\infty}^{\infty} |\eta| N_1(\eta | m(\underline{T}, \underline{E}), \sigma^2(\underline{E})) d\eta \right] \pi^*(\underline{E}) d\underline{E},$$

and  $\rho = (\chi_1 - 2\chi_0) / \chi_1$ ,  $0 \leq \rho < 1$ . In (3.2.3), if  $\Delta(\underline{T}) = \rho\Gamma(\underline{T})$ ,  $\varphi^{(\lambda)}(\underline{T})$  randomizes between the actions  $d^+$ ,  $d^0$ , while if  $\Delta(\underline{T}) = -\rho\Gamma(\underline{T})$ ,  $\varphi^{(\lambda)}(\underline{T})$  randomizes between  $d^0$  and  $d^-$ .

For any constant  $c$ ,  $-1 < c < 1$ , let

$$q_c(\underline{T}) = \Delta(\underline{T}) + c\Gamma(\underline{T}). \quad (3.2.4)$$

It follows from (3.1.7) and (3.1.8) that

$$q_c(\underline{T}) = \int_{\mathcal{E}} [\omega(\underline{T}, \underline{E}) + cQ(\omega(\underline{T}, \underline{E}))] \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E}, \quad (3.2.5)$$

where

$$\omega(\underline{T}, \underline{E}) = \frac{m(\underline{T}, \underline{E})}{\sigma(\underline{E})} = \sum_{i=1}^3 \frac{a_i(\underline{E})}{\sigma(\underline{E})} T_i. \quad (3.2.6)$$

Hence, from (3.2.3),



$$\varphi^{(\lambda)}(\underline{T}) = \begin{cases} (1,0,0), & \text{if } q_{-\rho}(\underline{T}) > 0, \\ (0,1,0), & \text{if } q_{-\rho}(\underline{T}) < 0, q_{\rho}(\underline{T}) > 0, \\ (0,0,1), & \text{if } q_{\rho}(\underline{T}) < 0. \end{cases} \quad (3.2.7)$$

Since  $\Gamma(\underline{T}) \geq 0$  and  $\rho \geq 0$ , it is clear from (3.2.4) that

$$q_{\rho}(\underline{T}) \geq q_{-\rho}(\underline{T}). \quad (3.2.8)$$

Some useful properties of the function  $q_c(\underline{T})$  follow directly from (3.2.5) and the results of Section 3.1.

Lemma 3.2.1. For all  $c$ ,  $-1 < c < 1$ ,  $q_c(T_1, T_2, T_3)$  is a strictly increasing function in each of its arguments  $T_1, T_2, T_3$ .

Proof. Since  $a_i(\underline{E}) > 0$ ,  $1 \leq i \leq 3$ , and  $\sigma(\underline{E}) > 0$ , it follows from (3.2.6) that  $\omega(\underline{T}, \underline{E})$  is strictly increasing in  $T_1, T_2, T_3$ . The assertion of the lemma now follows from (3.2.5) and Lemma 3.1.1.  $\square$

Lemma 3.2.2. For all  $c$ ,  $-1 < c < 1$ ,  $q_c(-\underline{T}) = -q_{-c}(\underline{T})$ .

Proof. The result is a direct consequence of (3.2.5), (3.2.6) and Lemma 3.1.3.  $\square$

Lemma 3.2.3. For all  $c$ ,  $-1 < c < 1$ , all fixed  $T_2, T_3$ ,

- (i)  $q_c(\underline{T}) = q_c(T_1, T_2, T_3)$  is continuous in  $T_1$ ,
- (ii)  $\lim_{T_1 \rightarrow \pm\infty} q_c(\underline{T}) = \pm\infty$ .

Proof. For any  $T_1, T_1^*, -\infty < T_1, T_1^* < \infty$ , it follows from (3.2.5), (3.2.6) and (3.1.12) that

$$\begin{aligned}
& |q_c(T_1, T_2, T_3) - q_c(T_1^*, T_2, T_3)| \\
&= \left| \int_{\mathcal{E}} \left[ \frac{a_1(\underline{E})}{\sigma(\underline{E})} (T_1 - T_1^*) + cQ(\omega(T_1, T_2, T_3, \underline{E})) - cQ(\omega(T_1^*, T_2, T_3, \underline{E})) \right] \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E} \right| \\
&\leq (1 + |c|) |T_1 - T_1^*| \left[ \int_{\mathcal{E}} a_1(\underline{E}) \pi^*(\underline{E}) d\underline{E} \right].
\end{aligned}$$

Since the integral of  $a_1(\underline{E})$  is finite, assertion (i) follows directly.

To verify (ii) when  $T_1 \rightarrow \infty$ , apply Fatou's lemma and (3.1.10) to (3.2.5), noting from (3.2.6) that  $\lim_{T_1 \rightarrow \infty} \omega(T_1, \underline{E}) = \infty$ . Thus

$$\begin{aligned}
\lim_{T_1 \rightarrow \infty} q_c(T_1) &\geq \int_{\mathcal{E}} \lim_{T_1 \rightarrow \infty} [\omega(T_1, \underline{E}) + cQ(\omega(T_1, \underline{E}))] \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E} \\
&= \infty.
\end{aligned}$$

For the case  $T_1 \rightarrow -\infty$ , apply Lemma 3.2.2 and the above result:

$$\begin{aligned}
\infty &= \lim_{T_1 \rightarrow \infty} q_{-c}(T_1, -T_2, -T_3) = \lim_{T_1 \rightarrow \infty} -q_c(-T_1, T_2, T_3) \\
&= -\overline{\lim}_{T_1 \rightarrow -\infty} q_c(T_1, T_2, T_3).
\end{aligned}$$

From these two results, assertion (ii) follows.  $\square$

Theorem 3.2.4. For fixed  $T_2, T_3$ , there exist unique constants  $T_1^L(T_2, T_3), T_1^U(T_2, T_3)$  satisfying

$$q_\rho(T_1^L(T_2, T_3), T_2, T_3) = 0 = q_{-\rho}(T_1^U(T_2, T_3), T_2, T_3) \quad (3.2.9)$$

and

$$T_1^L(T_2, T_3) \leq T_1^U(T_2, T_3).$$

Hence

$$\varphi^{(\lambda)}(\underline{T}) = \begin{cases} (1,0,0), & \text{if } T_1 > T_1^U(T_2, T_3), \\ (0,1,0), & \text{if } T_1^L(T_2, T_3) < T_1 < T_1^U(T_2, T_3), \\ (0,0,1), & \text{if } T_1 < T_1^L(T_2, T_3). \end{cases} \quad (3.2.10)$$

Proof. The existence of  $T_1^L(T_1, T_2), T_1^U(T_1, T_2)$  follows directly from Lemmas 3.2.1 and 3.2.3. The inequality between  $T_1^L(T_2, T_3)$  and  $T_1^U(T_2, T_3)$  follows directly from (3.2.8), Lemma 3.2.1, and (3.2.9). Finally, (3.2.10) follows directly from (3.2.7), Lemma 3.2.1, and (3.2.9).  $\square$

From (3.2.10) it is easily seen that  $\varphi^{(\lambda)}(\underline{T})$  is almost surely (with respect to  $p(X, SSE|\theta)$ , all  $\theta \in \Theta$ ) a nonrandomized rule. Indeed,  $\varphi^{(\lambda)}(\underline{T})$  will randomize between two actions if and only if

$$T_1 = T_1^L(T_2, T_2), T_1 = T_1^U(T_2, T_3). \quad (3.2.11)$$

Since the conditional distribution of  $T_1$  given  $T_2, T_3$ , as obtained from  $p(X, SSE|\theta)$ , is a continuous (indeed, normal) distribution for all  $\theta \in \Theta$ , the conditional probability of one of the equalities (3.2.11) holding is zero for all  $\theta \in \Theta$ . Hence, unconditionally the equalities (3.2.11) have probability zero to occur, all  $\theta \in \Theta$ . This proves the assertion made in Section 2.2 that the component Bayes rules are nonrandomized with probability one.

Having demonstrated the monotonicity of the component Bayes rules in the cell mean contrast  $T_1(X) = \bar{X}_{i_1 j_1} - \bar{X}_{i_2 j_2}$ , it would be very useful to determine the explicit functional forms of  $T_1^L(T_2, T_3), T_1^U(T_2, T_3)$  as functions of  $T_2, T_3$  (and also of  $SS^*$ ). Unfortunately, this has proven to be unfeasible, largely due to the extremely complicated nature of  $\Gamma(\underline{T})$

as a function of  $\underline{T}$ . Consequently, our implementation of the component Bayes rule proceeds through (3.2.7). This necessitates computation of  $\Gamma(\underline{T})$ , as can be seen from (3.2.4) and (3.2.7). Since such computation can be costly in terms of computer time, we seek a way to avoid this computation wherever possible. To do so, certain bounds on  $q_c(\underline{T})$  are developed in the remainder of this section.

Define

$$a_i = \int_{\mathcal{E}} a_i(\underline{E})\pi^*(\underline{E})d\underline{E}, \quad i=1,2,3 \quad (3.2.12)$$

and let  $\underline{a} = (a_1, a_2, a_3)'$ . Note that

$$\Delta(\underline{T}) = \int_{\mathcal{E}} m(\underline{T}, \underline{E})\pi^*(\underline{E}) = \sum_{i=1}^3 a_i T_i = \underline{a}'\underline{T} \quad (3.2.13)$$

is a linear function of  $\underline{T}$ . Define

$$\sigma_0 = \int_{\mathcal{E}} \sigma(\underline{E})\pi^*(\underline{E})d\underline{E} < \infty.$$

Lemma 3.2.5. For all  $\underline{T}$ ,

$$q_{-\rho}(\underline{T}) \leq \Delta(\underline{T}) - \rho\sigma_0 Q(\sigma_0^{-1}\Delta(\underline{T})), \quad q_{\rho}(\underline{T}) \geq \Delta(\underline{T}) + \rho\sigma_0 Q(\sigma_0^{-1}\Delta(\underline{T})).$$

Proof. Recall that

$$Q(\omega) = \int_{-\infty}^{\infty} |\eta| N_1(\eta|\omega, 1) d\eta. \quad (3.2.14)$$

Since the density  $N_1(\eta|\omega, 1)$  is of Polya Type III, and  $|\eta|$  is a convex function of  $\eta$ , it follows from Proposition 3.2, p.23 of Karlin (1968) that  $Q(\omega)$  is a convex function of  $\omega$ . Hence  $\omega + cQ(\omega)$  is convex in  $\omega$  when  $c \geq 0$ , concave in  $\omega$  when  $c < 0$ . Applying Jensen's Inequality to

$$\sigma_0^{-1} q_c(\underline{T}) = \int_{\mathcal{E}} [\omega(\underline{T}, \underline{E}) + cQ(\omega(\underline{T}, \underline{E}))] \left( \frac{\sigma(\underline{E})\pi^*(\underline{E})}{\sigma_0} \right) d\underline{E},$$

noting that  $\sigma_0^{-1} \sigma(\underline{E}) \pi^*(\underline{E})$  is a density, and that the expected value of  $\omega(\underline{T}, \underline{E})$  under that density is  $\sigma_0^{-1} \Delta(\underline{T})$ , yields the stated inequalities.  $\square$

Let  $t^*$  be the unique solution of

$$t^* - \rho Q(t^*) = 0,$$

guaranteed by Lemma 3.1.2. It follows from Lemmas 3.1.1, 3.1.2 and 3.2.5, and also Equations (3.2.7) and (3.2.13), that

$$\left| \sum_{i=1}^3 a_i T_i \right| \leq \sigma_0 t^* \Rightarrow \varphi^{(\lambda)}(\underline{T}) = (0, 1, 0). \quad (3.2.15)$$

Thus if  $\left| \sum_{i=1}^3 a_i T_i \right| \leq \sigma_0 t^*$ , we can make the decision  $d^0$  without having to compute  $r(\underline{T})$ ,  $q_{-\rho}(\underline{T})$  or  $q_{\rho}(\underline{T})$ .

On the other hand, note from (3.2.14) that

$$\begin{aligned} Q(\omega) &= \int_0^{\infty} \eta N_1(\eta | \omega, 1) d\eta + \int_{-\infty}^0 (-\eta) N_1(\eta | \omega, 1) d\eta \\ &= \int_{-\omega}^{\infty} (z + \omega) N_1(z | 0, 1) dz - \int_{-\infty}^{-\omega} (z + \omega) N_1(z | 0, 1) dz \\ &= \omega \left[ 1 - 2 \int_{-\infty}^{-\omega} N_1(z | 0, 1) dz \right] + 2N_1(-\omega | 0, 1) \end{aligned}$$

where we have let  $z = \eta - \omega$ , and used the fact that for all  $t$ ,

$$\int_t^{\infty} z N_1(z | 0, 1) dz = - \int_{-\infty}^t z N_1(z | 0, 1) dz = N_1(t | 0, 1).$$

Since it is also true that  $Q(\omega) = Q(-\omega)$ , it can straightforwardly be shown that

$$Q(\omega) = |\omega| \left[ 1 - 2 \int_{|\omega|}^{\infty} N_1(z | 0, 1) dz \right] + 2N_1(|\omega| | 0, 1).$$

Lemma 2, p. 175 in Feller (1968) can now be applied to yield the following inequality:

$$|\omega| \leq Q(\omega) \leq |\omega| + \frac{2 \exp\{-\frac{\omega^2}{2}\}}{\sqrt{2\pi} \omega^2} . \quad (3.2.16)$$

Define

$$\gamma(\underline{T}) = \frac{1}{\sqrt{2\pi}} \int \frac{\exp\{-\frac{1}{2}\omega^2(\underline{T}, \underline{E})\}}{\omega^2(\underline{T}, \underline{E})} \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E}. \quad (3.2.17)$$

Lemma 3.2.6. If  $T_1, T_2, T_3 \geq 0$ , then

$$-2\rho\gamma(\underline{T}) \leq q_{-\rho}(\underline{T}) - (1-\rho)\Delta(\underline{T}) \leq 0. \quad (3.2.18)$$

If  $T_1, T_2, T_3 \leq 0$ , then

$$0 \leq q_{\rho}(\underline{T}) - (1-\rho)\Delta(\underline{T}) \leq 2\rho\gamma(\underline{T}). \quad (3.2.19)$$

Proof. If  $T_1, T_2, T_3$  all have the same sign, then since  $a_i(\underline{E}) \geq 0$ ,  $1 \leq i \leq 3$ , and  $\sigma(\underline{E}) > 0$ , it follows from (3.2.6) that

$$|\omega(\underline{T}, \underline{E})| = \sum_{i=1}^3 \frac{a_i(\underline{E})}{\sigma(\underline{E})} |T_i| .$$

The inequalities (3.2.18) and (3.2.19) now are obtained from (3.2.5), (3.2.6), (3.2.16) and the definition of  $\Delta(\underline{T})$ .  $\square$

Lemma 3.2.6 is used as a screening device in the following fashion.

If  $T_1, T_2, T_3 \geq 0$ , then

$$\Delta(\underline{T}) = \sum_{i=1}^3 a_i T_i \geq 0,$$

and, as remarked in Section 2.2, choice can be restricted to the actions  $d^+$  and  $d^0$ . The lower bound

$$q_{-\rho}(\underline{T}) \geq (1-\rho)\Delta(\underline{T}) - 2\rho\gamma(\underline{T})$$

to  $q_{-\rho}(\underline{T})$  is now calculated. If this lower bound is positive, the action

$d^+$  can be taken [see (3.2.7)]. Otherwise, the sign of  $q_{-\rho}(\underline{T})$  must be determined by calculating the entire expression (3.2.5),  $c = -\rho$ , in order to decide between the actions  $d^+$  and  $d^0$ . Similarly, if  $T_1, T_2, T_3 \leq 0$ , then  $\Delta(\underline{T}) \leq 0$ , and choice can be restricted to the actions  $d^0, d^-$ . If the upper bound  $(1-\rho)\Delta(\underline{T}) + 2\rho\gamma(\underline{T})$  is negative, the action  $d^-$  can be taken. Otherwise, the entire function  $q_{\rho}(\underline{T})$  must be calculated to decide between  $d^0$  and  $d^-$ .

Since  $2\rho\gamma(\underline{T}) \geq 0$  serves as a correction factor to  $\Delta(\underline{T})$  in this screening procedure, it is worth noting that when  $T_1, T_2, T_3$  have the same sign,

$$\lim_{\max_{1 \leq i \leq 3} |T_i| \rightarrow \infty} \gamma(\underline{T}) = 0,$$

as can be easily shown. This fact leads us to expect that when all of the  $T_i$ 's are of the same sign, and at least one  $T_i$  is large in magnitude, the screening algorithm described above will allow us to make a decision ( $d^+$  when  $T_i \geq 0$ , all  $i$ ,  $d^-$  when  $T_i \leq 0$ , all  $i$ ) without the need to calculate the entire function  $q_{-\rho}(\underline{T})$  or  $q_{\rho}(\underline{T})$ . Use of both screening devices [(3.2.15) and Lemma 3.2.6] thus offers the hope of avoiding expensive calculations in all but marginal cases.

Algorithms for calculating  $a_1, a_2, a_3, \sigma_0$  and  $\gamma(\underline{T})$  are given in Chapter 4.

### 3.3. Mutual Consistency of the Component Bayes Rules

The purpose of this section is to show that the component Bayes rules  $\varphi_{t_1, t_2}^{(\lambda)}(X, SSE)$ ,  $1 \leq t_1 < t_2 \leq n$ , when applied simultaneously, yield consistent action vectors  $\underline{d}$  (see Section 2.1) with probability one. Our proof follows from the following fundamental result.

Lemma 3.3.1. Let  $x_i, a_i, 1 \leq i \leq 3$ , be any real numbers. For a real number  $t$  define

$$s(t) = |a_1 - x_1 t| + |a_2 - x_2 t| - |a_3 - x_3 t|, \quad (3.3.1)$$

and

$$q(t) = s(t) + s(-t). \quad (3.3.2)$$

If  $|x_1| + |x_2| \geq |x_3|, |a_1| + |a_2| \geq |a_3|$ , then  $q(t) \geq 0$  for all  $t$ .

Proof. It is straightforward to show that for any real numbers  $u$  and  $v$ ,

$$|u-v| + |u+v| = |u| + |v| + ||u| - |v|| = 2 \max\{|u|, |v|\}.$$

Thus

$$q(t) = 2[\max\{|a_1|, |x_1 t|\} + \max\{|a_2|, |x_2 t|\} - \max\{|a_3|, |x_3 t|\}].$$

If  $|a_3| \geq |x_3 t|$ , then since  $\max\{|a_i|, |x_i t|\} \geq |a_i|, i=1,2$ , it follows from the assumption that  $|a_1| + |a_2| \geq |a_3|$  that

$$q(t) \geq 2[|a_1| + |a_2| - |a_3|] \geq 0.$$

On the other hand, if  $|x_3 t| \geq |a_3|$ , then since  $\max\{|a_i|, |x_i t|\} \geq |x_i t|, i=1,2$ , and from the assumption that  $|x_1| + |x_2| \geq |x_3|$ ,

$$q(t) \geq 2[|x_1 t| + |x_2 t| - |x_3 t|] = 2|t| [|x_1| + |x_2| - |x_3|] \geq 0.$$

This completes the proof that  $q(t) \geq 0$  for all  $t$ .  $\square$

Lemma 3.3.2. If  $w(t)$  is a density function symmetric about  $t=0$  [i.e.,  $w(t) = w(-t)$ ], then under the conditions of Lemma 3.3.1,

$$\int_{-\infty}^{\infty} s(t)w(t)dt \geq 0,$$

where  $s(t)$  is defined by (3.3.1).



Proof. Using the change of variables  $t \rightarrow -t$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} s(t)w(t)dt &= \int_0^{\infty} s(t)w(t)dt + \int_{-\infty}^0 s(t)w(t)dt \\ &= \int_0^{\infty} [s(t)+s(-t)]w(t)dt \\ &= \int_0^{\infty} q(t)w(t) \geq 0, \end{aligned}$$

since  $q(t) \geq 0$ , all  $t$ , by Lemma 3.3.1.  $\square$

Lemma 3.3.3. Let  $Q(\omega)$  be defined by (3.2.14). Then for any real numbers  $x_i, a_i$ ,  $1 \leq i \leq 3$ , satisfying the conditions of Lemma 3.3.1,

$$|x_1|Q\left(\frac{a_1}{x_1}\right) + |x_2|Q\left(\frac{a_2}{x_2}\right) \geq |x_3|Q\left(\frac{a_3}{x_3}\right). \quad (3.3.3)$$

Proof. Note that

$$Q(\omega) = \int_{-\infty}^{\infty} |z+\omega|N_1(z|0,1)dz = \int_{-\infty}^{\infty} |-t+\omega|N_1(t|0,1)dt.$$

Also, recall that the standard normal density is symmetric about 0.

Since

$$|x|Q\left(\frac{a}{x}\right) = \int_{-\infty}^{\infty} |a-xt|N_1(t|0,1)dt, \quad (3.3.4)$$

the result (3.3.3) follows from (3.3.1), (3.3.4) and Lemma 3.3.2.  $\square$

Let  $\alpha$  be the component problem which compares  $u_{i_1j_1}$  to  $u_{i_2j_2}$ , let  $\beta$  be the component problem which compares  $u_{i_2j_2}$  to  $u_{i_3j_3}$ , and let  $\gamma$  be the component problem which compares  $u_{i_1j_1}$  to  $u_{i_3j_3}$ . Let  $T_{\sim\alpha}, T_{\sim\beta}, T_{\sim\gamma}$  be the respective  $T(X) = (T_1(X), T_2(X), T_3(X))$  statistics for these

problems, defined by (3.1.2). Note that  $\underline{SS}^*$ ,  $\underline{f}^*$  [and thus  $\pi^*(\underline{E}) = \pi(\underline{E}|\underline{SS}^*, \underline{f}^*)$ ] remain the same for all three component problems.

It is easily seen from (3.1.2) that

$$\underline{T}_\gamma = \underline{T}_\alpha + \underline{T}_\beta.$$

Thus, it follows from (3.2.1) that

$$m(\underline{T}_\gamma, \underline{E}) = m(\underline{T}_\alpha, \underline{E}) + m(\underline{T}_\beta, \underline{E}) \quad (3.3.5)$$

for all  $\underline{E} \in \mathcal{E}$ . Combining (3.3.5) with (2.2.19), it is easily seen that

$$\Delta(\underline{T}_\gamma) = \Delta(\underline{T}_\alpha) + \Delta(\underline{T}_\beta). \quad (3.3.6)$$

It also follows from (3.3.5) that

$$|m(\underline{T}_\gamma, \underline{E})| \leq |m(\underline{T}_\alpha, \underline{E})| + |m(\underline{T}_\beta, \underline{E})|. \quad (3.3.7)$$

Let  $\sigma_\alpha^2(\underline{E}), \sigma_\beta^2(\underline{E}), \sigma_\gamma^2(\underline{E})$  be the quantities defined for component problems  $\alpha, \beta, \gamma$  by (3.2.2). Since

$$i_1 \neq i_3 \Rightarrow i_1 \neq i_2 \text{ or } i_2 \neq i_3,$$

$$j_1 \neq j_3 \Rightarrow j_1 \neq j_2 \text{ or } j_2 \neq j_3,$$

it is easily seen from (3.2.2) that

$$\sigma_\gamma^2(\underline{E}) \leq \sigma_\alpha^2(\underline{E}) + \sigma_\beta^2(\underline{E}),$$

and thus that

$$\sigma_\gamma(\underline{E}) \leq \sigma_\alpha(\underline{E}) + \sigma_\beta(\underline{E}). \quad (3.3.8)$$

It follows from the definition of  $\Gamma(\underline{T})$  given at the beginning of Section 3.2, (3.1.7) and (3.1.8) that

$$\Gamma(\underline{T}) = \int_{\mathcal{E}} \sigma(\underline{E}) Q\left(\frac{m(\underline{T}, \underline{E})}{\sigma(\underline{E})}\right) \pi^*(\underline{E}) d\underline{E}. \quad (3.3.9)$$

Hence, from (3.3.7), (3.3.8), (3.3.9) and Lemma 3.3.3,

$$\Gamma(\underline{T}_\alpha) + \Gamma(\underline{T}_\beta) \geq \Gamma(\underline{T}_\gamma). \quad (3.3.10)$$

However, it also follows from (3.3.5) that

$$|m(\underline{T}_\beta, \underline{E})| \leq |m(\underline{T}_\gamma, \underline{E})| + |m(\underline{T}_\alpha, \underline{E})|. \quad (3.3.11)$$

Noting that

$$i_2 \neq i_3 \Rightarrow i_1 \neq i_3 \text{ or } i_1 \neq i_2,$$

$$j_2 \neq j_3 \Rightarrow j_1 \neq j_3 \text{ or } j_1 \neq j_2,$$

it can be shown that

$$\sigma_\beta^2(\underline{E}) \leq \sigma_\gamma^2(\underline{E}) + \sigma_\alpha^2(\underline{E}),$$

and hence that

$$\sigma_\beta(\underline{E}) \leq \sigma_\gamma(\underline{E}) + \sigma_\alpha(\underline{E}). \quad (3.3.12)$$

It now follows from (3.3.9), (3.3.11), (3.3.12) and Lemma 3.3.3 that

$$\Gamma(\underline{T}_\gamma) + \Gamma(\underline{T}_\alpha) \geq \Gamma(\underline{T}_\beta). \quad (3.3.13)$$

Using (3.2.4), (3.3.6), (3.3.10) and (3.3.13), the following inequalities can be obtained:

$$q_\rho(\underline{T}_\beta) + q_{-\rho}(\underline{T}_\alpha) \leq q_\rho(\underline{T}_\gamma) \leq q_\rho(\underline{T}_\alpha) + q_\rho(\underline{T}_\beta), \quad (3.3.14)$$

$$q_{-\rho}(\underline{T}_\alpha) + q_{-\rho}(\underline{T}_\beta) \leq q_{-\rho}(\underline{T}_\gamma) \leq q_{-\rho}(\underline{T}_\beta) + q_{-\rho}(\underline{T}_\alpha).$$

From (3.2.7) and (3.3.14), it can be seen that:

- (i) If the action  $d^+$  is taken in component problems  $\alpha$  and  $\beta$ , then  $q_{-\rho}(\underline{T}_\alpha) > 0$ ,  $q_{-\rho}(\underline{T}_\beta) > 0$ , which implies that  $q_{-\rho}(\underline{T}_\gamma) > 0$ , forcing action  $d^+$  to be taken in component problem  $\gamma$ .
- (ii) If the action  $d^-$  is taken in component problems  $\alpha$  and  $\beta$ , then  $q_\rho(\underline{T}_\alpha) < 0$ ,  $q_\rho(\underline{T}_\beta) < 0$ , which implies that  $q_\rho(\underline{T}_\gamma) < 0$ , forcing action  $d^-$  to be taken in component problem  $\gamma$ .
- (iii) If the actions  $d^+, d^0$  are taken in component problems  $\alpha, \beta$ ,

respectively, then

$$q_{-\rho}(\bar{T}_\alpha) > 0, \quad q_{-\rho}(\bar{T}_\beta) < 0 < q_\rho(\bar{T}_\beta),$$

implying that

$$q_\rho(\bar{T}_\gamma) \geq q_\rho(\bar{T}_\beta) + q_{-\rho}(\bar{T}_\alpha) > 0,$$

and thus that action  $d^-$  cannot be taken in component problem  $\gamma$ .

