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MELLIN TRANSFORMS FROM FOURIER TRANSFORMS[1]

by

Herman Rubin
Purdue University

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Department of Statistics
Division of Mathematical Sciences

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ABSTRACT

Let F be of bounded variation, $\hat{F}$ its Fourier-Stieltjes transform. Then if $0 < \Re(\alpha) < 1$, we obtain an explicit formula for $\int_{-\infty}^{\infty} x^{-\alpha} dF$ if that integral exists, and if $\Re(\alpha) = 0$, $\alpha \neq 0$, we give an explicit limit for the integral with no restrictions.

Let F be of bounded variation, $\hat{F}$ its Fourier-Stieltjes transform. Then

Theorem 1: If $0 < \lambda = \Re(\alpha) < 1$ and $|x|^{-\lambda}$ is F-integrable,

(I) $$e^{\frac{1}{2}\pi i\alpha}\int_0^\infty y^{-\alpha}dF(y) + e^{-\frac{1}{2}\pi i\alpha}\int_{-\infty}^{0}|y|^{-\alpha}dF(y) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\hat{F}(t)dt.$$

Theorem 2: If $\Re(\alpha) = 0$, $\alpha \neq 0$, and (δ,σ) approaches $(0,0)$ through positive pairs such that $\delta^{-\sigma}$ is bounded, then

(II) $$e^{\frac{1}{2}\pi i\alpha}\int_0^\infty y^{-\alpha}dF(y) + e^{-\frac{1}{2}\pi i\alpha}\int_{-\infty}^{0^-}|y|^{-\alpha}dF(y)$$

$$= \frac{1}{\Gamma(\alpha)}\lim\left[\int_0^\infty t^{\sigma+\alpha-1}e^{-\delta t}\hat{F}(t)dt - \delta^{-\sigma-\alpha}F\{0\}\right].$$

The results of the abstract follow easily, since $e^{\frac{1}{2}\pi i\alpha} \neq e^{-\frac{1}{2}\pi i\alpha}$, and since the argument of $\delta^{-\alpha}$ can be made arbitrary.

We recall the following results from classical analysis: Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(t) > 0$, μ not zero or a negative real number. Then without using analytic function theory, we can show that

(A) $$\Gamma(\alpha)t^{-\alpha} = \int_0^\infty x^{\alpha-1}e^{-tx}dx$$

and

(B) $$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\mu^{\beta}} = \int_0^\infty \frac{x^{\alpha-1}}{(x+\mu)^{\alpha+\beta}}\,dx,$$

where the powers are all principal, i.e., if $w = e^v$, v of imaginary part less than π in magnitude, then $w^\gamma = e^{\gamma v}$.

Let us now demonstrate the theorems. For the first, from (A)

$$(1) \qquad t^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha} e^{-tx} dx ,$$

and hence

$$(2) \qquad \int_\varepsilon^M t^{\alpha-1} \int_{-\infty}^\infty e^{ity} dF(y) dt$$

$$= \int_\varepsilon^M \int_0^\infty \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} e^{-tx} \int_{-\infty}^\infty e^{ity} dF(y) dx dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^\infty \int_0^\infty \frac{x^{-\alpha}}{x-iy} (e^{-\varepsilon(x-iy)} - e^{-M(x-iy)}) dx dF(y),$$

since the right side is absolutely integrable in all variables. Now if $y \neq 0$ and $\Re(\alpha) = \lambda$, the inner integral is bounded by $C_\lambda |y|^{-\lambda}$. Thus we can use the bounded convergence theorem and (B) to obtain

$$(3) \qquad \int_0^\infty t^{\alpha-1} \int_{-\infty}^\infty e^{ity} dF(y) dt$$

$$= \Gamma(\alpha) \int_{-\infty}^\infty (-iy)^{-\alpha} dF(y)$$

$$= \Gamma(\alpha)(e^{\frac{1}{2}\pi i\alpha} \int_0^\infty y^{-\alpha} dF(y) + e^{-\frac{1}{2}\pi i\alpha} \int_{-\infty}^0 |y|^{-\alpha} dF(y)).$$

If $\Re(\alpha) = 0$, it can be shown that the inner integral in the last expression of (2) is bounded, but no convergence is possible as $\varepsilon \to 0$. However

$$(4) \qquad \int_0^\infty t^{\sigma+\alpha-1} e^{-\delta t} \int_{-\infty}^\infty e^{ity} dF(y) dt$$

$$= \frac{1}{\Gamma(1-\sigma-\alpha)} \int_0^\infty \int_0^\infty x^{-\sigma-\alpha} e^{-t(x+\delta)} \int_{-\infty}^\infty e^{ity} dF(y) dx dt$$

$$= \frac{1}{\Gamma(1-\sigma-\alpha)} \int_{-\infty}^\infty \int_0^\infty \frac{x^{-\sigma-\alpha}}{x+\delta-iy} dx dF(y),$$

and the inner integral is equal to $\Gamma(\sigma+\alpha)\Gamma(1-\sigma-\alpha)(\delta-iy)^{-\sigma-\alpha}$. Thus

$$\text{(5)} \qquad \frac{1}{\Gamma(\sigma+\alpha)}\left[\int t^{\sigma+\alpha-1}e^{-\delta t}\int e^{ity}dF(y)dt - \delta^{-\sigma-\alpha}F\{0\}\right]$$

$$= \int_{y\neq 0} (\delta - iy)^{-\sigma-\alpha}dF(y).$$

The result then follows from the continuity of the gamma function and the application of the bounded convergence theorem to the right side of (5).