ON THE LIMIT BEHAVIOR OF CERTAIN QUANTITIES IN A SUBCRITICAL STORAGE MODEL

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ABSTRACT. The limit behavior of the content of a subcritical storage model defined on a semi-Markov process is examined. This is achieved by creating a renewal equation using a regeneration point $(i_0,0)$ of the process. By showing that the expected return time to $(i_0,0)$ is finite, the conditions needed for the basic renewal theorem are established. The joint asymptotic distribution of the content of the storage at time t and the accumulated amount of the unmet (lost) demands during (0,t] is then established by showing the asymptotic independence of these two.

KEY WORDS. STORAGE MODELS; TOTAL DEMAND LOST; MARKOV CHAINS; SEMI-MARKOV PROCESSES; ϕ-IRREDUCIBILITY; ERGODICITY; RENEWAL THEORY; STORAGE LEVEL.

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1. INTRODUCTION. The present work deals with a storage model which allows both inputs as well as releases occurring in random amounts and at random times according to an underlying semi-Markov process. While the reader may find other types of storage models elsewhere in the literature (see Moran [12], Prabhu [15], Lloyd [10], Ali Khan and Gani [1], for references) the present model is along the lines of Puri and Woolford [17], which itself is a generalization of a model considered previously by Senturia and Puri ([19], [20]) and Balagopal [3]. A special case of these models can be found in an earlier work (see Puri and Senturia [16]) which relates such models to a live sutuation arising in biology. The purpose of the present work is to answer a question left open by these authors in the so-called 'subcritical' case of these models and is concerned with the limit behavior of the storage level for the continuous-time case. As will become evident, in order to prove the main

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results (other more direct probabilistic approaches having failed), it was found essential to employ simultaneously a battery of several existing tools including a renewal theory argument. Needless to say, the approach adopted here is powerful enough to be found useful in other similar situations. To begin with we start with a brief description of our model.

All our random variables will be considered as defined on a given underlying basic probability space $(\Omega,\mathcal{A},P)$. Let $\mathcal{J}$ be a subset of nonnegative integers and $\{X_n, n=0,1,2,\ldots\}$ be a positive recurrent, aperiodic, irreducible Markov chain with state space $\mathcal{J}$, transition matrix $\underset{\sim}{P}=(p_{ij})$ and the stationary probability measure $\{\pi_i\}$. Let $\{T_n, n=0,1,2,\ldots\}$ be a nondecreasing sequence of random variables with $T_o\equiv 0$, such that for all $i,j\in\mathcal{J}$,

$$(1.1)\qquad P(X_n = j,\ T_n - T_{n-1} \le t \mid T_o,X_o,T_1,X_1,\ldots,T_n,X_n=i) = A_{ij}(t),$$

where $A_{ij}(t)$ is a nondecreasing right continuous function of t satisfying

$$(1.2)\qquad \begin{cases} \text{(i)} & A_{ij}(0) = 0 \\ \text{(ii)} & 0 \le p_{ij} = A_{ij}(\infty) \le 1 \\ \text{(iii)} & \sum_{j\in\mathcal{J}} p_{ij} = 1, \text{ for } i\in\mathcal{J}. \end{cases}$$

The process $\{(X_n,T_n), n=0,1,2,\ldots\}$ as defined above is the usual Markov renewal process with the semi-Markov matrix $\underset{\sim}{A}(t) = (A_{ij}(t))$ (see Çinlar [5]), with the distribution function of the sojourn times in a state $i \in \mathcal{S}$ being given by

$$(1.3) \qquad B_i(t) \equiv P(T_n - T_{n-1} \le t | X_{n-1} = i) = \sum_{j \in \mathcal{S}} A_{ij}(t),$$

