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ON BROWNIAN SLOW POINTS

by

Burgess Davis
Purdue University

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Department of Statistics
Purdue University

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Abstract

It is shown that, for a Wiener process X_t, both the quantities $\inf_t \overline{\lim}_{h\to 0+} |X_{t+h} - X_t|/\sqrt{h}$ and $\sup_t \underline{\lim}_{h\to 0+} (X_{t+h} - X_t)/\sqrt{h}$ are almost surely equal to 1.

1. Introduction. Let W_t, $t \geq 0$, be standard Brownian motion. In the 1932 paper where they showed that almost every Brownian path is nowhere differentiable ([7]), Paley, Wiener, and Zygmund proved the stronger result that, for each $\varepsilon > 0$,

$$P(\overline{\lim}_{h\to 0+} \frac{|W_{t+h} - W_t|}{h^{\frac{1}{2}+\varepsilon}} = \infty \ \forall\ t) = 1,$$

and in 1963 A. Dvoretzky ([1]) improved this by establishing

$$(1.1) \qquad P(\overline{\lim}_{h\to 0+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} > c_0 \ \forall\ t) = 1,$$

for a positive constant c_0. The natural question, whether (1.1) holds for all constants, was settled by J.-P. Kahane in 1974 ([2]). The answer is no. Kahane showed

$$(1.2) \qquad P(\exists\ t: \ \overline{\lim}_{h\to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} < c_1) = 1,$$

for a constant $c_1 < \infty$.

Here, note that h may be allowed to approach 0 from either the left or the right, giving a better result. (The two sided version of (1.1) is of course weaker than (1.1).) Kahane calls those t which satisfy

$$\overline{\lim}_{h\to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} < \infty$$

slow points.

The law of the iterated logarithm implies that the slow points almost surely have Lebesgue measure 0, but Kahane has proved ([2], [3]) that their Hausdorff dimension a.s. equals 1 and that the Hausdorff dimension of those slow points which are also zeros of W_t is a.s. $\frac{1}{2}$, so that the slow points are a fairly thick set. Kahane has very recently given another, simpler, proof of (1.2) and the two sided version of (1.1), together with related results for other Guassian processes, in [4].

Following Kahane, we will call a point t <u>slow from the right</u> if

$$\overline{\lim}_{h\to 0+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} < \infty.$$

In Section 2 we investigate the question: How slow from the right can a point be? It is shown that

$$\inf_t \overline{\lim}_{h\to 0+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} = 1 \quad \text{a.s.} \tag{1.3}$$

The proof that the expression to the left of the equality in (1.3) is no smaller than 1 is a refinement of Dvoretzky's proof in [1], while the proof that it is no larger than 1 is not related to Kahane's arguments.

Let z be the smallest positive 0 of $M(-\frac{1}{2}, \frac{1}{2}, x^2/2)$, where M is the confluent hypergeometric function ($z \approx 1.3069$). In Section 4 it is shown that

(1.4) $$P(\inf_t \overline{\lim}_{h\to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} < z) = 0,$$

but we cannot prove

$$P(\inf_t \overline{\lim}_{h\to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} = z) = 1.$$

Nonetheless, this is probably true. Not only does (1.4) hold, but also it is shown in Section 4 that, if X_t and Y_t are independent Brownian motions, and if $D_r^X = \{t: \overline{\lim}_{h\to 0+} \frac{|X_{t+h} - X_t|}{\sqrt{h}} < r\}$ and D_r^Y is defined similarly, then $D_r^X \cap D_r^Y$ is almost surely empty if $r < z$ and not empty if $r > z$.

