FURTHER RESULTS ON THE TRACE OF A NONCENTRAL WISHART MATRIX

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ABSTRACT

In an earlier article Mathai (1980) has given compact representations for the moments and cumulants of the trace of a noncentral Wishart matrix. He has also shown that $(\mathrm{tr}\,\underset{\sim}{A}-n\,\mathrm{tr}\,\underset{\sim}{\Sigma})/(2n\,\mathrm{tr}\,\underset{\sim}{\Sigma}^2)^{1/2}$ is asymptotically standard normal where $\underset{\sim}{A}$ is a noncentral Wishart matrix with n degrees of freedom and covariance matrix $\underset{\sim}{\Sigma}>\underset{\sim}{0}$. In the present article explicit expressions for the exact density of the trace are given in terms of confluent hypergeometric functions and in terms of zonal polynomials for the general case and as finite sums when the sample size is odd. As a consequence of some of these representations some summation formulae for zonal polynomials are also given.

1. INTRODUCTION

Let $x = \mathrm{tr}\,\underset{\sim}{A}$ where $\underset{\sim}{A}$ is a pxp noncentral Wishart matrix with n degrees of freedom, noncentrality parameter $\underset{\sim}{\Omega}$ and covariance matrix $\underset{\sim}{\Sigma}>\underset{\sim}{0}$. Let $M_x(t)$ denote the moment generating function of x. In Mathai (1980) it is shown that

$$M_x(t) = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \prod_{j=1}^{p} (1-2t\lambda_j)^{-n/2} e^{b_{jj}(1-2t\lambda_j)^{-1}} \tag{1}$$

where $\lambda_1,\ldots,\lambda_p$ are the eigen values of $\underset{\sim}{\Sigma}$, b_{jj}, $j=1,\ldots,p$ are the diagonal elements of $\underset{\sim}{Q}'\underset{\sim}{\Omega}\underset{\sim}{Q}$ where $\underset{\sim}{Q}$ is an orthogonal matrix such that $\underset{\sim}{Q}'\underset{\sim}{\Sigma}\underset{\sim}{Q} = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$. The aim of the present paper is to work out exact density of x and represent it in a number of different ways which are all suitable for computational purposes. One representation will be in terms of confluent hypergeometric function of several variables, another in terms of zonal polynomials and a third in terms of finite sums when the degrees of freedom n is even. By using some of these representations a few summation formulae for zonal polynomials will also be obtained.

From the structure in (1) it is evident that $\mathrm{tr}\underset{\sim}{A}$ can also be represented as a linear combination of independent noncentral chisquared variables. Linear combinations of independent chi-squared variables (central and noncentral) appear in a wide variety of problems such as queueing problems with gamma type inputs, study of quadratic forms, geometric probabilities, see for example Ruben (1962), study of regression residuals and certain time series problems, see for example MacNeill (1978).

2. DENSITY WHEN THE SAMPLE SIZE IS ODD

Expanding the expression $\exp\{\sum_{j=1}^{p} b_{jj}(1-2t\lambda_j)^{-1}\}$ as a power series one gets,

$$\mathrm{Exp}\{\sum_{j=1}^{p} b_{jj}(1-2t\lambda_j)^{-1}\} = \sum_{k=0}^{\infty}\sum_{\kappa} b_{11}^{k_1}\cdots b_{pp}^{k_p}(1-2t\lambda_1)^{-k_1}\cdots$$

$$\cdots(1-2t\lambda_p)^{-k_p}/\{k_1!\cdots k_p!\}$$

where $\kappa = (k_1,\ldots,k_p)$, $k_1+\ldots+k_p = k$. Now $M_x(t)$ of (1) can be written as

$$M_x(t) = e^{-tr\Omega} \sum_{k=0}^{\infty} \sum_{\kappa} b_{11}^{k_1} \cdots b_{pp}^{k_p} \prod_{j=1}^{p} \{(1-2t\lambda_j)^{-m_j}/k_j!\} \tag{2}$$

where $m_j = k_j + n/2$ and m_j is an integer when n is even or when the sample size is odd. In this case we will represent $\prod_{j=1}^{p}(1-2t\lambda_j)^{-m_j}$ as a finite sum by a generalized partial fraction technique. A convenient simplification, when the technique is applied, is developed in Mathai and Rathie (1971). When m_j's are integers we write

$$\prod_{j=1}^{p} (1-2t\lambda_j)^{-m_j} = \sum_{j=1}^{p} \sum_{r=1}^{m_j} a_{jr}(1-2t\lambda_j)^{-r}$$