and the corresponding moments by

$$(1.4) \qquad m_i^{(k)} = \int_0^\infty t^k dB_i(t), \; k=1,2,\ldots,$$

where for simplicity we shall write $m_i^{(1)} = m_i$. Again with each $i \in \mathcal{S}$ we associate a sequence $\{U_n(i),\ n=0,1,2,\ldots\}$ of I.I.D. real valued random variables, which are assumed to be independent of $\{(X_n, T_n),\ n=0,1,2,\ldots\}$ and of $\{U_n(j),\ n=0,1,2,\ldots\}$ for $j \ne i$, with $E|U_1(i)| < \infty$, $\forall\, i \in \mathcal{S}$. With these we define for $n \ge 1$,

$$(1.5) \qquad \begin{cases} Z_n = \max(0, Z_{n-1} + U_n(X_n)) \\ \\ L_n = L_{n-1} - \min(0, Z_{n-1} + U_n(X_n)), \end{cases}$$

with $L_0 \equiv 0$. From (1.5) it easily follows that

$$(1.6) \qquad \begin{cases} Z_n = \max[Z_0 + \sum_{i=1}^{n} U_i(X_i),\ \max_{1 \le j \le n} (\sum_{i=j+1}^{n} U_i(X_i))], \\ \\ L_n = Z_n - Z_0 - \sum_{i=1}^{n} U_i(X_i). \end{cases}$$

Note that if at time T_n we consider $(U_n(X_n))^+$ as an input into the storage and $(U_n(X_n))^-$ as a demand for output from the storage, then it follows from (1.5) that Z_n represents the storage level at time T_n and L_n represents the cumulative amount of demands that were not met and hence lost during the time interval $(0,T_n]$. Finally to define these quantities for an arbitrary time t, we let

$$(1.7) \qquad (X(t),Z(t),L(t)) \equiv (X_{M(t)},Z_{M(t)},L_{M(t)}),$$

where

$$(1.8) \qquad M(t) = \sup\{n: \; T_n \leq t\}.$$

Also, we let

$$(1.9) \qquad \beta = \sum_{i\in \mathcal{J}} \pi_i m_i,$$

and

$$(1.10) \qquad E_\pi U = \sum_{i\in \mathcal{J}} \pi_i E(U_1(i)).$$

We shall adopt the terminology of saying that we are in the <u>subcritical case</u>, <u>critical</u> or the <u>supercritical case</u> according as $E_\pi U$ is less than, equal to or greater than zero. In [3], [17], [19] and [20], various authors studied the limit behavior of quantities such as Z(t) and L(t) but only for the critical and supercritical cases. The methods used by these authors did not lend themselves to study the joint limit behavior of $(Z(t),L(t))$ for the subcritical case. Consequently this question was left open and will now be studied using a different approach based on a renewal equation argument. Thus throughout the paper we assume that $E_\pi U < 0$. Section 2 deals with some preliminary results to be used later. In section 3, we establish the ergodicity of the process $\{(X_n,Z_n)\}$. Section 4 deals with the study of asymptotic behavior of $\{X(t),Z(t)\}$ via a renewal equation. Finally in section 5, the asymptotic independence of $Z(t)$ and $L(t)$ appropriately normalized is established. The joint asymptotic behavior of $Z(t)$ and the normalized $L(t)$ follows then from those of their marginals.

2. SOME PRELIMINARY RESULTS.

The purpose of this section is to present several topics and techniques that will be used in later sections. Well known results are presented here for the sake of completeness, with references to where the appropriate proofs may be found.

Let $\{(\hat{X}_n,\hat{T}_n),\ n=0,1,\ldots\}$ be a Markov renewal process, taking values on $\mathcal{J} \times [0,\infty)$, and independent of any of the variables thus far defined. Let the associated semi-Markov matrix $\hat{A}(t) = \hat{A}_{ij}(t))$ be defined for each pair $i,j \in \mathcal{J}$ by

$$(2.1) \qquad \hat{A}_{ij}(t) = \frac{\pi_j}{\pi_i} A_{ji}(t).$$

Let the initial distribution of X_o be the stationary measure π.