S. Orey and J. Taylor have shown how rapid a point can be by proving (see [5])

$$\sup_t \overline{\lim}_{h\to 0+} \frac{|W_{t+h}-W_t|}{\sqrt{2h \log \frac{1}{h}}} = 1 \text{ a.s..}$$

This is equivalent to

(1.5) $$\sup_t \overline{\lim}_{h\to 0+} \frac{W_{t+h}-W_t}{\sqrt{2h \log \frac{1}{h}}} = 1 \text{ a.s.,}$$

giving a global upper bound for the lim sup, as $h \to 0+$, of $W_{t+h} - W_t$. In his book ([5], p.148) F. Knight asks for a global lower bound for this lim sup. Precisely, Knight asks for a function $\phi(h) > 0$ such that, almost surely, for all t

$$\overline{\lim}_{h\to 0+}(W_{t+h}-W_t)/\phi(h) \geq -1, \text{ and } \overline{\lim}_{h\to 0+}(W_{t+h}-W_t)/\phi(h) = -1 \text{ for some } t.$$

In Section 3 we come pretty close to solving this problem, and do give a lower bound in the sense that (1.5) gives an upper bound. We prove

$$(1.6)\qquad \inf_t \overline{\lim}_{h\to 0+} \frac{W_{t+h}-W_t}{\sqrt{h}} = -1 \text{ a.s..}$$

We do not know if the $\overline{\lim}$ equals -1 for some t.

The two sided version of (1.6) is essentially known and not hard to prove. It is

$$(1.7)\qquad \inf_t \overline{\lim}_{h\to 0} \frac{W_{t+h}-W_t}{\sqrt{|h|}} = 0 \text{ a.s.}$$

Note that the existence of times for which W_t has a local maximum shows that 0 can not be replaced by a larger number in (1.7). Only strict maxima need be considered, and, these being countable, the proof of (1.7) can be completed by examining the behavior of W_t around an absolute maximum. See [5] for a treatment of such ideas, which yields sharper results than (1.7).

Define the sets

$$A_c = \{\exists\, t \in [0,1]: |W_{t+h} - W_t| < c\sqrt{h}\ \forall\, h \in (0,1]\}$$

and

$$B_c = \{\exists\, t \in [0,1]: (W_{t+h} - W_t) > c\sqrt{h}\ \forall\, h \in (0,1]\}.$$

We will prove that $P(A_c)=0$ if $c < 1$ and $P(A_c) > 0$ if $c > 1$, implying (1.3), and that $P(B_c) = 0$ if $c > 1$ and $P(B_c) > 0$ if $c < 1$, implying $\sup_t \underline{\lim}_{h\to 0+}(W_{t+h}-W_t)/\sqrt{h} = 1$ a.s., which is equivalent to (1.6).

For the remainder of the paper the qualifier a.s. will usually be omitted.

2. Proof of (1.3). First the following lemma is established. Let $\wedge$ denote minimum.

Lemma 2.1. *If the nonnegative random variable* X *satisfies*

(2.1) $$\lim_{n\to\infty} nP(X \geq n)/EX\wedge n = 0,$$

then $EX^p < \infty$, $0 < p < 1$.

PROOF. Note that the example $P(X \geq t) = t^{-1}$, $t \geq 1$, shows that (2.1) can hold and $EX = \infty$.

Now for any nonnegative random variable Z and any $p > 0$, EZ^p is finite or infinite depending on whether $\sum_{n=1}^{\infty} 2^{np} P(Z \geq 2^n)$ is finite or infinite. Let $\gamma_n = 2^n P(X \geq 2^n)$. Then (2.1) implies

$$\lim_{n\to\infty} \gamma_n / \sum_{i=1}^{n-1} \gamma_i = 0.$$

For $\varepsilon > 0$ let $N(\varepsilon) = N$ satisfy $\gamma_n / \sum_{i=1}^{n-1} \gamma_i < \varepsilon$, $n > N$. Put $\sum_{i=1}^{N} \gamma_i = y$. Then $\gamma_{N+1} < \varepsilon y$ and $\gamma_{N+k} < \varepsilon(y + \sum_{j=1}^{k-1} \gamma_{N+j})$. If we put $\alpha_{N+k} = y(1 + \varepsilon)^k$, then $\alpha_{N+1} \geq \varepsilon y$ and $\alpha_{N+k} \geq \varepsilon(y + \sum_{j=1}^{k-1} \alpha_{N+j})$, and so, by induction,

$\gamma_{N+k} \le \alpha_{N+k} = y(1+\varepsilon)^k$, $k \ge 1$. Since ε is arbitrary, this implies $EX^p < \infty$, $p < 1$, as claimed.

