A NOTE ON THE LOWER LIMIT OF THE STANDARDIZED EMPIRICAL DISTRIBUTION FUNCTION

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ABSTRACT. Let $X_1, X_2, \ldots$ be a sequence of independent uniform (0,1) random variables and put $T_n(\delta_n) = \sup_{\delta_n \le x \le 1-\delta_n} (|F_n(x) - x|(x(1-x))^{-1/2})$ where $F_n(x)$ denotes the empirical distribution function of the variables $X_1, X_2, \ldots, X_n$. We show that $a_n = (\log (1 + (\log\log n)^{-1} \log 1/\delta_n))^{1/2}$ is a lim inf sequence for $n^{1/2} T_n(\delta_n)$.

KEY WORDS: EMPIRICAL DISTRIBUTION FUNCTION, KIEFER PROCESS, LIM INF SEQUENCE

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1. INTRODUCTION. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each uniformly distributed on the interval $(0,1)$. Denote by $F_n(x)$ the empirical distribution function of the variables $X_1, X_2, \ldots, X_n$. Let $\Psi_n(x)$, $0 \le x \le 1$ be a weight function (which may depend on n) and consider the random variables

$$Z_n = \sup_{0 \le x \le 1} |F_n(x) - x| \Psi_n(x). \tag{1.1}$$

The lim sup behaviour of Z_n has been studied intensively in the literature, while much less is known about the lim inf of Z_n. Mogulskii [11] and Kuelbs [10] have shown that

$$\liminf_{n\to\infty} (n \log\log n)^{1/2} \sup_{0 \le x \le 1} |F_n(x) - x| = \pi/\sqrt{8} \quad \text{a.s.} \tag{1.2}$$

and the author [3] has proved that

$$\liminf_{n\to\infty} (n/\log\log n)^{1/2} \sup_{0<x<1} \frac{|F_n(x)-x|}{(x(1-x))^{1/2}} = \sqrt{2} \quad \text{a.s.} \tag{1.3}$$

In this note we investigate the variables

$$n^{1/2}\, T_n(\delta_n) = n^{1/2} \sup_{\delta_n \le x \le 1-\delta_n} \frac{|F_n(x)-x|}{(x(1-x))^{1/2}} \tag{1.4}$$

and show that

$$a_n = \left(\log\left(1 + \frac{\log 1/\delta_n'}{\log\log n}\right)\right)^{1/2} \tag{1.5}$$

is a lim inf sequence for (1.4), where $\delta'_n = \max(\delta_n, 1/n)$. Note that for $\delta_n = \delta > 0$ (δ is a constant), $a_n \sim (\log 1/\delta)^{1/2}(\log\log n)^{-1/2}$, the same norming factor (but with different constants) as in (1.2), while for $\delta_n = 0$, we have $a_n \sim (\log\log n)^{1/2}$ as in (1.3). The norming factor given by (1.5) is a continuous link between these two extremal cases. This fact and the estimations given in Section 2 resemble those given in [4] by Révész and the author. In many respects the proofs of the present paper follow the same lines as given in [4].

In [8] Jaeschke investigates the limiting distribution of $T_n(\delta_n)$ suitably normalized and shows that it is the double exponential extreme value distribution. This proof is based on the strong approximation of Komlós, Major and Tusnády [9], on the equation

$$(1.6) \qquad P\Big(\sup_{\delta\le x\le 1-\delta} \frac{|K(x,n)|}{(nx(1-x))^{1/2}} < u\Big) = P\Big(\sup_{0\le t\le \log\frac{1-\delta}{\delta}} |U(t)| < u\Big),$$

where $K(x,n)$ is a Kiefer process, $U(t)$ is an Ornstein-Uhlenbeck process, and on the results of Darling and Erdös [6] concerning the limiting distribution of $\sup_{0\le t\le T} |U(t)|$, when $T \to \infty$. In our proof we use also these results.