DEFINITION. $\{(\hat{X}_n,\hat{T}_n)\}$ as defined above is said to be the dual Markov renewal process for $\{(X_n,T_n)\}$. Likewise $\{\hat{X}_n\}$ is called the dual Markov chain for $\{X_n\}$.

PROPOSITION 2.1. *Let the initial distribution of X_o be π. Then, for all $n \geq 0$ and $m \geq 1$, we have*

$$P(X_n = j_o,\ U_n(j_o) \in A_o,\ X_{n+1} = j_1, U_{n+1}(j_1) \in A_1,\ T_{n+1} - T_n \leq s_1,$$

$$\ldots, X_{n+m} = j_m,\ U_{n+m}(j_m) \in A_m,\ T_{n+m} - T_{n+m-1} \leq s_m)$$

$$= P(\hat{X}_n = j_m,\ U_n(j_o) \in A_m,\ \hat{X}_{n+1} = j_{m-1},\ U_{n+1}(j_{m-1}) \in A_{m-1},$$

$$\hat{T}_{n+1} - \hat{T}_n \leq s_m, \ldots, \hat{X}_{n+m} = j_o,\ U_{n+m}(j_o) \in A_o,\ \hat{T}_{n+m} - \hat{T}_{n+m-1} \leq s_1),$$

where $0 \leq s_\ell \leq \infty$, $j_\ell \in \mathcal{J}$, $A_\ell \in \mathcal{B}$, for $0 \leq \ell \leq m$, and $\mathcal{B}$ denotes the Borel σ-field on the real line. (For a proof see Woolford [22]).

From the above proposition it can be seen that, in some sense, the dual is a 'reversal' of the original process.

Let $\{\bar{X}_n, n=0,1,\ldots\}$ be another Markov chain defined on $(\Omega,\mathcal{G},P)$ with state space $\mathcal{J}$, transition matrix $\underset{\sim}{P}$, and initial distribution $\underset{\sim}{\pi}$. Then, if we define $T = \min\{n > 0:\ X_n = \bar{X}_n\}$, Hoel, Port and Stone [7] have established:

PROPOSITION 2.2. For any initial distribution of X_o, we have

i) $T < \infty$, a.s.

ii) $P(X_n = j_o, X_{n+1} = j_1, \ldots, X_{n+m} = j_m, T \leq n)$

$$= P(\bar{X}_n = j_o, \bar{X}_{n+1} = j_1, \ldots, \bar{X}_{n+m} = j_m, T \leq n)$$

for $n \geq 0$, $m \geq 0$ and $j_\ell \in \mathcal{S}$, $0 \leq \ell \leq m$.

From the above proposition, we note that after T, the chains $\{X_n\}$ and $\{\bar{X}_n\}$ become 'probabilistically indistinguishable'.

DEFINITION. The process $\{\bar{X}_n, n=0,1,\ldots\}$ as defined above will be referred to as the auxiliary Markov chain for the chain $\{X_n, n=0,1,\ldots\}$.

For the rest of this section, let $\{X_n\}$ be a Markov chain which takes values in some arbitrary space $(S, \mathfrak{F})$ with homogeneous transition probabilities

$$(2.2) \qquad P(x,A) = P(X_n \in A | X_{n-1} = x), \ \forall n \geq 1, \ x \in S, \ A \in \mathfrak{F}.$$

Let $P^n(x,A) = P(X_n \in A | X_o = x)$. Then for all n, we have

$$(2.3) \begin{cases} \text{i)} & P^n(\cdot,A) \text{ is a measurable function on } S, \ \forall A \in \mathfrak{F}, \\ \text{ii)} & P^n(x,\cdot) \text{ is a probability measure on the } \sigma\text{-field } \mathfrak{F}, \ \forall x \in S. \end{cases}$$

Let ϕ be a non-trivial σ-finite measure on $\mathfrak{F}$.