- (iv) If the actions  $d^-, d^0$  are taken in component problems  $\alpha, \beta$ , respectively, then

$$q_\rho(\bar{T}_\alpha) < 0, \quad q_{-\rho}(\bar{T}_\beta) < 0 < q_\rho(\bar{T}_\beta),$$

implying that

$$q_{-\rho}(\bar{T}_\gamma) \leq q_{-\rho}(\bar{T}_\beta) + q_\rho(\bar{T}_\alpha) < 0,$$

and thus that action  $d^+$  cannot be taken in component problem  $\gamma$ .

Comparing these conclusions with the definition of inconsistency in Section 2.1, it is seen that the component Bayes rules, when applied simultaneously, cannot produce inconsistent action vectors. This result completes the proof of Theorem 2.2.1.

Using arguments similar to those used to derive (3.3.13), but with the roles of component problems  $\alpha$  and  $\beta$  interchanged, it can be shown that

$$\Gamma(\bar{T}_\gamma) + \Gamma(\bar{T}_\beta) \geq \Gamma(\bar{T}_\alpha)$$

and hence that

$$q_\rho(\bar{T}_\alpha) + q_{-\rho}(\bar{T}_\beta) \leq q_\rho(\bar{T}_\gamma),$$

$$q_{-\rho}(\bar{T}_\gamma) \leq q_{-\rho}(\bar{T}_\alpha) + q_\rho(\bar{T}_\beta).$$

Hence

$$\max\{q_{\rho}(\tilde{T}_{\alpha})+q_{-\rho}(\tilde{T}_{\beta}),q_{-\rho}(\tilde{T}_{\alpha})+q_{\rho}(\tilde{T}_{\beta})\} \leq q_{\rho}(\tilde{T}_{\gamma}) \leq q_{\rho}(\tilde{T}_{\alpha}) + q_{\rho}(\tilde{T}_{\beta}),$$

$$q_{\rho}(\tilde{T}_{\alpha}) + q_{-\rho}(\tilde{T}_{\beta}) \leq q_{-\rho}(\tilde{T}_{\gamma}) \leq \min\{q_{\rho}(\tilde{T}_{\alpha})+q_{-\rho}(\tilde{T}_{\beta}),q_{-\rho}(\tilde{T}_{\alpha})+q_{\rho}(\tilde{T}_{\beta})\}.$$

The above inequalities can produce bounds for the quantities  $q_{\pm\rho}(\tilde{T})$  for one component problem in terms of similar quantities already calculated in other (related) component problems. Hence, these inequalities could be used to obtain screening procedures to replace or supplement those described in Section 3.2. Because the logic needed to program the computer to search for "related" component problems is complex, such screening procedures were not utilized in the computer program described in Section 4.2.

CHAPTER IV  
ALGORITHMS FOR COMPUTING THE BAYES RULE

In Chapter 3, it was shown that all component Bayes rules  $\varphi^{(\lambda)}(\underline{T})$  choose actions based upon the signs of two functions  $q_p(\underline{T})$  and  $q_{-p}(\underline{T})$ . The present chapter is devoted to deriving computational procedures for these functions, and also to devising an algorithm for implementing the full Bayes decision procedure of Theorem 2.2.1. Such a procedure will use the screening methods of Section 3.2 in order to avoid, as much as possible, the necessity of computing  $q_p(\underline{T})$  and  $q_{-p}(\underline{T})$  in any given component problem.

Section 4.1 gives an iterative procedure for computing the constants

$$a_i = \int_{\mathcal{E}} a_i(\underline{E}) \pi^*(\underline{E}) d\underline{E}, \quad 1 \leq i \leq 3.$$

These constants determine the posterior mean

$$\Delta(\underline{T}) = \sum_{i=1}^3 a_i T_i$$

of  $\eta = u_{i_1 j_1} - u_{i_2 j_2}$  in each comparison problem. In Section 2.2 it was noted that the sign of  $\Delta(\underline{T})$  allowed us to eliminate one action ( $d^-$  if  $\Delta(\underline{T}) > 0$ ,  $d^+$  if  $\Delta(\underline{T}) < 0$ ) from our choice of possible actions in any component problem. The  $a_i$ 's also play an important role in the first of the two screening procedures described in Section 3.2. Section 4.2 deals with a method for computing  $q_p(\underline{T})$  and  $q_{-p}(\underline{T})$ , and also the

quantities  $\sigma_0$  and  $\gamma(T)$  defined in Section 3.2. At the end of Section 4.2, the proposed algorithm for implementing the full Bayes rule is described.

#### 4.1. Algorithm for Computing $a_i$ , $1 \leq i \leq 3$ , and $\Delta(T)$ .

Define

$$M(\underline{d} ; \underline{S}) = M(d_1, d_2, d_3, d_4 ; S_1, S_2, S_3, S_4) \\ = \int_0^\infty \int_{v_4}^\infty \int_{v_3}^\infty \int_{v_3}^\infty \frac{\exp\left\{-\sum_{i=1}^4 \frac{S_i}{2v_i}\right\}}{\prod_{i=1}^4 \left[v_i^{\frac{1}{2}d_i+1}\right]} dv_1 dv_2 dv_3 dv_4, \quad (4.1.1)$$

where  $d_1, d_2, d_3, d_4$  are positive integers, and  $S_1, S_2, S_3, S_4$  are positive real numbers. From (2.1.13), (2.1.15), (2.1.17), (2.1.18), (3.1.3), the definition of  $\pi^*(E)$  as  $\pi(E|\underline{S}\underline{S}^*, \underline{f}^*)$ , and (3.2.12), it can be shown that

$$a_1 = 1 - \frac{M(f_A^*, f_B^*, f_{AB}^*+2, f_e^*-2 ; \underline{S}\underline{S}^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^* ; \underline{S}\underline{S}^*)}, \\ a_2 = \frac{M(f_A^*, f_B^*, f_{AB}^*+2, f_e^*-2 ; \underline{S}\underline{S}^*) - M(f_A^*, f_B^*+2, f_{AB}^*, f_e^*-2 ; \underline{S}\underline{S}^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^* ; \underline{S}\underline{S}^*)} \quad (4.1.2) \\ a_3 = \frac{M(f_A^*, f_B^*, f_{AB}^*+2, f_e^*-2 ; \underline{S}\underline{S}^*) - M(f_A^*+2, f_B^*, f_{AB}^*, f_e^*-2 ; \underline{S}\underline{S}^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^* ; \underline{S}\underline{S}^*)}$$

where  $\underline{f}^* = (f_A^*, f_B^*, f_{AB}^*, f_e^*)'$  and  $\underline{S}\underline{S}^* = (SSA^*, SSB^*, SSAB^*, SSE^*)'$  are defined by (2.1.19). The function  $\Delta(T)$  can then be calculated from (4.1.2) and (3.1.4).

The function  $M(\underline{d} ; \underline{S})$  is recognizable as an iterated multi-dimensional integral of incomplete gamma functions. Note that

$$f_e^* = rc(K-1) + f_e$$

is an even integer in any of the following cases:

- (i)  $r$  or  $c$  are even integers and  $f_e$  is an even integer,
- (ii)  $K$  is an odd integer and  $f_e$  is an even integer,
- (iii)  $r, c,$  and  $f_e$  are odd integers,  $K$  is an even integer.

When, as is often the case, a prior of indifference is used (in which case  $f_e = 0$  is even),  $f_e^*$  will be even in all cases except the one where  $r, c$  are odd and  $K$  is even. From the above considerations, finding an algorithm for  $M(\underline{d} ; \underline{S})$  when  $d_4$  is even will allow computation of the constants  $a_1, a_2, a_3$  (and of  $\Delta(\underline{T})$ ) in the majority of applications.

#### Algorithm for Computing $M(\underline{d} ; \underline{S})$ When $d_4$ is Even

Define

$$G_x(d_1, d_2, d_3 ; S_1, S_2, S_3) = \int_x^\infty \int_{v_3}^\infty \int_{v_3}^\infty \frac{\exp\left\{-\sum_{i=1}^3 \frac{S_i}{2v_i}\right\}}{3 \prod_{i=1}^3 [v_i^{\frac{1}{2}} d_i + 1]} dv_1 dv_2 dv_3. \quad (4.1.3)$$

Then

$$M(\underline{d} ; \underline{S}) = \int_0^\infty \frac{\exp\left\{-\frac{S_4}{2v_4}\right\}}{v_4^{\frac{1}{2}} d_4 + 1} G_{v_4}(d_1, d_2, d_3 ; S_1, S_2, S_3) dv_4. \quad (4.1.4)$$

Also note that if

$$I(x|d, S) = \int_x^\infty \frac{\exp\left\{-\frac{S}{2w}\right\}}{w^{\frac{1}{2}} d + 1} dw, \quad (4.1.5)$$

then



$$G_{v_4}(d_1, d_2, d_3 ; S_1, S_2, S_3) = \int_{v_4}^{\infty} \frac{\exp\left\{-\frac{S_3}{2v_3}\right\}}{v_3^{\frac{1}{2}} d_3 + 1} I(v_3|d_1, S_1) I(v_3|d_2, S_2) dv_3, \quad (4.1.6)$$

and

$$-\frac{d}{dv_4} G_{v_4}(d_1, d_2, d_3 ; S_1, S_2, S_3) = \frac{\exp\left\{-\frac{S_3}{2v_4}\right\}}{v_4^{\frac{1}{2}} d_3 + 1} I(v_4|d_1, S_1) I(v_4|d_2, S_2). \quad (4.1.7)$$

Lemma 4.1.1. For  $d_4 \geq 1$ ,

$$M(d_1, d_2, d_3, d_4 + 2 ; \underline{S}) = \frac{d_4}{S_4} M(\underline{d} ; \underline{S}) + \frac{2}{S_4} G_0(d_1, d_2, d_3 + d_4 ; S_1, S_2, S_3 + S_4).$$

Proof. From (4.1.4)

$$M(\underline{d} ; \underline{S}) = \int_0^{\infty} \left( \frac{1}{v_4^{\frac{1}{2}} d_4 + 1} \right) \left[ \exp\left\{-\frac{S_4}{2v_4}\right\} G_{v_4}(d_1, d_2, d_3 ; S_1, S_2, S_3) \right] dv_4.$$

Integrate by parts, using  $v_4^{-\left(\frac{1}{2} d_4 + 1\right)}$  as one term, and  $\exp\{-S_4/2v_4\} G_{v_4}(d_1, d_2, d_3 ; S_1, S_2, S_3)$  as the other. Use of (4.1.7), (4.1.4) and (4.1.6) then yields

$$M(\underline{d} ; \underline{S}) = \frac{S_4}{d_4} M(d_1, d_2, d_3, d_4 + 2 ; \underline{S}) - \frac{2}{d_4} G_0(d_1, d_2, d_3 + d_4 ; S_1, S_2, S_3 + S_4)$$

from which the stated result directly follows.  $\square$

Lemma 4.1.2.  $M(d_1, d_2, d_3, 2 ; \underline{S}) = (2/S_4) G_0(d_1, d_2, d_3 ; S_1, S_2, S_3 + S_4).$

Proof. After changing variables from  $v_4$  to  $x = v_4^{-1}$ ,

$$M(d_1, d_2, d_3, 2; S) = \int_0^{\infty} \exp\left\{-\frac{S_4 x}{2}\right\} \frac{G_1(d_1, d_2, d_3; S_1, S_2, S_3)}{x} dx.$$

Integrate by parts, using  $\exp\{-\frac{1}{2} S_4 x\}$  as one term and

$G_{x^{-1}}(d_1, d_2, d_3; S_1, S_2, S_3)$  as the other, and making use of (4.1.7).

Then, change the variable of integration back from  $x$  to  $v_4 = x^{-1}$ , and apply (4.1.6).  $\square$

Starting with  $d_4$  any even number,  $d_4 \geq 2$ , the recursive scheme provided by Lemmas 4.1.1 and 4.1.2 enables us to avoid calculation of the fourfold integral (4.1.1), since every term calculated in the recursion is a threefold integral of the form (4.1.3), with  $\varkappa = 0$ . However, these threefold integrals also do not have to be calculated, as shown below.

Define

$$J(t_1, t_2; C_1, C_2) = \int_0^{\infty} \frac{\exp\left\{-\frac{C_2}{2v}\right\}}{v^{\frac{1}{2}t_2+1}} I(v|t_1, C_1) dv. \quad (4.1.8)$$

Lemma 4.1.3. For all  $d_1, d_2, d_3 \geq 1$ , and  $S_1, S_2, S_3 > 0$ ,

$$\begin{aligned} & G_0(d_1, d_2, d_3+2; S_1, S_2, S_3) \\ &= \frac{d_3}{S_3} G_0(d_1, d_2, d_3; S_1, S_2, S_3) \\ &+ \frac{2}{S_3} \left[ J(d_1, d_2+d_3; S_1, S_2+S_3) + J(d_2, d_1+d_3; S_2, S_1+S_3) \right]. \end{aligned}$$

Proof. Integrate by parts in (4.1.6), using  $v_3^{-(\frac{1}{2}d_3+1)}$  as one term and  $\exp\{-S_3/2v_3\}I(v_3|d_1, S_1)I(v_3|d_2, S_2)$  as the other. Using the fact that

$$-\frac{d}{dv} I(v|d, S) = \frac{\exp\left\{-\frac{S}{2v}\right\}}{v^{\frac{1}{2}}d+1}$$

and (4.1.6), and rearranging terms as in the proof of Lemma 4.1.1, yields the stated result.  $\square$

Note: Lemma 4.1.3 also holds for  $d_3 = 0$ . The proof of this assertion is similar to that of Lemma 4.1.2. Thus,

$$G_0(d_1, d_2, 2; S_1, S_2, S_3) = \frac{2}{S_3} \left[ J(d_1, d_2; S_1, S_2+S_3) + J(d_2, d_1; S_2, S_1+S_3) \right]. \quad (4.1.9)$$

Define

$$B(\tau; \alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\tau y^{\alpha_1-1} (1-y)^{\alpha_2-1} dy, \quad 0 \leq \tau \leq 1, \quad (4.1.10)$$

and

$$H(m; T) = \frac{\Gamma(\frac{1}{2}m)}{\left(\frac{T}{2}\right)^{\frac{1}{2}m}}. \quad (4.1.11)$$

We recognize  $B(\tau; \alpha_1, \alpha_2)$  as being the incomplete Beta function.

Lemma 4.1.4.  $J(t_1, t_2; C_1, C_2) = H(t_1; C_1)H(t_2; C_2)B\left(\frac{C_1}{C_1+C_2}; \frac{1}{2}t_1, \frac{1}{2}t_2\right)$ .

Proof. From (4.1.5) and (4.1.8),

$$J(t_1, t_2; C_1, C_2) = \int_0^\infty \int_v^\infty \frac{\exp\left\{-\frac{C_2}{2v} - \frac{C_1}{2w}\right\}}{v^{\frac{1}{2}}t_2+1} \frac{1}{w^{\frac{1}{2}}t_1+1} dw dv.$$

Make the change of variables

$$x = \frac{C_2 w + C_1 v}{vw}, \quad y = \frac{C_1 v}{C_2 w + C_1 v}$$

in the above double integral, noting that the range restriction  $w \geq v$  implies that  $y \leq C_1/(C_1+C_2)$ . Integrating over  $x$  and applying the definitions (4.1.10), (4.1.11), completes the proof.  $\square$

The computation of  $M(d; \underline{S})$  has now been reduced to a recursion, where at every step the familiar incomplete Beta integral is computed. Although subroutines exist to give this computation, we have found it more feasible to calculate the function  $J(t_1, t_2; C_1, C_2)$  recursively. Our procedure is based upon the following well known facts about the incomplete Beta function and the complete gamma function.

Lemma 4.1.5.

- (i)  $J(t_1, t_2; C_1, C_2) = H(t_1; C_1)H(t_2; C_2) - J(t_2, t_1; C_2, C_1)$ ,
- (ii)  $J(t_1, t_2+2; C_1, C_2) = \frac{t_2}{C_2} J(t_1, t_2; C_1, C_2) + \frac{2}{C_2} H(t_1+t_2; C_1+C_2)$ ,
- (iii)  $J(t_1+2, t_2; C_1, C_2) = \frac{t_1}{C_1} J(t_1, t_2; C_1, C_2) - \frac{2}{C_1} H(t_1+t_2; C_1+C_2)$ ,
- (iv)  $H(m+2; T) = \frac{m}{T} H(m; T)$ ,

Lemma 4.1.6.

- (i)  $J(1, 1; C_1, C_2) = \frac{4}{\sqrt{C_1 C_2}} \arctan \left( \sqrt{\frac{C_1}{C_2}} \right)$ ,
- $J(2, 2; C_1, C_2) = \frac{4}{C_2(C_1+C_2)}$ ,

$$(ii) \quad J(1,2; c_1, c_2) = \frac{2}{c_2} \sqrt{\frac{2\pi}{c_1+c_2}},$$

$$J(2,1; c_1, c_2) = \frac{2\sqrt{2\pi}}{c_1\sqrt{c_2}} \left(1 - \sqrt{\frac{c_2}{c_1+c_2}}\right),$$

$$(iii) \quad H(2; T) = 2T^{-1}, \quad H(3; T) = \sqrt{2\pi} T^{-3/2}, \quad H(4; T) = 4T^{-2}.$$

Before describing the complete computational algorithm, two more useful results about the function  $G_0(d_1, d_2, d_3; S_1, S_2, S_3)$  are given.

Lemma 4.1.7. (Updating first and second indices of  $G_0$ .) For

$$d_1, d_2, d_3 \geq 1, \quad S_1, S_2, S_3 > 0,$$

$$(i) \quad G_0(d_1+2, d_2, d_3; S_1, S_2, S_3) = \frac{d_1}{S_1} G_0(d_1, d_2, d_3; S_1, S_2, S_3) \\ - \frac{2}{S_1} J(d_2, d_1+d_3; S_2, S_1+S_3),$$

$$(ii) \quad G_0(d_1, d_2+2, d_3; S_1, S_2, S_3) = \frac{d_2}{S_2} G_0(d_1, d_2, d_3; S_1, S_2, S_3) \\ - \frac{2}{S_2} J(d_1, d_2+d_3; S_1, S_2+S_3).$$

Proof. Both of these results follow from the relationship:

$$I(x|d, S) = \frac{S}{d} I(x|d+2, S) + \frac{2}{d} \frac{e^{-\frac{S}{2x}}}{x^{\frac{1}{2}} d},$$

which can be proven by using integration by parts in (4.1.5). Rewriting this equation in the form

$$I(x|d+2,S) = \frac{d}{S} I(x|d,S) - \frac{2}{S} \frac{e^{-\frac{S}{2x}}}{x^{\frac{1}{2}}d},$$

with  $x=v_3$ ,  $d=d_1$ ,  $S=S_1$  in (i),  $x=v_3$ ,  $d=d_2$ ,  $S=S_2$  in (ii), and substituting into (4.1.6) yields the desired results.  $\square$

Lemma 4.1.8. For  $S_1, S_2, S_3 > 0$ ,

$$(i) \quad G(2, d_2, d_3; S_1, S_2, S_3) = \frac{2}{S_1} \left[ J(d_2, d_3; S_2, S_3) - J(d_2, d_3; S_2, S_1 + S_3) \right],$$

$$(ii) \quad G(d_1, 2, d_3; S_1, S_2, S_3) = \frac{2}{S_2} \left[ J(d_1, d_3; S_1, S_3) - J(d_1, d_3; S_1, S_2 + S_3) \right].$$

Proof. In (4.1.3) carry out the integration over  $v_1$  [for (i)] or over  $v_2$  [for (ii)], and apply (4.1.8).  $\square$

Finally, the following results are useful for starting the recursion for  $G_0(d_1, d_2, d_3; S_1, S_2, S_3)$ .

Lemma 4.1.9. For  $S_1, S_2, S_3 > 0$ ,

$$G_0(1, 1, 1; S_1, S_2, S_3) = 4 \left( \frac{2\pi}{S_1 S_2 S_3} \right)^{\frac{1}{2}} \arctan \left[ \left( \frac{S_1 S_2}{[(S_1 + S_3)(S_2 + S_3) - S_1 S_2]} \right)^{\frac{1}{2}} \right]$$

Proof. From (4.1.3),

$$G_0(1, 1, 1; S_1, S_2, S_3) = \int_0^\infty \int_{v_3}^\infty \int_{v_3}^\infty \frac{\exp \left\{ - \sum_{i=1}^3 \frac{S_i}{2v_i} \right\}}{(v_1 v_2 v_3)^{3/2}} dv_1 dv_2 dv_3.$$

Make the change of variables

$$x_1 = \frac{S_1 v_3}{S_3 v_1 + S_1 v_3}, \quad x_2 = \frac{S_2 v_3}{S_3 v_2 + S_2 v_3}, \quad x_3 = \frac{S_3}{v_3},$$

and integrate over  $x_3$ . Then, with  $F_i = S_i(S_i + S_3)^{-1}$ ,  $i=1,2$ ,

$$G_0(1,1,1; S_1, S_2, S_3) = \left( \frac{2\pi}{S_1 S_2 S_3} \right)^{\frac{1}{2}} \int_0^{F_1} \int_0^{F_2} x_1^{-\frac{1}{2}} x_2^{-\frac{1}{2}} (1-x_1 x_2)^{-3/2} dx_2 dx_1.$$

Make a further change of variables to  $t_1 = F_2 x_1, t_2 = x_1 x_2$ . Then

$$\begin{aligned} G_0(1,1,1; S_1, S_2, S_3) &= \left( \frac{2\pi}{S_1 S_2 S_3} \right)^{\frac{1}{2}} \int_0^{F_1 F_2} \frac{1}{t_1} \left[ \int_0^{t_1} t_2^{-\frac{1}{2}} (1-t_2)^{-3/2} dt_2 \right] dt_1 \\ &= 2 \left( \frac{2\pi}{S_1 S_2 S_3} \right)^{\frac{1}{2}} \int_0^{F_1 F_2} [t_1(1-t_1)]^{-\frac{1}{2}} dt \\ &= 4 \left( \frac{2\pi}{S_1 S_2 S_3} \right)^{\frac{1}{2}} \arctan \left( \sqrt{\frac{F_1 F_2}{1-F_1 F_2}} \right), \end{aligned}$$

and the result follows by substituting for  $F_1, F_2$ , in terms of  $S_1, S_2, S_3$ .  $\square$

Lemma 4.1.10. For  $S_1, S_2, S_3 > 0$ ,

(i)  $G_0(2,1,1; S_1, S_2, S_3)$

$$= \frac{8}{S_1 \sqrt{S_2}} \left[ \frac{\arctan \left( \sqrt{\frac{S_2}{S_3}} \right)}{\sqrt{S_3}} - \frac{\arctan \left( \sqrt{\frac{S_2}{S_1 + S_3}} \right)}{\sqrt{S_1 + S_3}} \right],$$

$$(ii) G_0(1,2,1; S_1, S_2, S_3)$$

$$= \frac{8}{S_2 \sqrt{S_1}} \left[ \frac{\arctan\left(\sqrt{\frac{S_1}{S_3}}\right)}{\sqrt{S_3}} - \frac{\arctan\left(\sqrt{\frac{S_1}{S_2+S_3}}\right)}{\sqrt{S_2+S_3}} \right],$$

$$(iii) G_0(2,2,1; S_1, S_2, S_3)$$

$$= \frac{4\sqrt{2\pi}}{S_1 S_2} \left[ \frac{1}{\sqrt{S_3}} - \frac{1}{\sqrt{S_1+S_3}} - \frac{1}{\sqrt{S_2+S_3}} + \frac{1}{\sqrt{S_1+S_2+S_3}} \right].$$

Proof. The above equations follow directly from Lemmas 4.1.8 and 4.1.6.  $\square$

#### Algorithm for Computing $M(\underline{d}; \underline{S})$ When $d_4$ is Odd

When  $d_4$  is odd, computation of  $M(\underline{d}; \underline{S})$  is more complicated. However, if  $d_1$  and  $d_2$  are both even, it is possible to express  $M(\underline{d}; \underline{S})$  as a finite sum. First, by a change of variables it can be shown that

$$I(x|\underline{d}, \underline{S}) = H(\underline{d}; \underline{S}) \int_0^{\frac{S}{2x}} \frac{u^{\frac{d}{2}-1} e^{-u}}{\Gamma(\frac{d}{2})} du. \quad (4.1.12)$$

We recognize the integral on the right side of (4.1.12) as the incomplete gamma function. When  $d$  is even, we can apply the familiar expansion of the incomplete gamma function in a Poisson series, obtaining

$$I(x|\underline{d}, \underline{S}) = H(\underline{d}; \underline{S}) \left( 1 - \sum_{i=0}^{\frac{1}{2}d-1} \frac{\left(\frac{S}{2x}\right)^i}{i!} \exp\left\{-\frac{S}{2x}\right\} \right). \quad (4.1.13)$$

Substituting (4.1.13) with  $d = d_1$ ,  $S = S_1$  and also  $d = d_2$ ,  $S = S_2$  into (4.1.6), and this result into (4.1.4), yields

$$M(\underline{d}; \underline{S}) = H(d_1; S_1) H(d_2; S_2) \left[ \Sigma_0^{-\Sigma_1 - \Sigma_2 + \Sigma_{12}} \right], \quad (4.1.14)$$



where

$$\Sigma_0 = J(d_3, d_4; S_3, S_4),$$

$$\Sigma_1 = \sum_{i=0}^{\frac{1}{2}d_1-1} \frac{\left(\frac{S_1}{2}\right)^i}{i!} J(d_3+2i, d_4; S_1+S_3, S_4),$$

$$\Sigma_2 = \sum_{j=0}^{\frac{1}{2}d_2-1} \frac{\left(\frac{S_2}{2}\right)^j}{j!} J(d_3+2j, d_4; S_2+S_3, S_4),$$

$$\Sigma_{12} = \sum_{i=0}^{\frac{1}{2}d_1-1} \sum_{j=0}^{\frac{1}{2}d_2-1} \frac{\left(\frac{S_1}{2}\right)^i \left(\frac{S_2}{2}\right)^j}{i! j!} J(d_3+2(i+j), d_4; S_1+S_2+S_3, S_4).$$

Lemmas 4.1.5 and 4.1.6 can be used to evaluate the quantities

$J(t_1, t_2; C_1, C_2)$  appearing in the sums  $\Sigma_0, \Sigma_1, \Sigma_2$ , and  $\Sigma_{12}$  in (4.1.14).

However, if  $d_3$  is also even, further simplification can be done by using Lemma 4.1.4, and the well known representation of the incomplete beta function as a sum of binomial probabilities, to represent each  $J(t_1, t_2; C_1, C_2)$  as a finite sum of powers of  $C_1/(C_1+C_2)$ .