Our aim is to prove the following result:

THEOREM. <u>Let</u> $\delta_n \ge 0$, $n = 1, 2,\ldots$ <u>be a non-increasing sequence such that</u> $n(\log 1/\delta_n)^{-1}$ <u>is non-decreasing</u>. <u>Then</u>

$$(1.7) \qquad \liminf n^{1/2}\, T_n(\delta_n)/a_n = c \qquad \text{a.s.}$$

where $T_n(\delta_n)$ *is defined by* (1.4), a_n *is defined by* (1.5) *and* c *is a finite positive number*. *If, furthermore* $\lim(\log\log 1/\delta_n)/\log\log\log n = \infty$, *or* $\delta_n = 0$, *then* $c = \sqrt{2}$.

REMARK. We can not give the exact value of c in general. As mentioned earlier, our proof will be based on the equation (1.6) and to determine the exact value of c we would need an asymptotic value for the probability $P(\sup_{0\le t\le T} |U(t)| < u)$, when this is small. In Section 2 upper and lower estimations will be given to the probability above, but the upper and lower bounds are not close enough to yield the exact value of c in general.

2. UPPER AND LOWER BOUNDS FOR $P(\sup_{0\le t\le T} |U(t)| \le u)$.

Let U(t) be an Ornstein-Uhlenbeck process, i.e. stationary Gaussian process with mean zero and covariance function $E(U(0)U(t)) = e^{-|t|}$.

LEMMA 1. *There exist positive finite constants* c_1 *and* c_2 *such that for all* $u > 0$ *and* $T > 0$ *we have*

$$(2.1)\qquad -\frac{(T+1)}{2(\exp(u^2/c_2^2)-1} \le \log P(\sup_{0\le t\le T} |U(t)| \le u) \le -\frac{2(T-1)}{\exp(u^2/c_1^2)-1}.$$

The upper bound in (2.1) *is valid with* $c_1 = \sqrt{2}\,(1-\varepsilon)$ *for* $T \ge T_0(\varepsilon)$ *and* $u \ge u_0(\varepsilon)$.

PROOF. First we prove the upper part of the inequality (2.1). Assume that $T \geq 1$.

$$P(\sup_{0\le t\le T} |U(t)| \le u) =$$

$$= \int_{-u}^{u} P(\sup_{0\le t\le T} |U(t)| \le u/U(T-1) = z)\ \varphi(z)dz =$$

$$= \int_{-u}^{u} P(\sup_{0\le t\le T-1} |U(t)| \le u/U(T-1) = z)\ P(\sup_{T-1\le t\le T} |U(t)| \le u/U(T-1) = z)\ \varphi(z)dz$$

From stationarity, $P(\sup_{T-1\le t\le T} |U(t)| \le u/U(T-1) = z) = P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = z)$ and this conditional probability has its maximum at z = 0 (see Anderson [1]).

Hence

$$P(\sup_{0\le t\le T} |U(t)| \le u) \le$$

$$\le P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = 0) \int_{-u}^{u} P(\sup_{0\le t\le T-1} |U(t)| \le u/U(T-1) = z)\ \varphi(z)\ dz$$

$$= P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = 0)\ P(\sup_{0\le t\le T-1} |U(t)| \le u).$$

By repeating this procedure several times we obtain the inequality

$$P(\sup_{0\le t\le T} |U(t)| \le u) \le$$

$$\le (P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = 0))^{[T]}\ P(\sup_{[T]\le t\le T} |U(t)| \le u) \tag{2.2}$$

$$(P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = 0))^{T-1}$$

If $W(\cdot)$ is a Wiener process starting from 0, then $U(t) = e^{-t}\, W(e^{2t})$ is an Ornstein-Uhlenbeck process, and we can obtain

$$P(\sup_{0\le t\le 1} |U(t)| \le u/U(0) = 0) \le$$

$$\le P(|W(x)| \le u\sqrt{x},\ 1 \le x \le e^2/W(1) = 0) \le$$

$$P(\sup_{0\le s\le 1} |W(s)| \le u\sqrt{2})$$

From the distribution of $\sup_{0\le s\le 1} |W(s)|$ (see e.g. Feller [7]) the following estimations are straightforward:

$$(2.3)\qquad P(\sup_{0\le s\le 1} |W(s)| \le u\sqrt{2}) \le \frac{4}{\pi}\exp(-\pi^2/16u^2))$$

and

$$(2.4)\qquad P(\sup_{0\le s\le 1} |W(s)| \le u\sqrt{2}) \le 2\Phi\,(u\sqrt{2}) - 1$$