<u>DEFINITION</u>. $\{X_n\}$ is called ϕ-irreducible if, whenever $\phi(A) > 0$, for $A \in \mathfrak{F}$, then $\sum_{n=1}^{\infty} 2^{-n} P^n(x,A) > 0, \forall x \in S$.

<u>DEFINITION</u>. A σ-finite non-trivial measure μ on $\mathfrak{F}$ is called subinvariant for $\{X_n\}$ if $\mu(A) \geq \int \mu(dy)P(y,A), \forall A \in \mathfrak{F}$, and called invariant if strict equality holds.

The following theorem can be found in (Jain and Jamison [8]).

<u>THEOREM 2.3</u>. <u>If</u> $\{X_n\}$ <u>is a ϕ-irreducible Markov chain, then a subinvariant measure</u> μ <u>exists, where</u> $\mu \gg \phi$.

For a fixed subinvariant measure μ, define

$$\text{(2.4)} \qquad \mathfrak{F}_\mu = \{A \in \mathfrak{F} : \ 0 < \mu(A) < \infty\}.$$

The following lemmas can be found in (Tweedie [2]).

<u>LEMMA 2.4</u>. <u>If</u> $\{X_n\}$ <u>is</u> ϕ-<u>irreducible, then either</u>

(1) $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m(x,A) = 0, \forall x \in S, \forall A \in \mathfrak{F}_\mu$, <u>or</u>

(2) <u>there exists a finite invariant measure</u>. <u>If</u> (2) <u>is true, we call</u> $\{X_n\}$ <u>ergodic, and the finite invariant measure</u> $\pi(\cdot)$ <u>with</u> $\pi(S) = 1$, <u>we call the stationary measure</u>.

LEMMA 2.5. If $\{X_n\}$ is ergodic, then for any $A \in \mathfrak{F}$ with $\phi(A) > 0$, there is a ϕ-null set $N(A)$ such that $\forall\, x \notin N(A)$, $P(\bigcup_{n=1}^{\infty} X_n \in A | X_o = x) = 1$.

Let $\{(X_n, T_n)\}$ be a semi-Markov process where $\{X_n\}$ is defined on an arbitrary state space $(S, \mathfrak{F})$, (for details see Çinlar [6]).

Let

(2.5) $$P(X_n \in A,\ T_n - T_{n-1} \leq t | T_o, X_o, \ldots, T_{n-1}, X_{n-1} = y)$$
$$= P(X_n \in A,\ T_n - T_{n-1} \leq t | X_{n-1} = y) = H_{yA}(t).$$

Let $\phi(\cdot)$ be a σ-finite measure defined on $(S, \mathfrak{F})$. Then $H_{yA}(t)$ must satisfy the following regularity conditions.

(2.6)

- i) $H_{xA}(t)$ as a function of x is ϕ-measurable, for all $A \in \mathfrak{F}$ with $\phi(A) > 0$, $\forall\, t$,
- ii) $H_{xA}(t)$ as a function of t is right continuous, nondecreasing $\forall\, x \in S$, $\forall\, A \in \mathfrak{F}$ with $\phi(A) > 0$,
- iii) $H_{xA}(t) \leq 1$, $\forall\, t$, $\forall\, x$, $\forall\, A \in \mathfrak{F}$ with $\phi(A) > 0$, and $H_{xS}(\infty) = 1$, $\forall\, x \in S$,
- iv) $H_{xA}(t)$ as a function of A is a positive, finite measure on $(S, \mathfrak{F})$ $\forall\, x \in S$, $\forall\, t$.

