ON THE RATE OF CONVERGENCE
FOR THE WEAK LAW OF LARGE NUMBERS

by

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1. INTRODUCTION

Let $X, X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(t) = P(X \leq t)$, and let $S_n = X_1 + \ldots + X_n$ $(n \geq 1)$. In studing the rate of convergence in weak laws of large numbers, the convergence of the series

$$(1.1) \qquad \sum_{n=1}^{\infty} P(|S_n| \geq n\varepsilon)$$

for some $\varepsilon > 0$, was found to be connected with the existence of second moment of X (see Hsu and Robbins [6], Erdös [3] or Révész [9]). In particular, Erdös [2] has shown that the series (1.1) converges for some $\varepsilon > 0$, if and only if, EX^2 and $|EX| < \varepsilon$.

Subsequently, number of authors (notably Heyde and Rohatgi [5], Chow and Lai [2] and Lai and Lan [8]) analysed the convergence of the series

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of the form

(1.2) $$\sum_{n=1}^{\infty} c_n P(|S_n| \geq a_n)$$

for various $\{c_n\}$ and $\{a_n\}$, again connecting it with the appropriate moment conditions.

Certain considerations arising in stochastic modeling for the growth of cancer tumors (see [1]), led us to the analysis of convergence of series of type (1.1) with the index of summation restricted to a subsequence.

The problem of this note may be formulated as follows. Let $\{K_n\}$ be a sequence of integers satisfying

(1.3) $$1 \leq K_1 \leq K_2 \leq \cdots$$

and

(1.4) $$\lim_{n\to\infty} K_n = \infty .$$

Consider the series

(1.5) $$\sum_{n=1}^{\infty} P\{|S_{K_n}| \geq \varepsilon K_n\}$$

for some $\varepsilon > 0$. By grouping the terms corresponding to identical indices K_n, we may write (1.5) as

(1.6) $$\sum_{n=1}^{\infty} c_n P\{|S_{b_n}| \geq \varepsilon b_n\}$$

where the sequences $\{b_n\}$ and $\{c_n\}$ are defined by

(1.7) $$c_0 = 0, \; c_{n+1} = \min\{r: K_r > K_{c_n+1}\} - 1 - c_0 - \cdots - c_n$$

and

$$b_{n+1} = K_{c_0+c_1+\dots+c_n+1} \tag{1.8}$$

for $n = 0,1,\dots$.

Note that since $\lim K_n = \infty$ we have $1 \le c_n < \infty$ for all $n \ge 1$, and $1 \le b_1 < b_2 < \dots$.

We shall now drop the condition that c_n's are integers, and consider generally the problem of convergence of series (1.6), where $\{c_n\}$ is some sequence of positive real numbers and $\{b_n\}$ is a strictly increasing sequence of positive integers.

Clearly, we have here an interplay of three types of conditions: (i) convergence of series (1.6), (ii) an appropriate moment condition and (iii) a condition imposing constraints on the behaviour of the sequences $\{c_n\}$ and $\{b_n\}$. We shall prove three theorems, in each of them two among (i)-(iii) implying the third, with theorem 1 being valid for the general case where the random variables involved are not necessarily independent and identically distributed (I.I.D.).

2. THE RESULTS

We start by presenting a lemma due to von Bahr and Esséen, which will be needed below.

LEMMA 1. *Let* $Y_1,\dots,Y_n$ *be a finite sequence of random variables. Denote* $S_i = Y_1 + \dots Y_i$ *and assume that* $E(Y_i|S_{i-1}) = 0$, $E|Y_i|^{1+\lambda} < \infty$, $i=1,\dots,n$, *for some* λ *with* $0 < \lambda \le 1$. *Then there exists a constant* $C(\lambda) > 0$, *such that*

$$E|S_n|^{1+\lambda} \le C(\lambda) \sum_{i=1}^{n} E|Y_i|^{1+\lambda} \tag{2.1}$$

In fact, as pointed out by Rubin [10], we have

$$(2.2) \qquad C(\lambda) = \sup_x \left[\frac{|1+x|^{1+\lambda} - 1 - (1+\lambda)x}{|x|^{1+\lambda}} \right]$$

with $1 \leq C(\lambda) \leq 2$ for $0 \leq \lambda \leq 1$.

We shall first prove

THEOREM 1. *Let* $Y_1, Y_2, \ldots$ *be a sequence of random variables with* $E(Y_i | S_{i-1}) = 0$, $i=1,2,\ldots$, *where* $S_0 = 0$, $S_i = Y_1 + \ldots + Y_i$, $i=1,2,\ldots$.

