A NOTE ON THE BEHAVIOR OF SAMPLE STATISTICS WHEN THE POPULATION MEAN IS INFINITE

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ABSTRACT: Let $X_i \geq 0$ be i.i.d. random variables with $E(X_i) = \infty$. Then for suitable functions φ we have $\frac{\overline{\varphi(X)}}{\varphi(\overline{X})} \to 0$ a.s. We give some applications of this result.

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1. The Main Theorem.

Let us prove the following:

Theorem 1: If $X_1, X_2, \ldots$ are i.i.d., $X_i \geq 0$. $EX_1 = \infty$, and if φ is a function such that

(1.1) There exist constants A and B such that $a_i \geq B$, $i = 1,2,\ldots n$, implies

$$\sum_{i=1}^{n} \frac{\varphi(a_i)}{n} \leq A \ \varphi\left(\frac{\sum_{i=1}^{n} a_i}{n}\right),$$

(1.2) $\varphi(x) \to \infty$ as $x \to \infty$,

(1.3) There exist constants C, x_0 and α, $\alpha < 1$. Such that $\varphi(\lambda x)/\varphi(x) \leq C\lambda^{\alpha}$ for $\lambda \geq 1$, $x \geq x_0$, and $\varphi(x)$ is bounded for $x \leq x_0$.

then

$$R_n = \frac{\frac{1}{n}\sum_{i=i}^{n} \varphi(X_i)}{\varphi(\frac{1}{n}\sum_{i=i}^{n} X_i)} \xrightarrow{a.s.} 0.$$

Note: (a) Following the same argument of Mulholland [4], Theorem 1, we have: (1.1) is equivalent to

(1.4) there exist constants A and B and a concave function ψ, such that

$$\psi(x) \leq \varphi(x) \leq A\,\psi(x) \quad \text{for all} \quad x \geq B.$$

(b) Condition (1.3), according to the terminology of Bingham and Goldie [3], is:

(1.5) The upper Matuszewska index of φ is less than 1.

For properties connected with (1.3), see Drasin and Shea [1] and Bingham and Goldie [2], [3].

Proof:

Let d be a positive number. Let p_n be the proportion of i's, $i \leq n$, such that $X_i > d$. For n sufficiently large, $p_n > 0$. Let us assume this is the case.

Then:

$$R_n = \frac{\frac{1}{n} \sum_{X_i \leq d} \varphi(X_i) + \frac{1}{n} \sum_{X_i > d} \varphi(X_i)}{\varphi\left(\frac{1}{n} \sum_{i=i}^{n} X_i\right)}$$

$$\leq \frac{\frac{1}{n} \sum_{X_i \leq d} K_d}{\varphi\left(\frac{1}{n} \sum_{i=i}^{n} X_i\right)} + \frac{\frac{1}{j} \sum_{X_i > d} \varphi(X_i)}{\varphi\left(\frac{1}{j} \sum_{X_i > d} X_i\right)} \; \frac{\varphi\left(\frac{1}{j} \sum_{X_i > d} X_i\right)}{\varphi\left(\frac{1}{n} \sum_{i=i}^{n} X_i\right)} \cdot \frac{j}{n}$$

$$= T_1 + T_2 \cdot T_3 \cdot \frac{j}{n} , \quad \text{say,}$$

where K_d comes from condition (1.3) since (1.3) implies φ is bounded in any finite interval.

Since $E(X_1) = \infty$, T_1 approaches 0 a.s. as $n \to \infty$ by condition (1.2) and the strong law of large numbers.

Let $j = n\, p_n = \# \{i : X_i > d,\ i = 1,2,\ldots, n\}$. Then condition (1.1) implies T_2 is bounded by A.