where the coefficients a_{jr} are to be determined. Thus the density of x, denoted by f(x), is available by inverting the moment generating function. Term by term inversion is valid in this case and thus we have for $\kappa=(k_1,\ldots,k_p)$, $k_1+\ldots+k_p = k$,

$$f(x) = e^{-tr\Omega} \sum_{k=0}^{\infty} \sum_{\kappa} \prod_{i=1}^{p}(b_{ii}^{k_i}/k_i!) \sum_{j=1}^{p} \sum_{r=1}^{m_j} a_{jr} \frac{x^{r-1} e^{-x/(2\lambda_j)}}{(2\lambda_j)^r (r-1)!}$$

for $x > 0$ and $f(x)=0$ for $x \leq 0$. In order to compute the coefficients a_{jr} we proceed as follows. If some λ_j's are equal then we combine the corresponding factors. Hence in the following discussion we assume that all the λ_j's are distinct and nonzero. Now

$$\prod_{j=1}^{p} (1-2t\lambda_j)^{-m_j} = \prod_{j=1}^{p} (-2\lambda_j)^{-m_j} \prod_{j=1}^{p} (t-1/(2\lambda_j))^{-m_j}.$$

Let

$$\prod_{j=1}^{p} (t-1/(2\lambda_j))^{-m_j} = \sum_{j=1}^{p} \sum_{r=1}^{m_j} c_{jr}(t-1/(2\lambda_j))^{-r}.$$

Then

$$c_{jr} = \lim_{t\to 1/(2\lambda_j)} \frac{1}{(m_j-r)!} \frac{\partial^{m_j-r}}{\partial t^{m_j-r}} \{(t-1/(2\lambda_j))^{m_j} \prod_{i=1}^{p}(t-1/(2\lambda_i))^{-m_i}\}.$$

For details see Mathai and Rathie (1971). An outline of the technique will be given here. Let

$$B(t) = \prod_{\substack{i=1\\ i\neq j}}^{p} (t-1/(2\lambda_i))^{-m_i} \text{ and } A(t) = \frac{\partial}{\partial t}\log B(t) = -\sum_{\substack{i=1\\ i\neq j}}^{p} m_i/(t-\frac{1}{2\lambda_i}).$$

Then

$$\frac{\partial}{\partial t} B(t) = A(t)\, B(t),$$

$$\frac{\partial^{m_j-r}}{\partial t^{m_j-r}} B(t) = \frac{\partial^{m_j-r-1}}{\partial t^{m_j-r-1}} \{A(t)B(t)\}$$

$$= \sum_{j_1=0}^{m_j-r-1} \binom{m_j-r-1}{j_1} A^{(m_j-r-1-j_1)} B^{(j_1)}, \qquad (3)$$

where for example $A^{(q)}$ and $B^{(q)}$ denote the qth partial derivative with respect to t of A(t) and B(t) respectively with $A^{(0)}=A(t)$ and $B^{(0)}=B(t)$ and for example $\binom{m}{n} = m!/(n!(m-n)!)$, $0!=1$. Let

$$A_j^{(q)} = \lim_{t\to 1/(2\lambda_j)} A^{(q)} \text{ and } B_j = \lim_{t\to 1/(2\lambda_j)} B(t).$$

Continuing the process in (3) and then evaluating the final expression at $t=1/(2\lambda_j)$ one has

$$c_{jr} = \frac{1}{(m_j-r)!} \{ \sum_{j_1=0}^{m_j-r-1} \binom{m_j-r-1}{j_1} A_j^{(m_j-r-1-j_1)} \sum_{j_2=0}^{j_1-1} \binom{j_1-1}{j_2} A^{(j_1-1-j_2)}$$

$$\ldots\} B_j, \qquad (4)$$

where

$$B_j = \prod_{\substack{i=1\\ i\neq j}}^{p} (\frac{1}{2\lambda_j} - \frac{1}{2\lambda_i})^{-m_i} \qquad \text{and}$$

$$A_j^{(q)} = (-1)^{q+1}\, q! \sum_{\substack{i=1\\ i\neq j}}^{p} m_i (\frac{1}{2\lambda_j} - \frac{1}{2\lambda_i})^{-(q+1)} .$$

Hence the density is

$$f(x) = e^{-\mathrm{tr}\,\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \{\prod_{i=1}^{p} (-2\lambda_i)^{-m_i} (b_{ii}^{k_i}/k_i!)\} \sum_{j=1}^{p} \sum_{r=1}^{m_j} (-1)^r c_{jr}$$

$$x^{r-1} e^{-x/(2\lambda_j)}/(r-1)! \tag{5}$$

for $x > 0$ and $f(x)=0$ elsewhere where c_{jr} is given in (3).