both inequalities being valid for all $u > 0$. It is then easy to find a constant c_1 such that

$$(2.5)\qquad \log\frac{4}{\pi} - \frac{\pi^2}{16u^2} \le - \frac{2}{\exp(u^2/c_1^2)-1} \qquad \text{for } u \le 1$$

and

$$(2.6)\qquad \log(2\Phi\,(u\sqrt{2}) - 1) \le - \frac{2}{\exp(u^2/c_1^2)-1} \qquad \text{for } u \ge 1,$$

establishing the upper part of (2.1). (Note that $c_1 = 1/3$ works.)

Now let $\varepsilon > 0$ be small and put $\alpha_0 = \log\dfrac{8 + 4\varepsilon + \varepsilon^2}{4\varepsilon + \varepsilon^2}$. In [3] we have shown that

$$(2.7)\qquad P(\sup_{0\le t\le T} |U(t)| < u) \le (\Phi(u(1 + \tfrac{\varepsilon}{2})))^{[\frac{T}{\alpha_0}]}$$

If $T \ge 2\alpha_0$, then $[\frac{T}{\alpha_0}] \ge \frac{T-1}{2\alpha_0}$, thus

$$\log P(\sup_{0\le t\le T} |U(t)| < u) \le \frac{T-1}{2\alpha_0}\log\Phi(u(1 + \tfrac{\varepsilon}{2}))$$

Furthermore, if $u \to \infty$, then

$$\log \Phi(u(1+\tfrac{\varepsilon}{2})) \sim -(1-\Phi(u(1+\tfrac{\varepsilon}{2}))) \sim -\frac{1}{u(1+\frac{\varepsilon}{2})\sqrt{2\pi}}\, e^{-\frac{u^2}{2}(1+\frac{\varepsilon}{2})^2}$$

therefore for large enough u,

$$\frac{1}{2\alpha_0}\log \Phi(u(1+\tfrac{\varepsilon}{2})) \le -\frac{2}{\exp(\frac{u^2}{2(1-\varepsilon)^2})-1} \tag{2.8}$$

completing the upper part estimation of LEMMA 1.

Our next step is to obtain the lower estimation in (2.1).

$$P(\sup_{0\le t\le T}|U(t)| \le u) \ge P(\sup_{0\le t\le [T]+1}|U(t)| \le u,\ \max_{1\le i\le [T]+1}|U(i)| \le \tfrac{u}{2}) =$$

$$= \int_{-u/2}^{u/2} P(\sup_{0\le t\le [T]}|U(t)| \le u,\ \max_{1\le i\le [T]}|U(i)| \le \tfrac{u}{2} / U[T]) = z)\ \varphi(z)$$

$$P(\sup_{[T]\le t\le [T]+1}|U(t)| \le u,\ |U([T]+1)| \le \tfrac{u}{2} / U([T]) = z)dz.$$

By stationarity,

$$P(\sup_{[T]\le t\le [T]+1}|U(t)| \le u,\ |U([T]+1)| \le \tfrac{u}{2} / U([T]) = z) =$$

$$= P(\sup_{0\le t\le 1}|U(t)| \le u,\ |U(1)| \le \tfrac{u}{2} / U(0) = z)$$

and using again the fact that $U(t) = e^{-t}W(e^{2t})$ is an Ornstein-Uhlenbeck process if $W(\cdot)$ is a Wiener process,

$$P(\sup_{0\le t\le 1}|U(t)| \le u,\ |U(1)| \le \tfrac{u}{2} / U(0) = z) \ge$$

$$\ge P(\sup_{1\le s\le e^2}|W(s)| \le u/W(1) = z) \ge$$

$$\ge P(\sup_{0\le s\le e^2-1}|W(s)| \le u/2)$$

for $-u/2 \le z \le u/2$.