Let

(2.7) $$T_x^A = \inf\{t > T_1 :\ X(t) \in A | X(0) = x\}, \ \forall\, x \in S, \ \forall\, A \in \mathfrak{F}.$$

Let us define

$$(2.8)\quad \begin{cases} \text{i)} \quad P(x,A) = P(X_n \in A \mid X_{n-1}=x) = H_{xA}(\infty), \\ \text{ii)} \quad {}_B p^n(x,A) = P(X_n \in A,\ X_{n-1} \notin B, \ldots, X_1 \notin B \mid X_o = x) \end{cases}$$

$$= \int_{B^c} \int_{B^c} \cdots \int_{B^c} P(x,dy_1)P(y_1,dy_2)\ldots P(y_{n-2},dy_{n-1})P(y_{n-1},A).$$

Also, for all $x \in S$, let Y_x be a random variable such that for all t,

$$(2.9)\qquad P(Y_x \leq t) = P(T_n - T_{n-1} \leq t \mid X_{n-1} = x) = H_{xS}(t).$$

The following theorem appears new, as we were unable to find it in the literature.

THEOREM 2.5. *For a semi-Markov process* $\{(X_n,T_n)\}$, *if* $\{X_n\}$ *is* ϕ-*irreducible, and if* $A \in \mathfrak{F}$ *with* $\phi(A) > 0$, *then*

$$ET_x^A = EY_x + \sum_{n=1}^{\infty} \left(\int_{A^c} {}_A p^n(x,dy)EY_y\right), \ \forall x \in S.$$

PROOF. Let us define

$$(2.10)\quad\begin{cases} \text{i)} & E(A,\lambda|x) = \int_0^\infty e^{-\lambda t} P(T_x^A > t)dt, \\ \text{ii)} & \theta_{xA}(\lambda) = \int_0^\infty e^{-\lambda t} H_{xA}(dt) \\ \text{iii)} & \psi_x(\lambda) = \int_0^\infty e^{-\lambda t} P(Y_x > t)dt \\ \text{iv)} & {}_B\theta_{xA}^{(n)}(\lambda) \\ & = \int_{B^c}\int_{B^c}\cdots\int_{B^c}\theta_{xdy_1}(\lambda)\theta_{y_1dy_2}(\lambda)\cdots\theta_{y_{n-2}dy_{n-1}}(\lambda)\theta_{y_{n-1}A}(\lambda). \end{cases}$$

Then

$$(2.11)\qquad P(T_x^A > t) = P(Y_x > t) + \int_{A^c}\int_0^t H_{xdy}(d\tau)\, P(T_y^A > t-\tau).$$

Thus

$$(2.12)\qquad E(A,\lambda|x) = \int_0^\infty e^{-\lambda t} P(Y_x > t)dt + \int_0^\infty e^{-\lambda t}\left(\int_{A^c}\int_0^t H_{xdy}(d\tau) P(T_y^A > t-\tau)\right)dt$$

$$= \psi_x(\lambda) + \int_{A^c}\int_0^\infty e^{-\lambda t}\left(\int_\tau^\infty e^{-\lambda(t-\tau)}\, P(T_y^A > t-\tau)dt\right)H_{xdy}(d\tau)$$

$$= \psi_x(\lambda) + \int_{A^c} \theta_{xdy}(\lambda)\, E(A,\lambda|y).$$

By iteration, we obtain

$$(2.13)\qquad E(A,\lambda|x) = \psi_x(\lambda) + \int_{A^c} \theta_{xdy}(\lambda)\psi_y(\lambda) + \int_{A^c}\int_{A^c} \theta_{xdy}(\lambda)\theta_{ydz}(\lambda)E(A,\lambda|z).$$

More generally, we have

(2.14) $$E(A,\lambda|x) = \psi_x(\lambda) + \sum_{n=1}^{N} \int_{A^c} {}_A\theta_{xdy}^{(n)}(\lambda)\psi_y(\lambda) + \int_{A^c} {}_A\theta_{xdy}^{(N+1)}(\lambda)E(A,\lambda|y).$$

Since ${}_A\theta_{xB}^{(n)}(\lambda) \leq {}_AP^n(x,B)$, $\forall\ \lambda \geq 0$, if we let