#### Description of Algorithm

In this section we have described methods for constructing algorithms to compute  $M(\underline{d}, \underline{S})$  in two cases: (i) when  $d_4$  is even; and (ii) when  $d_4$  is odd (or even), and  $d_1, d_2$  are even. Algorithms for computing  $M(\underline{d}; \underline{S})$  in the remaining cases [where  $d_4$  and either  $d_1$  or  $d_2$  (or both) are odd] can be constructed. However, such algorithms are likely to be considerably more cumbersome than those expositied above. The two algorithms described above are sufficient to handle a large number of practical problems, and permit us to apply the Bayes procedure

described in Chapters 2 and 3 to real data in Chapter 5. If the Bayes procedure proves to be a useful statistical tool, we will then have some justification for improving and extending our algorithms for calculating  $M(\underline{d}; \underline{S})$ . A flow chart to compute  $M(\underline{d}; \underline{S})$  in the aforementioned two cases is given in Figure 4.1.

#### 4.2. Recursive Method for Computing $\Gamma(\underline{T})$

The functions  $\Delta(\underline{T})$  and  $\Gamma(\underline{T})$  together determine the component Bayes rules; see, for example, Equations (3.2.5) and (3.2.7). Recall from (3.3.9) that

$$\Gamma(\underline{T}) = \int_{\underline{E}} Q\left(\frac{m(\underline{T}, \underline{E})}{\sigma(\underline{E})}\right) \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E}, \quad (4.2.1)$$

where  $m(\underline{T}, \underline{E})$  is defined by (3.2.1),  $\sigma^2(\underline{E})$  is defined by (3.2.2),  $\pi^*(\underline{E}) = \pi(\underline{E} | \underline{S} \underline{S}^*, \underline{f}^*)$ , and  $Q(\omega)$  is defined by (3.1.8).

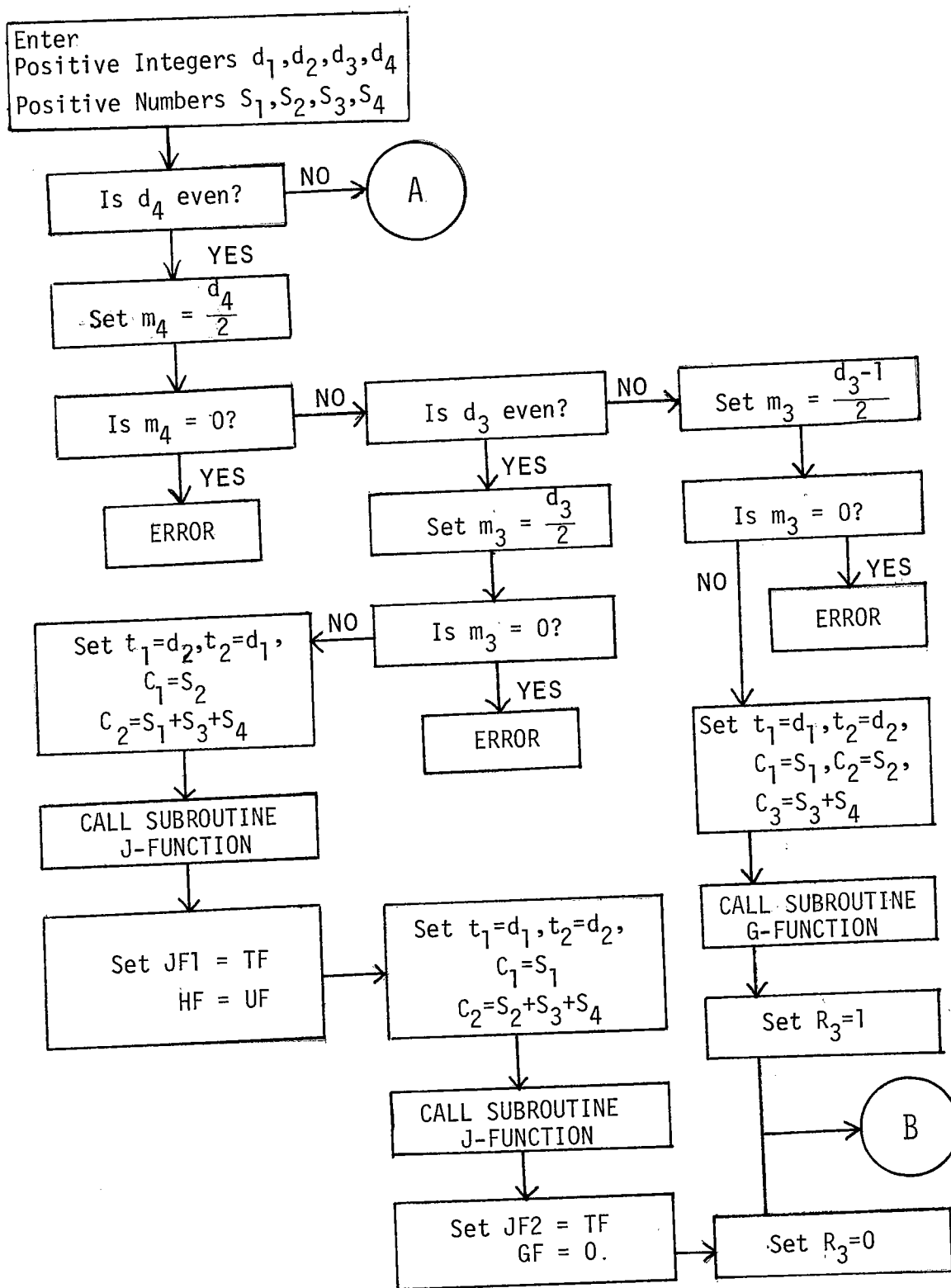
Lemma 4.2.1. For all  $\omega$ ,  $-\infty < \omega < \infty$ ,

$$Q(\omega) = \sum_{j=0}^{\infty} \frac{\sqrt{2}}{\Gamma(j+\frac{1}{2})} \left(\frac{\omega^2}{2}\right)^j \exp\left\{-\frac{1}{2}\omega^2\right\}.$$

Proof. Note from (3.1.8) that  $Q(\omega)$  is the expected value of  $|V|$ , where  $V \sim N_1(\omega, 1)$ . It is well known that in this case  $U = V^2$  has a noncentral  $\chi^2$  distribution with noncentrality parameter  $\frac{1}{2}\omega^2$ . Thus

$$Q(\omega) = E(|V|) = E(\sqrt{U}) = \int_0^{\infty} \sqrt{u} \left[ \sum_{j=0}^{\infty} \frac{\left(\frac{\omega^2}{2}\right)^j \exp\left\{-\frac{1}{2}\omega^2\right\}}{j!} \frac{u^{\frac{1}{2}+j-1} \exp\left\{-\frac{1}{2}u\right\}}{\Gamma(j+\frac{1}{2})2^{\frac{1}{2}+j}} \right] du,$$

and the stated result follows after interchanging sum and integral, and integrating term by term.  $\square$

Figure 4.1: Algorithm for Computing  $M(d, S)$

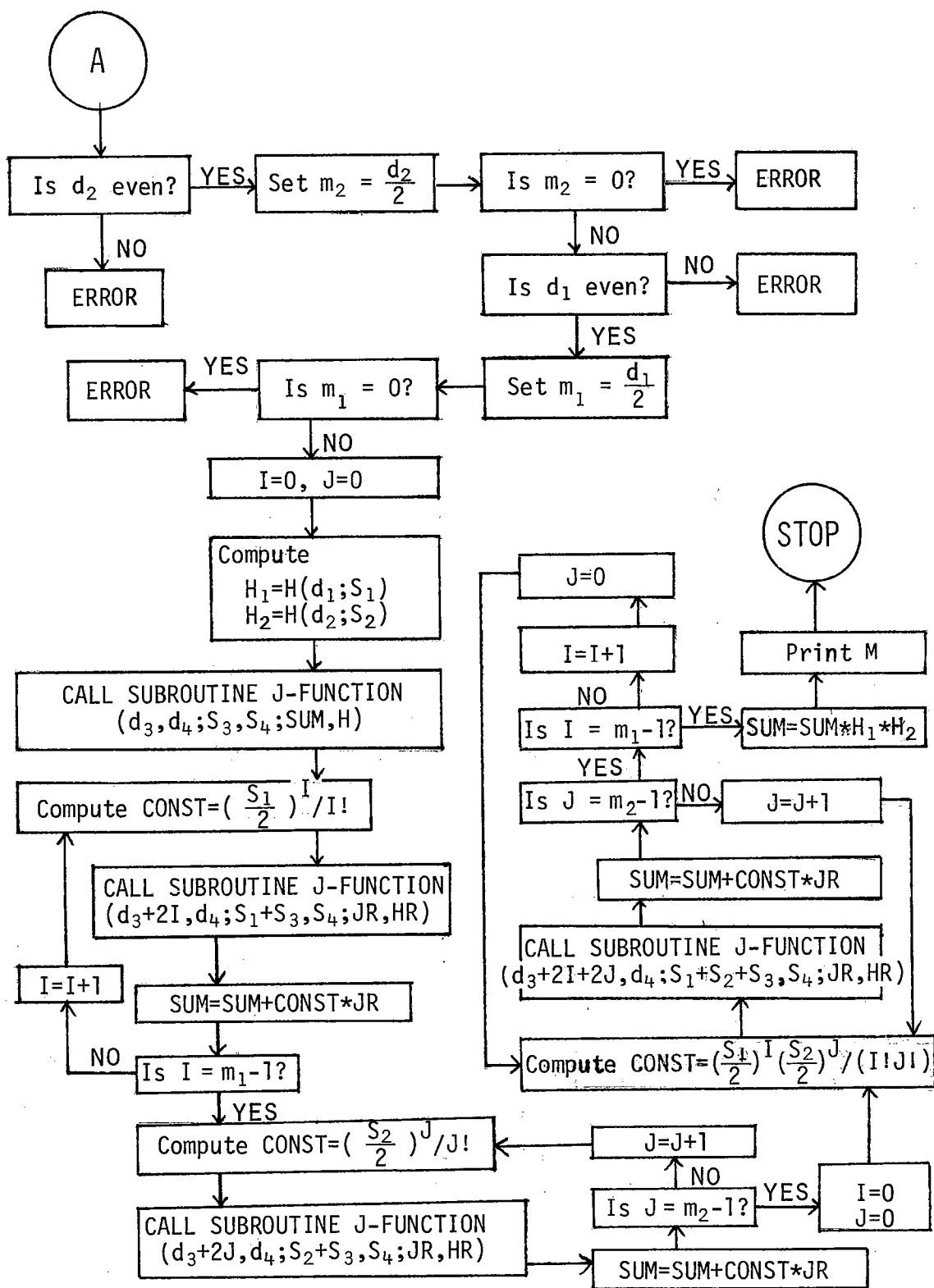


Figure 4.1: Continued

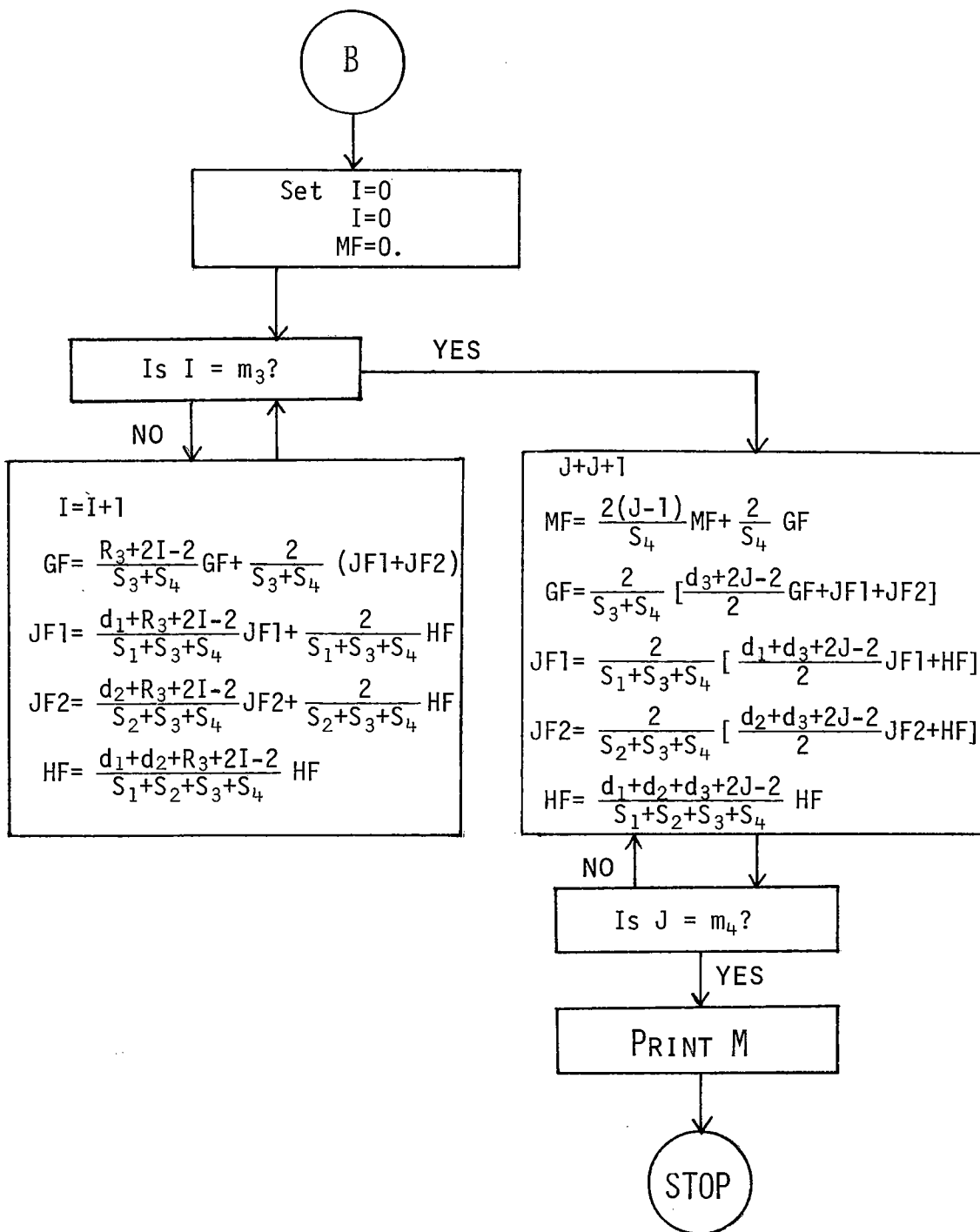


Figure 4.1: Continued

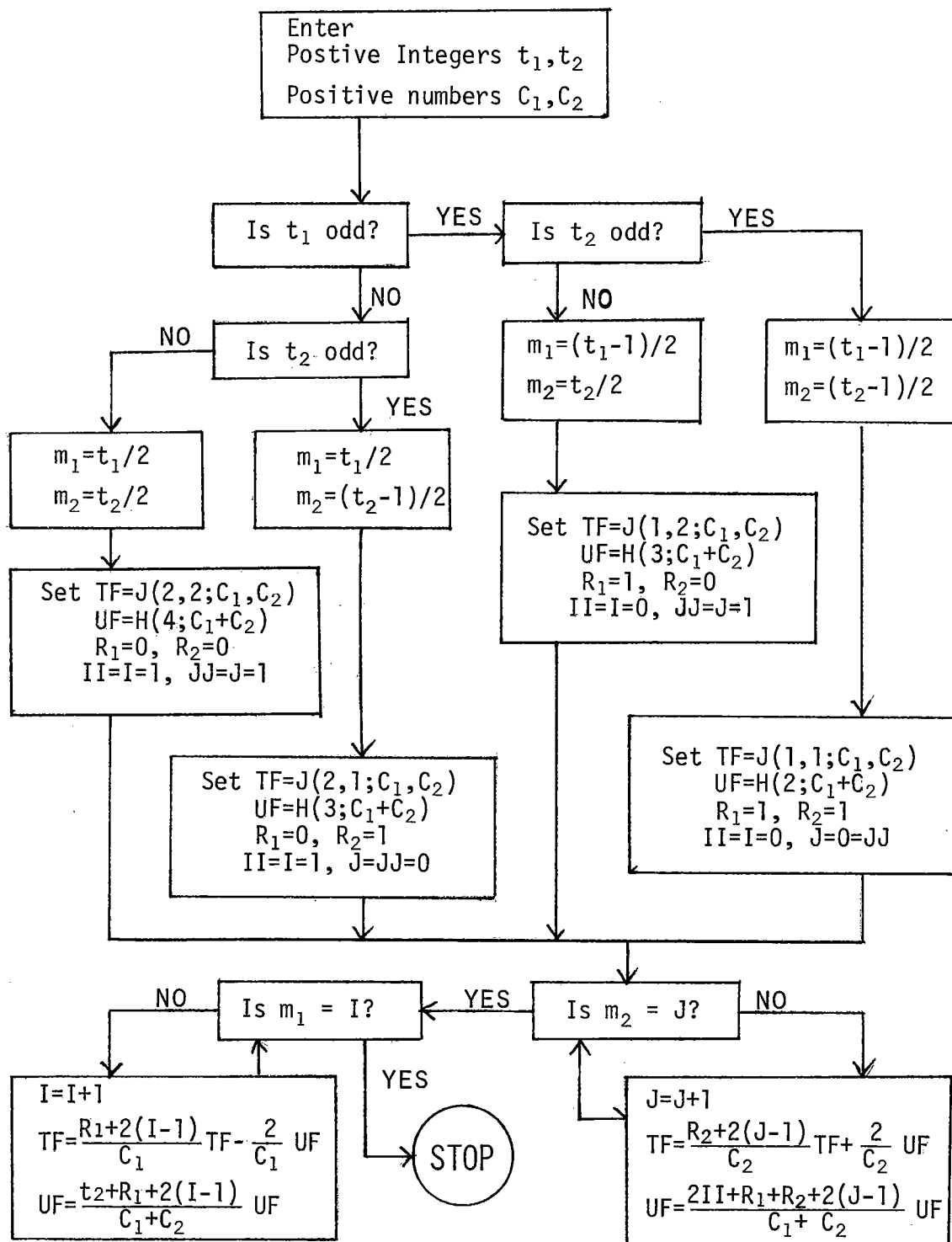
SUBROUTINE J-FUNCTION ( $t_1, t_2; C_1, C_2; TF, UF$ )Compute  $TF=J(t_1, t_2; C_1, C_2)$ ,  $UF=H(t_1+t_2; C_1+C_2)$ 

Figure 4.1: Continued

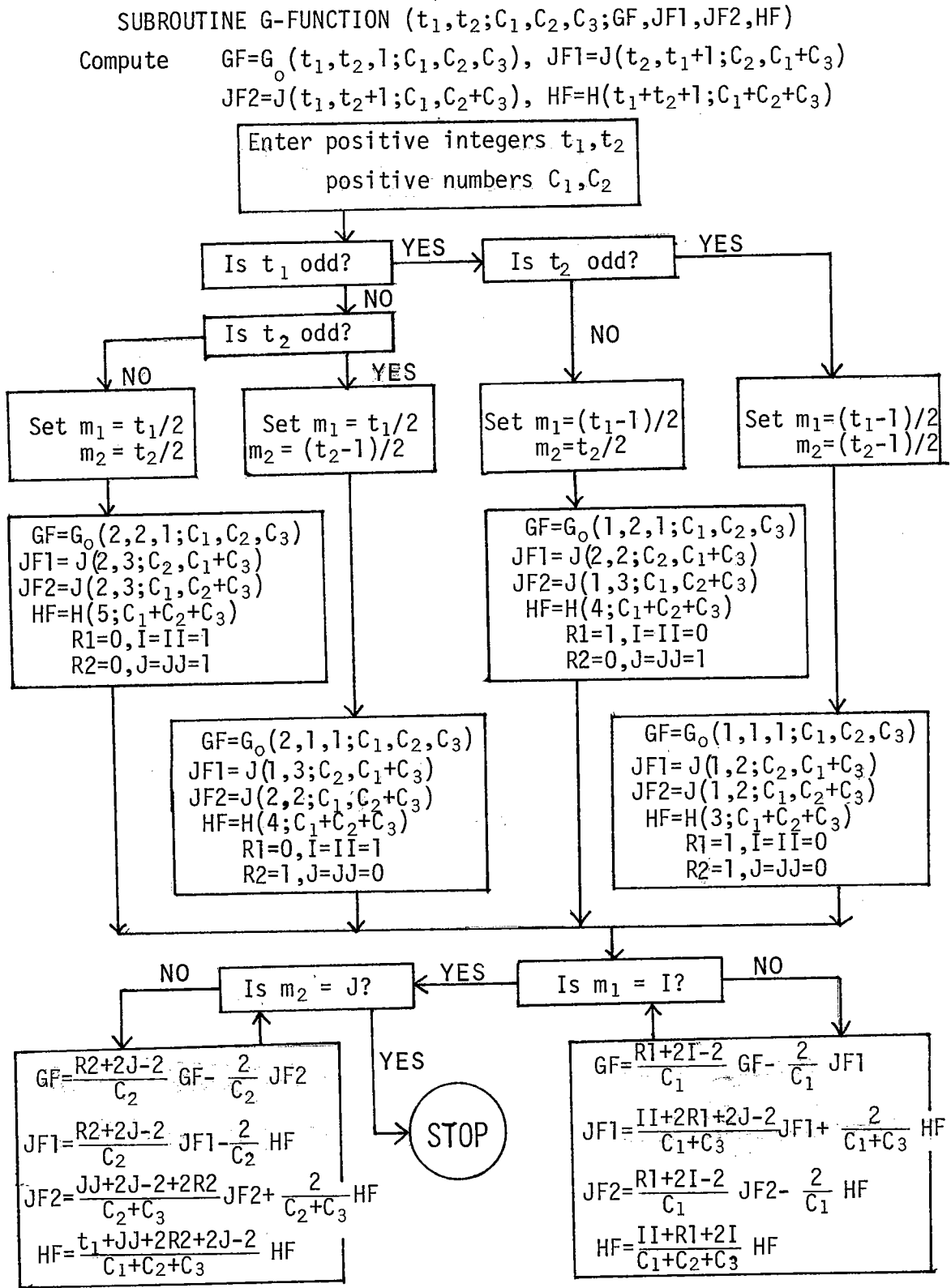


Figure 4.1: Continued

Let

$$\xi_j(\omega) = (\omega^2)^j \exp\left\{-\frac{1}{2}\omega^2\right\}. \quad (4.2.2)$$

Using Lemma 4.2.1 and (4.2.1), it follows that in the component problem where  $u_{i_1 j_1}$  is compared to  $u_{i_2 j_2}$ ,

$$\Gamma(\underline{T}) = \frac{\sqrt{2}}{M(\underline{f}^*, \underline{S}S^*)} \sum_{j=0}^{\infty} [2^j \Gamma(j+\frac{1}{2})]^{-1} H_j(\underline{T}; \underline{f}^*; \underline{S}S^*), \quad (4.2.3)$$

where the function  $M(\underline{d}; \underline{S})$  of  $\underline{d} = (d_1, d_2, d_3, d_4)'$  and  $\underline{S} = (S_1, S_2, S_3, S_4)'$  has previously been defined in (4.1.1), while the function  $H_j(\underline{T}; \underline{d}; \underline{S})$  is defined by

$$\begin{aligned} & H_j(\underline{T}; \underline{d}; \underline{S}) \\ &= \int_0^{\infty} \int_{v_4}^{\infty} \int_{v_3}^{\infty} \int_{v_3}^{\infty} \xi_j \left( \frac{m(\underline{T}, \underline{y})}{\sigma(\underline{y})} \right) \sigma(\underline{y}) \left[ \frac{\exp \left\{ - \sum_{i=1}^4 \frac{S_i}{2v_i} \right\}}{4 \prod_{i=1}^4 v_i^{\frac{1}{2}d_i+1}} \right] dv_1 dv_2 dv_3 dv_4, \end{aligned} \quad (4.2.4)$$

where  $\underline{y} = (v_1, v_2, v_3, v_4)'$ ,

$$m(\underline{T}, \underline{y}) = \left(1 - \frac{v_4}{v_3}\right) T_1 + \left(\frac{v_4}{v_3} - \frac{v_4}{v_2}\right) T_2 + \left(\frac{v_4}{v_3} - \frac{v_4}{v_1}\right) T_3, \quad (4.2.5)$$

$$\sigma^2(\underline{T}, \underline{y}) = \frac{2v_4}{K} \left[ \left(1 - \frac{v_4}{v_3}\right) + \frac{1}{r} (1 - \delta_{j_1 j_2}) \left(\frac{v_4}{v_3} - \frac{v_4}{v_2}\right) + \frac{1}{c} (1 - \delta_{i_1 i_2}) \left(\frac{v_4}{v_3} - \frac{v_4}{v_1}\right) \right].$$

#### Type I and Type II Component Problems

In Type I component problems,  $m(\underline{T}, \underline{y})$  and  $\sigma^2(\underline{y})$  are functionally independent of  $v_1$ . Similarly, in Type II component problems,  $m(\underline{T}, \underline{y})$  and  $\sigma^2(\underline{y})$  are functionally independent of  $v_2$ . In these cases, (4.2.4) can be simplified.



