EXACT SOLUTIONS FOR THE FULL INFORMATION BEST CHOICE PROBLEM

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Abstract

Arguments are given to justify three aspects of a classical optimal stopping problem presented somewhat informally by Gilbert and Mosteller in 1966: Why the optimal stopping rule must be of the type they considered; why the optimal probability of best choice decreases with increasing sample size; and how the limiting optimal best choice probability can be expressed both analytically and probabilistically.

KEY WORDS: Optimal stopping; Backward induction; Secretary problem.

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1. PROBLEM AND SOLUTION

This is a footnote to a classic problem which was elegantly presented in (Gilbert and Mosteller 1966, section 3), but with a few loose ends. Recently it has turned out that filling in those gaps has been helpful for both understanding and extending the problem. The hope is that it will be even more helpful to have the clarifications put down all together in one place.

Here is how Gilbert and Mosteller introduced the problem:

> "One by one, a sample of n measurements is drawn from a population with continuous cumulative distribution F. The continuity assures that ties have probability zero. After each draw, the player, who knows F and n, is informed of its value, whereupon he must decide whether or not to choose that draw. He is to maximize the probability of choosing the draw with the largest measurement in the sample."

A complete solution to this problem can be described as follows: Let the measurements be $X_1, X_2, \ldots, X_n$, and the optimal stopping rule for each n be τ_n; n=1,2,... Then there is a single sequence of decision numbers $\{b_m: m=0,1,2,\ldots\}$, not depending on n, such that

$$\tau_n = \min_{1\le i\le n}\{i: X_i=\max(X_1,\ldots,X_i) \text{ and } F(X_i)\ge b_{n-i}\}. \tag{1.1}$$

Naturally b_0 is zero; for $m \geq 1$, the b_m's are the solutions to

$$1 = \sum_{j=1}^{m} j^{-1}\binom{m}{j}(b_m^{-1}-1)^j \tag{1.2a}$$

or, equivalently,

$$\sum_{j=1}^{m} j^{-1}b_m^{-j} = 1 + \sum_{j=1}^{m} j^{-1} \quad m=1,2,\ldots \tag{1.2b}$$

As one would expect, $b_1 = 1/2$ and the b_m's are increasing to one as m (which represents the number of draws remaining) becomes infinite. In fact

$$m(b_m^{-1}-1) \to c \approx .80435 \tag{1.3}$$

where c is the solution to

$$\sum_{j=1}^{\infty} c^j/j!j = 1. \tag{1.4}$$

Let $\{w_n\}$ be the probabilities of "winning"; i.e.,

$$w_n = P(X_{\tau_n}=\max\{X_1,\ldots,X_n\}).$$

Then

$$w_n \text{ is strictly } \underline{\text{decreasing}} \text{ in } n \tag{1.5}$$

$$\lim_{n\to\infty} w_n = e^{-c} - (e^c-c-1)\int_1^{\infty} x^{-1}e^{-cx}dx \tag{1.6}$$

$$\approx .580164.$$

Indeed

$$\lim_{n\to\infty} w_n = P(Z(1-T)<c<[Z+Z'/T][1-TT']) \qquad (1.7)$$

where Z, Z', T, and T' are mutually independent random variables with Z and Z' each exponentially distributed with parameter one, and T and T' each uniformly distributed on the interval (0,1).

2. OPTIMAL STOPPING RULES

Gilbert and Mosteller examined only stopping rules of the following form:

(a) "Corresponding to each draw, assign a decision number. As the drawing proceeds, choose as the largest the first candidate [largest value seen so far] whose value exceeds its decision number."

(b) "The decision numbers obviously decrease as we go through the draws, because with fewer draws to go, we have less chance of getting a high number."

The best rules satisfying constraints (a) and (b), they showed, are given by (1.1) and (1.2). To actually demonstrate that these are the best of <u>all</u> stopping rules is quite easy and rewarding to do by following the standard method of backward induction, as explained in, e.g. (Chow, Robbins, and Siegmund 1971).

To begin with, since the distribution is known exactly, and since the largest measurement in a sample remains the largest under all monotonic transformations of its variable, we lose no generality by assuming that F is the standard uniform: $F(x)=x$ on $0 \leq x \leq 1$. We let τ denote any stopping rule; $M_0 \equiv 0$, and, for $i=1,2,\ldots,n$,

$$M_i = \max(X_1,\ldots,X_i) \quad i=1,2,\ldots,n$$

$$w_n = \max_\tau P(X_\tau=M_n)$$

$$Z_i = P(X_i=M_n|X_1,\ldots,X_i) = X_i^{n-i} \quad \text{if} \quad X_i > M_{i-1}$$
$$= 0 \quad \text{if} \quad X_i < M_{i-1} .$$