$M(A) = \{x:P(\bigcup_{n=1}^{\infty} X_n \in A|X_o = x) = 1\}$, we obtain $\forall\ x \in M(A)$, as $n \to \infty$,

$${}_A\theta_{xB}^{(n)}(\lambda) \leq {}_AP^n(x,S) = P(\bigcap_{m=1}^{n-1} X_m \notin A|X_o = x) \to 0.$$

Thus, $\forall\ x \in M(A)$, since $E(A,\lambda|y) \leq 1/\lambda$, $\forall\ y$, $\lambda > 0$, we have

(2.15) $$E(A,\lambda|x) = \psi_x(\lambda) + \sum_{n=1}^{\infty} \int_{A^c} {}_A\theta_{xdy}^{(n)}(\lambda)\psi_y(\lambda).$$

Now by letting $\lambda \downarrow 0$, we obtain for all $x \in M(A)$,

(2.16) $$ET_x^A = EY_x + \sum_{n=1}^{\infty} \int_{A^c} {}_AP^n(x,dy)EY_y.$$

For $x \notin M(A)$, we have that as $n \to \infty$,

$${}_AP^n(x,A^c) = P(\bigcap_{m=1}^{n}(X_m \notin A)|X_o = x) \downarrow c > 0.$$

Thus, except for the trivial case for which $A^c = B_1 \cup B_2$ and where ${}_AP^n(x,B_1) \to 0$ and for all $y \in B_2$, $EY_y = 0$, we have $\underline{\lim} \int_{A^c} {}_AP^n(x,dy)EY_y > 0$. Consequently, $ET_x^A = \infty$ and $\sum_{n=1}^{\infty} \int_{A^c} {}_AP^n(x,dy)EY_y = \infty$, so that (2.16) still holds.□

<u>THEOREM 2.6</u>. <u>If the Markov chain</u> $\{X_n\}$ <u>is ergodic, then for</u> $A\in \mathfrak{F}$, <u>we have</u> $\int_A ET_x^A\ \mu(dx) \leq \int_S EY_y\ \mu(dy)$, <u>for a finite invariant measure</u> μ, <u>with equality if for</u> $C = \{x\in A^c\colon \lim_{n\to\infty} {}_AP^n(x,A^c) = 0\}$, <u>we have</u> $\mu(C) = \mu(A^c)$.

<u>PROOF</u>. From Lemma 2.4, $\exists$ a σ-finite measure μ satisfying $\mu(E) = \int\mu(dy)P(y,E)$, $\forall\ E\in \mathfrak{F}$. From (Orey [14], p.33) we have, for $N\geq 1$ and $E\in \mathfrak{F}$,

$$\mu(E) = \sum_{n=1}^{N} \int_A {}_AP^n(x,E)\ \mu(dx) + \int_{A^c} {}_AP^{N+1}(x,E)\ \mu(dx). \tag{2.17}$$

Thus it follows that $\mu(E) \geq \int_A \left(\sum_{n=1}^{\infty} {}_AP^n(x,E)\right) \mu(dx)$. Using Theorem 2.5, we obtain

$$\int_A ET_x^A\ \mu(dx) = \int_A EY_x\ \mu(dx) + \int_{A^c} \left[\int_A \left(\sum_{n=1}^{\infty} {}_AP^n(x,dy)\right) \mu(dx)\right] EY_y \tag{2.18}$$

$$\leq \int_A EY_x\ \mu(dx) + \int_{A^c} \mu(dy)\ EY_y = \int_S EY_x\ \mu(dx).$$

Now, if $C = \{x\in A^c\colon \lim_{n\to\infty} {}_AP^n(x,A^c) = 0\}$ and $\mu(C) = \mu(A^c)$, it is easy to see that for all $B\subset A$, $\mu(B) = \int_A \left(\sum_{n=1}^{\infty} {}_AP^n(x,B)\right) \mu(dx)$, from which it follows that (2.18) is a strict equality.□

Note: This result generalizes a well known result (see Orey [14]), which states if $\{X_n\}$ is an ergodic Markov Chain on an arbitrary state space then $\int_A ET_x^A \pi(dx) = \int_S \pi(dx) = 1$; and also that of Çinlar [5], which, for $\{(X_n, T_n)\}$, a semi-Markov chain on a denumerable state space $\mathcal{J}$, gives $\pi_i ET_i^i = \sum_{j \in \mathcal{J}} EY_j \pi_j$, for $i \in \mathcal{J}$.