Lemma 4.2.2. For any integer  $d > 2$ ,

$$\int_x^\infty \frac{\exp\left\{-\frac{S}{2v}\right\}}{v^{\frac{1}{2}(d+1)}} dv = \left(\frac{d-2}{S}\right) \int_x^\infty \frac{\exp\left\{-\frac{S}{2v}\right\}}{v^{\frac{1}{2}d}} dv - \left(\frac{2}{S}\right) \frac{\exp\left\{-\frac{S}{2x}\right\}}{x^{\frac{1}{2}(d-1)}}$$

while

$$\int_x^\infty \frac{\exp\left\{-\frac{S}{2v}\right\}}{v^2} dv = \frac{2}{S} \left[1 - \exp\left\{-\frac{S}{2x}\right\}\right].$$

Proof. The first result follows from an integration by parts, while the second result is established by direct integration.  $\square$

For  $\underline{t} = (t_1, t_2, t_3)$ ;  $\underline{c} = (c_1, c_2, c_3)$ ; define

$$F_j(\underline{T}; \underline{t}; \underline{c})$$

$$= \int_0^\infty \int_{w_3}^\infty \int_{w_2}^\infty \xi_j \left( \frac{m(\underline{T}, \underline{w})}{\sigma(\underline{w})} \right) \sigma(\underline{w}) \frac{\exp\left\{-\sum_{i=1}^3 \frac{c_i}{2w_i}\right\}}{3 \prod_{i=1}^3 w_i^{\frac{1}{2}t_i+1}} dw_1 dw_2 dw_3, \quad (4.2.6)$$

where in Type I component problems,

$$m(\underline{T}, \underline{w}) = \left(1 - \frac{w_3}{w_2}\right)T_1 + \left(\frac{w_3}{w_2} - \frac{w_3}{w_1}\right)T_2, \quad \sigma^2(\underline{w}) = \frac{2w_3}{K} \left[\left(1 - \frac{w_3}{w_2}\right) + \frac{1}{r} \left(\frac{w_3}{w_2} - \frac{w_3}{w_1}\right)\right],$$

and in Type II component problems,

$$m(\underline{T}, \underline{w}) = \left(1 - \frac{w_3}{w_2}\right)T_1 + \left(\frac{w_3}{w_2} - \frac{w_3}{w_1}\right)T_3, \quad \sigma^2(\underline{w}) = \frac{2w_3}{K} \left[\left(1 - \frac{w_3}{w_2}\right) + \frac{1}{c} \left(\frac{w_3}{w_2} - \frac{w_3}{w_1}\right)\right].$$

Lemma 4.2.3. Suppose that  $d_1, d_2, d_3, d_4 \geq 1$ , and  $S_1, S_2, S_3, S_4 > 0$ . In Type I component problems,

$$\begin{aligned}
& H_j(\underline{T}; d_1+2, d_2, d_3, d_4; \underline{S}) \\
&= \frac{d_1}{S_1} H_j(\underline{T}; \underline{d}; \underline{S}) - \frac{2}{S_1} F_j(\underline{T}; d_2, d_1+d_3, d_4; S_2, S_1+S_3, S_4),
\end{aligned}$$

and

$$\begin{aligned}
& H_j(\underline{T}; 2, d_2, d_3, d_4; \underline{S}) \\
&= \frac{2}{S_1} [F_j(\underline{T}; d_2, d_3, d_4; S_2, S_3, S_4) - F_j(\underline{T}; d_2, d_3, d_4; S_2, S_1+S_3, S_4)].
\end{aligned}$$

In Type II component problems,

$$\begin{aligned}
& H_j(\underline{T}; d_1, d_2+2, d_3, d_4; \underline{S}) \\
&= \frac{d_2}{S_2} H_j(\underline{T}; \underline{d}; \underline{S}) - \frac{2}{S_2} F_j(\underline{T}; d_1, d_2+d_3, d_4; S_1, S_2+S_3, S_4),
\end{aligned}$$

and

$$\begin{aligned}
& H_j(\underline{T}; d_1, 2, d_3, d_4; \underline{S}) \\
&= \frac{2}{S_2} [F_j(\underline{T}; d_1, d_3, d_4; S_1, S_3, S_4) - F_j(\underline{T}; d_1, d_3, d_4; S_1, S_2+S_3, S_4)].
\end{aligned}$$

Proof. These results follow directly from the previously mentioned fact that  $m(\underline{T}; \underline{v})$  and  $\sigma^2(\underline{v})$  are functionally independent of  $v_1$  in Type I component problems, and of  $v_2$  in Type II component problems. This allows us to use Lemma 4.2.2 to provide the recursion formulas given above.  $\square$

The triple integral in (4.2.6) can be reduced to a double integral by making the transformation:

$$x = \frac{w_3}{w_1}, \quad y = \frac{w_3}{w_2}, \quad z = \frac{1}{w_3},$$

and integrating out  $z$ . Thus,

$$F_j(\underline{T}; \underline{t}; \underline{C}) = \frac{(K)^{j-\frac{1}{2}} \Gamma(\frac{1}{2}) \sum_{i=1}^3 t_i^{j-\frac{1}{2}}}{(\frac{1}{2})^{\frac{1}{2}} \sum_{i=1}^3 t_i} \cdot \int_0^1 \int_0^y \frac{(m(x,y))^{2j} x^{\frac{1}{2}t_1-1} y^{\frac{1}{2}t_2-1} dx dy}{(\sigma^2(x,y))^{j-\frac{1}{2}} \left[ K \frac{m^2(x,y)}{\sigma^2(x,y)} + C_1 x + C_2 y + C_3 \right]^{\frac{1}{2} \sum_{i=1}^3 t_i^{j-\frac{1}{2}}}}$$

(4.2.7)

where

$$m(x,y) = (1-y)T_1 + (y-x)T_2, \quad \sigma^2(x,y) = (1-y) + \frac{1}{r}(y-x),$$

in Type I component problems, while

$$m(x,y) = (1-y)T_1 + (y-x)T_3, \quad \sigma^2(x,y) = (1-y) + \frac{1}{c}(y-x),$$

in Type II component problems.

For  $j=0,1,2,\dots$ ,  $F_j(\underline{t}; \underline{C})$  defines a double iterated integral over a triangular area bounded by the vertices  $(0,0)$ ,  $(0,1)$ ,  $(1,0)$ . A numerical quadrature procedure called TRINT [Mustard, Lyness and Blatt (1963)], in the possession of the TRW Program Library, is used to evaluate this integral. The number of subintervals of equal length partitioning the interval  $[0,1]$  on each axis is chosen to be 32. In this case, it is known that the maximum error made by the quadrature formula in approximating  $F_j(\underline{T}; \underline{t}; \underline{C})$  is

$$3.3114 \times 10^{-10} \max_{0 < x < y < 1} \left[ \frac{\partial^4 f(x,y)}{\partial x^4} + \frac{\partial^4 f(x,y)}{\partial y^4} + 10 \frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} \right]$$

where  $f(x,y)$  is the integrand of the double integral (4.2.7).

In using the recursion provided by Lemma 4.2.3 for  $H_j(\underline{T}; \underline{d}; \underline{S})$ , initial values  $H_j(\underline{T}; 1, d_2, d_3, d_4; \underline{S})$ ,  $H_j(\underline{T}; d_1, 1, d_3, d_4; \underline{S})$  are needed for Type I component problems with  $d_1 = f_A^*$  odd, and Type II component problems with  $d_2 = f_B^*$  odd, respectively. These can be obtained by means of the general algorithm for  $H_j(\underline{T}; \underline{d}; \underline{S})$  to be described for use in Type III component problems. Of course, for Type I component problems with  $d_1 = f_A^*$  even, and Type II component problems with  $d_2 = f_B^*$  even, the initial values  $H_j(\underline{T}; 2, d_2, d_3, d_4; \underline{S})$  and  $H_j(\underline{T}; d_1, 2, d_3, d_4; \underline{S})$  are specified by Lemma 4.2.3.

### Type III Component Problems

In Type III component problems,  $m(\underline{T}, \underline{y})$  and  $\sigma^2(\underline{y})$  depend upon every element of  $\underline{y}$ , complicating the problem of calculation. One simplification can be achieved by making the transformation:

$$x = \frac{v_4}{v_1}, \quad y = \frac{v_4}{v_2}, \quad z = \frac{v_4}{v_3}, \quad w = \frac{1}{v_4}$$

in (4.2.4), and integrating over  $w$ . Thus

$$H_j(\underline{T}; \underline{d}; \underline{S}) = \frac{K^{j-\frac{1}{2}} \Gamma\left(\frac{1}{2} \sum_{i=1}^4 d_i + j - \frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2} \sum_{i=1}^4 d_i}} \cdot \int_0^1 \int_0^z \int_0^z \frac{(m(x,y,z))^{2j} x^{\frac{1}{2}d_1-1} y^{\frac{1}{2}d_2-1} z^{\frac{1}{2}d_3-1} dx dy dz}{(\sigma^2(x,y,z))^{j-\frac{1}{2}} \left[ K \frac{m^2(x,y,z)}{\sigma^2(x,y,z)} + S_1 x + S_2 y + S_3 z + S_4 \right]^{\frac{1}{2} \sum_{i=1}^4 d_i + j - \frac{1}{2}}}, \quad (4.2.8)$$

where

$$m(x,y,z) = (1-z)T_1 + (z-y)T_2 + (z-x)T_3,$$

$$\sigma^2(x,y,z) = (1-z) + \frac{1}{r} (z-y) + \frac{1}{c} (z-x).$$

To evaluate this three-dimensional iterated integral, a subroutine INT3D is obtained from Stroud (1973). The procedure is based on a 16-point product Gauss-Legendre polynomial formula. The computer program is given in Appendix B. References giving estimates of the numerical errors made by such a procedure are Ahlin (1962), Chawla (1968), Stenger (1966), and Stroud and Secrest (1966).

#### Calculation of $\gamma(\underline{T})$ and $\sigma_0$

Recall that

$$\gamma(\underline{T}) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{E}} \frac{(\sigma(\underline{E}))^3 \exp\left\{-\frac{m^2(\underline{T}, \underline{E})}{2\sigma^2(\underline{E})}\right\}}{m^2(\underline{T}, \underline{E})} \pi^*(\underline{E}) d\underline{E}$$

plays a role in one of the screening methods described in Section 3.2. Using an obvious extension of the definition of  $H_j(\underline{T}; \underline{d}; \underline{S})$  to the case  $j = -1$ , we can show that

$$\gamma(\underline{T}) = \frac{H_{-1}(\underline{T}; \underline{f}^*; \underline{SS}^*)}{\sqrt{2\pi} M(\underline{f}^*; \underline{SS}^*)}.$$

Further, in Type I and Type II component problems,  $H_{-1}(\underline{T}; \underline{d}; \underline{S})$  obeys obvious analogues of the recursion formulas of Lemma 4.2.3, and  $F_{-1}(\underline{T}; \underline{t}; \underline{C})$  is computable from (4.2.7) with  $j = -1$ . It is also always the case that  $H_{-1}(\underline{T}; \underline{d}; \underline{S})$  can be calculated from (4.2.8) with  $j = -1$  for Type III component problems. Thus,  $\gamma(\underline{T})$  can be calculated by means of the same techniques used to calculate  $\Gamma(\underline{T})$ , except that  $\Gamma(\underline{T})$  is an

infinite weighted sum of  $H_j(\underline{T}; \underline{f}^*; \underline{SS}^*)$ ,  $j=0,1,2,\dots$ , while  $\gamma(\underline{T})$  only involves  $H_{-1}(\underline{T}; \underline{f}^*; \underline{SS}^*)$ .

The constant

$$\sigma_0 = \sigma_0(\underline{f}^*; \underline{SS}^*) = \int_{\mathcal{E}} \sigma(\underline{E}) \pi^*(\underline{E}) d\underline{E},$$

also has a role in a screening procedure. Again, in Type I and Type II component problems,  $\sigma_0(\underline{d}; \underline{S})$  obeys a recursion relationship of the form of Lemma 4.2.3. That is, in Type I component problems

$$\sigma_0(d_1+2, d_2, d_3, d_4; \underline{S}) = \frac{d_1}{S_1} \sigma_0(\underline{d}; \underline{S}) - \frac{2}{S_1} G(d_2, d_1+d_3, d_4; S_2, S_1+S_3, S_4),$$

and

$$\sigma_0(2, d_2, d_3, d_4; \underline{S}) = \frac{2}{S_1} [G(d_2, d_3, d_4; S_2, S_3, S_4) - G(d_2, d_3, d_4; S_2, S_1+S_3, S_4)],$$

while in Type II component problems,

$$\sigma_0(d_1, d_2+2, d_3, d_4; \underline{S}) = \frac{d_2}{S_2} \sigma_0(\underline{d}; \underline{S}) - \frac{2}{S_2} G(d_1, d_2+d_3, d_4; S_1, S_2+S_3, S_4)$$

and

$$\sigma_0(d_1, 2, d_3, d_4; \underline{S}) = \frac{2}{S_2} [G(d_1, d_3, d_4; S_1, S_3, S_4) - G(d_1, d_3, d_4; S_1, S_2+S_3, S_4)],$$

where  $G(t_1, t_2, t_3; C_1, C_2, C_3)$  can be obtained from the right side of (4.2.7) by setting  $T_1 = T_2 = T_3 = j = 0$ . Similarly,  $\sigma_0(\underline{d}; \underline{S})$  can always be computed from the right side of (4.2.8) by setting  $T_1 = T_2 = T_3 = j = 0$  for Type III component problems.

#### Computation of $\Gamma(\underline{T})$ , $q_{\pm 0}(\underline{T})$ , and Implementation of the Bayes Rule.

The function  $\Gamma(\underline{T})$  can be computed exactly using the results given in this section. In each component problem, the algorithm will decide the type of component problem (Type I, II or III) and determine, in the

case of Type I and Type II component problems, what initial values to use for the computation of the  $H_j(\underline{T}; \underline{f}^*, \underline{SS}^*)$ 's. However, the exact value of  $\Gamma(\underline{T})$  is not really needed in order for the component Bayes rule to make a decision. The infinite summation (4.2.3) defining  $\Gamma(\underline{T})$  consists of nonnegative terms. Thus, if we calculate term by term of this summation, starting at  $j=0$ , we need only carry out the summation until the signs of

$$q_{-\rho} = \Delta(\underline{T}) - \rho\Gamma(\underline{T}), \quad q_{\rho} = \Delta(\underline{T}) + \rho\Gamma(\underline{T})$$

are determined. Thus, in every computation of  $\Gamma(\underline{T})$  only a finite number of terms in the sum (4.2.3) needs to be computed.

Even this computation is unnecessary if the screening methods described in Section 3.2 allow us to arrive at a decision. For this reason, the algorithm for implementing each component Bayes rule starts by calculating  $\Delta(\underline{T})$ , using the methods described in Section 4.1, plus  $\sigma_0 = \sigma_0(\underline{f}^*; \underline{SS}^*)$  and  $\gamma(\underline{T})$ , using the methods given in the present section. These quantities are then used to implement the screening procedures of Section 3.2. Only if these procedures fail to identify an action to be taken, is  $\Gamma(\underline{T})$  calculated. Since if  $\Delta(\underline{T}) > 0$ , the decision of what action to take is restricted to choosing  $d^0$  or  $d^+$ , only  $q_{-\rho}(\underline{T})$  is calculated. The expression (4.2.3) is calculated summing term by term until it either exceeds  $\rho^{-1}\Delta(\underline{T})$  (in which case  $q_{-\rho}(\underline{T}) < 0$  and action  $d^0$  is taken), or until a term is reached in the summation which is so small in magnitude that it is clear that further terms will not materially change the value of the sum. (A comparison value of  $10^{-6}$  is used for this decision.) Since the  $H_j(\underline{T}; \underline{f}^*; \underline{SS}^*)$  terms are bounded, this rule can be justified as yielding an exact procedure, since

the weights  $[2^{j_{\tau}(j+\frac{1}{2})}]^{-1}$  determine a convergent power series. If a term less than  $10^{-6}$  is reached and the sum up to that point is less than  $\rho^{-1}\Delta(\underline{T})$ , action  $d^+$  is taken. Similarly, if  $\Delta(\underline{T}) < 0$ , calculation of (4.2.3) proceeds by summing term by term until either the sum exceeds  $\rho^{-1}|\Delta(\underline{T})|$ , in which case action  $d^0$  is taken, or until summation is stopped by a term less than  $10^{-6}$ , in which case action  $d^-$  is taken.

A flow chart of the full computer program MCP (Multiple Comparison Problem) is given in Figure 4.2. The data from an  $r \times c$  two way balanced design with  $K$  observations per cell are first analyzed in the usual fashion, producing an ANOVA table, sample cell mean matrix  $X$ , and other informational data. After pooling the ANOVA sums of squares and degrees of freedom with the corresponding sums of squares and degrees of freedom provided by the user's prior distribution, the algorithm calculates the constants  $a_1, a_2, a_3$  (see Section 4.1), which are common to all component problems, and then proceeds to implement the component Bayes rules. Because of their greater computational simplicity, Type I and Type II comparisons are done first, followed by Type III comparisons. In every component problem, the approach mentioned above [using the screening procedures first, after which full calculation of  $q_{\rho}(\underline{T})$  or  $q_{-\rho}(\underline{T})$  is done only if necessary] is followed. After all comparisons have been completed, a printout of relevant data summaries and the component decisions is given.

Examples of the use of the Bayes rule on four actual data sets are given in Chapter 5. In addition, Chapter 5 compares the Bayes rule with appropriately modified classical procedures (Scheffé, Newman-Keuls, LSD, Tukey) in terms of the decisions made on the given data sets. Finally,



an approximation to the Bayes rule based on the approximate normality of posterior densities in large samples is described and compared to the exact Bayes rule.

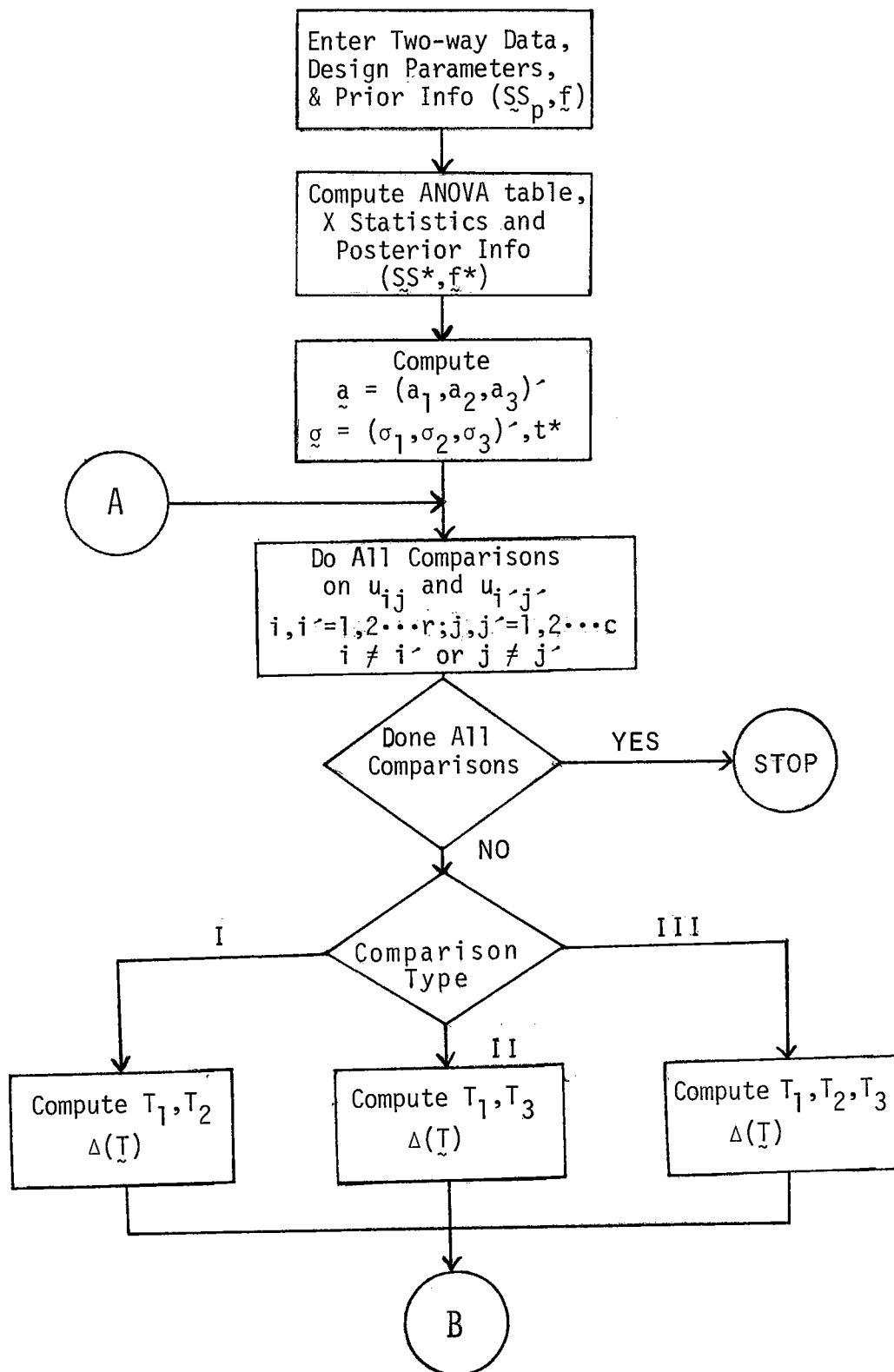


Figure 4.2: Bayesian Decision Making Process Incorporating Screening Criteria

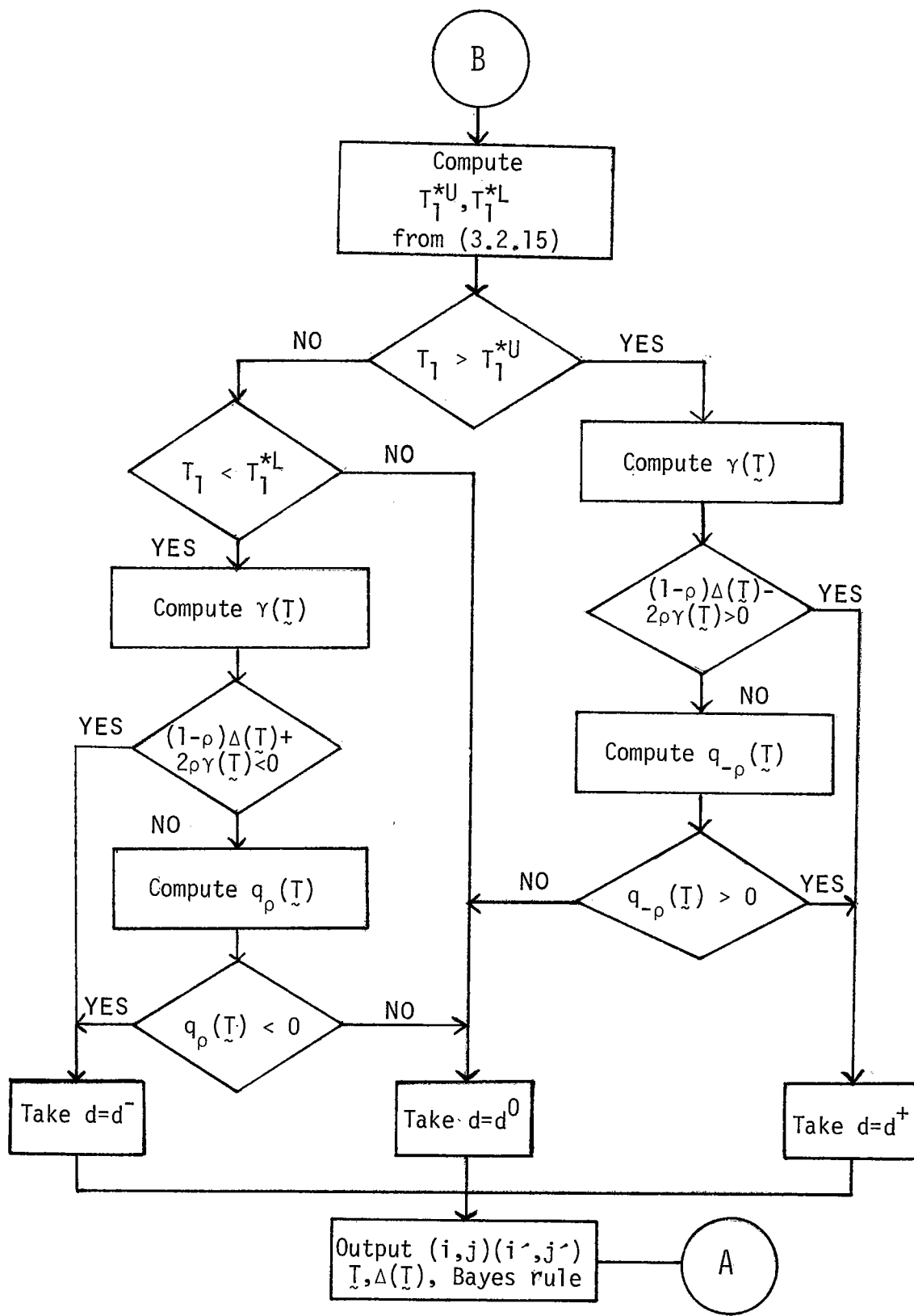


Figure 4.2: Continued

## CHAPTER V

### APPLICATION OF THE BAYES RULE

In order to assess the general performance of the Bayes rule proposed in this study, in comparison with the usual non-Bayesian multiple comparison procedures available in the two-way design, four examples are presented with the intention that they will be somewhat representative of situations found in most two-factor experiments. The examples are chosen to represent various combinations of significant or nonsignificant main effects (A,B) and interactions (AB). The results of applying the Bayes rule in these examples, and a discussion of the performance of the Bayes rule in comparison to other available multiple comparisons procedures, is presented in Section 5.1. Section 5.2 describes an approximate Bayes rule based on the asymptotic posterior distribution of the true mean difference. Since computation of the exact Bayes rule can be costly in terms of computer time, such an approximation provides an economical alternative. Some concluding remarks about the exact Bayes rule are also included in Section 5.2.

#### 5.1. Full Application Procedure with Description and Illustration

Using the computer program listed in Appendix B and flowcharted in Section 4.2, the Bayes rule proposed here can be computed for the data from any balanced rxc design.

To use the Bayes procedure, we must first choose an appropriate value of the error-weight-ratio  $\chi = \chi_1/\chi_0$ . This ratio can be interpreted as a measure of relative seriousness of comparisonwise Type I to comparisonwise Type II errors.

In Waller and Duncan's (1969, 1974) work, it is suggested that for a given  $\alpha$ -level of 0.10, 0.05 or 0.01, the value for the  $\chi$ -ratio can be taken at 50, 100 or 500, respectively. Such a correspondence between  $\alpha$  and  $\chi$  was suggested by considering the case of 2 treatments in a one-way design. Taking  $F^* = 4.0$  as a representative F-ratio, and using  $F^* = 4.0$  in Equation (1.2.16) with values of  $\chi = 50, 100, 500$ , the resulting Bayes t-values ( $t=1.715, 1.988$  or  $2.577$ ) are very close to the critical t-values ( $t=1.645, 1.960$  or  $2.576$ ) obtained from the t-distribution at  $\alpha$ -levels of 10%, 5% or 1%. Thus with these correspondences as reference points, lower, intermediate or higher values of  $\chi$  may be chosen accordingly. For simplicity, in our applications, all examples are run at  $\chi=100$ .

The examples we use here are taken from well-known textbooks on the design of experiments. To specify any prior distribution other than the prior of indifference would reflect knowledge not available to us. For this reason, the prior distribution chosen is the prior of indifference (for which  $SSA_p = SSB_p = SSAB_p = SSE_p = 0$ ,  $f_A = f_B = f_{AB} = f_e = 0$ ).

Our first two examples are designed to illustrate the adaptive nature of our Bayes rule. If the F-value for the interactions is significant, the two-stage approach (see Section 1.3) views the cell means as if they came from a one-factor design, thus permitting various classical procedures described in Section 1.1, Chapter I to be employed.

Example 1. The data for this example are taken from Anderson and McLean (1974), p. 63. Five levels ( $r=5$ ) of the factor A (time of automatic welding cycle) and three levels ( $c=3$ ) of the factor B (gage bar setting) are considered and  $K=2$  measurements of the dependent variable (breaking strength of the weld) are taken for each treatment combination. The data and its ANOVA table are given in Table 5.1 and Table 5.2 respectively.

Table 5.1: Data of Example 1

Factor A	Factor B			Row Mean
	$B_1$	$B_2$	$B_3$	
$A_1$	11.00	17.00	9.00	12.3333
$A_2$	15.00	13.00	10.50	12.8333
$A_3$	25.50	34.00	7.50	22.3333
$A_4$	17.00	13.00	14.50	14.8333
$A_5$	19.00	13.00	15.00	15.6667
Column mean	17.50	18.00	11.30	15.6000 (grand mean)

Table 5.2: ANOVA of Data in Table 5.1

Source	SS	d.f.	MS	F-Ratio	$F_{crit}^{(0.05)}$
A	385.53	4	96.38	8.82	> 3.06
B	278.60	2	139.30	12.74	> 3.68
AB	597.07	8	74.63	6.82	> 2.64
Error	164.00	15	10.93		

The results of use of the Bayes rule and various classical rules to compare the means is summarized in the following tabular form (Table 5.3). Note that the column headings are the sample cell means arrayed in decreasing order of value.

Table 5.3: Comparison Results of Five Multiple Comparisons Procedures on Example 1.