Now probability equals expectation of conditional probability, so

$$P(X_\tau=M_n) = EZ_\tau ,$$

hence

$$w_n = \max_\tau EZ_\tau .$$

It is well known that w_n is attained by the stopping rule

$$\tau_n = \min_{1\leq i\leq n}\{i: Z_i \geq \gamma_i^{(n)}\}$$

where $\gamma_n^{(n)} \equiv 0$ and, for $i=n-1,n-2,\ldots,1,0$,

$$\gamma_i^{(n)} \equiv \gamma_i^{(n)}(X_1,\ldots,X_i) = \max_{\tau>i} E(Z_\tau|X_1,\ldots,X_i);$$

and that the $\gamma_i^{(n)}$'s obey:

$$\gamma_{i-1}^{(n)} = E\{\max(Z_i,\gamma_i^{(n)})|X_1,\ldots,X_{i-1}\}$$
$$= E\{\max(X_i^{n-i},\gamma_i^{(n)})I_{\{X_i>M_{i-1}\}}|M_{i-1}\} + E\{\gamma_i^{(n)}I_{\{X_i<M_{i-1}\}}|M_{i-1}\}$$

for $i=n,n-1,\ldots,1$. In particular, we get

$$\gamma_{n-1}^{(n)} = P(X_n=M_n|X_1,\ldots,X_{n-1}) = 1 - M_{n-1}$$

and

$$\gamma_0^{(n)} = w_n = E\{\max[X_1^{n-1},\gamma_1^{(n)}(X_1)]$$

$$= \int_0^1 \max[x^{n-1},\gamma_1^{(n)}(x)]dx.$$

One can easily check that the solutions to these systems of equations are of the form

$$\gamma_i^{(n)}(X_1,\ldots,X_i) = f_{n-i}(M_i) \quad i=n-1,n-2,\ldots,1$$

$$\gamma_0^{(n)} = f_n(0)$$

where

$$f_1(x) = 1 - x$$

$$f_{k+1}(x) = xf_k(x) + \int_x^1 \max[y^k,f_k(y)]dx. \tag{2.1}$$

We see at once that the $f_k(x)$'s are strictly decreasing from w_k to zero as x increases from zero to one. So for each k there is a unique b_k in (0,1) such that $f_k(b_k) = b_k^k$. Moreover the b_k's are strictly increasing in k (decreasing in n - k) because

$$f_{k+1}(b_k) > f_k(b_k) > b_k^{k+1}.$$

Thus

$$\{Z_i \geq \gamma_i^{(n)}\} = \{X_i=M_i \geq b_{n-i}\}$$

which shows that the optimal stopping rule does indeed satisfy constraints (a) and (b).

Actual computation of the b_k's is a straightforward matter once we know they are a monotone sequence. Either analytically, from (2.1), or probabilistically from the fact that

$$M_i \geq b_{n-i} \text{ and } \tau_n > i$$

$$\Longrightarrow$$

$$\tau_n = \text{first } j > i \text{ (if any) such that } X_j > M_i,$$

we get

$$f_k(x) = \sum_{j=1}^{k} j^{-1}\binom{k}{j}(1-x)^j x^{k-j} \quad \text{if} \quad x \geq b_k$$

$$= EN^{-1} I_{\{N \geq 1\}}$$

where

$$N = \sum_{i=1}^{k} I_{\{X_i > x\}}.$$

Setting $x = b_k$ gives (1.2a).

Another approach, via Markov chains, rather than backward induction, was given in (Bojdecki 1978).

Formulas (1.3) and (1.4) are contained in Gilbert and Mosteller's paper and can be obtained by letting $c_m = m(b_m^{-1}-1)$ so (1.2a) becomes

$$1 = \sum_{j=1}^{\infty} [(c_m)^j \prod_{i=1}^{j} (1-im^{-1})]/j!j \ .$$

The square bracket terms are all bounded by one and $\Sigma(j!j)^{-1}$ is convergent so the dominated convergence theorem can be employed to show that for any convergent subsequence $c_{m_k} \to c$, c must satisfy (1.4).

3. MONOTONICITY OF OPTIMAL BEST CHOICE PROBABILITIES

That the probabilities w_n are decreasing in n was noticed (for their tabled values) by Gilbert and Mosteller, but not proved. It is not a trivial matter because, while the bigger n is the more chances we have to choose, to "win" we have to choose the biggest of a larger sample. Indeed, no direct analytic proof is known.