3. DENSITY IN THE GENERAL CASE

In this case we start with the moment generating function in (2) where m_j's are no longer integers. Consider the following representations for the various factors.

$$(1-2\lambda_i t)^{-m_i} = (1-2\lambda_1 t)^{-m_i} (\lambda_1/\lambda_i)^{m_i} \sum_{r_i=0}^{\infty} (m_i)_{r_i} (1-\lambda_1/\lambda_i)^{r_i}$$

$$(1-2\lambda_1 t)^{-r_i}/r_i!$$

for $|(1-\lambda_1/\lambda_i)/(1-2\lambda_1 t)| < 1$, $i=1,\ldots,p$. A sufficient condition for the expansions is that $t < \min\{1/(2\lambda_1),\ldots,1/(2\lambda_p)\}$ and λ_1 is the smallest of the λ_j's such that $1-\lambda_1/\lambda_i < 1-2\lambda_1 t$, $i-1,\ldots,p$. Hence

$$\prod_{j=1}^{p}(1-2\lambda_j t)^{-m_j} = \{\lambda_1^{m-m_1}/\prod_{j=2}^{p}\lambda_j^{m_j}\} \sum_{r_2=0}^{\infty} \cdots \sum_{r_p=0}^{\infty} (m_2)_{r_2} \cdots (m_p)_{r_p}$$

$$(1-\lambda_1/\lambda_2)^{r_2} \cdots (1-\lambda_1/\lambda_p)^{r_p} (1-2\lambda_1 t)^{-(m+r)}$$

$$/(r_2! \cdots r_p!), \tag{6}$$

where $m = m_1+\ldots+m_p$ and $r=r_2+\ldots+r_p$. Term by term inversion is possible in this case and the density corresponding to the moment generating function $(1-2\lambda_1 t)^{-(m+r)}$ is

$$x^{m+r-1} e^{-x/(2\lambda_1)} /\{(2\lambda_1)^{m+r}\, \Gamma(m+r)\}$$

$$= \{x^{m-1} e^{-x/(2\lambda_1)}/((2\lambda_1)^m\, \Gamma(m))\}\{x^r/((2\lambda_1)^r (m)_r)\}, \tag{7}$$

where $(m)_r = m(m+1)\ldots(m+r-1)$, $(m)_0 = 1$. The inverse corresponding to (6) is the following:

$$\{x^{m-1}e^{-x/(2\lambda_1)}/(2^m\lambda_1^{m_1}\ldots\lambda_p^{m_p}\ \Gamma(m))\} \sum_{r_2=0}^{\infty}\ldots\sum_{r_p=0}^{\infty}(m_2)_{r_2}\ldots(m_p)_{r_p}$$

$$\{(1/\lambda_1-1/\lambda_2)(x/2)\}^{r_2}\ldots\{(1/\lambda_1-1/\lambda_p)(x/2)\}^{r_p}/\{(m)_r r_2!\ldots r_p!\}$$

$$=\{x^{m-1}\ e^{-x/(2\lambda_1)}/(2^m\ \lambda_1^{m_1}\ldots\lambda_p^{m_p}\ \Gamma(m)\}\ \emptyset_2(m_2,\ldots,m_p;m;$$

$$(1/\lambda_1\ -1/\lambda_2)x/2,\ldots,(1/\lambda_1\ -1/\lambda_p)x/2),$$

where $\emptyset_2$ is a confluent hypergeometric function of p-1 variables. This function and its properties are will-known in the theory of Special Functions. This function behaves like a $_1F_1$ and the series form is convergent for all values of the arguments. For a definition see Mathai and Saxena (1978, p.163). Hence the density of $y = \rho x$, $\rho > 0$, in the general case is as follows:

$$\rho g(y) = e^{-tr\underset{\sim}{\Omega}} \sum_{k=0}^{\infty}\sum_{\kappa}\{\prod_{i=1}^{p} b_{ii}^{k_i}/k_i!\}\ y^{m-1}\ e^{-y/(2\rho\lambda_1)}$$

$$\emptyset_2(m_2,\ldots,m_p;m;\ ((\rho\lambda_1)^{-1}-(\rho\lambda_2)^{-1})\ \frac{y}{2}\ ,$$