	$\bar{X}_{32}$	$\bar{X}_{31}$	$\bar{X}_{51}$	$\bar{X}_{41}$	$\bar{X}_{12}$	$\bar{X}_{21}$	$\bar{X}_{53}$	$\bar{X}_{43}$	$\bar{X}_{22}$	$\bar{X}_{42}$	$\bar{X}_{52}$	$\bar{X}_{11}$	$\bar{X}_{23}$	$\bar{X}_{13}$	$\bar{X}_{33}$	
	34	25.5	19	17	17	15	15	14.5	13	13	13	11	10.5	9	7.5	
$\bar{X}_{32}$		L NB	LT NB	LT NB	LT NB	LT NB	LT NB	LST NB	LST NB	LST NB	LST NB	LST NB	LST NB	LST NB	LST NB	
$\bar{X}_{31}$			L B	L B	L B	L B	L B	L B	L NB	L NB	L NB	LT NB	LT NB	LT NB	LT NB	
$\bar{X}_{51}$									L	L	L	L B	L B	L B	L B	
$\bar{X}_{41}$												L	L B	L B	L B	
$\bar{X}_{12}$												L	L B	L B	L B	
$\bar{X}_{21}$														L	L	
$\bar{X}_{53}$														L	L	
$\bar{X}_{43}$															L	
⋮																

A "B" in a cell of the above array denotes that the two entries (cell means) which define the cell are significantly different by Bayes rule. Similarly, an "S" denotes a significant difference by Scheffé's

method, a "T" denotes a significant difference by Tukey's HSD method, an "N" denotes a significant difference by Newman-Keuls test and an "L" indicates a significant difference by the LSD procedure. Looking at the above array, it can be seen that the Scheffé method gives the most conservative conclusions. Tukey's test gives more significant differences than Scheffé's method. This is not surprising since the HSD method generally is more sensitive than the Scheffé method in situations where one is only interested in pairwise comparisons.

It is worth noting that Newman-Keuls' method, also being an adaptive procedure, performs more similarly to the Bayes rule than any of the other experimentwise procedures. The only comparisonwise procedure used (the LSD at the  $\alpha = 5\%$  level), besides all the significant differences declared by the Bayes rule, adds ten more pairs of significance differences (such as  $(\bar{X}_{51}, \bar{X}_{22})$ ,  $(\bar{X}_{51}, \bar{X}_{42})$ ,  $(\bar{X}_{51}, \bar{X}_{52})$ ,  $(\bar{X}_{41}, \bar{X}_{11})$ , ..., etc.) Comparing the results of the LSD, HSD and Bayes rules, gives some support to the claim that the Bayes rule effectively provides a compromise between the comparisonwise (such as LSD) and experimentwise (such as HSD) approaches.

Note. In Examples 1 through 4 it should be noted that our ranking of the means in descending order offers a convenient way to display significant differences without specifying whether these differences are significantly positive or significantly negative. However, it is possible that the two observed cell means show  $\bar{X}_{ij} > \bar{X}_{i-j}$ , and yet the Bayes rule concludes that  $u_{ij} < u_{i-j}$ . Fortunately this event did not happen in any of the given four examples.



Example 2. The data for this example are taken from Lindman (1976).

The data and its ANOVA table are displayed in Table 5.4 and Table 5.5.

Table 5.4: Data of Example 2

Factor A	Factor B					Row mean
	B <sub>1</sub>	B <sub>2</sub>	B <sub>3</sub>	B <sub>4</sub>	B <sub>5</sub>	
A <sub>1</sub>	35.00	53.00	68.00	75.00	70.00	60.20
A <sub>2</sub>	62.00	76.00	84.00	83.00	82.00	77.40
A <sub>3</sub>	48.00	65.00	76.00	79.00	76.00	68.80
Column mean	48.33	64.67	76.00	79.00	76.00	68.80 (grand mean)

Table 5.5: ANOVA of Data in Table 5.4

Source	SS	d.f.	MS	F-Ratio	F(0.05)
A	2218.80	2	1109.40	78.13	> 3.32
B	5793.20	4	1448.30	101.99	> 2.69
AB	365.20	8	45.65	3.21	> 2.27
Error	426.00	30	14.20		

To be able to visually see the comparison of various results by these different procedures, a tabular form similar to that used in Example 1 is utilized.



The comparative performances of the Bayes, LSD, Scheffé, Tukey, and Newman-Keuls procedures are similar to that in Example 1. However, note that there are now 9 comparisons where the Bayes procedure declares a significant difference, while the LSD procedure does not, while there are no cases where the converse occurs. Thus, the Bayes procedure is actually more sensitive than the LSD in this example. A possible explanation is that the low (but significant) level of interaction allows the Bayes procedure to use row and column means in arriving at a decision.

In the above example, only 16 pairs of nonsignificantly different means are found from a total of 105 pairwise comparisons. This small portion of nonsignificant differences is not surprising considering that all the main effects are highly significant.

The next two examples illustrate application of the Bayes rule when the interactions are not significant. It was noted in Section 1.3 that classical procedures can be modified so as to compare cell means in a two-factor additive model. For example, it is possible to modify the Scheffé and LSD rules to apply in such situations. Because the Scheffé rule is very conservative (as can be seen in Examples 1 and 2), only the LSD rule has been used to compare to the Bayes rule. Due to a lack of funds (and time to write and correct a computer program), the computation for the LSD rule had to be done by hand.

Example 3. The data are taken from Hicks (1973), p. 136, in which the effect of both depth and position in a tank on the concentration of a cleaning solution in ounces per gallon is to be determined. Concentrations are measured at three depths ( $c=3$ ) from the surface of the tank, 0 inches, 15 inches and 30 inches. At each depth two measurements

(K=2) are taken at each of the five different lateral positions in the tank (r=5). The data collected are shown in Table 5.7 and the ANOVA table in Table 5.8.

Table 5.7: Cleaning Solution Concentration Data

Position	Depth from Top of Tank			Row mean
	B <sub>1</sub>	B <sub>2</sub>	B <sub>3</sub>	
A <sub>1</sub>	590.50	589.50	587.00	589.00
A <sub>2</sub>	590.50	589.00	579.00	586.17
A <sub>3</sub>	592.00	591.00	584.50	589.17
A <sub>4</sub>	592.00	592.00	586.00	590.00
A <sub>5</sub>	588.50	592.00	584.50	588.33
Column mean	590.70	590.70	584.20	588.53 (grand mean)

Table 5.8: ANOVA of Data in Table 5.7

Source		d. f.		F-Ratio	F (0.05) crit
A	50.47	4	12.62	1.00 <	3.06
B	281.67	2	140.83	11.18 >	3.68
AB	58.33	8	7.29	0.58 <	2.64
Error	189.00	15	12.60		

When the means are arranged in descending order, the Bayes rule declares two nonsignificant groups. The first group consists of the first 10 means -  $\bar{X}_{31}, \bar{X}_{41}, \bar{X}_{42}, \bar{X}_{52}, \bar{X}_{32}, \bar{X}_{11}, \bar{X}_{21}, \bar{X}_{12}, \bar{X}_{22}, \bar{X}_{51}$ . The second group consists of the remaining five means -  $\bar{X}_{13}, \bar{X}_{43}, \bar{X}_{33}, \bar{X}_{53}$  and  $\bar{X}_{23}$ . Any mean from the first group is significantly larger than any mean from



Note that the Bayes rule declares significance in some cases where the modified LSD does not and vice versa. A possible explanation for such disagreement is that even if the interaction is not significant, the Bayes rule can still use the information provided by the cell means together with row and column means in arriving at a decision. The modified LSD can only use row and column means.

Example 4. The data are taken from Lindman (1976). The data with  $r=3$ ,  $c=5$ ,  $K=3$  and its ANOVA table are shown in Table 5.10 and 5.11, respectively.

Table 5.10: Data of Example 4

Factor A	Factor B					Row mean
	B <sub>1</sub>	B <sub>2</sub>	B <sub>3</sub>	B <sub>4</sub>	B <sub>5</sub>	
A <sub>1</sub>	17.00	9.00	15.00	14.00	14.00	13.80
A <sub>2</sub>	12.00	10.00	4.00	6.00	11.00	8.60
A <sub>3</sub>	12.00	6.00	11.00	4.50	12.00	9.10
Column mean	13.67	8.33	10.00	8.17	12.33	10.50 (grand mean)

Table 5.11: ANOVA of Data in Table 5.10

Source	SS	d.f.	MS	F-Ratio	$F_{crit}^{(0.05)}$
A	246.90	2	123.45	10.69	> 3.32
B	214.00	4	53.50	4.63	> 2.69
AB	185.60	8	23.20	2.01	< 2.27
Error	346.50	30	11.55		



In Examples 3 and 4, the interaction is not significant by the usual F-test. Nonetheless the Bayes rule would still use the information provided by the cell means data to determine the decisions. The modified LSD does not use any cell data at all. Thus the effect from the cell mean differences for the Bayes rule can alter the decisions made differently by the LSD rule.

Most of the computation time for Examples 1-4 is spent on the numerical procedures for calculating double or triple integrals for  $P(I)$  in determining the component Bayes rules. The execution time in terms of the CPU (Central Processor Unit) for the above four examples are estimated to be 1270, 1080, 1510, 1620 seconds, respectively, when CDC 6500 of the Purdue MACE system is used. For our four examples, the following table indicates how the screening procedures of Section 3.2 have succeeded in reducing the total amount of computation time.

Table 5.13: Summary Statistics of Examples

Component Type	Number of Component Problems Computed by:			Total
	1 <sup>st</sup> Screening	2 <sup>nd</sup> Screening	$q_p$ or $q_{-p}$	
Example 1	I	11	3	15
	II	21	8	30
	III	34	15	60
Example 2	I	5	22	30
	II	0	15	15
	III	12	42	60
Example 3	I	5	10	15
	II	30	0	30
	III	20	20	60
Example 4	I	19	6	30
	II	6	5	15
	III	30	19	60



It can be seen that due to the high computation cost of implementing the exact Bayes rule, an extensive study of its properties on simulated data is unfeasible. However, an approximate Bayes rule which behaves similarly to the Bayes rule is given in the next section.

## 5.2. An Approximate Bayes Rule; Summary and Conclusion

The posterior distribution of  $\eta$ , given the sufficient statistic  $(X, SSE)$  is a normal distribution with posterior mean

$$E(\eta|X, SSE) = \Delta(T(X), SS^*, f^*),$$

and variance

$$\text{Var}(\eta|X, SSE) = \int_{\mathcal{E}} \sigma^2(E) \pi(E|SS^*, f^*) dE \equiv \sigma^2(SS^*, f^*). \quad (5.2.1)$$

It is known from Johnson (1967, 1970) that after suitably centering and scaling, the scaled posterior distribution of

$$[\sigma^2(SS^*, f^*)]^{-\frac{1}{2}} (\eta - \Delta(T(X), SS^*, f^*))$$

has an asymptotic (as  $rcK \rightarrow \infty$ ) expansion with the standard normal density as the leading term. Thus the posterior distribution of  $\eta$  given  $(X, SSE)$  can be approximated by the  $N_1(\Delta(T(X), SS^*, f^*), \sigma^2(SS^*, f^*))$  distribution.

Using this approximation for the posterior density, the functions  $q_\rho(\underline{T})$  and  $q_{-\rho}(\underline{T})$  in (3.2.4) can be approximated by

$$\begin{aligned} q_c(\underline{T}) &= \int_{-\infty}^{\infty} \eta N_1(\eta | \Delta(\underline{T}(X), SS^*, f^*), \sigma^2(SS^*, f^*)) d\eta \\ &+ c \int_{-\infty}^{\infty} |\eta| N_1(\eta | \Delta(\underline{T}(X), SS^*, f^*), \sigma^2(SS^*, f^*)) d\eta, \quad (5.2.2) \end{aligned}$$

with  $c = \rho, -\rho$ .

Since both  $\Delta(\underline{T}(X), \underline{SS}^*, \underline{f}^*), \sigma^2(\underline{SS}^*, \underline{f}^*)$  are constants given the data  $(X, SSE)$ , the approximate Bayes rule can be derived exactly as in the case of Section 3.1 where all the  $\sigma_A^2, \sigma_B^2, \sigma_{AB}^2$  and  $\sigma_e^2$  were assumed known. Thus from (3.1.14) the approximate Bayes rule  $\varphi^*(\underline{T}(X))$  has the form

$$\varphi^*(\underline{T}(X)) = \begin{cases} (1, 0, 0), & \Delta > t^*(\sigma^2)^{\frac{1}{2}} \\ (0, 1, 0), & |\Delta| < t^*(\sigma^2)^{\frac{1}{2}}, \\ (0, 0, 1), & \Delta < -t^*(\sigma^2)^{\frac{1}{2}}, \end{cases} \quad (5.2.3)$$

where  $\Delta = \Delta(\underline{T}(X), \underline{SS}^*, \underline{f}^*), \sigma^2 = \sigma^2(\underline{SS}^*, \underline{f}^*)$ , and  $t^*$  is the unique solution of the equation given in (3.1.15).

The computation of such an approximate Bayes rule  $\varphi^*$  is straightforward since the computations of  $\Delta$  and  $\sigma^2$  are direct applications of the algorithms described in Section 4.1. Indeed, it can be seen that

$$\sigma^2 = \frac{2}{K} [b_1 + \frac{1}{r} (1 - \delta_{j_1, j_2}) b_2 + \frac{1}{c} (1 - \delta_{i_1, i_2}) b_3], \quad (5.2.4)$$

where,

$$\begin{aligned} b_1 &= \frac{M(f_A^*, f_B^*, f_{AB}^*, f_e^* - 2; SS^*) - M(f_A^*, f_B^*, f_{AB}^* + 2, f_e^* - 4; SS^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^*; SS^*)} \\ b_2 &= \frac{M(f_A^*, f_B^*, f_{AB}^* + 2, f_e^* - 4; SS^*) - M(f_A^*, f_B^* + 2, f_{AB}^*, f_e^* - 4; SS^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^*; SS^*)} \\ b_3 &= \frac{M(f_A^*, f_B^*, f_{AB}^* + 2, f_e^* - 4; SS^*) - M(f_A^* + 2, f_B^*, f_{AB}^*, f_e^* - 4; SS^*)}{M(f_A^*, f_B^*, f_{AB}^*, f_e^*; SS^*)} \end{aligned} \quad (5.2.5)$$

with  $f^*, SS^*$  defined by (2.1.19) and  $M(\underline{d}, \underline{S})$  by (4.1.1).

The approximate Bayes rule  $\varphi^*$  of (5.2.3) offers us a very economical, alternative way to arrive at insight into the relationships of the cell means, since the expensive computation of  $\Gamma(\underline{T})$  for the exact Bayes rule  $\varphi$  can be avoided. Indeed, such an approximate Bayes rule  $\varphi^*$

performs quite similarly to the exact Bayes rule  $\varphi$ . The amount of discrepancy between the decisions made by the rules  $\varphi^*$  and  $\varphi$  depends largely on how frequently states of nature occur which result in marginal cases ( $q_p(\bar{T})$  and  $q_{-p}(\bar{T})$  close to 0) for the Bayes rule in the component problems. In particular, the approximate Bayes rule appears not to be very sensitive in detecting small differences among cell means. The disagreement between  $\varphi$  and  $\varphi^*$  can be substantial when one or more of the main effects or interactions are not strongly significant.

We have compared the approximate Bayes rule  $\varphi^*$  with the exact Bayes rule on each of the four examples discussed in Section 5.1. The extent of disagreement in their conclusions is summarized below.

Table 5.14: Discrepancy Between  $\varphi$  and  $\varphi^*$

	# of Disagreements in			Total # of Component Comparisons	Total # of Disagreements
	Type I	Type II	Type III		
Example 1	0	0	2	105	2
Example 2	3	0	1	105	4
Example 3	3	1	3	105	7
Example 4	0	0	3	105	3

Note that in Example 3, where the greatest disagreement occurs, both the main effect of A and the interaction term are insignificant.

We have previously mentioned that due to the expensive computation of the Bayes rule, a simulation study to assess the general performances of the Bayes rule in comparison with the classical multiple comparisons procedures is not feasible. However, a simulation study on the approximate Bayes rule could be done with relatively little cost. Since the

approximate Bayes rule  $\varphi^*$  seems to possess the same properties as the exact rule  $\varphi$ , such a simulation would give insight into the properties of both rules,  $\varphi$  and  $\varphi^*$ .

Summary and Conclusion. In this study we have shown that the Bayesian procedure proposed by Waller and Duncan (1969, 1974) for the symmetric multiple comparisons problem in one-factor designs can be extended to give an adaptive Bayes procedure in two-factor designs (or in higher-factorial designs if the pairwise comparisons problem is of interest to the user). In the exact formulation of the Bayes rule, the Bayesian solution involves exceedingly complex integrations which are costly (in terms of computer time) to compute. By analytic simplification and by the introduction of two screening procedures (given in Sections 3.2), computation time has been reduced by an order of magnitude 1/2400. Nevertheless, one disadvantage of the Bayes rule that has prevented studies of its properties on simulated data is the high cost of computing time for implementing the procedure.

Nevertheless, it has been shown by examples that in cases where there is highly significant interaction, the Bayes rule performs similarly to the Waller-Duncan procedure in one-way designs in terms of its sensitivity relative to Tukey's HSD, the Scheffé method, the LSD and the Newman-Keuls test. But when the interaction is only weakly present, the Bayes rule can take advantage of the information provided by the cell mean differences  $\bar{X}_{ij} - \bar{X}_{i-j}$ , so as to be more sensitive in some cases than the LSD classical one-factor rule, or even the modified LSD rule for an additive model. Because of the high computational cost of using the Bayes rule, we have offered an approximate Bayes rule which is

considerably easier to compute. The algorithms that are given in Section 4.1 suffice to compute this approximate Bayes rule in almost all practical cases. Since the approximate Bayes rule appears to behave very similarly to the Bayes rule, it is hoped that this rule will prove to be an attractive alternative to the exact Bayesian solution to the symmetric multiple comparisons problem.

Questions for Future Research. The Bayes procedure proposed in this thesis exhibits two characteristics which are worth emphasizing again. The first characteristic is the dependence of the Bayes rule on the experimental  $F^*$ -values rather than on the number of treatment cell means to be compared. The second characteristic of interest for the proposed Bayes rule is that prior information may be easily incorporated into the analysis by pooling techniques. With these advantages in mind, the following questions of practical interest invite similar Bayesian treatment.

In experiments comparing  $rc$  treatments in which one of the treatments is considered to be a control, it is assumed that one is interested in comparing the control with each of the other  $(rc-1)$  treatments. The proposed Bayes rule will not be applicable here since the exchangeable prior assumption is not satisfied. But if the control is not one of the treatment combinations considered in the experiment, it is possible to derive a Bayes rule for the multiple comparisons of the control with each of the  $rc$  treatment combinations. Here, the grand mean difference  $\mu_c - \mu$  would have to be assigned an appropriate prior, which subsequently results in a more complicated integral equation to be solved in order to determine the Bayes rule.

We have so far considered univariate populations only. Quite often, we may collect several kinds of measurements (say,  $p$ ) from each experimental unit. Suppose one is interested in pairwise comparison of treatment mean vectors. It is possible to carry out  $p$  separate univariate analyses of variance, one for each of the  $p$  characteristics considered, but we sacrifice some power in not making use of the correlations among the  $p$  variates. If a multivariate ( $p$ -dimensional) analysis of variance is performed, we now have two different kinds of multiple comparison problems to be considered: (1) Which populations differ? (2) With respect to which of the  $p$  characteristics do the populations differ? The latter problem does not, of course, occur in the univariate ( $p=1$ ) situation.

The approximate Bayes rule developed in the beginning of the present section can be used to give a procedure for constructing simultaneous interval estimates for all the true mean differences, using their asymptotic joint posterior distribution.

Other research topics related to the present Bayesian development of the multiple comparisons problem that are worth exploring are: ranking the main effects in additive analysis of variance models, multiple comparisons of regression functions, multiple comparisons of the tetrad differences or interactions in two-way balanced designs, pairwise comparisons of tetrad differences.

## LIST OF REFERENCES

## REFERENCES

- Anderson, V. L. and McLean, R. A. (1974). Design of Experiments. Marcel Dekker, Inc., New York.
- Ahlin, A. C. (1962). On error bounds for Gaussian cubature. SIAM Rev., 4, 25-39.
- Audo, A. and Kaufmans, G. M. (1965). Bayesian analysis of the independent multivariate normal process-neither mean nor precision known. J. Amer. Statist. Assoc., 60, 347-358.
- Balaam, L. N. (1963). Multiple comparisons - a sampling experiment. Austral. Jour. Statist., 5, 62-85.
- Bancroft, T. A. (1968). Topics in Intermediate Statistical Methods, Vol. 1. Iowa State Univ. Press, Ames, Iowa.
- Bickel, P. J. and Doksum, K. A. (1977). Mathematical Statistics: Basic Ideas and Selected Topics. Holden-Day, Inc., San Francisco.
- Bleistein, N. and Handelsman, R. A. (1975). Asymptotic Expansions of Integrals. Holt, Rinehart and Winston, New York.
- Boardmann, T. J. and Moffitt, D. R. (1971). Graphical Monte Carlo Type I error rates for multiple comparison procedures. Biometrics, 27, 738-744.
- Box, G. E. P. and Tiao, G. C. (1973). Bayesian Inference in Statistical Analysis. Addison-Wesley, Reading, Mass.
- Bradu, D. and Gabriel, K. R. (1974). Simultaneous statistical inference on interactions in two-way analysis of variance. J. Amer. Statist. Assoc., 69, 428-436.
- Byrd, P. F. and Friedman, M. D. (1971). Handbook of Elliptic Integrals for Engineers and Scientists. Springer-Verlag, New York.
- Carmer, S. G. and Swanson, M. R. (1971). Detection of differences between means: A Monte Carlo study of five pairwise multiple comparisons procedures. Agronomy Journal, 63, 940-945.



- Carmer, S. G. and Swanson, M. R. (1973). Evaluation of ten pairwise multiple comparisons procedures by Monte Carlo methods. J. Amer. Statist. Assoc., 68, 66-74.
- Chawla, M. M. (1968). On the estimation of errors of Gaussian cubature formulas, SIAM J. Numer. Anal., 5, 172-181.
- Chen, T. C. (1960). Multiple Comparisons of Population Means, MS Thesis, Iowa State Univ., Ames, Iowa.
- Conte, S. D. (1965). Elementary Numerical Analysis - An Algorithmic Approach. McGraw-Hill Book Co., New York.
- Davis, P. J. and Rabinowitz, P. (1975). Methods of Numerical Integration. Academic Press, New York.
- DeGroot, M. H. (1970). Optimal Statistical Decisions. McGraw-Hill, Inc., New York.
- Ditkin, V. A. and Prudnikov, A. P. (1965). Integral Transforms and Operational Calculus. Translated by D. E. Brown. Pergamon Press, New York.
- Dixon, D. O. and Duncan, D. B. (1975). Minimum Bayes risk t-intervals for multiple comparisons. J. Amer. Statist. Assoc., 70, 822-831.
- Duncan, D. B. (1947). Significance Tests for Differences Between Ranked Variates Drawn from Normal Populations. Iowa State College Ph.D. Thesis, 117 pp.
- Duncan, D. B. (1951). A significance test for differences between ranked treatments in an analysis of variance. Virginia Journal of Science, 2, 171-189.
- Duncan, D. B. (1955). Multiple range and multiple F-tests. Biometrics, 11, 1-42.
- Duncan, D. B. (1961). Bayes rule for a common multiple comparisons problem and related Student t-problems. Ann. Math. Statist., 32, 1013-1033.
- Duncan, D. B. (1965). A Bayesian approach to multiple comparisons. Technometrics, 7, 171-222.
- Duncan, D. B. (1975). t-tests and intervals for comparisons suggested by the data. Biometrics, 31, 339-359.
- Dunn, O. J. (1961). Multiple comparisons among means. J. Amer. Statist. Assoc., 56, 52-64.
- Federer, W. T. (1955). Experimental Design, Theory and Application. MacMillan & Co., New York.

- Feller, W. (1968). An Introduction to Probability Theory and Its Application, Vol. I. John Wiley and Sons, Inc., New York.
- Fisher, R. A. (1951). The Design of Experiments. 6th ed. Oliver and Boyd, London.
- Fisher, R. A. (1959). Statistical Methods and Scientific Inference. Oliver and Boyd, London.
- Gabriel, K. R. (1969). Simultaneous test procedures - Some theory of multiple comparisons. Ann. Math. Statist., 40, 224-250.
- Gabriel, K. R. (1978). A simple method of multiple comparisons of means. J. Amer. Statist. Assoc., 73, 724-729.
- Gill, J. L. (1973). Current status of multiple comparisons of means in designed experiments. Jour. Dairy Sci., 56, 973-977.
- Graybill, F. A. (1976). Theory and Application of the Linear Model. Duxbury Press, Mass.
- Handscomb, D. C. (1966). Methods of Numerical Approximation. Pergamon Press, New York.
- Harter, H. L. (1957). Error rates and sample sizes for range tests in multiple comparisons. Biometrics, 13, 511-536.
- Harter, H. L. (1970). Multiple comparison procedures for interactions. Amer. Statist., 24, 30-32.
- Hicks, C. R. (1973). Fundamental Concepts in the Design of Experiments. 2nd ed., Holt, Rinehart and Winston, Inc., New York.
- Hill, M. (1965). Inference about variance components in the one-way model. J. Amer. Statist. Assoc., 60, 806-825.
- Hochberg, Y. (1975). An extension of the T-method to general unbalanced models of fixed effects. J. Roy. Statist. Soc. Series B, 37, 426-433.
- Hochberg, Y. (1976). A modification of the T-method of multiple comparisons for a one-way layout with unequal variances. J. Amer. Statist. Assoc., 71, 200-203.
- Johnson, D. E. and Graybill, F. A. (1972). An analysis of a two-way model with interaction and no replication. J. Amer. Statist. Assoc., 67, 862-868.
- Johnson, L. W. and Riess, R. D. (1977). Numerical Analysis. Addison-Wesley Publishing Co., Menlo Park, Calif.

- Johnson, R. A. (1967). An asymptotic expansion for posterior distributions. Ann. Math. Statist., 38, 1899-1906.
- Johnson, R. A. (1970). Asymptotic expansions associated with posterior distributions. Ann. Math. Statist., 41, 851-864.
- Karlin, S. (1968). Total Positivity. Stanford University Press, Stanford, California.
- Keselman, H. J. and Rogan, J. C. (1978). A comparison of the modified-Tukey and Scheffé methods of multiple comparisons for pairwise contrasts. J. Amer. Statist. Assoc., 73, 47-52.
- Keselman, H. J., Games, P. A. and Rogan, J. C. (1979). An addendum to "A Comparison of the Modified-Tukey and Scheffé methods of multiple comparisons for pairwise contrasts". J. Amer. Statist. Assoc., 74, 626-627.
- Keuls, M. (1952). The use of the Studentized range in connection with an analysis of variance. Euphytica, 1, 112-122.
- Kramer, C. Y. (1956). Extension of multiple range tests to group means with unequal numbers of replications. Biometrics, 12, 309-310.
- Kramer, C. Y. (1957). Extension of multiple range tests to group correlated adjusted means. Biometrics, 13, 13-18.
- Kurtz, T. E., Link, R. F., Tukey, J. W. and Wallace, D. L. (1965). Short-cut multiple comparisons for balanced single and double classifications: Part I, Results. Technometrics, 7, 95-161.
- Lehmann, E. L. (1959). Testing Statistical Hypothesis. John Wiley and Sons, New York.
- Lehmann, E. L. (1957). A theory of some multiple decision problems, I. Ann. Math. Statist., 28, 1-25.
- Lindman, H. R. (1974). Analysis of Variance in Complex Experimental Designs. W. H. Freeman and Co., San Francisco.
- McCormick, J. M. and Salvadori, M. G. (1964). Numerical Methods in Fortran. Prentice-Hall, Inc., New Jersey.
- Miller, R. G. (1966). Simultaneous Statistical Inference. McGraw-Hill Inc., New York.
- Miller, R. G. (1977). Developments in multiple comparisons 1966-1977. J. Amer. Statist. Assoc., 72, 779-788.