Assume that for some sequence $\{\lambda_n\}$ *with* $0 < \lambda_n \leq 1$ *we have* $E|Y_i|^{1+\lambda} < \infty$, $i=1,2,\ldots$, *where* $\lambda = \sup_n \lambda_n$, *and the sequences* $\{c_n\}$ *and* $\{b_n\}$ *satisfy the condition*

$$(2.3) \qquad \sum_{n=1}^{\infty} c_n \bar{\theta}_n b_n^{-\lambda_n} < \infty$$

where

$$(2.4) \qquad \bar{\theta}_n = \frac{1}{b_n} \sum_{j=1}^{b_n} E|Y_j|^{1+\lambda_n} .$$

Then for every $\varepsilon > 0$ *we have*

$$(2.5) \qquad \sum_{n=1}^{\infty} c_n P\{|Y_1 + \ldots + Y_{b_n}| \geq \varepsilon b_n\} < \infty .$$

Proof. We may estimate the terms of the series in (2.5), using Markov inequality and Lemma 1, as follows

(2.6) $$c_n P\{|Y_1+\ldots+Y_{b_n}| \geq \varepsilon b_n\} = c_n P\{|S_{b_n}|^{1+\lambda_n} \geq (\varepsilon b_n)^{1+\lambda_n}\}$$

$$\leq c_n \frac{E|S_{b_n}|^{1+\lambda_n}}{(\varepsilon b_n)^{1+\lambda_n}}$$

$$\leq c_n \; C(\lambda_n)\, \bar{\theta}_n b_n^{-\lambda_n} \cdot \varepsilon^{-(1+\lambda_n)} \quad .$$

The theorem now follows from (2.3), since $\sup_n C(\lambda_n) \leq 2$, and $\varepsilon^{-(1+\lambda_n)} \leq \varepsilon^{-1}$ or ε^{-2}, depending on whether $\varepsilon < 1$ or $\varepsilon \geq 1$. □

In particular, in the case of I.I.D. random variables $X, X_1, X_2, \ldots$ we obtain

COROLLARY 1. *Assume that* $E|X|^{1+\lambda} < \infty$ *for some* λ *with* $0 < \lambda \leq 1$. *Moreover, let* $EX = 0$, *and assume that the sequences* $\{b_n\}$ *and* $\{c_n\}$ *satisfy the condition*

(2.7) $$\sum_{n=1}^{\infty} c_n b_n^{-\lambda} < \infty \; .$$

Then the series (1.6) *converges for every* $\varepsilon > 0$.

Observe that for $\tau < 0$, if we put $c_n = n^{\tau}$, $b_n = n$ and $\lambda > 1 + \tau$, we obtain the sufficiency part of Theorem 1 of Katz [7].

We shall now prove

THEOREM 2. Assume that $\liminf_{n\to\infty} c_n > 0$. If for some $\lambda > 0$ we have

$$\limsup_{n\to\infty} \frac{b_{n+1}^{\lambda}(b_{n+1}-b_n)}{c_n b_n} < \infty \tag{2.8}$$

and the series (1.6) converges for some $\varepsilon > 0$, then $E|X|^{1+\lambda} < \infty$ and $|EX| < \varepsilon$.

Proof. Using the inequality (see Feller [4], p.149)

$$P\{|X_1+\ldots+X_n| \geq t\} \geq \tfrac{1}{2}\left(1-e^{-n[1-F(t)+F(-t)]}\right) \tag{2.9}$$

we infer from the convergence of series (1.6) that

$$\sum_{n=1}^{\infty} c_n\left(1-e^{-b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]}\right) < \infty . \tag{2.10}$$

Since $b_n \uparrow \infty$ and c_n's are bounded away from 0 for n large enough, we must have

$$\lim_{n\to\infty} b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)] = 0 \tag{2.11}$$

and hence

$$\sum_{n=1}^{\infty} c_n b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)] < \infty . \tag{2.12}$$

Again, (see Feller [4], p.151), we have $E|X|^{1+\lambda} < \infty$ iff

$$\int_0^{\infty} x^{\lambda}[1-F(x)+F(-x)]dx < \infty . \tag{2.13}$$

Also, from (2.8) it follows that for some constant M we have

$$b_{n+1}^{\lambda}(b_{n+1}-b_n) \leq Mc_n b_n, \quad n=1,2,\ldots . \tag{2.14}$$

Since the sequence $\{b_n\}$ is strictly increasing, while $1 - F(t) + F(-t)$ is nonincreasing, we bound the integral in (2.13) as follows:

$$(2.15)\qquad \int_0^\infty x^\lambda[1-F(x)+F(-x)]dx \le \sum_{n=1}^\infty (\varepsilon b_{n+1})^\lambda[1-F(\varepsilon b_n)+F(-\varepsilon b_n)](b_{n+1}-b_n)+(\varepsilon b_1)^{1+\lambda}$$

$$\le M\varepsilon^\lambda \sum_{n=1}^\infty c_n b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)] + (\varepsilon b_1)^{1+\lambda}$$

The fact that the last sum is finite in view of (2.12), implies that $E|X|^{1+\lambda} < \infty$.