Let $\tau_r(W) = \tau_r = \inf\{t \ge 1: |X_t| = r\sqrt{t}\}$. Precise information concerning the moments of τ_r may be found in Shepp, [8]. For our purposes the following lemma suffices. The notation $P_{a,b}$ and $E_{a,b}$ will signify probability and expectation associated with W_t given $W_a = b$.

Lemma 2.2. If $r > 1$ <u>there is a</u> $p = p(r) < 1$ <u>such that</u> $E\tau_r^p = \infty$. <u>Furthermore</u> $E_{1,0}\tau_1 = \infty$.

PROOF. First the well known proof of the second statement will be supplied. For a stopping time T we have

(2.2) $$EW_T^2 = ET \quad \text{if} \quad ET < \infty,$$

and applying this to the Wiener process W_{t+1} under $P_{1,0}$ yields

$$E_{1,0}W_{\tau_1}^2 = E_{1,0}(\tau_1 - 1) \quad \text{if} \quad E_{1,0}(\tau_1 - 1) < \infty.$$

Since $P_{1,0}(W_{\tau_1}^2 = \tau_1) = 1$, this implies $E_{1,0}(\tau_1 - 1) = \infty$, so $E_{1,0}\tau_1 = \infty$.

To prove the rest of the lemma, fix $r > 1$ and define

$$\gamma_1 = \inf\{t \ge 1: \ W_t = 0 \ \text{ or } |W_t| \ge r\sqrt{t}\},$$

and in general

$$\gamma_{2k} = \inf\{t \ge \gamma_{2k-1}: |W_t| \ge \sqrt{t}\}, \text{ and}$$

$$\gamma_{2k+1} = \inf\{t \ge \gamma_{2k}: \ W_t = 0 \ \text{ or } |W_t| \ge r\sqrt{t}\}.$$

Note that on $\{W_{\gamma_{2k-1}} = 0\}$, if $\lambda > 0$,

$$P(\gamma_{2k} - \gamma_{2k-1} > \lambda\gamma_{2k-1} \mid W_{\gamma_{2k-1}}) = P_{1,0}(\tau_1 > 1 + \lambda)$$

using the strong Markov property and Brownian scaling. Furthermore, if $\varepsilon = P_{1,1}(W_t = 0 \text{ before } |W_t| = r\sqrt{t})$, then, on $\{\gamma_{2k-2} < \tau_r\}$, we have, for $k \geq 2$ and $\lambda > 0$,

$$\begin{aligned}
&P(\gamma_{2k} - \gamma_{2k-1} > \lambda\gamma_{2k-2} | W_{\gamma_{2k-2}}) \\
&\quad \geq P(\gamma_{2k} - \gamma_{2k-1} > \lambda\gamma_{2k-1} | W_{\gamma_{2k-2}}, W_{\gamma_{2k-1}} = 0)\, P(W_{\gamma_{2k-1}} = 0 | W_{\gamma_{2k-2}}) \\
&\quad = P_{1,0}(\tau_1 > 1 + \lambda)\varepsilon.
\end{aligned}$$

Thus, since $\{\gamma_{2k-2} < \tau_r\} = \{\gamma_{2k-3} < \gamma_{2k-2}\}$, this gives

$$\begin{aligned}
E(\gamma_{2k} - \gamma_{2k-1})^p &\geq \varepsilon E_{1,0}(\tau_1 - 1)^p \cdot E\gamma_{2k-2}^p I(\gamma_{2k-2} < \tau_r) \\
&\geq \varepsilon E_{1,0}(\tau_1 - 1)^p E(\gamma_{2k-2} - \gamma_{2k-3})^p
\end{aligned}$$

and iteration gives

$$E(\gamma_{2k} - \gamma_{2k-1})^p \geq (\varepsilon E_{1,0}(\tau_1 - 1)^p)^{k-1} E(\gamma_2 - \gamma_1)^p$$

Pick $p < 1$ such that $\varepsilon\, E_{1,0}(\tau_1 - 1)^p > 1$. This is possible since $E_{1,0}\tau_1 = \infty$. Then $E\tau_r^p \geq E\gamma_{2k}^p \geq E(\gamma_{2k} - \gamma_{2k-1})^p \to \infty$ as $k \to \infty$.