- Mustard, D., Lyness, J. N. and Blatt, J. M. (1963). Numerical quadrature in  $n$  dimensions. Computer Journal, 6, 75-87.
- Newman, D. (1939). The distribution of the range in samples from a normal population, expressed in terms of an independent estimate of the standard deviation. Biometrics, 31, 20-30.
- O'Neill, R. and Wetherill, G. B. (1971). The present state of multiple comparisons methods. J. Roy. Statist. Soc. Series B, 33, 218-241.
- Ostle, B. (1963). Statistics in Research. 2nd Ed. The Iowa State University Press, Ames, Iowa.
- Raiffa, H. and Schlaifer, R. (1961). Applied Statistical Decision Theory. Cambridge: Harvard Grad. School of Business Administration.
- Rice, J. R. (1964). The Approximation of Functions, Vol. I; Linear Theory. Addison-Wesley, Reading, Mass.
- Scheffé, H. (1953). A method for judging all contrasts in the analysis of variance. Biometrika, 40, 87-104.
- Scheffé, H. (1959). The Analysis of Variance. John Wiley & Sons, Inc., New York.
- Seale, S. R. (1971). Linear Models. John Wiley & Sons, Inc. New York.
- Spjotvill, E. and Stoline, M. R. (1973). An extension of the T-method of multiple comparison to include the cases with unequal sample size. J. Amer. Statist. Assoc., 68, 975-978.
- Stenger, F. (1966). Error bounds for the evaluation of integrals by repeated Gauss-type formulae, Numer. Math., 9, 200-213.
- Stoline, M. R. and Ury, H. K. (1979). Table of the Studentized maximum modulus distribution and an application to multiple comparisons among means. Technometrics, 21, 87-94.
- Stroud, A. H. and Secrest, D. (1966). Gaussian Quadrature Formulas. Prentice-Hall, Inc., New Jersey.
- Stroud, A. H. (1971). Approximate Calculation of Multiple Integrals. Prentice-Hall, Inc., New Jersey.
- Tamhane, A. C. (1979). A comparison of procedures for multiple comparisons of means with unequal variances. J. Amer. Statist. Assoc., 74, 471-480.
- Thomas, D. A. (1973). Multiple comparisons among means - a review. Statistician, 22, 16-42.

- Thomas, D. A. (1974). Error rates in multiple comparisons among means - Results of a simulation exercise. J. Roy. Statist. Soc. Series C, 23, 284-294.
- Tiao, G. C. and Tan, W. Y. (1966). Bayesian analysis of random effect models in the analysis of variance, II. Effect of autocorrelated errors. Biometrika, 53, 477-492.
- Tukey, J. W. (1949). Comparing individual means in the analysis of variance. Biometrics, 5, 99-114.
- Tukey, J. W. (1951). Quick and dirty methods in statistics, Part II, Simple analyses for standard designs. Proc. Fifth Ann. Convention, Amer. Soc. for Quality Control, 189-197.
- Tukey, J. W. (1960). Conclusion vs. decisions. Technometrics, 4, 423-434.
- Ury, H. K. (1976). A comparison of four procedures for multiple comparisons among means (pairwise contrasts) for arbitrary sample size. Technometrics, 18, No. 1, 89-97.
- Ury, H. K. and Wiggins, A. D. (1976). A comparison of three procedures for multiple comparisons among means. British Jour. of Math. and Statist. Psychology, 28, 88-102.
- Waller, R. A. (1967). A Bayes Solution to the Symmetric Multiple Comparisons Problem. John Hopkins University Ph.D. Thesis, 138 pp.
- Waller R. A. and Duncan, D. B. (1969). A Bayes rule for the symmetric multiple comparisons problem, I. J. Amer. Statist. Assoc., 64, 1484-1503.
- Waller, R. A. and Duncan, D. B. (1972). Corrigenda. J. Amer. Statist. Assoc., 67, 253-255.
- Waller, R. A. and Duncan, D. B. (1974). A Bayes rule for the symmetric multiple comparisons problem, II. Ann. of the Institute of Statist. Math., 26, 247-264.
- Waller, R. A. and Kemp, K. E. (1975). Computations of Bayesian t-values for multiple comparisons. Jour. of Computation and Simulation, 4, 169-171.

## APPENDICES

## APPENDIX A

THIS APPENDIX PROVIDES T\* VALUES FOR VARIOUS LEVELS OF K, THE ERROR WEIGHT RATIO. THE PROGRAM WHICH COMPUTES THESE VALUES IS LISTED AFTER THE TABLEAU OF DATA.

ERROR-WEIGHT RATIO	T-STAR VALUE
5.000	.6360272846
10.000	.9014615963
20.000	1.1589215716
30.000	1.3053535081
50.000	1.4851958086
70.000	1.6007734702
100.000	1.7207832624
150.000	1.8540994349
200.000	1.9467111563
300.000	2.0745128132
500.000	2.2311160714
700.000	2.3316784725
1000.000	2.4361181524
2000.000	2.6330191921
3000.000	2.7446790662





## APPENDIX B

```
PROGRAM MAIN(INPUT,OUTPUT,TAPES=INPUT,TAPES=OUTPUT)
*
*      *****
*      *              *
*      * MAIN PREGRAM *
*      *              *
*      *****
*
* *****
*
* THIS PROGRAM COMPUTES THE COMPONENT BAYES RULES FOR THE
* MULTIPLE COMPARISONS PROBLEM IN A TWO-WAY BALANCED DESIGN.
* IT WILL READ DATA,PRODUCE AN ANOVA TABLE,SAMPLE CELL MEAN
* MATRIX, AND OTHER INFORMATIONAL DATA. AFTER POOLING THE
* ANOVA SUMS OF SQUARES AND DEGREES OF FREEDOMS WITH THE
* CORRESPONDING SS'S AND D.F.'S PROVIDED BY THE USER'S
* PRIOR DISTRIBUTION. THE FULL IMPLEMENTATION OF THE
* COMPONENT BAYES RULES FOR EACH OF THE TYPE I,TYPE II,
* TYPE III COMPARISONS WILL PROCEED. THIS LISTING IS
* INTERNALLY DOCUMENTED.
*
* *****
*
* DIMENSION Y(10),CMEAN(10,10),CUAR(10,10)
* DIMENSION ROWMEAN(10),COLMEAN(10)
* DIMENSION FCRT(3)
*
* GLOBAL VARIABLES      *
*
* INTEGER      ILEVEL,JLEVEL,KCELL
* INTEGER      IDECIDE,MULTI,MULT2
*
* REAL      QA,QB,QC,QE,SSA,SSB,SSC,SSE,
* !          T1,T2,T3,FI,FJ,FK,FKK,FAP,FBP,FCP,FEP,
* !          EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
* REAL      WEIG1,WEIG2,WEIG3,RMM,RMA,RMB,RMC
* REAL      SIGMA1,SIGMA2,SIGMA3
* REAL      RESULT1,RESULT2,RMFC1,RMFC2
* REAL      SSAP,SSBP,SSCP,SSEP,SSSUB,SSSUM
* REAL      TSUB,EXSUB,FSUB,SQSUB,FLAG1,FLAG2
* REAL      CONST1,CONST2,CONST3
*
* COMMON /STATS/  QA,QB,QC,QE,SSA,SSB,SSC,SSE,
* !              T1,T2,T3,FI,FJ,FK,FKK,KK
* COMMON /EXPON/  EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
* COMMON /WEIGH/  WEIG1,WEIG2,WEIG3,RMM
* COMMON /SIGMA/  SIGMA1,SIGMA2,SIGMA3
* COMMON /DRULE/  IDECIDE,RESULT1,RESULT2
* COMMON /SUBST/  TSUB,FSUB,EXSUB,SSSUB,SSSUM,SQSUB
* COMMON /MULTS/  CONST1,CONST2,CONST3,MULT1,MULT2
* COMMON /FLAGS/  FLAG1,FLAG2
*
*
* EXTERNAL      G,GS,FFO,FFOS,FFKK,FFKS,P,PP,Z
*
* LOCAL VARIABLES      *
```

```

*
*   INTEGER      INCRMNT, ILIMIT, JLIMIT, NDIU
*
*   REAL         CONST, OK1, OK2, FDDD, FMM
*   REAL         SUM, AVE, SUM2, ASUM2, BSUM2, CSUM2, GRAND,
!             MSA, MSB, MSC, MSE, FUALA, FUALB, FUALC
*   REAL         INT3D
*
*   READ IN THE DESIGN MATRIX AND # OF OBS PER CELL
*
*   READ (5,500) ILEVEL, JLEVEL, KCELL
*   READ (5,519) (FCRIT(I), I=1,3)
*
*   WRITE (6,501)
*   WRITE (6,502)
*   WRITE (6,503)
*
*   PRODUCE FLOATING POINTS TO ACHIEVE EFFICIENCY
*
*   FI=FLOAT(ILEVEL)
*   FJ=FLOAT(JLEVEL)
*   FK=FLOAT(KCELL)
*
*   COMPUTE DEGREES OF FREEDOM FOR ANOVA
*
*   QA=FI-1.
*   QB=FJ-1.
*   QC=QA*QB
*   QE=FI*FJ*(FK-1.)
*
*   COMPUTE OFFTEN-USED CONSTANTS TO SAVE COMPUTING TIME
*
*   EXPA=QA/2.-1.
*   EXPB=QB/2.-1.
*   EXPC=QC/2.-1.
*   EXPE=QE/2.-1.
*   SUMQ1=(QB+QC+QE-1.)/2.
*   SUMQ2=(QA+QC+QE-1.)/2.
*   SUMQ3=(QA+QB+QC+QE-1.)/2.
*
*   READ IN SAMPLES FOR EACH CELL
*   SAMPLE CELL MEANS ARE COMPUTED
*
*   DO 100 I=1, ILEVEL, 1
*   DO 200 J=1, JLEVEL, 1
*
*   READ (5,519) (Y(K), K=1, KCELL)
*
*   CALCULATE CELL SAMPLE MEANS
*
*   SUM=0.
*   DO 1 K=1, KCELL, 1
*   SUM=SUM+Y(K)
* 1 CONTINUE
*   AVE=SUM/FK
*
*   CALCULATE CELL ERROR VARIANCES
*
*   SUM2=0.
*   DO 2 K=1, KCELL, 1
*   SUM2=SUM2+Y(K)*Y(K)
* 2 CONTINUE
*
*   CVAR(I, J)=SUM2-FK*AVE*AVE
*   CMEAN(I, J)=AVE
*
* 200 CONTINUE
* 100 CONTINUE

```

```

*
* COMPUTE ROW AND COLUMN MEANS FOR THE TWO-WAY
*
DO 5 I=1, ILEVEL, 1
  ROWMEAN(I)=0.
5 CONTINUE
*
DO 6 J=1, JLEVEL, 1
  COLMEAN(J)=0.
6 CONTINUE
*
DO 8 I=1, ILEVEL, 1
DO 9 J=1, JLEVEL, 1
  ROWMEAN(I)=ROWMEAN(I)+CMEAN(I, J)
9 CONTINUE
  ROWMEAN(I)=ROWMEAN(I)/FJ
8 CONTINUE
*
GRAND=0.
DO 11 J=1, JLEVEL, 1
DO 12 I=1, ILEVEL, 1
  COLMEAN(J)=COLMEAN(J)+CMEAN(I, J)
12 CONTINUE
  COLMEAN(J)=COLMEAN(J)/FI
  GRAND=GRAND+COLMEAN(J)
11 CONTINUE
  GRAND=GRAND/FJ
*
* COMPUTE ALL SUMS OF SQUARES FOR THE ANOVA
*
SSA=0.
SSB=0.
SSC=0.
SSE=0.
ASUM2=0.
BSUM2=0.
CSUM2=0.
*
DO 20 I=1, ILEVEL, 1
DO 30 J=1, JLEVEL, 1
  BSUM2=BSUM2+COLMEAN(J)*COLMEAN(J)
  CSUM2=CSUM2+(CMEAN(I, J)-COLMEAN(J)-ROWMEAN(I)+GRAND)**2
  SSE=SSE+CVAR(I, J)
30 CONTINUE
  ASUM2=ASUM2+ROWMEAN(I)*ROWMEAN(I)
20 CONTINUE
  SSA=FJ*FK*(ASUM2-FI*GRAND*GRAND)
  SSB=FI*FK*(BSUM2/FI-FJ*GRAND*GRAND)
  SSC=FK*CSUM2
*
* COMPUTE THE MEAN-SQUARES FOR THE ANOVA
*
MSA=SSA/QA
MSB=SSB/QB
MSC=SSC/QC
MSE=SSE/QE
*
* COMPUTE THE F-VALUES FOR THE ANOVA
*
FUALA=MSA/MSE
FUALB=MSB/MSE
FUALC=MSC/MSE
*
* PRINT THE ANOVA TABLE FOR THE TWO-WAY
*
WRITE (6, 510)
WRITE (6, 511)
WRITE (6, 512) SSA, QA, MSA, FUALA, FCRIT(1)

```

```

WRITE (6,513) SSB,QB,MSB,FUALB,FCRIT(2)
WRITE (6,514) SSC,QC,MSC,FUALC,FCRIT(3)
WRITE (6,515) SSE,QE,MSE
WRITE (6,526)
*
DO 40 I=1,ILEVEL,1
WRITE (6,520) (CMEAN(I,J),J=1,JLEVEL),ROWMEAN(I)
40 CONTINUE
WRITE (6,527)
WRITE (6,520) (COLMEAN(J),J=1,JLEVEL),GRAND
*
* INPUT THE USER'S PRIOR INFORMATION IN TERMS OF SUMS OF SQUARES
* SQUARES ESTIMATES AND DEGREES OF FREEDOMS FROM THE PREVIOUS
* EXPERIMENTS
*
READ (5,504) SSAP,SSBP,SSCP,SSEP
READ (5,504) FAP,FBP,FCP,FEP
*
* COMBINE THE SUMS OF SQUARES FROM DATA AND FROM PRIOR
* INFORMATION BY POOLING TECHNIQUES.
*
SSA=SSA+SSAP
SSB=SSB+SSBP
SSC=SSC+SSCP
SSE=SSE+SSEP
*
QA=QA+FAP
QB=QB+FBP
QC=QC+FCP
QE=QE+FEP
*
* COMPUTE THE EXPECTATION OF THE MEAN FUNCTION
*
* THE ALGORITHM TO CALCULATE M-FUNCTION DEPENDS ON THE INTEGER
* VALUE OF QE. WHEN QE IS EVEN, SUBROUTINE RMFC1 IS USED.
* OTHERWISE, RMFC2 IS USED FOR ODD QE.
*
IF (AMOD(QE,2.).EQ.1.) GO TO 45
RMM=RMFC1(QA,QB,QC,QE)
RMC=RMFC1(QA,QB,QC+2.,QE-2.)
RMB=RMFC1(QA,QB+2.,QC,QE-2.)
RMA=RMFC1(QA+2.,QB,QC,QE-2.)
GO TO 48
*
45 RMM=RMFC2(QA/2.,QB/2.,QC/2.,QE/2.)
RMC=RMFC2(QA/2.,QB/2.,QC/2.+1.,QE/2.-1.)
RMB=RMFC2(QA/2.,QB/2.+1.,QC/2.,QE/2.-1.)
RMA=RMFC2(QA/2.+1.,QB/2.,QC/2.,QE/2.-1.)
*
* THE MATCHING CONSTANT FOR THE POSTERIOR DISTRIBUTION.
* IS D-STAR=1.0/RMM.
*
* COMPUTE THE THREE WEIGHTING FACTORS.
*
48 WEIG1=(RMM-RMC)/RMM
WEIG2=(RMC-RMB)/RMM
WEIG3=(RMC-RMA)/RMM
*
* COMPUTE THE EXPECTATIONS OF THE THREE SIGMA FUNCTIONS
*
* COMPUTE THE INTEGRAL OF SIGMA I FUNCTION
*
NDIV=32
FLAG1=1.
FLAG2=0.
TSUB=T2
SSSUB=SSB

```

```

EXSUB=EXPB
SQSUB=SUMQ1
SSSUM=SSA+SSC
FSUB=FI
*
IF (AMOD(QA,2.).EQ.0.) GO TO 150
MULT1=(INT(QA)-1)/2
CONST1=GAMMA(SUMQ3)/SQRT(FK)
SIGMA1=CONST1*INT3D(Z)/RMM
GO TO 180
*
150 MULT1=INT(QA)/2-1
CONST1=GAMMA(SUMQ1)/SQRT(FK)
OK1=2.*CONST1*(TRINT(G,NDIV)-TRINT(GS,NDIV))/SSA
IF (MULT1.EQ.0) GO TO 175
*
CONST=CONST1
DO 170 MM=1,MULT1,1
FMM=FLOAT(MM)
CONST=SQSUB*CONST
EXPC=EXPC+1.
SQSUB=SQSUB+1.
OK2=2.0*(FMM*OK1-CONST*TRINT(GS,NDIV))/SSA
OK1=OK2
170 CONTINUE
*
175 SIGMA1=OK1/RMM
EXPC=EXPC-FLOAT(MULT1)
*
* COMPUTE THE INTEGRAL OF SIGMA II FUNCTION
*
180 FLAG1=0.
FLAG2=1.
TSUB=T3
FSUB=FJ
EXSUB=EXPA
SSSUB=SSA
SSSUM=SSB+SSC
SQSUB=SUMQ2
*
IF (AMOD(QB,2.).EQ.0.) GO TO 190
MULT2=(INT(QB)-1)/2
CONST2=GAMMA(SUMQ3)/SQRT(FK)
SIGMA2=CONST2*INT3D(Z)/RMM
GO TO 250
*
190 MULT2=INT(QB)/2-1
CONST2=GAMMA(SUMQ2)/SQRT(FK)
OK1=2.*CONST2*(TRINT(G,NDIV)-TRINT(GS,NDIV))/SSB
IF (MULT2.EQ.0) GO TO 220
*
CONST=CONST2
DO 215 MM=1,MULT2,1
FMM=FLOAT(MM)
CONST=SQSUB*CONST
EXPC=EXPC+1.
SQSUB=SQSUB+1.
OK2=2.0*(FMM*OK1-CONST*TRINT(GS,NDIV))/SSB
OK1=OK2
215 CONTINUE
*
220 SIGMA2=OK1/RMM
EXPC=EXPC-FLOAT(MULT2)
*
* COMPUTE THE INTEGRAL OF SIGMA III FUNCTION
*
250 FLAG1=1.
FLAG2=1.

```

```

CONST3=GAMMA(SUMQ3)/SQRT(FK)
SIGMA3=CONST3*INT3D(Z)/RMM
*
* WRITE OUT THE POSTERIOR MEANS AND SIGMA FUNCTIONS.
*
WRITE (6,528)
WRITE (6,529) RMM,RMC,RMB,RMA
WRITE (6,530)
WRITE (6,531) WEIG1,WEIG2,WEIG3
WRITE (6,532)
WRITE (6,533) SIGMA1,SIGMA2,SIGMA3
*
* PREPARE T1,T2,T3 STATISTICS FOR MULTIPLE COMPARISONS
* FOR EACH PAIR IN EACH TYPE
*
* PERFORM TYPE I COMPARISONS FIRST
*
WRITE (6,516)
WRITE (6,523)
*
FLAG1=1.
FLAG2=0.
FSUB=FI
SSSUB=SSB
SSSUM=SSA+SSC
EXSUB=EXPB
SQSUB=SUMQ1
*
DO 50 I=1,ILEVEL,1
  JLIMIT=JLEVEL-1
  DO 51 J=1,JLIMIT,1
    INCRMNT=J+1
    DO 52 JJ=INCRMNT,JLEVEL,1
      T1=(CMEAN(I,J)-CMEAN(I,JJ))/1.414213562
      T2=(COLMEAN(J)-COLMEAN(JJ))/1.414213562
      TSUB=T2
      T3=0.
      CALL TYPEI
      WRITE (6,522) I,J,I,JJ,T1,T2,RESULT1,RESULT2,IDEcide
52    CONTINUE
51 CONTINUE
50 CONTINUE
*
* DO ALL TYPE II COMPARISONS
*
WRITE (6,517)
WRITE (6,524)
*
FLAG1=0.
FLAG2=1.
FSUB=FJ
EXSUB=EXPA
SSSUB=SSA
SSSUM=SSB+SSC
SQSUB=SUMQ2
*
DO 60 J=1,JLEVEL,1
  ILIMIT=ILEVEL-1
  DO 61 I=1,ILIMIT,1
    INCRMNT=I+1
    DO 62 II=INCRMNT,ILEVEL,1
      T1=(CMEAN(I,J)-CMEAN(II,J))/1.414213562
      T3=(ROWMEAN(I)-ROWMEAN(II))/1.414213562
      TSUB=T3
      T2=0.
      CALL TYPEII
      WRITE (6,522) I,J,II,J,T1,T3,RESULT1,RESULT2,IDEcide

```

```

62     CONTINUE
61 CONTINUE
60 CONTINUE
*
*     DO ALL TYPE III COMPARISONS
*
*     WRITE (6,518)
*     WRITE (6,521)
*     FLAG1=1.
*     FLAG2=1.
*
*     DO 70 I=1,ILIMIT,1
*     DO 71 J=1,JLEVEL,1
*       INCRMNT=I+1
*       DO 72 II=INCRMNT,ILEVEL,1
*       DO 73 JJ=1,JLEVEL,1
*         IF (JJ.EQ.J) GO TO 73
*         T1=(CMEAN(I,J)-CMEAN(II,JJ))/1.414213562
*         T2=(COLMEAN(J)-COLMEAN(JJ))/1.414213562
*         T3=(ROWMEAN(I)-ROWMEAN(II))/1.414213562
*         CALL TYPEIII
*         WRITE (6,525)I,J,II,JJ,T1,T2,T3,RESULT1,RESULT2,IDEcide
73     CONTINUE
72     CONTINUE
71 CONTINUE
70 CONTINUE
*
*     FOLLOWINGS ARE FORMAT STATEMENTS
*
500 FORMAT (1X,3I2)
501 FORMAT (1X,*THIS PROGRAM PERFORMS MONTE CARLO SIMULATIONS*)
502 FORMAT (1X,* ON THE MULTIPLE COMPARISONS PROBLEM*)
503 FORMAT (1X,* OF THE TWO-WAY CLASSIFICATION BALANCED DESIGN*)
504 FORMAT (1X,4F8.2)
510 FORMAT (//,1X,*ANOVA TABLE FOR THE TWO-WAY LAYOUT*)
511 FORMAT (6X,*SOURCE*,14X,*SS*,9X,*D.F.*,11X,*MS*,9X,
!      *F-VALUE*,8X,*F-CRITICAL*)
512 FORMAT (1X,*A MAIN EFFECTS*,F15.4,F10.1,2X,3(F14.4))
513 FORMAT (1X,*B MAIN EFFECTS*,F15.4,F10.1,2X,3(F14.4))
514 FORMAT (1X,*AB INTERSECTION*,F14.4,F10.1,2X,3(F14.4))
515 FORMAT (1X,* ERROR*,8X,F14.4,F10.1,2X,F14.4)
516 FORMAT (//,1X,*ALL PAIRWISE COMPARISONS OF TYPE I*)
517 FORMAT (//,1X,*ALL PAIRWISE COMPARISONS OF TYPE II*)
518 FORMAT (//,1X,*ALL PAIRWISE COMPARISONS OF TYPE III*)
519 FORMAT (1X,10F6.2)
520 FORMAT (1X,5F10.4,5X,F10.4)
521 FORMAT (//,1X,* COMPARISON OF  *,1X,*T1*,5X,*T2*,5X,
!      *T3*,7X,*L-1*,5X,*L-2*,6X,*DECISION*)
522 FORMAT (/,1X,1H(,I2,1H,,I2,1H),2X,1H(,I2,1H,,I2,
!      1H),2(F12.6),2(E14.6),2X,I2)
523 FORMAT (//,1X,* COMPARISON OF  *,6X,*T1*,10X,*T2*,
!      10X,*L-1*,10X,*L-2*,3X,*DECISION*)
524 FORMAT (//,1X,* COMPARISON OF  *,6X,*T1*,10X,*T3*,
!      10X,*L-1*,10X,*L-2*,3X,*DECISION*)
525 FORMAT (/,1X,1H(,I2,1H,,I2,1H),2X,1H(,I2,1H,,I2,1H),
!      3(F11.6),2(E14.6),2X,I2)
526 FORMAT (//,1X,*THE OBSERVED MEANS ARE IN MATRIX FORM*)
527 FORMAT (1X,/)
528 FORMAT (//,5X,*THE MEAN COMPONENTS INFORMATION IS GIVEN*)
529 FORMAT (10X,*RMM=*,E13.6,10X,*RMC=*,E13.6,10X,
!      *RMB=*,E13.6,10X,*RMA=*,E13.6)
530 FORMAT (//,5X,*THE THREE WEIGHTING FACTORS ARE GIVEN*)
531 FORMAT (10X,*WEIGHT#1=*,F13.8,10X,*WEIGHT#2=*,F13.8,10X,
!      *WEIGHT#3=*,F13.8)
532 FORMAT (//,5X,*THE SIGMA FUNCTIONS ARE GIVEN*)
533 FORMAT (10X,*SIGMA1=*,E13.6,10X,*SIGMA2=*,E13.6,10X,
!      *SIGMA3=*,E13.6)
STOP