By repeating this procedure several times, we finally get

$$(2.9)\qquad P(\sup_{0\le t\le T}|U(t)|\le u)\ge (P(\sup_{0\le s\le e^2-1}|W(s)|\le u/2))^{T+1}$$

The following inequalities are valid for all u:

$$(2.10)\qquad P(\sup_{0\le s\le e^2-1}|W(s)|\le u/2)\ge \frac{4}{\pi}(\exp(-\frac{\pi^2(e^2-1)}{2u^2}-\frac{1}{3}\exp(-\frac{9\pi^2(e^2-1)}{2u^2}))$$

$$(2.11)\qquad P(\sup_{0\le s\le e^2-1}|W(s)|\le u/2)\ge 4\Phi(\frac{u}{2(e^2-1)^{1/2}})-3.$$

Using (2.10) for $u \le 5$ and (2.11) for $u > 5$, and taking (2.9) into account, it is not difficult to find a constant c_2 that satisfies the lower part of the inequality (2.1).

The proof of Lemma 1 is complete.

3. LOWER LIMITS FOR STANDARDIZED KIEFER PROCESS

A. Kiefer process $K(x, y)$ $(0 \le x \le 1,\ 0 \le y < \infty)$ is defined as a two-parameter Gaussian process with mean zero and covariance $E(K(x_1, y_1)K(x_2, y_2)) = (x_1 \wedge x_2 - x_1x_2)\ (y_1 \wedge y_2)$. Note that for integral n, $K(x,n)$ is equal to the sum of n independent Brownian bridges, hence $n^{-1/2}K(x,n)$ is a Brownian bridge itself. Define

$$(3.1)\qquad n^{1/2}\, T_n'(\delta_n) = \sup_{\delta_n\le x\le 1-\delta_n}\frac{|K(x,n)|}{(nx(1-x))^{1/2}}\ .$$

Let δ_n be a non-increasing sequence such that $1/n \le \delta_n < 1/2$ and $n^{1/2}\, a_n$ is non-decreasing, where a_n is defined by (1.5). Assume furthermore that $n(\log 1/\delta_n)^{-1}$ is non-decreasing.

LEMMA 2. Let c_1 and c_2 be the same constants as in LEMMA 1. If $\lim_{n\to\infty} \delta_n = 0$, then

$$c_1 \leq \liminf_{n\to\infty} n^{1/2}\, T_n'\, (\delta_n)/a_n \leq c_2 \qquad \text{a.s.} \tag{3.2}$$

If $\lim \delta_n = \delta > 0$, then

$$\frac{\pi}{\sqrt{8}}\left(\frac{1 - 2\delta}{\log 1/\delta}\right)^{1/2} \leq \liminf_{n\to\infty} n^{1/2}\, T_n'\, (\delta_n)/a_n \leq c_2 \qquad \text{a.s.} \tag{3.3}$$

PROOF. We show first the lower estimation in (3.2), i.e.

$$c_1 \leq \liminf_{n\to\infty} n^{1/2}\, T_n'\, (\delta_n)/a_n \qquad \text{a.s.} \tag{3.4}$$

Let $n_k = \exp(k/(\log k)^3)$ and define the events B_k by

$$B_k = \{ \min_{n_{k-1}\leq n<n_k} (nT_n'(\delta_{n_{k-1}})) < (c_1 - \varepsilon) n_k^{1/2}\, a_{n_k} \} \tag{3.5}$$

We show that $\sum_k P(B_k) < \infty$ and hence by Borel-Cantelli lemma we have $P(B_k \text{ i.o.}) = 0$ for all $\varepsilon > 0$, which in turn implies (3.4). We use the following inequality of Mogulskii [11]: Let S_n, $n = 1, 2, \ldots$ be partial sums of i.i.d. Banach space-valued random variables; $v > 0$, $y > 0$, $\gamma = \min_{m\leq n\leq r} P(||S_{r-n}|| \leq y)$. Then

$$P(\min_{m\leq n\leq r} ||S_n|| \leq v) \leq \frac{1}{\gamma} P(||S_r|| \leq v + y). \tag{3.6}$$

We apply this inequality with $m = n_{k-1}$, $r = n_k$, $S_n = (x(1 - x))^{-1/2} K(x,n)$, $||S_n|| = n\, T_n'\, (\delta_{n_{k-1}})$, $v = (c_1 - \varepsilon) n_k^{1/2}\, a_{n_k}$, $y = 2\, ((n_k - n_{k-1}) \operatorname{loglog}\, (1/\delta_{n_{k-1}}))^{1/2}$.