Next put $M_W = M = \max_{0 \leq t \leq 1} |W_t|$, and $T_r^W = T_r = \inf\{t \geq 1 : |W_t| = M + r\sqrt{t}\}$.

Lemma 2.3. *If* $c < 1$, $ET_c < \infty$.

PROOF. For $t \geq 1$ the equality (2.2) gives

$$\begin{aligned}
ET_c \wedge t &= EW^2_{T_c \wedge t} \\
&\leq E(M + c\sqrt{T_c \wedge t})^2 \\
&= EM^2 + 2cEM\sqrt{T_c \wedge t} + c^2 ET_c \wedge t.
\end{aligned}$$

Thus $(1-c^2)ET_c \wedge t \leq EM^2 + 2cEM\sqrt{T_c \wedge t}$

$$\leq EM^2 + 2c(EM^2)^{\frac{1}{2}}(ET_c \wedge t)^{\frac{1}{2}}.$$

Since EM^2 is finite, $ET_c \wedge t$ must stay bounded as $t \to \infty$, so $ET_c < \infty$.

As has been mentioned, the following theorem implies (1.3).

Theorem 2.1. *If* $c < 1$, $P(A_c) = 0$, *and if* $c > 1$, $P(A_c) > 0$.

PROOF. Fix $c < 1$, and for a subinterval $[a,b] = I$ of $[0,1]$ let $\Delta_I = \{\exists\, t \in I\colon\ |X_{t+h} - X_t| < c\sqrt{h},\ 0 < h \leq 1\}$. Note that, if $M_I = \max_{a \leq t \leq b} |W_t - W_a|$, then $\Delta_I \subset \{|W_{a+h} - W_a| < M_I + c\sqrt{h},\ b-a \leq h \leq 1\}$, by a geometrical argument. Thus, conditioning on W_a and changing scale, we have

$$P(\Delta_I) \leq P(T_c \geq (b-a)^{-1}),$$

and especially, if I has length n^{-1}, $P(\Delta_I) \leq P(T_c \geq n)$. Divide $[0,1]$ into intervals I_k of length n^{-1}. Then $P(A_c) \leq \sum P(\Delta_{I_k}) \leq nP(T_c \geq n)$. Lemma 2.3 gives $ET_c < \infty$, so $nP(T_c \geq n) \to 0$, proving $P(A_c) = 0$.

Now fix $c > 1$ and put $\Gamma_n = \{\exists\, t \in [0,1]\colon\ |W_{t+h} - W_t| < c\sqrt{h},\ n^{-1} \leq h \leq 1\}$. Note that $\Gamma_n \subseteq \Gamma_m$ if $n \geq m$. We will show that $\underline{\lim}_{n\to\infty} P(\Gamma_n) > 0$, implying $P(\bigcap_{n=1}^{\infty} \Gamma_n) = P(A_c) > 0$. Put $v_{0,n} = v_0 = 0$, and, if $i \geq 1$,

$$v_{i,n} = v_i = (v_{i-1} + 1) \wedge \inf\{t \geq v_{i-1} + n^{-1}\colon\ |W_t - W_{v_{i-1}}| \geq c\sqrt{t - v_{i-1}}\}.$$

Then

$$P(v_{i+1} - v_i = 1 \mid W_{v_i}) = P(\tau_c \geq n), \text{ and} \tag{2.3}$$

$$E(v_{i+1} - v_i \mid W_{v_i}) = n^{-1} E\tau_c \wedge n. \tag{2.4}$$

Of course $v_k \geq 1$ if $v_i - v_{i-1} = 1$ for some $i \leq k$. Thus

$$\varphi_{n,m} = P(v_{i+1} - v_i = 1 \text{ for some } i \leq m \text{ such that } v_i \leq 1\}$$