The following proof adapts a trick from (Chow, et al 1964) which is also used in (Samuels 1981, page 195). The trick is to exhibit a randomized rule, for picking the best of n, which has success probability bigger than w_{n+1}, but (of course) no bigger than w_n. The way to do it is to first consider n + 1 observations, and then condition on (i.e., be "told" the values of) the arrival time, say σ, of the worst (i.e., smallest) of $X_1,\ldots,X_{n+1}$, and on its value, X_σ. Now, given σ and X_σ, the other n X_i's are conditionally independent, each uniform on $(X_\sigma, 1)$, so among all stopping rules τ adapted to σ, X_σ and the remaining X_i's the best one clearly has success probability precisely w_n. Within this class is the rule

$$\begin{aligned} \tilde{\tau} &= \tau_{n+1} \quad \text{if} \quad \tau_{n+1} \neq \sigma \\ &= n + 1 \quad \text{if} \quad \tau_{n+1} = \sigma \end{aligned}$$

which chooses the exact same observation as τ_{n+1} except when τ_{n+1} chooses the worst one (this, of course, can only happen on a subset of the event $\{\sigma=1\}$) so it has a slightly greater chance than τ_{n+1} of choosing the best. So $\tilde{\tau}$ is as advertised.

4. LIMITING OPTIMAL BEST CHOICE PROBABILITY

Gilbert and Mosteller obtained the numerical value .580164 for $\lim_{n\to\infty} w_n$ by passing to the limit in a formula for the probability of winning at any given draw, which led to some computer-assisted numerical integrations. Essentially the same argument they used to get their formula can be employed to get (1.7), as follows:

For convenience we consider an infinite sequence $X_1, X_2, \ldots$ of independent standard uniform random variables and let σ_n and σ_n' be, respectively, the "arrival times" of the largest of the first n X_i's and of the largest prior to σ_n; i.e.,

$$X_{\sigma_n} = M_n \text{ and } X_{\sigma_n'} = M_{\sigma_n - 1} .$$

Then, because b_{n-i} is decreasing in i while M_i is increasing in i, it follows immediately that

$$P(X_{\tau_n} = M_n) = P(M_n > b_{n-\sigma_n} \text{ and } M_{\sigma_n - 1} > b_{n-\sigma_n'}). \tag{4.1}$$

Now we make the change of variables:

$$Z_n = n(1-M_n) \ ; \ T_n = \sigma_n/n$$

$$Z_n' = (\sigma_n - 1)(1-M_{\sigma_n - 1}/M_n) \ ; \ T_n' = \sigma_n'/(\sigma_n - 1)$$

and let $c_m = m(b_m^{-1}-1)$ again so that (4.1) becomes

$$P(X_{\tau_n}=M_n) = P(A_n \cap B_n)$$

where

$$A_n = \{Z_n(1-T_n+n^{-1}c_{n(1-T_n)})<c_{n(1-T_n)}\}$$

and, letting

$$K_n = n(1-T_nT_n') + T_n' ,$$

$$B_n = \{c_{K_n}<[Z_n+(T_n-\frac{1}{n})^{-1}(1-n^{-1}Z_n][1-T_nT_n'+n^{-1}(c_{K_n}+T_n')]\} .$$

Using familiar properties of the uniform distribution, one can verify the weak convergence result

$$(Z_n,Z_n',T_n,T_n') \xrightarrow{\mathcal{D}} (Z,Z',T,T')$$

where Z,Z',T, and T' are as described following (1.7). This in turn implies (1.7) itself.

To get from (1.7) to (1.6) we first condition on $Z = z$, $T = t$, and $T'= t'$; the conditional probability is

$$e^{-t(c/(1-tt')-z)^+} I_{\{z<c/(1-t)\}}.$$

Integrating this multiplied by the exponential density of Z yields the conditional probability given $T = t$ and $T'= t'$ which is

$$(1-t)^{-1}e^{-ct/(1-tt')}(1-e^{-c(1-t)/(1-tt')}) + e^{-c/(1-tt')} - e^{-c/(1-t)}.$$

The final step of integrating this expression over the unit square requires the change of variables

$$u = (1-t)/(1-tt'), \quad v = 1/(1-tt')$$

on all but the last term. This, with the help of (1.4) and the identity

$$\int_0^1 u^{-1}(e^{cu}-1) = \sum_{k=1}^{\infty} c^k/k!k ,$$

yields the expression

$$\int_0^{\infty} \int_0^1 v^{-2}(v-u)^{-1}e^{-c(v-u)}dudv.$$

Letting $w = v - u$ and interchanging the order of integration then leads directly to (1.6). Numerical evaluation of (1.6) is easy from the identity

$$\int_1^{\infty} x^{-1}e^{-cx}dx = |\log c| - \gamma - \sum_{j=1}^{\infty} (-c)^j/j!j$$

where γ is Euler's constant $\approx$ .577216.

The technique just described has been used in (Petrucelli 1980 and 1982) and (Campbell and Samuels 1981) for generalizations of this best choice problem.

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