Here

$$\gamma = \gamma_k = \min_{n_{k-1}\le n<n_k} P\Big(\sup_{\delta_{n_{k-1}}\le x\le 1-\delta_{n_{k-1}}} \Big|\frac{K(x,n_k-n)}{(x(1-x))^{1/2}}\Big| \le$$

$$\le 2((n_k - n_{k-1}) \text{ loglog } (1/\delta_{n_{k-1}}))^{1/2}\Big) =$$

$$= P\Big(\sup_{\delta_{n_{k-1}}\le x\le 1-\delta_{n_{k-1}}} \Big|\frac{K(x,\, n_k-n_{k-1})}{((n_k-n_{k-1})x(1-x))^{1/2}}\Big| \le$$

$$\le 2(\text{loglog } (1/\delta_{n_{k-1}}))^{1/2}\Big) =$$

$$= P\Big(\sup_{0\le t\le \log \frac{1-\delta_{n_{k-1}}}{\delta_{n_{k-1}}}} |U(t)| \le 2(\text{loglog } 1/\delta_{n_{k-1}})^{1/2}\Big),$$

where in the last step we used (1.6). If follows from Darling and Erdös [6] that $\gamma_k \to 1$ as $k \to \infty$ and hence there exists $\gamma_0 > 0$ such that for all k large enough, $\gamma_k \ge \gamma_0$. It can be seen furthermore that for large k,

$$2((n_k - n_{k-1}) \text{ loglog } 1/\delta_{n_{k-1}})^{1/2} < \varepsilon n_k^{1/2}\, a_{n_k}$$

and hence,

$$P(B_k) \le \frac{1}{\gamma_0} P(n_k\, T'_{n_k}\, (\delta_{n_{k-1}}) \le c_1\, n_k^{1/2}\, a_{n_k}) =$$

$$= \frac{1}{\gamma_0} P\Big(\sup_{0\le t\le \log \frac{1-\delta_{n_{k-1}}}{\delta_{n_{k-1}}}} |U(t)| \le c_1\, a_{n_k}\Big).$$

On applying LEMMA 1, we can see that

$$\log(\gamma_0 P(B_k)) \le -2 \log\log n_k \left(\frac{\log \frac{1-\delta_{n_{k-1}}}{\delta_{n_{k-1}}} - 1}{\log 1/\delta_{n_k}}\right) \le$$

(3.7)
$$\log(\log n_k)^{-3/2}$$

for large k, showing that $\sum_k P(B_k) < \infty$. Hence the lower estimation in (3.2) follows. In the case when $\lim_{n\to\infty} \delta_n = \delta > 0$, then without loss of generality we may assume that $\delta_n = \delta$, n = 1, 2,... and apply the same procedure as above. In this case $\gamma_k = \gamma_0 > 0$ for $k \ge 1$, and we have the inequality

(3.8)
$$P(B_k) \le \frac{1}{\gamma_0} P\left(\sup_{0\le t\le \log \frac{1-\delta}{\delta}} |U(t)| \le c_1' a_{n_k}\right),$$

where c_1 in (3.5) should be replaced by $c_1' = \frac{\pi}{\sqrt{8}} \left(\frac{1-2\delta}{\log 1/\delta}\right)^{1/2}$. Instead of LEMMA 1, use the estimation

$$P\left(\sup_{0\le t\le T} |U(t)| \le u\right) = P\left(\sup_{1\le s\le e^{2T}} \left|\frac{W(s)}{s^{1/2}}\right| \le u\right) \le$$

$$P\left(\sup_{1\le s\le e^{2T}} |W(s)| \le ue^T\right) \le \frac{4}{\pi} \exp\left(-\frac{\pi^2(e^{2T}-1)}{8u^2 e^{2T}}\right),$$

and the asymptotic relation

$$a_n \sim (\log 1/\delta)^{1/2} (\log\log n)^{-1/2}, \quad n \to \infty.$$

This proves $\sum_k P(B_k) < \infty$ in this case too, i.e. the lower estimation in (3.3) is established.