$$\ldots,((\rho\lambda_1)^{-1}-(\rho\lambda_p)^{-1})\frac{y}{2})/\{(2\rho)^m\ \Gamma(m)\lambda_1^{m_1}\ldots\lambda_p^{m_p}\}, \quad (8)$$

for $y > 0$ and $g(y)=0$ where $\kappa=(k_1,\ldots,k_p), k=k_1+\ldots+k_p$, $m_i = k_i + n/2$, $i=1,\ldots,p$, $m = m_1 + \ldots + m_p$ and $\emptyset_2$ is the confluent hypergeometric function of p-1 variables.

When one starts with ρx instead of x for $\rho > 0$ this will make a change of λ_i in $\emptyset_2$ to $\rho\lambda_i$, $i=1,\ldots,p$. This form is given in (8). Now by choosing ρ one can make the convergence of $\emptyset_2$ faster. In the central case the density of $y=\rho x$ reduces to the following form.

$$\rho g(y) = y^{(np/2)-1}\ e^{-y/(2\rho\lambda_1)}\ \emptyset_2(n/2,\ldots,n/2;np/2;$$

$$((\rho\lambda_1)^{-1}-(\rho\lambda_2)^{-1})y/2,\ldots,((\rho\lambda_1)^{-1}-(\rho\lambda_p)^{-1})y/2)$$

$$/\{(2\rho)^{np/2}\ \lambda_1^{n/2}\ldots\lambda_p^{n/2}\ \Gamma(np/2)\}, \quad (9)$$

for $y > 0$ and $g(y) = 0$ elsewhere.

4. DENSITY IN ZONAL POLYNOMIALS

In the central case the density has a simple representation which will be given first followed by the general noncentral case. In the central case one can write the moment generating function in terms of a determinant as follows.

$$\prod_{j=1}^{p}(1-2\lambda_j t)^{-n/2} = |\underset{\sim}{I}-2t\underset{\sim}{\Sigma}|^{-n/2},$$

where $\lambda_1,\ldots,\lambda_p$ are the eigen values of $\underset{\sim}{\Sigma}$. Expanding the determinant in terms of zonal polynomials one has

$$|\underset{\sim}{I}-2t\underset{\sim}{\Sigma}|^{-n/2} = |\underset{\sim}{I}-\underset{\sim}{\Sigma}/\eta + \underset{\sim}{\Sigma}(1-2\eta t)/\eta|^{-n/2}$$

$$= \eta^{np/2}|\underset{\sim}{\Sigma}|^{-n/2}(1-2\eta t)^{-np/2}|\underset{\sim}{I}-(\underset{\sim}{I}-\eta\underset{\sim}{\Sigma}^{-1})/(1-2\eta t)|^{-n/2}$$

$$= \eta^{np/2}|\underset{\sim}{\Sigma}|^{-n/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_{\kappa}}{k!}C_{\kappa}(\underset{\sim}{I}-\eta\underset{\sim}{\Sigma}^{-1})(1-2\eta t)^{(k+np/2)}, \tag{10}$$

where $\kappa=(k_1,\ldots,k_p)$, $k_1 \geq\ldots\geq k_p \geq 0$, $k_1+\ldots+k_p=k$ and C_κ is a zonal polynomial of order k. For a discussion of the zonal polynomials see James (1964). The notations are the following. $(a)_m=a(a+1)\ldots(a+m-1)$, $(a)_0=1$, $(a)_\kappa = \prod_{i=1}^{p}(a-(i-1)/2)_{k_i}$. The series in (10) is valid for a norm of the matrix $(\underset{\sim}{I}-\eta\underset{\sim}{\Sigma}^{-1})/(1-2\eta t)$ is less than unity and a sufficient condition is that $t < 1/(2\eta)$ and $\max|1-\eta/\lambda_i| < 1-2\eta t$. The validity of the expansion in (10) is guarenteed due to the presence of η and term by term inversion is possible. Hence the density function can be written as follows:

$$f(x)=\eta^{np/2}|\underset{\sim}{\Sigma}|^{-n/2}\sum_{k=0}^{\infty}\sum_{\kappa}(n/2)_{\kappa}C_{\kappa}(\underset{\sim}{I}-\eta\underset{\sim}{\Sigma}^{-1})\, x^{k-1+np/2}e^{-x/(2\eta)}$$

$$/\{k!(2\eta)^{k+np/2}\Gamma(k+np/2)\} = \{2^{np/2}|\underset{\sim}{\Sigma}|^{n/2}\,\Gamma(np/2)\}^{-1}$$

$$x^{-1+np/2}\, e^{-x/(2\eta)}\sum_{k=0}^{\infty}\sum_{\kappa}(n/2)_{\kappa}C_{\kappa}(\underset{\sim}{I}/\eta-\underset{\sim}{\Sigma}^{-1})(x/2)^{k}$$

$$/\{k!(np/2)_k\} \tag{11}$$

for $x > 0$ and $f(x)=0$ elsewhere.