$$= \sum_{i=1}^{m} P(\tau_c \geq n)\ P(v_i \leq 1)$$

$$\geq m\ P(\tau_c \geq n)\ P(v_m \leq 1).$$

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of integers approaching infinity such that $n_k P(\tau_c \geq n_k)/E\tau_c\hat{\ }n_k \geq \alpha > 0$ for all k, such a choice being possible by Lemmas 2.1 and 2.2. We also assume

$$E\tau_c\hat{\ }n_k/n_k \leq 1/6.$$

Let the integer m_k satisfy

$$1/3 \leq (m_k/n_k)E\tau_c\hat{\ }n_k \leq 1/2. \tag{2.5}$$

By (2.4),

$$Ev_{m_k,n_k} = (m_k/n_k)E\tau_c\hat{\ }n_k,$$

so

$$P(v_{m_k,n_k} \geq 1) \leq 1/2,$$

and, using the left inequality in (2.5), we have

$$P(\Gamma_{n_k}) \geq \varphi_{n_k,m_k}$$

$$\geq m_k P(\tau_c \geq n_k)P(v_{m_k,n_k} \leq 1)$$

$$\geq m_k P(\tau_c \geq n_k)/2$$

$$\geq m_k \alpha E\tau_c\hat{\ }n_k/2n_k$$

$$\geq \alpha/6.$$

3. Proof of (1.6). The arguments involving B_c are very similar to those of the last section and proofs will just be sketched. Let $\eta_a = \inf\{t \geq 0 : W_t \leq a\sqrt{t}\}$. Precise information on the moments of η_a has been supplied by Novikov in [6]. Here we need only the following analog of Lemma 2.2.

Lemma 3.1. *If* $r < 1$ *there is a* $q = q(r) < 1$ *such that* $E\eta_r^q = \infty$. *Furthermore* $E_{1,2}\eta_1 = \infty$.

PROOF. That $E_{0,1}\eta_1 = \infty$ follows from $E_{0,1}\eta_1 = E_{0,1}(W_{\eta_1} - 1)^2$ if $E_{0,1}\eta_1 < \infty$, because $P_{0,1}(W^2_{\eta_1} = \eta_1) = 1$. Since $P_{1,2}(\eta_1 - 1 > \lambda) > P_{0,1}(\eta_1 > \lambda)$, $\lambda > 0$, we get $E_{1,2}\eta_1 \geq E_{0,1}\eta_1 = \infty$.

The proof of the first assertion of Lemma 3.1 can be patterned on the proof of the first assertion of Lemma 2.2. The analogs of the times γ_i here are

$$\tilde{\gamma}_1 = \inf\{t \geq 1: \;\; W_t = 2\sqrt{t} \;\text{ or }\; W_t \leq r\sqrt{t}\},$$

and in general

$$\tilde{\gamma}_{2k} = \inf\{t \geq \tilde{\gamma}_{2k-1}: \;\; W_t \leq \sqrt{t}\}, \text{ and}$$

$$\tilde{\gamma}_{2k+1} = \inf\{t \geq \tilde{\gamma}_{2k}: \;\; W_t = 2\sqrt{t} \;\text{ or }\; W_t \leq r\sqrt{t}\}.$$

Now let $M^- = \min_{0 \leq t \leq 1} W_t$, and let $U_r = \inf\{t \geq 1: W_t = r\sqrt{t-1} + M^-\}$

Lemma 3.2. *If* $c > 1$, $EU_c < \infty$.

PROOF. For each $t > 1$, (2.2) gives

$$EU_c \wedge t = E(W_{U_c \wedge t})^2$$
$$\geq E(c\sqrt{U_c \wedge t - 1} + M^-)^2,$$

and the rest of the proof resembles the proof of Lemma 2.3.

Theorem 3.1. If $c > 1$, $P(B_c) = 0$ and, if $c < 1$, $P(B_c) \neq 0$.