```





```

*
* COMPUTE THE UPPER AND LOWER BOUNDS.
*
TUPPER=(TSTAR*SIGMA1-T2*WEIG2)/WEIG1
TLOWER=(-TSTAR*SIGMA1-T2*WEIG2)/WEIG1
*
WRITE (6,2210) TUPPER,TLOWER,FIRST
*
CBOUND=2.*FIRST/101.
CUBIC=0.
IF (T1.GE.TUPPER) GO TO 2000
IF (T1.LE.TLOWER) GO TO 2001
IDECIDE=0
RESULT1=0.
RESULT2=0.
GO TO 2100
*
* THIS BLOCK COMPUTES AN UPPER AND LOWER BOUNDS FOR L'S.
*
2000 IF (T2.LT.0.) GO TO 2010
    IFIX=1
    GO TO 2002
*
2001 IF (T2.GT.0.) GO TO 2020
    IFIX=-1
*
2002 KK=-1
    FKK=FLOAT(KK)
*
    IF (AMOD(QA,2.).EQ.0.) GO TO 2003
    CONNS=CONST1/FK/(SUMQ3-1.)
    CUBIC=CUBIC+CONNS*INT3D(PP)/RMM/2.506628274631
    GO TO 2004
*
2003 CONNS=CONST1/FK/(SUMQ1-1.)
    OKKK1=2.*CONNS*(TRINT(FFKK,NDIV)-TRINT(FFKKS,NDIV))/SSA
    IF (MULT1.EQ.0.) GO TO 2008
*
    DO 2005 MM=1,MULT1,1
    FMM=FLOAT(MM)
    CONNS=(SQSUB-1.)*CONNS
    EXPC=EXPC+1.
    SQSUB=SQSUB+1.
    OKKK2=2.*(FMM*OKKK1-CONNS*TRINT(FFKKS,NDIV))/SSA
    OKKK1=OKKK2
2005 CONTINUE
*
2008 CUBIC=CUBIC+OKKK1/RMM/2.506628274631
    EXPC=EXPC-FLOAT(MULT1)
    SQSUB=SUMQ1
2004 SCREEN=CBOUND-1.9603960396*CUBIC*FLOAT(IFIX)
    WRITE (6,2220) CUBIC, SCREEN
*
    IF (IFIX.EQ.1) GO TO 2007
    IF (SCREEN.GT.0.) GO TO 2020
    GO TO 2009
2007 IF (SCREEN.LT.0.) GO TO 2010
2009 IDECIDE=1*IFIX
    RESULT1=1.000
    RESULT2=1.000
    GO TO 2100
*
* THIS BLOCK COMPUTES L1-FUNCTION TO MAKE DECISION D-0 OR D+.
*
2010 CFUNC=0.72139177*FIRST
    IFIX=1
    GO TO 2030
*

```

```

*      THIS BLOCK COMPUTES L2-FUNCTION TO MAKE DECISION D=0 OR D=.
*
2020 CFUNC=-0.72139177*FIRST
      IFIX=-1
*
*      THIS BLOCK COMPUTES THE INFINITE SERIES IN THE L-FUNCTION.
*      THE FIRST TERM OF THE SERIES WHEN KK=0 IS COMPUTED
*      SEPARATELY FROM THE REST OF THE TERMS.
*      DO-LOOP IS USED FOR KK=1,2,3,.....
*
*      THE RECURSIVE FORMULA DEPENDS ON THE INTEGER VALUE OF QA.
*
*      THE BLOCK DOES RECURSIVE METHOD FOR BOTH EVEN OR ODD QA.
*
2030 IF (AMOD(QA,2.).EQ.0.) GO TO 2034
      CONST=CONST1/1.7724538509
      OK1=CONST*INT3D(P)/RMM
*
      UPDATE=SUMQ3*CONST*FK
      DO 2032 KK=1,20,1
        FKK=FLOAT(KK)
        CONST=UPDATE
        OK1=CONST*INT3D(PP)
        UPDATE=UPDATE*FK*(SUMQ3+FKK)/(2.0*FKK+1.)
        SERIS=SERIS+OK1/RMM
        WRITE (6,2200) OK1/RMM, KK
        IF (SERIS.GE.CFUNC) GO TO 2070
        IF (OK1/RMM.LE.BOUND) GO TO 2075
2032 CONTINUE
      GO TO 2075
*
2034 CONST=CONST1/1.7724538509
      OK1=2.*CONST*(TRINT(FFD,NDIV)-TRINT(FFOS,NDIV))/SSA
      IF (MULT1.EQ.0) GO TO 2038
*
      DO 2035 MM=1,MULT1,1
        FMM=FLOAT(MM)
        CONST=SQSUB*CONST
        EXPC=EXPC+1.
        SQSUB=SQSUB+1.
        OK2=2.0*(FMM*OK1-CONST*TRINT(FFOS,NDIV))/SSA
        OK1=OK2
2035 CONTINUE
*
2038 SERIS=SERIS+OK1/RMM
      WRITE (6,2200) OK1/RMM
      EXPC=EXPC-FLOAT(MULT1)
      SQSUB=SUMQ1
      UPDATE=CONST1*FK*SUMQ1/1.7724538509
*
      DO 2045 KK=1,20,1
        FKK=FLOAT(KK)
        CONST=UPDATE
        OK1=2.*CONST*(TRINT(FFKK,NDIV)-TRINT(FFKKS,NDIV))/SSA
        IF (MULT1.EQ.0) GO TO 2043
*
      DO 2040 MM=1,MULT1,1
        FMM=FLOAT(MM)
        CONST=(SQSUB+FMM)*CONST
        SQSUB=SQSUB+1.0
        EXPC=EXPC+1.0
        OK2=2.0*(FMM*OK1-CONST*TRINT(FFKKS,NDIV))/SSA
        OK1=OK2
2040 CONTINUE
*
2043 EXPC=EXPC-FLOAT(MULT1)
      SQSUB=SUMQ1
      UPDATE=(SQSUB+FMM)*FK*UPDATE/(2.0*FMM+1.0)

```

```
SERIS=SERIS+OK1/RMM
WRITE (6,2200) OK1/RMM, KK
IF (SERIS.GE.CFUNC) GO TO 2070
IF (OK1/RMM.LE.BOUND) GO TO 2075
2045 CONTINUE
GO TO 2075
*
2070 IDECIDE=0
GO TO 2080
*
2075 IDECIDE=1*IFIX
*
2080 IF (IFIX.NE.1) GO TO 2085
RESULT1=FIRST-1.386209332*SERIS
RESULT2=0.
GO TO 2100
*
2085 RESULT1=0.
RESULT2=FIRST+1.386209332*SERIS
*
2200 FORMAT(5X, F18.12, I3)
2210 FORMAT(//, 5X, *HI=*, F13.6, 5X, *LO=*, F13.6, 5X, *P-MEAN*, F13.8)
2220 FORMAT(5X, *CUBIC TERM=*, E13.6, 5X, *SCREEN=*, E13.6)
2100 RETURN
END
```

## SUBROUTINE TYPEII

```
*****  
*                                 *  
*          TYPE II              *  
*                                 *  
*****
```

```
*****
```

```
THIS SUBROUTINE COMPUTES THE COMPONENT BAYES RULES OF  
TYPE II PROBLEMS. TWO SCREENING PROCEDURES ARE UTILIZED  
TO AVOID THE COMPUTATION OF THE CRITICAL EQUATIONS  
DEFINING THE BAYES DECISION. THE COMPONENT BAYES RULE  
IS PRINTED FOR EACH COMPARISON PROBLEM.
```

```
*****
```

## GLOBAL VARIABLES \*

```
INTEGER      IDECIDE, MULT1, MULT2
```

```
REAL         QA, QB, QC, QE, SSA, SSB, SSC, SSE,  
!           T1, T2, T3, FI, FJ, FK, FKK,  
!           EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3  
REAL         WEIG1, WEIG2, WEIG3, RMM, RMA, RMB, RMC  
REAL         SIGMA1, SIGMA2, SIGMA3, RESULT1, RESULT2  
REAL         TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM  
REAL         CONST1, CONST2, CONST3, FLAG1, FLAG2
```

```
COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,  
!             T1, T2, T3, FI, FJ, FK, FKK, KK
```

```
COMMON /EXPON/ EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
```

```
COMMON /WEIGH/ WEIG1, WEIG2, WEIG3, RMM
```

```
COMMON /SIGMA/ SIGMA1, SIGMA2, SIGMA3
```

```
COMMON /DRULE/ IDECIDE, RESULT1, RESULT2
```

```
COMMON /SUBST/ TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB
```

```
COMMON /MULTS/ CONST1, CONST2, CONST3, MULT1, MULT2
```

```
COMMON /FLAGS/ FLAG1, FLAG2
```

```
EXTERNAL     FFO, FFO5, FFKK, FFKKS, P, PP
```

## LOCAL VARIABLES \*

```
REAL         TSTAR, BOUND, CFUNC, FIRST, SERIS,  
!           OK1, OK2, UPDATE, CONST, FMM, FODD  
REAL         CBOUND, CUBIC, SCREEN  
REAL         OKK1, OKK2, CONNS, INT3D
```

```
INTEGER      NDIU, IFIX
```

```
THE SUBPROGRAM WILL COMPARE T1 WITH ITS COMPUTED UPPER  
AND LOWER BOUNDS FOR A SHORTCUT PROCEDURE. IF T1 DOES NOT  
FALL IN-BETWEEN THE BOUNDS, EITHER L1 OR L2 FUNCTION  
WILL BE COMPUTED. THE RESULTING DECISION WILL BE TAKEN.
```

```
THE PROGRAM WILL GIVE THE DECISION AND L1 OR L2 VALUE.
```

```
AT C-LEVEL=100, THE T-STAR VALUE IS 1.7207832624.
```

```
TSTAR=1.72078326
```

```
NDIU=32
```

```
BOUND=1.E-6
```

```
SERIS=0.0
```

```
WE COMPUTE THE FIRST TERM OF THE L-FUNCTION.
```

```
FIRST=T1*WEIG1+T3*WEIG3
```

```

*
*   COMPUTE THE UPPER AND LOWER BOUNDS.
*
*   TUPPER=(TSTAR*SIGMA2-T3*WEIG3)/WEIG1
*   TLOWER=(-TSTAR*SIGMA2-T3*WEIG3)/WEIG1
*
*   WRITE (6,2210) TUPPER,TLOWER,FIRST
*
*   CBOUND=2.*FIRST/101.
*   CUBIC=0.
*   IF (T1.GE.TUPPER) GO TO 2000
*   IF (T1.LE.TLOWER) GO TO 2001
*   IDECIDE=0
*   RESULT1=0.
*   RESULT2=0.
*   GO TO 2100
*
*   THIS BLOCK COMPUTES AN UPPER AND LOWER BOUNDS FOR L"S.
*
*   2000 IF (T3.LT.0.) GO TO 2010
*       IFIX=1
*       GO TO 2002
*
*   2001 IF (T3.GT.0.) GO TO 2020
*       IFIX=-1
*
*   2002 KK=-1
*       FKK=FLOAT(KK)
*
*       IF (AMOD(QB,2.).EQ.0.) GO TO 2003
*       CONNS=CONST2/FK/(SUMQ3-1.)
*       CUBIC=CUBIC+CONNS*INT3D(PP)/RMM/2.506628274631
*       GO TO 2004
*
*   2003 CONNS=CONST2/FK/(SUMQ2-1.)
*       OKK1=2.*CONNS*(TRINT(FFKK,NDIU)-TRINT(FFKKS,NDIU))/SSB
*       IF (MULT2.EQ.0.) GO TO 2008
*
*       DO 2005 MM=1,MULT2,1
*           FMM=FLOAT(MM)
*           CONNS=(SQSUB-1.)*CONNS
*           EXPC=EXPC+1.
*           SQSUB=SQSUB+1.
*           OKK2=2.*(FMM*OKK1-CONNS*TRINT(FFKKS,NDIU))/SSB
*           OKK1=OKK2
*   2005 CONTINUE
*
*   2008 CUBIC=CUBIC+OKK1/RMM/2.506628274631
*       EXPC=EXPC-FLOAT(MULT2)
*       SQSUB=SUMQ2
*   2004 SCREEN=CBOUND-1.9603960396*CUBIC*FLOAT(IFIX)
*       WRITE (6,2220) CUBIC, SCREEN
*       IF (IFIX.EQ.1) GO TO 2007
*       IF (SCREEN.GT.0.) GO TO 2020
*       GO TO 2009
*   2007 IF (SCREEN.LT.0.) GO TO 2010
*   2009 IDECIDE=1*IFIX
*       RESULT1=1.0
*       RESULT2=1.0
*       GO TO 2100
*
*   THIS BLOCK COMPUTES L1-FUNCTION TO MAKE DECISION D=0 OR D+.
*
*   2010 CFUNC=0.72139177*FIRST
*       IFIX=1
*       GO TO 2030
*
*   THIS BLOCK COMPUTES L2-FUNCTION TO MAKE DECISION D=0 OR D-.

```

```

*
2020 CFUNC=-0.72139177*FIRST
      IFIX=-1
*
* THIS BLOCK COMPUTES THE INFINITE SERIES IN THE L-FUNCTION.
* THE FIRST TERM OF THE SERIES WHEN KK=0 IS COMPUTED
* SEPARATELY FROM THE REST OF THE TERMS.
* DO-LOOP IS USED FOR KK=1,2,3,....
*
* THE BLOCK DOES THE RECURSIVE FORMULA FOR BOTH EVEN OR ODD QA.
*
2030 IF (AMOD(QB,2.).EQ.0.) GO TO 2034
      CONST=CONST2/1.7724538509
      OK1=CONST*INT3D(P)/RMM
*
      UPDATE=SUMQ3*CONST*FK
      DO 2032 KK=1,20,1
        FKK=FLOAT(KK)
        CONST=UPDATE
        OK1=CONST*INT3D(PP)
        UPDATE=UPDATE*FK*(SUMQ3+FKK)/(2.*FKK+1.)
        SERIS=SERIS+OK1/RMM
        WRITE (6,2200) OK1/RMM, KK
        IF (SERIS.GE.CFUNC) GO TO 2070
        IF (OK1/RMM.LE.BOUND) GO TO 2075
2032 CONTINUE
      GO TO 2075
*
*
2034 CONST=CONST2/1.7724538509
      OK1=2.*CONST*(TRINT(FFO,NDIV)-TRINT(FFOS,NDIV))/SSB
      IF (MULT2.EQ.0) GO TO 2038
*
      DO 2035 MM=1,MULT2,1
        FMM=FLOAT(MM)
        CONST=SQSUB*CONST
        EXPC=EXPC+1.
        SQSUB=SQSUB+1.
        OK2=2.*(FMM*OK1-CONST*TRINT(FFOS,NDIV))/SSB
        OK1=OK2
2035 CONTINUE
*
2038 SERIS=SERIS+OK1/RMM
      WRITE (6,2200) OK1/RMM
      EXPC=EXPC-FLOAT(MULT2)
      SQSUB=SUMQ2
      UPDATE=CONST2*FK*SUMQ2/1.7724538509
*
      DO 2045 KK=1,20,1
        FKK=FLOAT(KK)
        CONST=UPDATE
        OK1=2.0*CONST*(TRINT(FFKK,NDIV)-TRINT(FFKKS,NDIV))/SSB
        IF (MULT2.EQ.0) GO TO 2043
*
      DO 2040 MM=1,MULT2,1
        FMM=FLOAT(MM)
        CONST=(SQSUB+FKK)*CONST
        SQSUB=SQSUB+1.0
        EXPC=EXPC+1.0
        OK2=2.*(FMM*OK1-CONST*TRINT(FFKKS,NDIV))/SSB
        OK1=OK2
2040 CONTINUE
*
2043 EXPC=EXPC-FLOAT(MULT2)
      SQSUB=SUMQ2
      UPDATE=(SQSUB+FKK)*FK*UPDATE/(2.0*FKK+1.0)
      SERIS=SERIS+OK1/RMM
      WRITE (6,2200) OK1/RMM, KK

```

```
IF (SERIS.GE.CFUNC) GO TO 2070
IF (OK1/RMM.LE.BOUND) GO TO 2075
2045 CONTINUE
GO TO 2075
*
2070 IDECIDE=0
GO TO 2080
*
2075 IDECIDE=1*IFIX
*
2080 IF (IFIX.NE.1) GO TO 2085
RESULT1=FIRST-1.386209332*SERIS
RESULT2=0.
GO TO 2090
*
2085 RESULT1=0.
RESULT2=FIRST+1.386209332*SERIS
*
2090 IF (AMOD(QB,2.).EQ.0.) GO TO 2100
EXPC=EXPC+0.5
SUMQ2=SUMQ2+0.5
*
2200 FORMAT (5X,F18.8,I3)
2210 FORMAT(//,5X,*HI=*,F13.6,5X,*LO=*,F13.6,5X,*P-MEAN*,F13.8)
2220 FORMAT(5X,*CUBIC TERM=*,E13.6,5X,*SCREEN=*,E13.6)
2100 RETURN
END
```

```

*
*
* *****
*
*
*     *
*     *
*     *
*     *
* *****
*
*
* REAL FUNCTION G(U,V)
*
*   GLOBAL   V A R I A B L E S
*           *
*
* REAL      QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!          T1, T2, T3, FI, FJ, FK, FKK
* REAL      EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* REAL      TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM
*
* COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!              T1, T2, T3, FI, FJ, FK, FKK, KK
* COMMON /EXPON/ EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* COMMON /SUBST/ TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB
*
*   G=0.
*   IF (U.EQ.0.) GO TO 320
*   G=(SQRT(1.-U*(1.-1./FSUB)-U/FSUB)*U**EXSUB*U**EXPC)/
!     ((SSSUB*U+SSC*U+SSE)/2.)**SQSUB
320 RETURN
*   END
*
*
* REAL FUNCTION GS(U,V)
*
*   GLOBAL   V A R I A B L E S
*           *
*
* REAL      QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!          T1, T2, T3, FI, FJ, FK, FKK
* REAL      EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* REAL      TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM
*
* COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!              T1, T2, T3, FI, FJ, FK, FKK, KK
* COMMON /EXPON/ EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* COMMON /SUBST/ TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB
*
*   GS=0.
*   IF (U.EQ.0.) GO TO 330
*   GS=(SQRT(1.-((1.-1./FSUB)*U-U/FSUB)*U**EXSUB*U**EXPC)/
!     ((SSSUB*U+SSSUM*U+SSE)/2.)**SQSUB
330 RETURN
*   END
*
*
* REAL FUNCTION FFO(U,V)
*
*   GLOBAL   V A R I A B L E S
*           *
*
* REAL      QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!          T1, T2, T3, FI, FJ, FK, FKK
* REAL      EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* REAL      TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM
*
* COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!              T1, T2, T3, FI, FJ, FK, FKK, KK
* COMMON /EXPON/ EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* COMMON /SUBST/ TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB
*
*   LOCAL   V A R I A B L E S
*           *

```



```

*
*   REAL          TEMP
*
  FFO=0.
  IF (U.EQ.0.) GO TO 320
  TEMP=(1.-(1.-1./FSUB)*U-U/FSUB)
  FFO=(SQRT(TEMP)*U**EXSUB*U**EXPC)/
  !   (((1.-U)*T1+(U-U)*TSUB)**2*FK/TEMP+
  !   SSSUB*U+SSC*U+SSE)/2.)**SQSUB
320 RETURN
  END
*
*
*   REAL FUNCTION FFK(U,U)
*
*   GLOBAL VARIABLES   *
*
  REAL          QA,QB,QC,QE,SSA,SSB,SSC,SSE,
  !             T1,T2,T3,FI,FJ,FK,FKK
  REAL          EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
  REAL          TSUB,EXSUB,FSUB,SQSUB,SSSUB,SSSUM
*
  COMMON /STATS/ QA,QB,QC,QE,SSA,SSB,SSC,SSE,
  !             T1,T2,T3,FI,FJ,FK,FKK,KK
  COMMON /EXPON/ EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
  COMMON /SUBST/ TSUB,FSUB,EXSUB,SSSUB,SSSUM,SQSUB
*
*   LOCAL VARIABLES   *
*
  REAL          TEMP1,TEMP2
*
  FFK=0.
  IF (U.EQ.0.) GO TO 325
  TEMP1=(1.-U)*T1+(U-U)*TSUB
  TEMP2=1.-(1.-1./FSUB)*U-U/FSUB
  FFK=(TEMP1**(2*K)*U**EXSUB*U**EXPC)/(TEMP2**(FKK-.5))/
  !   ((TEMP1**2*FK/TEMP2+SSSUB*U+SSC*U+SSE)/2.)**(SQSUB+FKK)
325 RETURN
  END
*
*
*   REAL FUNCTION FFOS(U,U)
*
*   GLOBAL VARIABLES   *
*
  REAL          QA,QB,QC,QE,SSA,SSB,SSC,SSE,
  !             T1,T2,T3,FI,FJ,FK,FKK
  REAL          EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
  REAL          TSUB,EXSUB,FSUB,SQSUB,SSSUB,SSSUM
*
  COMMON /STATS/ QA,QB,QC,QE,SSA,SSB,SSC,SSE,
  !             T1,T2,T3,FI,FJ,FK,FKK,KK
  COMMON /EXPON/ EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
  COMMON /SUBST/ TSUB,FSUB,EXSUB,SSSUB,SSSUM,SQSUB
*
*   LOCAL VARIABLES   *
*
  REAL          TEMP
*
  FFOS=0.
  IF (U.EQ.0.) GO TO 330
  TEMP=1.-(1.-1./FSUB)*U-U/FSUB
  FFOS=(SQRT(TEMP)*U**EXSUB*U**EXPC)/
  !   (((1.-U)*T1+(U-U)*TSUB)**2*FK/TEMP
  !   +SSSUB*U+SSSUM*U+SSE)/2.)**SQSUB
330 RETURN

```

```

END
*
*
*
REAL FUNCTION FFKKS(U,U)
*
*   GLOBAL VARIABLES   *
*
REAL      QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!         T1, T2, T3, FI, FJ, FK, FKK
REAL      EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
REAL      TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM

COMMON /STATS/  QA, QB, QC, QE, SSA, SSB, SSC, SSE,
!              T1, T2, T3, FI, FJ, FK, FKK, KK
COMMON /EXPON/  EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
COMMON /SUBST/  TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB

*
*   LOCAL VARIABLES   *
*
REAL      TEMP1, TEMP2

FFKKS=0.
IF (U.EQ.0.) GO TO 335
TEMP1=(1.-U)*T1+(U-U)*TSUB
TEMP2=(1.-(1.-1./FSUB)*U-U/FSUB)
FFKKS=(TEMP1**(2*KK)*U**EXSUB*U**EXPC)/(TEMP2**(FKK-.5))/
!      ((TEMP1**2*FK/TEMP2+SSSUB*U+SSSUM*U+SSE)/2.)*(SQSUB+FKK)
335 RETURN
END

```

## SUBROUTINE TYPEIII

```

*
*
*          *****
*          *
*          *   TYPE III   *
*          *
*          *****
*
* *****
*
* THIS SUBROUTINE COMPUTES THE COMPONENT BAYES RULES OF
* TYPE III PROBLEMS. TWO SCREENING PROCEDURES ARE UTILIZED
* TO AVOID THE COMPUTATION OF THE CRITICAL EQUATIONS
* DEFINING THE BAYES DECISION. THE COMPONENT BAYES RULE
* IS PRINTED FOR EACH COMPARISON PROBLEM.
*
* *****
*
* G L O B A L   V A R I A B L E S .   *
*
* INTEGER      IDECIDE, MULT1, MULT2
*
* REAL         QA, QB, QC, QE, SSA, SSB, SSC, SSE,
* !           T1, T2, T3, FI, FJ, FK, FKK,
* !           EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* REAL         WEIG1, WEIG2, WEIG3, RMM, RMA, RMB, RMC
* REAL         SIGMA1, SIGMA2, SIGMA3, RESULT1, RESULT2
* REAL         TSUB, EXSUB, FSUB, SQSUB, SSSUB, SSSUM
* REAL         CONST1, CONST2, CONST3, FLAG1, FLAG2
*
* COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,
* !           T1, T2, T3, FI, FJ, FK, FKK, KK
* COMMON /EXPON/ EXPA, EXPB, EXPC, EXPE, SUMQ1, SUMQ2, SUMQ3
* COMMON /WEIGH/ WEIG1, WEIG2, WEIG3, RMM
* COMMON /SIGMA/ SIGMA1, SIGMA2, SIGMA3
* COMMON /DRULE/ IDECIDE, RESULT1, RESULT2
* COMMON /SUBST/ TSUB, FSUB, EXSUB, SSSUB, SSSUM, SQSUB
* COMMON /MULTS/ CONST1, CONST2, CONST3, MULT1, MULT2
* COMMON /FLAGS/ FLAG1, FLAG2
*
* EXTERNAL     P, PP
*
* L O C A L   V A R I A B L E S   *
*
* REAL         TSTAR, BOUND, CFUNC, SERIS, INT3D,
* !           OK, UPDATE, CONST, FIRST
* REAL         CBOUND, CUBIC, SCREEN, CONNS
*
* INTEGER      IFIX
*
* THE SUBPROGRAM WILL COMPARE T1 WITH ITS COMPUTED UPPER
* AND LOWER BOUNDS FOR A SHORTCUT PROCEDURE. IF T1 DOES NOT
* FALL IN-BETWEEN THE BOUNDS, EITHER L1 OR L2 FUNCTION
* WILL BE COMPUTED. THE RESULTING DECISION WILL BE TAKEN.
*
* THE PROGRAM WILL GIVE THE DECISION AND L1 OR L2 VALUE.
*
* AT C-LEVEL=100, THE T-STAR VALUE IS 1.7207832624.
*
* TSTAR=1.72078326
* BOUND=1.E-6
* SERIS=0.0
*
* WE COMPUTE THE FIRST TERM OF THE L-FUNCTION.
*
* FIRST=T1*WEIG1+T2*WEIG2+T3*WEIG3
*
* COMPUTE THE UPPER AND LOWER BOUNDS.