For the noncentral case, using the method of factoring suggested by Pillai for deriving the joint distribution of the characteristic roots of a noncentral Wishart matrix (See Davis (1980), Eq.(6.2)) we can write

$$E(e^{t\,\mathrm{tr}\underset{\sim}{A}})=\{e^{-\mathrm{tr}\underset{\sim}{\Omega}}/|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}\,\Gamma_p(\tfrac{1}{2}n)\}\int_{\underset{\sim}{A}>\underset{\sim}{0}} e^{-\frac{1}{2}\mathrm{tr}(q-2t)\underset{\sim}{A}}|\underset{\sim}{A}|^{\frac{1}{2}(n-p-1)}$$

$$\sum_{\kappa,\lambda:\phi}^{\infty}\frac{\theta_{\phi}^{\kappa,\lambda}\,C_{\phi}^{\kappa,\lambda}}{2^{f}\,k!\ell!(\frac{1}{2}n)_{\lambda}}((q\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})\underset{\sim}{A},\underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{\Omega}\,\underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{A})\,d\underset{\sim}{A}. \tag{12}$$

Reader may refer to Davis (1980 for the definitions of $C_{\phi}^{\kappa,\lambda}(\underset{\sim}{X},\underset{\sim}{Y})$, an invariant polynomial with two matrix arguments extending the zonal polynomials, $\theta_{\phi}^{\kappa,\lambda}$, $f=k+\ell$, and other symbols. Further, Eq (5.8) of Davis (1980) has been used to obtain the form in (12). Alternately, (12) may be derived from Eq (6.2) of that paper by dediagonalization, noting the omission of the factor $e^{-\mathrm{tr}\underset{\sim}{\Omega}}$ there.

Now consider the transformation $\underset{\sim}{B}=(q-2t)\underset{\sim}{A}$; then $J(\underset{\sim}{A};\underset{\sim}{B}) = (q-2t)^{-p(p+1)/2}$. After making the above transformation in (12) and integrating w.r.t. $\underset{\sim}{B}$ using (5.5) of Davis (1980) we get

$$E(e^{t\,\mathrm{tr}\underset{\sim}{A}})=e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\sum_{\kappa,\lambda:\phi}^{\infty}\frac{\theta_{\phi}^{\kappa,\lambda}(\frac{1}{2}n)_{\phi}}{(q-2t)^{\frac{1}{2}np+f}}\,\frac{C_{\phi}^{\kappa,\lambda}(q\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1},\underset{\sim}{\Omega}\underset{\sim}{\Sigma}^{-1})}{k!\ell!(\frac{1}{2}n)_{\lambda}}. \tag{13}$$

Hence term by term inversion, guaranteed in view of the convergence factor q, gives

$$f(x)=e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\sum_{\kappa,\lambda:\phi}^{\infty}\theta_{\phi}^{\kappa,\lambda}(\tfrac{1}{2}n)_{\phi}\,C_{\phi}^{\kappa,\lambda}(q\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1},\underset{\sim}{\Omega}\underset{\sim}{\Sigma}^{-1})\frac{e^{-\frac{1}{2}qx}x^{\frac{1}{2}np+f-1}}{2^{\frac{1}{2}np+f}\Gamma(\frac{1}{2}np+f)}, \tag{14}$$

for $x > 0$ and $f(x) = 0$ elsewhere, When $\underset{\sim}{\Omega} = \underset{\sim}{0}$ (14) reduces to (11) with $q = \eta^{-1}$.

5. SOME SUMMATION FORMULAE FOR ZONAL POLYNOMIALS

Since the density of x is unique one can get some interesting summation formulae for zonal polynomials by comparing the expressions in (9) and (11). These summation formulae also establish a connection between certain zonal polynomials and hypergeometric functions of many variables. Consider the case $\rho=1$ in (9).