Since

$$(1.7)\qquad \frac{1}{n}\sum_{X_i > d} X_i \le \frac{1}{n}\sum_{i=1}^{n} X_i \le \frac{1}{j}\sum_{X_i > d} X_i ,$$

for n sufficiently large with probability 1,

$$(1.8)\qquad 1 \le \frac{\frac{1}{j}\sum_{X_i > d} X_i}{\frac{1}{n}\sum_{i=1}^{n} X_i} \le \frac{n}{j} .$$

Apply (1.3) and (1.8)

$$(1.9)\qquad T_3 = \frac{\varphi\left(\frac{1}{j}\sum_{X_i > d} X_i\right)}{\varphi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)} \le C\left(\frac{n}{j}\right)^{\alpha} .$$

Hence

$$(1.10)\qquad T_2 \cdot T_3 \cdot \frac{j}{n} \le A\, C \left(\frac{j}{n}\right)^{1-\alpha} .$$

Notice that since $\frac{j}{n} \to P(X_i > d)$ a.s., it follows that if we choose d large enough then R_n is eventually less than any positive number with probability 1, q.e.d.

2. Some applications.

If $\varphi(X) = X^{\mu}L(X)$ where $0 < \mu < 1$ and $L(X)$ is a slowly varying function, i.e. $\lim_{X\to\infty} \frac{L(\lambda X)}{L(X)} = 1$ for all $\lambda > 0$, then $\varphi(X)$ satisfies (1.2), 1.3) and (1.4).

Theorem 2: <u>If</u> $X_1, X_2, \ldots$ <u>are i.i.d.</u>, $X_i \geq 0$, $EX_1 = \infty$, <u>and</u> $\varphi(x) = x^{\mu} L(x)$ <u>for some</u> $0 < \mu < 1$ and slowly varying function L <u>then</u>:

$$(2.1) \qquad R_n = \frac{\frac{1}{n} \sum_{i=1}^{n} \varphi(X_i)}{\varphi(\frac{1}{n} \sum_{i=1}^{n} X_i)} \xrightarrow{\text{a.s.}} 0.$$

An easy corollary of Theorem 2 is:

Corollary 3. <u>If</u> $X_1, X_2, \ldots$ <u>are i.i.d.</u>, $X_i \geq 0$, $EX_1 = \infty$, <u>and</u> $0 < \mu < 1$, <u>then</u>

$$(2.2) \qquad \frac{\frac{1}{n} \sum_{i=1}^{n} X_i^{\mu}}{(\frac{1}{n} \sum_{i=1}^{n} X_i)^{\mu}} \xrightarrow{\text{a.s.}} 0.$$

Corollary 4: <u>If $Y_1, Y_2, \ldots$ are i.i.d.</u>, $p > 1$, <u>and</u> $E|Y_1|^p = \infty$, <u>then</u>

$$(2.3) \qquad \frac{(\frac{1}{n} \sum_{i=i}^{n} Y_i)^p}{\frac{1}{n} \sum_{i=i}^{n} |Y_i|^p} \xrightarrow{\text{a.s.}} 0.$$

Proof: Let $\varphi(x) = x^{1/p}$ and apply Theorem 2 to the i.i.d. random variables $|Y_1|^p, |Y_2|^p, \ldots, |Y_n|^p, \ldots$. We have

$$(2.4) \qquad \frac{\frac{1}{n} \sum_{i=i}^{n} |Y_i|}{(\frac{1}{n} \sum_{i=i}^{n} |Y_i|^p)^{1/p}} \xrightarrow{\text{a.s.}} 0. \qquad \text{q.e.d.}$$

For the special case p = 2, it is easy to see from Corollary 4 that when the second moment of the population does not exist, the ratio of the sample mean to the sample standard deviation approaches 0 almost surely as the sample size increases.

It is possible to apply the theorem to compare the growth rates of some familiar statistics.