```

```

*
* TUPPER=(TSTAR*SIGMA3-T2*WEIG2-T3*WEIG3)/WEIG1
* TLOWER=(-TSTAR*SIGMA3-T2*WEIG2-T3*WEIG3)/WEIG1
* WRITE (6,2210) TUPPER,TLOWER,FIRST
*
* CBOUND=2.*FIRST/101.
* CUBIC=0.
* IF (T1.GE.TUPPER) GO TO 2000
* IF (T1.LE.TLOWER) GO TO 2001
* IDECIDE=0
* RESULT1=0.
* RESULT2=0.
* GO TO 2100
*
* THIS BLOCK COMPUTES AN UPPER BOUND AND LOWER BOUND FOR L'S.
*
* 2000 IF (T2.LT.0.0.OR.T3.LT.0.) GO TO 2010
* IFIX=1
* GO TO 2002
*
* 2001 IF (T2.GT.0.0.OR.T3.GT.0.) GO TO 2020
* IFIX=-1
*
* 2002 KK=-1
* FKK=FLOAT(KK)
* CONNS=CONST3/FK/(SUMQ3-1.)
* CUBIC=CUBIC+CONNS*INT3D(PP)/RMM/2.506628274631
* SCREEN=CBOUND-CUBIC*FLOAT(IFIX)*1.9603960396
* WRITE (6,2220) CUBIC, SCREEN
* IF (IFIX.EQ.1) GO TO 2003
* IF (SCREEN.GT.0.) GO TO 2020
* GO TO 2009
* 2003 IF (SCREEN.LT.0.) GO TO 2010
* 2009 IDECIDE=1*IFIX
* RESULT1=1.0
* RESULT2=1.0
* GO TO 2100
*
* THIS BLOCK COMPUTES L1-FUNCTION TO MAKE DECISION D=0 OR D+.
*
* 2010 CFUNC=0.72139177*FIRST
* IFIX=1
* GO TO 2030
*
* THIS BLOCK COMPUTES L2-FUNCTION TO MAKE DECISION D=0 OR D-.
*
* 2020 CFUNC=-0.72139177*FIRST
* IFIX=-1
*
* THIS BLOCK COMPUTES THE INFINITE SERIES IN THE L-FUNCTION.
* THE FIRST TERM OF THE SERIES WHEN KK=0 IS COMPUTED
* SEPARATELY FROM THE REST OF THE TERMS.
* DO-LOOP IS USED FOR KK=1,2,3,....
*
* 2030 CONST=CONST3/1.7724538509
* SERIS=CONST*INT3D(P)/RMM
* WRITE (6,2200) SERIS
*
* UPDATE=SUMQ3*CONST*FK
* DO 2040 KK=1,20,1
* FKK=FLOAT(KK)
* CONST=UPDATE
* OK=CONST*INT3D(PP)
* UPDATE=(SUMQ3+FKK)*FK*UPDATE/(2.0*FKK+1.0)
* SERIS=SERIS+OK/RMM
* WRITE (6,2200) OK/RMM, KK
* IF (SERIS.GE.CFUNC) GO TO 2050
* IF (OK/RMM.LE.BOUND) GOTO 2060

```

```
2040 CONTINUE
      GO TO 2060
*
2050 IDECIDE=0
      GO TO 2070
*
2060 IDECIDE=1*IFIX
*
2070 IF (IFIX.NE.1) GO TO 2080
      RESULT1=FIRST-1.386209332*SERIS
      RESULT2=0.
      GO TO 2100
*
2080 RESULT1=0.
      RESULT2=FIRST+1.386209332*SERIS
*
2200 FORMAT (5X,F18.8,I3)
2210 FORMAT(//,5X,*HI=*,F13.6,5X,*LO=*,F13.6,5X,*P-MEAN*,F13.8)
2220 FORMAT(5X,*CUBIC TERM=*,E13.6,5X,*SCREEN=*,E13.6)
2100 RETURN
      END
```

```

*
*
*          *****
*          *
*          *          FUNCTIONS          *
*          *
*          *****
*
REAL FUNCTION Z(W,U,U)
*
* GLOBAL VARIABLES *
*
REAL      QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!         T1,T2,T3,FI,FJ,FK,FKK,
!         EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
REAL      FLAG1,FLAG2
*
COMMON /STATS/  QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!              T1,T2,T3,FI,FJ,FK,FKK,KK
COMMON /EXPON/  EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
COMMON /FLAGS/  FLAG1,FLAG2
*
Z=0.
IF (U.EQ.0.) GO TO 700
IF (U.EQ.0.) GO TO 700
Z=(SQRT((1.-W)+FLAG1*(W-U)/FI+FLAG2*(W-U)/FJ)
!  *U**EXPA*U**EXPB*W**EXPC)/
!  ((SSA*U+SSB*U+SSC*W+SSE)/2.))*SUMQ3
700 RETURN
END
*
*
*
REAL FUNCTION P(W,U,U)
*
* GLOBAL VARIABLES *
*
REAL      QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!         T1,T2,T3,FI,FJ,FK,FKK,
!         EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
REAL      FLAG1,FLAG2
*
COMMON /STATS/  QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!              T1,T2,T3,FI,FJ,FK,FKK,KK
COMMON /EXPON/  EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
COMMON /FLAGS/  FLAG1,FLAG2
*
* LOCAL VARIABLES *
*
REAL      TEMP
*
P=0.
IF (U.EQ.0.) GO TO 700
IF (U.EQ.0.) GO TO 700
TEMP=(1.-W)+FLAG1*(W-U)/FI+FLAG2*(W-U)/FJ
P=(SQRT(TEMP)*U**EXPA*U**EXPB*W**EXPC)/
!  (((1.-W)*T1+(W-U)*T2+(W-U)*T3)**2*FK/
!  TEMP+SSA*U+SSB*U+SSC*W+SSE)/2.))*SUMQ3
700 RETURN
END
*
*
*
REAL FUNCTION PP(W,U,U)
*
* GLOBAL VARIABLES *
*
REAL      QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!         T1,T2,T3,FI,FJ,FK,FKK

```

```

REAL          EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
REAL          FLAG1,FLAG2
*
COMMON /STATS/ QA,QB,QC,QE,SSA,SSB,SSC,SSE,
!             T1,T2,T3,F1,FJ,FK,FKK,KK
COMMON /EXPON/ EXPA,EXPB,EXPC,EXPE,SUMQ1,SUMQ2,SUMQ3
COMMON /FLAGS/ FLAG1,FLAG2
*
* LOCAL VARIABLES *
*
REAL          TEMP1,TEMP2
*
PP=0.
IF (U.EQ.0.) GO TO 800
IF (V.EQ.0.) GO TO 800
TEMP1=(1.-W)*T1+(W-U)*T2+(W-U)*T3
TEMP2=(1.-W)+FLAG1*(W-U)/F1+FLAG2*(W-U)/FJ
PP=(TEMP1**(2*KK)*U**EXPA*U**EXPB*W**EXPC)/
!   (TEMP2**(FKK-.5))/(((TEMP1**2*FK/TEMP2)
!   +SSA*U+SSB*V+SSC*W+SSE)/2.)*(SUMQ3+FKK)
800 RETURN
END

```

```
REAL FUNCTION RMFC1(Q1,Q2,Q3,Q4)
*
*                *****
*                *
*                *   M-ALGORITHM   *
*                *
*                *****
*
* *****
*
* THIS ALGORITHM CALCULATES THE EXPECTATION OF THE POSTERIOR
* MEAN WITH THE INPUT PARAMETERS Q1,Q2,Q3, AND Q4 DENOTING
* FOUR DEGREES OF FREEDOMS, SSA,SSB,SSC,SSE ARE THE FOUR
* GIVEN SUMS OF SQUARES ESTIMATES.
*
* ALL FOUR D.F.'S MUST BE POSITIVE INTEGERS AND SS'S MUST BE
* POSITIVE REAL NUMBERS. THIS ALGORITHM IS DESIGNED TO
* COMPUTE M-FUNCTION ONLY WHEN Q4 IS EVEN INTEGER.
*
* *****
*
* G L O B A L   V A R I A B L E S           *
*
* INTEGER       ILEVEL, JLEVEL, KCELL
*
* REAL          QA, QB, QC, QE, SSA, SSB, SSC, SSE,
* !            T1, T2, T3, FI, FJ, FK, FKK
*
* COMMON /STATS/ QA, QB, QC, QE, SSA, SSB, SSC, SSE,
* !            T1, T2, T3, FI, FJ, FK, FKK, KK
*
* L O C A L   V A R I A B L E S           *
*
* INTEGER       MULT3, MULT4
*
* REAL          TOL0, TOL1, TOL2, TOL3, GF, JF1, JF2, UF,
* !            RK, RL, R3, Q1, Q2, Q3, Q4
*
* TOL0=SSC+SSE
* TOL1=SSA+SSC+SSE
* TOL2=SSB+SSC+SSE
* TOL3=SSA+SSB+SSC+SSE
* MULT4=INT(Q4)/2
*
* M-FUNCTION IS COMPUTED FROM G-FUNCTION WHEN Q3 IS ODD.
* ELSE FROM J-FUNCTION WHEN Q3 IS EVEN.
*
* IF (AMOD(Q3,2).EQ.1.) GO TO 10
* MULT3=INT(Q3)/2
* R3=0.0
* CALL FUNCTNJ(Q2,Q1,SSB,TOL1,JF1,HF)
* CALL FUNCTNJ(Q1,Q2,SSA,TOL2,JF2,UF)
* GF=0.0
* GO TO 15
10 MULT3=(INT(Q3)-1)/2
* R3=1.0
* CALL FUNCTNG(Q1,Q2,SSA,SSB,SSC,SSE,GF,JF1,JF2,HF)
*
* RMFC1 IS COMPUTED RECURSIVELY OVER MULT3 AND MULT4 TIMES.
*
15 RMFC1=0.0
* IF (MULT3.EQ.0) GO TO 25
* DO 20 K=1,MULT3,1
* RK=FLOAT(K)
* GF=((R3+2.0*RK-2.0)*GF+2.0*(JF1+JF2))/TOL0
* JF1=((Q1+R3+2.0*RK-2.0)*JF1+2.0*HF)/TOL1
* JF2=((Q2+R3+2.0*RK-2.0)*JF2+2.0*HF)/TOL2
* HF=((Q1+Q2+R3+2.0*RK-2.0)*HF)/TOL3
```



```
20 CONTINUE
*
25 DO 30 L=1,MULT4,1
  RL=FLOAT(L)
  RMFC1=2.0*((RL-1.0)*RMFC1+GF)/SSE
  GF=2.0*((Q3/2.0+RL-1.0)*GF+JF1+JF2)/TOL0
  JF1=2.0*((Q1+Q3)/2.0+RL-1.0)*JF1+HF)/TOL1
  JF2=2.0*((Q2+Q3)/2.0+RL-1.0)*JF2+HF)/TOL2
  HF=(Q1+Q2+Q3+2.0*RL-2.0)*HF/TOL3
30 CONTINUE
*
  RETURN
  END
```

```

SUBROUTINE FUNCTNJ(AB1,AB2,S1,S2,TFUNC,UFUNC)
*
*          *****
*          *          *
*          *  J-SUBROUTINE  *
*          *          *
*          *****
*
* *****
*
* THIS PROGRAM COMPUTES THE DOUBLE INTEGRAL OF INCOMPLETE
* GAMMA PRODUCT WITH AB1 AND AB2 DEGREES OF FREEDOMS.
* Sqrt(2*PI) IS REPLACED BY A CONSTANT 2.506628274631.
*
* *****
*
* LOCAL VARIABLES
*
* REAL      S1,S2,TFUNC,UFUNC,R1,R2,QSUM,SQRT2PI,
*          RI,RJ,AB1,AB2
*
* INTEGER   MULT1,MULT2,ITEMP,JTEMP,I,J
*
* QSUM=S1+S2
* SQRT2PI=2.506628274631
* I=0
* J=0
* IF (AMOD(AB1,2.).EQ.0.) GO TO 105
*
* MULT1=(INT(AB1)-1)/2
* IF (AMOD(AB2,2.).EQ.0.) GO TO 100
* MULT2=(INT(AB2)-1)/2
* TFUNC=4.0*ATAN(SQRT(S1/S2))/SQRT(S1*S2)
* UFUNC=2.0/QSUM
* GO TO 115
*
* 100 MULT2=INT(AB2)/2
* TFUNC=2.0*SQRT2PI/(SQRT(QSUM)*S2)
* UFUNC=SQRT2PI/(QSUM*SQRT(QSUM))
* J=J+1
* GO TO 115
*
* 105 MULT1=INT(AB1)/2
* IF (AMOD(AB2,2.).EQ.0.) GO TO 110
* MULT2=(INT(AB2)-1)/2
* TFUNC=2.0*SQRT2PI*(1.0-SQRT(S2/QSUM))/(S1*SQRT(S2))
* UFUNC=SQRT2PI/(QSUM*SQRT(QSUM))
* I=I+1
* GO TO 115
*
* 110 MULT2=INT(AB2)/2
* TFUNC=4.0/(S2*QSUM)
* UFUNC=4.0/(QSUM*QSUM)
* I=I+1
* J=J+1
*
* 115 RI=FLOAT(I)
* RJ=FLOAT(J)
* R1=1.0-RI
* R2=1.0-RJ
* IF (MULT2.EQ.J) GO TO 125
* JTEMP=J+1
* DO 120 JJ=JTEMP,MULT2,1
* TFUNC=((R2+2.0*FLOAT(JJ-1))*TFUNC+2.0*UFUNC)/S2
* UFUNC=(2.0*RI+R1+R2+2.0*FLOAT(JJ-1))*UFUNC/QSUM
* 120 CONTINUE
*
* 125 IF (MULT1.EQ.I) GO TO 135

```

```
ITEMP=I+1
DO 130 II=ITEMP,MULT1,1
TFUNC=((R1+2.0*FLOAT(II-1))*TFUNC-2.0*UFUNC)/S1
UFUNC=(AB2+R1+2.0*FLOAT(II-1))*UFUNC/QSUM
130 CONTINUE
*
135 RETURN
END
```



```
I=I+1
J=J+1
*
215 RI=FLOAT(I)
    RJ=FLOAT(J)
    R1=1.0-RI
    R2=1.0-RJ
    IF (MULT1.EQ.I) GO TO 225
    ITEMP=I+1
    DO 220 II=ITEMP,MULT1,1
    GF=((R1+2.*FLOAT(II-1))*GF-2.*JF1)/S1
    JF1=((RI+2.*FLOAT(II-1)+2.*R1)*JF1+2.*HF)/P1
    JF2=((R1+2.*FLOAT(II-1))*JF2-2.*HF)/S1
    HF=(RJ+R1+2.*FLOAT(II))*HF/P3
220 CONTINUE
*
225 IF (MULT2.EQ.J) GO TO 235
    JTEMP=J+1
    DO 230 JJ=JTEMP,MULT2,1
    GF=((R2+2.*FLOAT(JJ-1))*GF-2.*JF2)/S2
    JF1=((R2+2.*FLOAT(JJ-1))*JF1-2.*HF)/S2
    JF2=((RJ+2.*FLOAT(JJ-1)+2.*R2)*JF2+2.*HF)/P2
    HF=(AB1+RJ+2.*R2+2.*FLOAT(JJ-1))*HF/P3
230 CONTINUE
*
235 RETURN
    END
```

```

REAL FUNCTION RMFC2(PQA,PQB,PQC,PQE)
*
*
*      *****
*      *                               *
*      *   M-ALGORITHM                 *
*      *                               *
*      *****
*
*      *****
*
*      THIS ALGORITHM CALCULATES THE EXPECTATION OF THE POSTERIOR
*      MEAN WITH THE INPUT PARAMETERS PQA,PQB,PQC,PQE DENOTING
*      FOUR POSITIVE INTEGERS, SSA,SSB,SSC,SSE ARE THE FOUR
*      GIVEN SUMS OF SQUARES ESTIMATES.
*      THIS ALGORITHM IS DESIGNED TO COMPUTE THE M-FUNCTION
*      ONLY WHEN PQE IS ODD INTEGER AND PQA,PQB ARE EVEN.
*
*      *****
*
*      DIMENSION TLL(10,10),TMM(10,10),SUM(10,10)
*
*      G L O B A L   V A R I A B L E S   *
*
*      INTEGER      ILEVEL,JLEVEL,KCELL
*
*      REAL          QA,QB,QC,QE,SSA,SSB,SSC,SSE,
*      !            T1,T2,T3,FI,FJ,FK,FKK
*
*      COMMON /STATS/ QA,QB,QC,QE,SSA,SSB,SSC,SSE,
*      !            T1,T2,T3,FI,FJ,FK,FKK,KK
*
*      L O C A L   V A R I A B L E S   *
*
*      INTEGER      IMM,INN,IM,IN
*
*      REAL          C1,C2,C3,C4,C5,C6,C7,C8,
*      !            C9,C10,C11,C12,C13,C14,
*      !            C15,C16,C17,PQA,PQB,
*      !            PQC,PQE,SSTOT,SUML,SUMM,ALL,AMM,ANN,
*      !            TL,TM,FL,FM,CONST1,CONST2,SUMN,CONS
*
*      IF (AMOD(PQC,1.).NE.0.) GO TO 50000
*      C1=SSC/(SSC+SSE)
*      C2=(SSE/(SSC+SSE))*PQE
*      C3=SSA/(SSA+SSC)
*      C4=(SSC/(SSA+SSC))*PQC
*      C5=(SSA+SSC)/(SSA+SSC+SSE)
*      C6=(SSE/(SSA+SSC+SSE))*PQE
*      C7=SSB/(SSB+SSC)
*      C8=(SSC/(SSB+SSC))*PQC
*      C9=(SSB+SSC)/(SSB+SSC+SSE)
*      C10=(SSE/(SSB+SSC+SSE))*PQE
*      SSTOT=SSA+SSB+SSC+SSE
*      C11=SSA/(SSA+SSB+SSC)
*      C12=SSB/(SSA+SSB+SSC)
*      C13=(SSC/(SSA+SSB+SSC))*PQC
*      C14=(SSA+SSB+SSC)/SSTOT
*      C15=(SSE/SSTOT)*PQE
*      C16=GAMMA(PQC)
*      C17=GAMMA(PQE)
*
*      SUML=1.
*      ALL=PQC-1.
*      IF (ALL.LE.0.) GO TO 11000
*      FL=1.
*      TL=1.
* 10000 TL=C1*(PQE+FL-1.)*TL/FL

```

```

      SUML=SUML+TL
      FL=FL+1.
      IF (FL.LE.ALL) GO TO 10000
11000 RMFC2=1.-C2*SUML
*
      SUML=1.
      TL=1.
      FL=1.
20000 TL=C5*(PQE+FL-1.)*TL/FL
      SUML=SUML+TL
      FL=FL+1.
      IF (FL.LE.ALL) GO TO 20000
      SUML=SUML*C6
*
      TM=1.
      AMM=PQA-1.
      SUMM=TM*(SUML-1.)
      IF (AMM.LE.0.) GO TO 23000
      FM=1.
21000 TM=C3*(PQC+FM-1.)*TM/FM
      TL=C5*(PQE+PQC+FM-2.)*TL/(PQC+FM-1.)
      SUML=SUML+TL*C6
      SUMM=SUMM+TM*(SUML-1.)
      FM=FM+1.
      IF (FM.LE.AMM) GO TO 21000
23000 RMFC2=RMFC2+SUMM*C4
*
      SUML=1.
      TL=1.
      FL=1.
30000 TL=C9*(PQE+FL-1.)*TL/FL
      SUML=SUML+TL
      FL=FL+1.
      IF (FL.LE.ALL) GO TO 30000
      SUML=SUML*C10
*
      TM=1.
      AMM=PQB-1.
      SUMM=TM*(SUML-1.)
      IF (AMM.LE.0.) GO TO 33000
      FM=1.
31000 TM=C7*(PQC+FM-1.)*TM/FM
      TL=C9*(PQE+PQC+FM-2.)*TL/(PQC+FM-1.)
      SUML=SUML+TL*C10
      SUMM=SUMM+TM*(SUML-1.)
      FM=FM+1.
      IF (FM.LE.AMM) GO TO 31000
33000 RMFC2=RMFC2+SUMM*C8
*
      SUML=1.
      TL=1.
      FL=1.
40000 TL=C14*(PQE+FL-1.)*TL/FL
      SUML=SUML+TL
      FL=FL+1.
      IF (FL.LE.ALL) GO TO 40000
      SUML=SUML*C15
*
      DO 42000 I=1,10,1
        DO 41000 J=1,10,1
          TLL(I,J)=0.
          TMM(I,J)=0.
          SUM(I,J)=0.
41000 CONTINUE
42000 CONTINUE
*
      IMM=INT(PQA)-1
      INN=INT(PQB)-1

```

```

TLL(1,1)=TL
SUM(1,1)=SUML
TMM(1,1)=1.
SUMM=1.-SUML
IM=0
IN=0
FIM=0.
FIN=0.
IF (IMM.EQ.0.AND.INN.EQ.0) GO TO 49000
GO TO 44000
43000 TMM(IM+1,1)=TMM(IM,1)*(PQC+FIM-1.)*C11/FIM
TLL(IM+1,1)=TLL(IM,1)*C14*(PQE+PQC+FIM-2.)/(PQC+FIM-1.)
SUM(IM+1,1)=SUM(IM,1)+TLL(IM+1,1)*C15
SUMM=SUMM+TMM(IM+1,1)*(1.-SUM(IM+1,1))
44000 IF (INN.EQ.0) GO TO 48000
IN=1
46000 FIM=FLOAT(IN)
TMM(IM+1,IN+1)=TMM(IM+1,IN)*(PQC+FIM+FIN-1.)*C12/FIN
TLL(IM+1,IN+1)=TLL(IM+1,IN)*(PQE+PQC+FIM+FIN-2.)*C14/
! (PQC+FIM+FIN-1.)
SUM(IM+1,IN+1)=SUM(IM+1,IN)+TLL(IM+1,IN+1)*C15
SUMM=SUMM+TMM(IM+1,IN+1)*(1.-SUM(IM+1,IN+1))
IN=IN+1
IF (IN.LE.INN) GO TO 46000
48000 IM=IM+1
FIM=FLOAT(IM)
IF (IM.LE.IMM) GO TO 43000
49000 RMFC2=RMFC2+SUMM*C13
*
CONS=(SSA/2.0)**PQA*(SSB/2.0)**PQB*(SSC/2.0)**PQC
CONS=CONS*(SSE/2.0)**PQE
RMFC2=GAMMA(PQA)*GAMMA(PQB)*C16*C17*RMFC2/CONS
GO TO 59000
*
50000 IMM=INT(PQA)-1
INN=INT(PQB)-1
TM=SSA+SSB+SSC
CALL FUNCTNJ(2.*PQC,2.*PQE,SSC,SSE,SUML,TL)
CALL FUNCTNJ(2.*PQC,2.*PQE,SSA+SSC,SSE,SUMM,TL)
CALL FUNCTNJ(2.*PQC,2.*PQE,TM,SSE,SUMN,TL)
SUML=SUML-SUMM+SUMN
*
CONS=1.
IF (IMM.LE.0) GO TO 52000
DO 51000 MM=1,IMM,1
AMM=FLOAT(MM)
CALL FUNCTNJ(2.*PQC+2.*AMM,2.*PQE,SSA+SSC,SSE,SUMM,TL)
CALL FUNCTNJ(2.*PQC+2.*AMM,2.*PQE,TM,SSE,SUMN,TL)
CONS=CONS*(SSA/2.)/AMM
SUML=SUML+CONS*SUMN-CONS*SUMM
51000 CONTINUE
*
52000 CONS=1.
CALL FUNCTNJ(2.*PQC,2.*PQE,SSB+SSC,SSE,SUMM,TL)
SUML=SUML-SUMM
IF (INN.LE.0) GO TO 54000
DO 53000 NN=1,INN,1
ANN=FLOAT(NN)
CALL FUNCTNJ(2.*PQC+2.*ANN,2.*PQE,SSB+SSC,SSE,SUMM,TL)
CALL FUNCTNJ(2.*PQC+2.*ANN,2.*PQE,TM,SSE,SUMN,TL)
CONS=CONS*(SSB/2.)/ANN
SUML=SUML+CONS*SUMN-CONS*SUMM
53000 CONTINUE
*
54000 CONST1=1.
IF (IMM.LE.0.OR.INN.LE.0) GO TO 57000
DO 56000 MM=1,IMM,1
AMM=FLOAT(MM)

```



```
CONST1=CONST1*(SSA/2.)/AMM
CONST2=1.
DO 55000 NN=1, INN, 1
ANN=FLOAT(NN)
CONST2=CONST2*(SSB/2.)/ANN
CALL FUNCTNJ(2.*(PQC+AMM+ANN), 2.*PQE, TM, SSE, SUMN, TL)
SUML=SUML+SUMN*CONST1*CONST2
55000 CONTINUE
56000 CONTINUE
*
57000 CONS=(SSA/2.)**PQA*(SSB/2.)**PQB
RMFC2=GAMMA(PQA)*GAMMA(PQB)*SUML/CONS
*
59000 RETURN
END
```

```
REAL FUNCTION TRINT(F,NDIU)
*
*
*      *****
*      *                *
*      *       TRINT   *
*      *                *
*      *****
*
*****
*
* THIS SUBPROGRAM INTEGRATES A GIVEN FUNCTION F OVER A
* TRIANGULAR REGION, DEFINED BY THE VERTICES (0,0),(1,0)
* AND (1,1). F(U,W) IS THE FUNCTION SUBROUTINE TO BE
* INTEGRATED. NDIU IS THE NUMBER OF SUBDIVISIONS OF THE
* UNIT INTERVAL (0,1). THE VALUE OF THE INTEGRAL IS
* GIVEN BY TRINT.
*
*****
*
EXTERNAL   F
ND=NDIU
H=0.5/FLOAT(ND)
TWOH=H+H
TOT=0.
IF(ND.EQ.1) GO TO 130
W1=2.
IF(ND.EQ.2) W1=1.
W2=8.
W3=4.
* WILL SUM ALONG DIAGONALS FROM (0,YF) TO (XL,1)
XL=TWOH
YF=1.-XL
*
* TAKE CARE OF FOUR SPECIAL POINTS NEAR (0,1)
TOT=F(0.,1.)+W1*(F(0.,YF)+F(XL,1.))+W2*F(H,YF+H)
IF(ND.EQ.2) GO TO 130
NDM2=ND-2
* DOUBLE LOOP OVER INTERNAL POINTS
DO 120 I=1,NDM2
XL=XL+TWOH
YF=1.-XL
IF(I.LT.NDM2) GO TO 100
W1=W1-1
W3=W3-1
100 X=0.
Y=YF
TOT=TOT+W1*F(0.,YF)
C=F(H,YF+H)
U=0.
DO 110 J=1,I
X=X+TWOH
Y=Y+TWOH
U=U+F(X,Y)
110 C=C+F(X+H,Y+H)
120 TOT=TOT+W3*U+W2*C+W1*F(XL,1.)
*
* SUM ALONG DIAGONAL STARTING AT (0,H)
130 C=0.
X=0.
Y=H
ND2=ND+ND
DO 140 I=1,ND2
C=C+F(X,Y)
X=X+H
140 Y=Y+H
TOT=TOT+C+C
*
* SUM ALONG DIAGONAL Y=X
```

```
150 W2=2.  
    W3=1.  
    X=0.  
    Y=0.  
    C=F(H,H)  
    U=0.  
    IF(ND.EQ.1) GO TO 170  
    NDM1=ND-1  
    DO 160 J=1,NDM1  
    X=X+TWOH  
    Y=Y+TWOH  
    U=U+F(X,Y)  
160 C=C+F(X+H,Y+H)  
170 TOT=TOT+W3*U+W2*C  
    TRINT=TOT*(H**2/3.)  
    RETURN  
    END
```



```

X(NI)=DMC*XX(I)+DPC
A(I)=DMC*AA(I)
444 A(NI)=DMC*AA(I)
RETURN
END

*
REAL FUNCTION QMULT2(FCN,AA,BB,FU1,FL1,FU2,FL2,X,A,MM)
*
3-DIMENSIONAL ITERATED INTEGRAL
DIMENSION X(40,3), A(40,3), MM(3)
H1=(BB-AA)/2.
G1=(BB+AA)/2.
Q1=0.
M1=MM(1)
M2=MM(2)
M3=MM(3)
DO 66 I=1,M1
UI=H1*X(I,1)+G1
AI=H1*A(I,1)
D1=FU1(UI)
C1=FL1(UI)
H2=(D1-C1)/2.
G2=(D1+C1)/2.
Q2=0.
DO 44 J=1,M2
UJ=H2*X(J,2)+G2
AJ=H2*A(J,2)
D=FU2(UI,UJ)
C=FL2(UI,UJ)
H=(D-C)/2.
G=(D+C)/2.
Q=0.
DO 22 K=1,M3
WK=H*X(K,3)+G
22 Q=Q+A(K,3)*FCN(UI,UJ,WK)
44 Q2=Q2+AJ*H*Q
66 Q1=Q1+AI*Q2
QMULT2=Q1
RETURN
END

*
*
REAL FUNCTION FUP1(W)
FUP1 = W
RETURN
END

*
*
REAL FUNCTION FLO1(W)
FLO1 = 0E0
RETURN
END

*
*
REAL FUNCTION FUP2(W,U)
FUP2 = W
RETURN
END

*
*
REAL FUNCTION FLO2(W,U)
FLO2 = 0E0
RETURN
END

```