Let $\mu = EX$. In the case $|\mu| > \varepsilon$, we can find an interval of the form $(\mu-\delta,\mu+\delta) \subset (-\varepsilon,\varepsilon)^c$ for some $\delta > 0$, such that by the weak law of large numbers we have

$$(2.16)\qquad 1 = \lim_{n\to\infty} P\left\{\left|\frac{S_{b_n}}{b_n} - \mu\right| < \delta\right\} \le \lim_{n\to\infty} P\left\{\left|\frac{S_{b_n}}{b_n}\right| \ge \varepsilon\right\} .$$

This means that the series (1.6) cannot converge, since $\liminf_{n\to\infty} c_n > 0$, leading thereby to a contradiction. The argument in the case with $|\mu| = \varepsilon$ being similar, proves that we must have $|EX| < \varepsilon$. □

For the next theorem we shall use the following lemma (see Feller [4], p.277).

LEMMA 2. Suppose that $\lambda_{n+1}/\lambda_n \to 1$ and $a_n \to \infty$ as $n \to \infty$. If U is a monotone function such that

$$(2.17)\qquad \lim_{n\to\infty}\left[\lambda_n U(a_n x)\right] = \chi(x) \le \infty$$

exists on a dense set, and χ is finite and positive in some interval, then U varies regularly and $\chi(x) = cx^\rho$ for some $-\infty < \rho < \infty$.

We shall now prove

THEOREM 3. Let $b_n/b_{n+1} \to 1$. Assume that for some $\lambda > 0$

$$(2.18)\qquad \lim_{x\to\infty} x^{1+\lambda}[1-F(x)+F(-x)]$$

exists and is positive, say equal to c. Then the convergence of series (1.6) for some $\varepsilon > 0$ implies (2.7).

Proof. As in the proof of Theorem 2, convergence of (1.6) implies (2.12). Let us write the series in (2.12) as

$$(2.19)\qquad \sum c_n b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)] = \sum (c_n b_n^{-\lambda})\left\{b_n^{1+\lambda}[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]\right\}.$$

We now apply Lemma 2 with $\lambda_n = b_n^{1+\lambda}$, $a_n = b_n$, $U(t) = 1 - F(t) + F(-t)$ and $x = \varepsilon$. As a result, $\lim_{n\to\infty} \lambda_n U(a_n x)$ becomes $\lim_{n\to\infty} b_n^{1+\lambda}[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]$, which exists and is positive in view of the assumption of the theorem. Consequently, the latter limit equals $c\varepsilon^{\rho}$ for some ρ. In fact, replacing x by εx in (2.18) we infer that $\rho = -(1+\lambda)$. From the convergence of (2.19) it follows now that $\Sigma c_n b_n^{-\lambda} < \infty$, as asserted. □

As an example, consider the case when X has the central t-distribution with 2 degrees of freedom, so that $EX^2 = \infty$ and $E|X| < \infty$. Here the limit (2.18) exists with $\lambda = 1$ and $c = 1/2$, so that Theorem 3 applies.

Note that since the sequence $\{b_n\}$ is strictly increasing, the condition (2.8) may be written as

$$(2.20)\qquad \liminf_{n\to\infty} \frac{c_n b_n^{-\lambda}}{(b_{n+1}/b_n)^{\lambda}\left(\frac{b_{n+1}}{b_n} - 1\right)} > 0.$$

Now, if (2.7) holds, then $c_n b_n^{-\lambda} \to 0$, so that condition (2.20) (and hence (2.8)) may hold only if $b_{n+1}/b_n \to 1$.

Let us also note that the existence of the positive limit (2.18) implies $E|X|^{1+\lambda} = \infty$, although $E|X|^{1+\sigma} < \infty$, for all $0 < \sigma < \lambda$. Conversely, if

(2.21) $$\sigma_0 = \sup\left\{\sigma: \int_0^\infty x^\sigma [1-F(x)+F(-x)]dx < \infty\right\}$$

and

(2.22) $$\int_0^\infty x^{\sigma_0} [1-F(x)+F(-x)]dx = \infty ,$$

then

(2.23) $$\lim_{x\to\infty} x^{1+\sigma}[1-F(x)+F(-x)] = 0$$

for all $\sigma < \sigma_0$. Here we cannot say that the limit (2.23) is positive or 0 in the case with $\sigma = \sigma_0$.

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