PROOF. Note that if $[a,b] = I$ is a subinterval of $[0,1]$, and if $M_I^- = \min_{a \leq t \leq b} W_t - W_a$, then

$$\{\exists\, t \in [a,b]:\ W_{t+h} - W_t > c\sqrt{h}\ \forall\, h \in (0,1]\}$$
$$\subseteq \{W_{t+h} - W_t > c\sqrt{h - (b-a)} + M_I^-,\ b - a \leq h \leq 1\},$$

and the rest of the proof that $P(B_c) = 0$, $c > 1$, follows from Lemma 3.2 just like the proof that $P(A_c) = 0$, $c < 1$, followed from Lemma 3.3. Furthermore, the proof that $P(B_c) > 0$, $c < 1$, is almost the same as the proof that $P(A_c) > 0$, $c > 1$.

4. Independent Wiener processes. The arguments in this section are similar to those of Section 2, but we make use of Shepp's results in [8]. Fix $r > 0$, and for $0 < t < 2$ and $|s| < r\sqrt{t}$ let $f_{t,s}$ be the continuous version of the density of $W_2 I(\tau_r > 2)$ under $P_{t,s}$. Of course f vanishes off $(-\sqrt{2}r, \sqrt{2}r)$. Then if $\alpha(r) = \alpha = 2/P_{1,0}(\tau_r < 2)$, we have

$$(4.1) \qquad f_{1,y}(s)/f_{1,0}(s) \leq \alpha, \quad -\sqrt{2}r < s < \sqrt{2}r.$$

To see this let I be a closed subinterval of $(-r\sqrt{2}, r\sqrt{2})$ and define the set $F \subset \{(t,s):\ 1 \leq t \leq 2, |s| < r\sqrt{t}\}$ by $(t,s) \in F$ if $g(t,s) \geq g(1,y)$, where

$$g(a,b) = P_{a,b}(W_2 \in I \text{ and } \tau_r > 2).$$

Then F is a closed set containing a curve joining $(1,y)$ and the midpoint of I. Let v be the first time $(t, W_t) \in F$. Using the Strong Markov Property, we get $g(1,0) \geq g(1,y)\ P_{1,0}(v < \tau_r \wedge 2)$. For $y > 0$ we have $P_{1,0}(v < \tau_r \wedge 2) \geq P_{1,0}(\tau_r < 2,\ W_{\tau_r} > 0)$ with a similar formula for $y < 0$, so $g(1,0) \geq \alpha g(1,y)$, implying (4.1).

Similarly, we can prove that for each $y \in (-r,r)$ there is a $K(r,y) = K > 0$ such that

$$(4.2) \qquad f_{1,y}(s)/f_{1,0}(s) \geq K, \ -\sqrt{2}r < s < \sqrt{2}r.$$

Shepp shows in [8] that if z is as in Section 1 and $r > z$, there exists $\gamma(r) = \gamma \in (0, \frac{1}{2})$ such that $E_{1,0}\tau_r^\gamma = \infty$, and so, using (4.2) and conditioning on $W_{\tau_r \wedge 2}$, we have $E_{1,y}\tau_r^\gamma = \infty$ for each $y \in (-r,r)$, implying

$$(4.3) \qquad \overline{\lim}_{\lambda\to\infty} P(\tau_r > \lambda)\lambda^p = \infty \text{ for each } p > \gamma.$$

Now let X_t and Y_t be independent Wiener processes. Put $\theta_r = \tau_r(X)\wedge\tau_r(Y)$. Then $P(\theta_r > \lambda) = P(\tau_r > \lambda)^2$, so, by (4.3), $\overline{\lim}_{\lambda\to\infty} P(\theta_r > \lambda)\lambda^{2p} = \infty$ if $p > \gamma$. In particular, there is an $\alpha = \alpha(r) < 1$ such that $E\theta_r^\alpha = \infty$. Now, methods similar to those employed in Section 2 show that, for $r > z$, $P(D_r^X \cap D_r^Y \neq \emptyset) = 1$. Note the set corresponding to A_c is

$$\{\exists\ t \in [0,1]:\ |X_{t+h} - X_t| \vee |Y_{t+h} - Y_t| < r\sqrt{h}\ \forall\ h \in (0,1]\},$$

and that

$$\theta_r = \inf\{t \geq 1:\ |X_{t+h} - X_t| \vee |Y_{t+h} - Y_t| \geq r\sqrt{h}\}.$$

The sets A_r and D_r are defined in Section 1.