Proposition 5: Let $X_1, X_2, \ldots X_n, \ldots$ be i.i.d. random variables. If $EX_1^2 = \infty$ then

$$(2.5) \qquad \frac{\frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} X_i X_j}{\frac{1}{n} \sum X_i^2} \xrightarrow{a.s.} 0.$$

Proof: $(X_1+X_2+\ldots+X_n)^2 = \sum X_i^2 + 2 \sum_{1 \le i<j \le n} X_i X_j$

Hence: $$\frac{\left(\frac{X_1+X_2+\ldots+X_n}{n}\right)^2}{\frac{1}{n}\sum X_i^2} = \frac{1}{n} + \frac{2\binom{n}{2} \frac{1}{\binom{n}{2}} \sum_{1 \le i<j \le n} X_i X_j}{n^2 \frac{1}{n} \sum X_i^2}$$

Applying corollary 4, we get (2.5). q.e.d.

Notice that $\frac{1}{\binom{n}{2}} \sum_{1 \le i<j \le n} X_i X_j$ is the U - statistic of the kernel $\Phi(x_1,x_2) = x_1 x_2$, and $\frac{1}{\binom{n}{2}} \sum X_i^2$ is the U - statistic of the kernel $\tilde{\Phi}(x) = \tilde{\Phi}(x,x) = x^2$. Following the same type of argument, we have the following theorem.

Theorem 6: If k > 1,

$\Phi(x_1, x_2, \ldots, x_k) = x_1 x_2 \ldots x_k$

$\tilde{\Phi}(x) = \Phi(x,x,\ldots,x)$,

$U_n(\Phi)$, $U_n(\tilde{\Phi})$ are the U - statistics of the kernel function Φ and $\tilde{\Phi}$ respectively.

If $E\,\tilde{\Phi}\,(X_1) = \infty$, *then*

$$(2.6)\qquad \frac{U_n(\Phi)}{U_n(\tilde{\Phi})} \xrightarrow{\text{a.s.}} 0.$$

Corollary 7. *Let* $\Phi(x_1,\ldots,x_k)$ *be a symmetric polynomial in* $x_1,\ldots,x_k$, *with all coefficients* ≤ 0, *and*

$$(2.7)\qquad \frac{\Phi(x,1,\ldots,1)}{\tilde{\Phi}(x)} \to \mu \text{ if } x \to \infty.$$

Let $X_1,\ldots,X_n,\ldots$ *be i.i.d. non-negative random variables with* $E(\tilde{\Phi}(X_1)) = \infty$. *Then*

$$(2.8)\qquad \frac{U_n(\Phi)}{U_n(\tilde{\Phi})} \to k\mu \leq 1 \quad \text{a.s.}$$

Another application of Theorem 1 to compare the growth rates of statistics is:

Theorem 8: Let $X_1,X_2,\ldots,X_n,\ldots$ *be* i.i.d. *with* $E|X_1|^p = \infty$ *for some* $p > 1$. *Then*

$$(2.9)\qquad \frac{\sum_{i=1}^{n} |X_i-\overline{X}|^p}{\sum_{i=1}^{n} |X_i|^p} \xrightarrow{\text{a.s.}} 1$$

where $\quad \overline{X} \equiv \frac{1}{n}\sum_{i=1}^{n} X_i.$

(The result (2.9) does not always hold for $p = 1$; by a different argument the ratio is asymptotically between $1-\varepsilon$ and 2 a.s. for all $\varepsilon > 0$).

Proof: Since

$$(2.10)\qquad \left(\sum_{i=1}^{n}|X_i|^p\right)^{1/p} - \left(\sum_{i=1}^{n}|\overline{X}|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}|X_i-\overline{X}|^p\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{n}|X_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|\overline{X}|^p\right)^{1/p} ,$$

and

$$(2.11)\qquad \frac{\left(\sum_{i=1}^{n}|\overline{X}|^p\right)^{1/p}}{\left(\sum_{i=1}^{n}|X_i|^p\right)^{1/p}} \le \left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^p}{\frac{1}{n}\sum_{i=1}^{n}|X_i|^p}\right]^{1/p} \xrightarrow{a.s.} 0,$$

the result follows.