To show the upper part of (3.2), let $n_k = k^k$. We have the following inequality:

$$\sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_k)|}{(x(1-x))^{1/2}} \le$$

$$\le \sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_k-n_{k-1})|}{(x(1-x))^{1/2}} + \sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_{k-1}|}{(x(1-x))^{1/2}}$$

Define the events C_k by

(3.9) $$C_k = \{ \sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_k-n_{k-1})|}{(x(1-x))^{1/2}} \le c_2\, n_k^{1/2}\, a_{n_k} \}.$$

Then by (2.1),

$$P(C_k) = P(\sup_{0\le t\le \log \frac{1-\delta_{n_k}}{\delta_{n_k}}} |U(t)| \le c_2\, a_{n_k}) \ge$$

$$\ge \exp(- \frac{1}{2} \frac{\log \frac{1-\delta_{n_k}}{\delta_{n_k}} + 1}{\log \frac{1}{\delta_{n_k}}} \operatorname{loglog} n_k)$$

$$\ge \exp(- \operatorname{loglog} n_k) = \frac{1}{k \log k} ,$$

i.e. $\sum_k P(C_k) = \infty$. Since the events C_k are independent, Borel-Cantelli lemma implies that

(3.10) $$P(C_k \text{ i.o.}) = 1.$$

On the other hand,

$$P\Big(\sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_{k-1})|}{(x(1-x))^{1/2}} \ge \varepsilon\, n_2^{1/2}\, a_{n_k}\Big) =$$

$$= P\Big(\sup_{0\le t\le \log \frac{1-\delta_{n_k}}{\delta_{n_k}}} |U(t)| \ge \varepsilon\Big(\frac{n_k}{n_{k-1}}\Big)^{1/2} a_{n_k}\Big) \le$$

$$\le (\log(1/\delta_{n_k}) + 1)\, e^{-c' \frac{n_k a_{n_k}^2}{n_{k-1}}}$$

with some constant c'. This is a term of a convergent sum, because $\log 1/\delta_{n_k} \le \log n_k = k \log k$, $n_k/n_{k-1} \ge k$ and $a_{n_k}^2 \ge (2 \log\log n_k)^{-1} = (2 \log(k \log k))^{-1}$ for large k. Hence

$$\lim_{k\to\infty} n_k^{-1/2} a_{n_k}^{-1} \sup_{\delta_{n_k}\le x\le 1-\delta_{n_k}} \frac{|K(x,n_{k-1})|}{(x(1-x))^{1/2}} = 0 \qquad \text{a.s.} \tag{3.11}$$

which together with (3.10) yields the upper estimations in both (3.2) and (3.3).

In certain cases we can give the exact value of the lim inf.

LEMMA 3. *Let δ_n satisfy the conditions of LEMMA 2 and moreover*

$$\lim_{n\to\infty} \frac{\log\log\log n}{\log\log 1/\delta_n} = 0. \tag{3.12}$$

Then

$$\liminf_{n\to\infty} n^{1/2}\, T_n'(\delta_n)/a_n = \sqrt{2} \qquad \text{a.s.}$$

PROOF. LEMMA 1 and LEMMA 2 imply

$$\liminf n^{1/2} T_n'(\delta_n)/a_n \geq \sqrt{2} \qquad \text{a.s.}$$

To prove the $\leq$ part, it follows from Darling and Erdös [6], that

$$\frac{\sup_{0<t<T} |U(t)|}{(\log T)^{1/2}} \overset{P}{\to} \sqrt{2}, \qquad \text{as } T \to \infty$$

where $\overset{P}{\to}$ means convergence in probability. This is equivalent to

$$\frac{n^{1/2} T_n'(\delta_n)}{(\log\log 1/\delta_n)^{1/2}} \overset{P}{\to} \sqrt{2}, \qquad \text{as } n \to \infty,$$

provided that $\lim_{n\to\infty} \delta_n = 0$. The condition (3.12) implies that $a_n \sim (\log\log 1/\delta_n)^{1/2}$ and also $\lim_{n\to\infty} \delta_n = 0$, therefore

$$\frac{n^{1/2} T_n'(\delta_n)}{a_n} \overset{P}{\to} \sqrt{2}, \qquad \text{as } n \to \infty$$

which implies the $\leq$ part in (3.13). The proof of LEMMA 3 is complete.