Shepp also proves that, if $s < z$, there exists a $\delta = \delta(s) > \frac{1}{2}$ such that $E_{1,0}\tau_s^\delta < \infty$. Conditioning on $X_{\tau_s \wedge 2}$ and using (4.1), this gives

$$(4.4) \qquad \lim_{\lambda\to\infty} \sup_{y\in(-s,s)} P_{1,y}(\tau_s > \lambda)\lambda^{\delta} < \infty$$

Now fix $r \in (0,z)$ and let $s = (r + z)/2$. Put

$$\Gamma = \Gamma_{k,n} = \{\exists\, t \in [(k/n),(k+1)/n]: \ |X_{t+h} - X_t| \vee |Y_{t+h} - Y_t| < r\sqrt{h}\ \forall\, h \in (0,1]\}.$$

Let M be the smallest integer such $(s-r)M \geq r$. Define the events $C_{j,k,n} = C_j$, $-M \leq j \leq M$, and $G_{i,k,n} = G_i$, $-M \leq i \leq M$, by

$$C_j = \{(t,X_t) \in \{(t,x): \ |x-\alpha_j| \leq s\sqrt{t-(k/n)},\ (k+1)/n \leq t \leq (k/n)+1\},$$

where $\alpha_j = X_{(k+1)/n} + (s-r)j/\sqrt{n}$, and

$$G_i = \{(t,Y_t) \in \{(t,x): \ |x-\beta_i| \leq s\sqrt{t-(k/n)}\},\ (k+1)/n \leq t \leq (k/n)+1\},$$

where $\beta_i = Y_{(k+1)/n} + (s-r)j/\sqrt{n}$.

Conditioning on $X_{(k+1)/n}$, and using Brownian scaling, we see both $P(C_j)$ and $P(G_i)$ are maximized by $\sup_{y\in(-r,r)} P_{1,y}(\tau_s > n)$ so that $P(C_j \cap G_i) = O(n^{-2\delta}) = o(n^{-1})$ by (4.4). A geometrical argument gives $\Gamma \subset \bigcup_{i,j} C_j \cap G_i$, so that $P(\Gamma) = o(n^{-1})$, yielding

$$P\Big(\bigcup_{k=0}^{n-1} \Gamma_{k,n}\Big) = o(1),$$

which implies

$$P(D_r^X \cap D_r^Y) = 0.$$

These arguments easily generalize to n independent Wiener processes, with the aid of the results in [8]. Let z_n be the smallest positive zero of $M(-1/n, 1/2, x^2/2)$, where M is the confluent hypergeometric function. Then we have

Theorem 4.1 $\bigcap_{i=1}^{n} D_r^{X_i}$ _is a.s. empty if_ $r < z_n$ _and not empty if_ $r > z_n$.

A proof of (1.4) can be made which is very similar to that of Theorem 4.1. Here it is convenient to work with a Brownian motion Z_t, $t \in (-\infty,\infty)$. We note that $S_t = Z_{(k+1)/n+t} - Z_{(k+1)/n}$, $t \geq 0$, and $R_t = Z_{k/n-t} - Z_{k/n}$, $t \geq 0$, are independent Wiener processes. Furthermore

$$\{\exists\ t \in [\tfrac{k}{n}, \tfrac{k+1}{n}]: \ |Z_{t+h} - Z_t| < r\sqrt{|h|}\ \forall\ h \in (0,1] \cup [-1,0)\}$$

can be shown to be contained in a set defined in terms of S_t and R_t in a manner similar to the way $UC_i \cap G_j$ was defined in terms of X_t and Y_t earlier, and thereby shown to have probability equal to $o(1/n)$ if $r < z$, from which we get

$$P(\exists\ t \in [0,1]: \ |Z_{t+h} - Z_t| < r\sqrt{|h|}\ \forall\ h \in (0,1] \cup [-1,0)) = 0,$$

which is equivalent to (1.4).

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