Theorem 1. $$e^{-x/(2\eta)} \sum_{k=0}^{\infty} \sum_{\kappa} (n/2)_\kappa C_\kappa(\underset{\sim}{I}/\eta - \underset{\sim}{\Sigma}^{-1})(x/2)^k/\{k!(np/2)_k\}$$

$$= e^{-x/(2\lambda_1)} \emptyset_2(n/2,\ldots,n/2;np/2;(1/\lambda_1-1/\lambda_2)x/2,$$
$$\ldots,(1/\lambda_1-1/\lambda_p)x/2),\ x>0,\ 0<\eta\le\lambda_i,\ 0<\lambda_1\le\lambda_i,$$

$i=1,\ldots,p$, $\emptyset_2$ is a confluent hypergeometric function of p-1 variables, $\lambda_1,\ldots,\lambda_p$ are the eigen values of the symmetric positive definite matrix $\underset{\sim}{\Sigma}$.

One can always rewrite a multiple series as follows:

$$\sum_{r_2=0}^{\infty} \cdots \sum_{r_p=0}^{\infty} (.) = \sum_{r=0}^{\infty} \sum_{r_2+\ldots+r_p=r} (.)\ .$$

Also theorem 1 is true for all $x>0$. Hence by comparing the coefficient of $(x/2)^k$ on both sides when $\eta=\lambda_1$ we have

Theorem 2. $$\sum_{\kappa} (\alpha)_\kappa C_\kappa(\underset{\sim}{I}/\lambda_1 - \underset{\sim}{\Sigma}^{-1}) = \sum_{r_2+\ldots+r_p=k} \prod_{i=2}^{p} (\alpha)_{r_i} \frac{(1/\lambda_1-1/\lambda_i)^{r_i}}{r_i!}$$

for $0<\lambda_1\le\lambda_i$, $i=2,\ldots,p$, $\kappa=(k_1,\ldots,k_p)$, $k_1+\ldots+k_p=k$, $k_1\ge\ldots\ge k_p\ge 0$, $\alpha>0$ and $(\alpha)_\kappa$ and C_κ are defined in (10).

In the central case when n is even the density of x is available from (5) as follows:

$$f(x) = \prod_{j=1}^{p}(1-2\lambda_j)^{-n/2} \sum_{j=1}^{p} \sum_{r=1}^{n/2} (-1)^r c_{jr}\, x^{r-1}\, e^{-x/(2\lambda_j)}/(r-1)! \tag{15}$$

for $x>0$ and $f(x)=0$ elsewhere, where c_{jr} is given in (4) with

$m_j = n/2$, $j=1,\ldots,p$. Comparing this with (11) we have

Theorem 3. For n an even integer, $0 < n \leq \lambda_i$, $i=1,\ldots,p$, $x > 0$

$$\sum_{j=1}^{p} \sum_{r=1}^{n/2} (-1)^{r+np/2} c_{jr} x^{r-1} e^{-x/(2\lambda_j)}/(r-1)!$$

$$= \{\Gamma(np/2)\}^{-1} x^{-1+np/2} e^{-x/(2\eta)} \sum_{k=0}^{\infty} \sum_{\kappa} (n/2)_\kappa C_\kappa(\underset{\sim}{I}-\eta\underset{\sim}{\Sigma}^{-1})(x/2\eta)^k$$

$$/\{k!(np/2)_k\},$$

where c_{jr} is given in (4) with m_j replaced by $n/2$ for all j, κ, $(n/2)_\kappa$ and C_κ are defined in (10).

Now comparing (15) with (9) for $\rho=1$ we have the following

Theroem 4. For n an even integer, λ_1 the smallest of $\lambda_i > 0$, $i=1,\ldots,p$ and for $x > 0$,

$$\sum_{j=1}^{p} \sum_{r=1}^{n/2} (-1)^{r+np/2} c_{jr} x^{r-1} e^{-x/(2\lambda_j)}/(r-1)!$$

$$= \{\Gamma(np/2)\}^{-1} x^{-1+np/2} e^{-x/(2\lambda_1)} \emptyset_2(n/2,\ldots,n/2;np/2;$$

$$(1/\lambda_1-1/\lambda_2)x/2,\ldots,(1/\lambda_1-1/\lambda_p)x/2),$$

where $\emptyset_2$ is a confluent hypergeometric function of $p-1$ variables and c_{jr} is as given in (4) with $m_j=n/2$ for all j.

Thus by making comparisons of the various representations of the density of x one can get a number of results of the types discussed above.

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