4. PROOF OF THE THEOREM

Our theorem is in fact a consequence of LEMMA 2, LEMMA 3 and the strong invariance theorem of Komlós, Major and Tusnády [9] (see also Csörgö and Révész [5]). On a suitably rich probability space one can construct a Kiefer process $K(\cdot,\cdot)$ such that

$$(4.1) \qquad \sup_{0\leq x\leq 1} |n(F_n(x) - x) - K(x,n)| = O(\log^2 n) \qquad \text{a.s.}$$

This theorem implies also

$$(4.2)\qquad \sup_{\delta_n\le x\le 1-\delta_n} \left| n^{1/2}\frac{F_n(x)-x}{(x(1-x))^{1/2}} - \frac{K(x,n)}{(nx(1-x))^{1/2}}\right| = O\left(\frac{\log^2 n}{(n\delta_n)^{1/2}}\right) \quad \text{a.s.}$$

If

$$(4.3)\qquad \lim_{n\to\infty} \frac{\log^2 n}{(n\delta_n)^{1/2} a_n} = 0,$$

Then the statement of our theorem follows from the zero-one law, LEMMA 2, LEMMA 3 and (4.2). This is the case e.g. if $\delta_n \ge n^{-1}(\log n)^4$ for $n \ge n_0$. In the case when there is a subsequence $\{n_k\}$, $k = 1, 2, \ldots$ such that $\delta_{n_k} < n_k^{-1}(\log n_k)^4$, we use the inequality

$$(4.4)\qquad T_n(\delta_n'') \le T_n(\delta_n) \le T_n(0),$$

where $\delta_n'' = \max(\delta_n, n^{-1}\log^4 n)$. Then on one hand.

$$(4.5)\qquad \liminf_{n\to\infty} \frac{n^{1/2}\, T_n(\delta_n)}{a_n} \ge \liminf_{n\to\infty} \frac{n^{1/2}\, T_n(\delta_n'')}{a_n} \ge c_1 \qquad \text{a.s.}$$

On the other hand, (see Jaeschke [8])

$$(4.6)\qquad \frac{n_k^{1/2}\, T_{n_k}(0)}{(\log\log n_k)^{1/2}} \xrightarrow{P} \sqrt{2} \qquad \text{as } k\to\infty$$

and hence

$$(4.7)\qquad \liminf_{k\to\infty} \frac{n_k^{1/2}\, T_{n_k}(0)}{(\log\log n_k)^{1/2}} \le \sqrt{2} \qquad \text{a.s.}$$

Since $a_{n_k} \sim (\log\log n_k)^{1/2}$, we have also

$$(4.8) \qquad \liminf_{n\to\infty} \frac{n^{1/2}\, T_n(\delta_n)}{a_n} \le \liminf_{n\to\infty} \frac{n_k^{1/2}\, T_{n_k}(0)}{a_{n_k}} \le \sqrt{2} \qquad \text{a.s.}$$

This completes the proof of the THEOREM.

REMARKS. 1. By comparing the present theorem with Theorem 3.1 in [2], we have the following results: if

$$(4.9) \qquad \lim_{n\to\infty} \frac{n\delta_n}{\log\log n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\log\log 1/\delta_n}{\log\log n} = c, \; (0 < c \le 1)$$

then

$$(4.10) \qquad \limsup_{n\to\infty} \frac{n^{1/2}\, T_n(\delta_n)}{(\log\log n)^{1/2}} = (2(c+1))^{1/2} \qquad \text{a.s.}$$

and

$$(4.11) \qquad \liminf_{n\to\infty} \frac{n^{1/2}\, T_n(\delta_n)}{(\log\log n)^{1/2}} = (2c)^{1/2} \qquad \text{a.s.}$$

2. It would be interesting to investigate further the lim inf of the more general quantities

$$(4.12) \qquad Z_n = \sup_{0<x<1} |F_n(x) - x|\Psi(x)$$

and

$$(4.13) \qquad Z_n^+ = \sup_{0<x<1} (F_n(x) - x)\, \Psi(x),$$

where $\Psi(x)$ is a weight function (which perhaps may also depend on n). It is an open problem to determine the lower limit of z_n^+ even in the case $\Psi(x) = 1$, $0 \le x \le 1$.

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