Corollary 9: Let $X_1, X_2, \ldots X_n, \ldots$ be i.i.d., S_n be the sample standard deviation. $h_n = c\, S_n\, n^{-\lambda}$, $\lambda > 0$.

Then

$$(2.12)\qquad h_n \xrightarrow{a.s.} 0$$

if and only if

$$(2.13)\qquad E|X_1|^{\frac{2}{1+2\lambda}} < \infty.$$

Proof: For $E|X_1| < \infty$.

$$(2.14)\qquad \frac{\sum(X_i-\overline{X})^2}{n^{1+2\lambda}} = \frac{\sum X_i^2}{n^{1+2\lambda}} - \frac{n(\overline{X})^2}{n^{1+2\lambda}} ,$$

Hence

(2.15) $$\frac{\Sigma(X_i-\overline{X})^2}{n^{1+2\lambda}} \xrightarrow{a.s.} 0 \quad \text{iff} \quad \frac{\Sigma X_i^2}{n^{1+2\lambda}} \xrightarrow{a.s.} 0$$

For $E|X_1| = \infty$

(2.16) $$\frac{\Sigma(X_i-\overline{X})^2}{n^{1+2\lambda}} = \frac{\Sigma X_i^2}{n^{1+2\lambda}} \quad \frac{\Sigma(X_i-\overline{X})^2}{\Sigma X_i^2} .$$

Applying Theorem 2 with $p = 2$

(2.17) $$\frac{\Sigma(X_i-\overline{X})^2}{n^{1+2\lambda}} \xrightarrow{a.s.} 0 \text{ iff } \frac{\Sigma X_i^2}{n^{1+2\lambda}} \xrightarrow{a.s.} 0.$$

Then apply the Marcinkiewicz-Zygmund Strong Law of large numbers.

(2.18) $$\frac{\Sigma X_i^2}{n^{1+2\lambda}} \xrightarrow{a.s.} 0 \quad \text{iff} \quad E|X^2|^{1/1+2\lambda} < \infty \qquad \text{q.e.d.}$$

Finally, if we regard Corollary 4 as a strengthened result of the Cauchy-Schwartz Inequality under stronger conditions, the following theorem strengthens the familiar arithmetic mean-geometric mean inequality.

Theorem 10: Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d., $X_1 \geq 0$, then a necessary and sufficient condition for

(2.19) $$\frac{(X_1 X_2 \ldots X_n)^{1/n}}{\frac{1}{n}\sum_{i=1}^{n} X_i} \xrightarrow{a.s.} 0$$

is

(2.20) $$E(X_1 - \log X_1) = \infty.$$

Proof:

Condition (2.20) is equivalent to

(2.21) $$EX_1 = \infty \text{ or } E \log X_1 = -\infty.$$

If $EX_1 = \infty$, then

(2.22) $$(X_1 X_2 \ldots X_n)^{1/n} = [(X_1^{1/2} X_1^{1/2} X_2^{1/2} X_2^{1/2} \ldots X_n^{1/2} X_n^{1/2})^{1/2n}]^2$$

$$\leq [\frac{1}{2n} \; 2 \sum_{i=1}^{n} X_i^{1/2}]^2$$

$$= [\frac{1}{n} \sum_{i=1}^{n} X_i^{1/2}]^2.$$

and

$$\frac{[\frac{1}{n} \sum_{i=1}^{n} X_i^{1/2}]^2}{\frac{1}{n} \sum_{i=1}^{n} X_i} \xrightarrow{a.s.} 0.$$

Hence (2.19) holds.

If $EX_1 < \infty$ and $E \log X_1 = -\infty$, apply the strong law of large numbers to the logarithm of the numerator, and (2.19) also holds in this case.

Suppose (2.19) is true, then either $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} \infty$ or $(X_1 X_2, \ldots, X_n)^{1/n} \xrightarrow{a.s.} 0$; applying the strong law of large numbers, we get $EX_1 = \infty$ or $E \log X_1 = -\infty$.

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