3. THE ERGODICITY OF $\{(X_n, Z_n)\}$.

We begin by defining

$$(3.1)\quad \begin{cases} S_n = \sum_{i=1}^{n} U_i(X_i) \\ \hat{S}_n = \sum_{i=1}^{n} U_i(\hat{X}_i) \\ \bar{S}_n = \sum_{i=1}^{n} U_i(\bar{X}_i). \end{cases}$$

Using this and (1.6) it follows that

$$Z_n = \max(Z_0 + S_n, \max_{1 \leq j \leq n} (S_n - S_j)).$$

The following lemma is needed for establishing the ergodicity of the process $\{(X_n, Z_n)\}$.

LEMMA 3.1. *If* $E_\pi U < 0$, *then there exists an* $i_0, n, \varepsilon > 0$ *and* $\delta > 0$ *such that*

$$P(S_n < -\varepsilon, \max(0, S_1, \ldots, S_n) = 0, X_n = i_0 | X_0 = i_0) > \delta.$$

PROOF. From (Chung [4]), it follows that for an arbitrary initial distribution of X_n, $n^{-1}S_n \to E_\pi U$, a.s., as $n \to \infty$, where $E_\pi U < 0$. Thus for all i, there is an n such that

$$(3.2) \qquad P(S_n < -\varepsilon,\ X_n = i \mid X_o = i) > \varepsilon.$$

As such, there must exist a sequence $j_1, j_2, \ldots, j_{n-1}$ such that

$$(3.3) \qquad P(S_n < -\varepsilon,\ X_n = i,\ X_{n-1} = j_{n-1}, \ldots, X_1 = j_1 \mid X_o = i) > \eta,$$

for some $\eta > 0$. Thus, there is an m, $0 \le m \le n$ such that

$$(3.4) \qquad P(S_n < -\varepsilon,\ S_m = \max_{1 \le j \le n} (S_j),\ X_n = i,\ X_{n-1} = j_{n-1}, \ldots, X_1 = j_1 \mid X_o = i) > \eta/n.$$

It follows from (3.4) that

$$(3.5) \qquad P(S_n < -\varepsilon,\ X_n = i, \ldots, X_1 = j_1,\ \bigcap_{\ell=m}^{n} [S_\ell - S_m \le 0],\ \bigcap_{\ell=1}^{m} [S_n - S_m + S_\ell \le 0] \mid X_o = i)$$

$$= P(S_n \le -\varepsilon,\ \max_{0 \le j \le n} (S_j) = 0,\ X_n = j_m, X_{n-1} = j_{m-1},$$

$$\ldots, X_{n-m} = i,\ X_{n-m-1} = j_{n-1}, \ldots, X_1 = j_{m+1} \mid X_o = j_m) > \eta/n.$$

Finally, letting $i_o = j_m$, $\delta = \eta/n$, the lemma follows.□

COROLLARY 3.1. If $E_\pi U < 0$, then for i_o, n, ε, and δ as in Lemma 3.1., we have

$$P(S_{nN} < -\varepsilon N, \max_{0\leq j\leq nN}(S_j) = 0, X_{nN} = i_o | X_o = i_o) > \delta^N.$$

PROOF. Evidently, using Lemma 3.1, we have for any N

$$P(S_{nN} < -\varepsilon N, \max_{0\leq j\leq nN}(S_j) = 0, X_{nN} = i_o | X_o = i_o)$$

$$\geq P(S_n \leq -\varepsilon, \max_{0\leq j\leq n}(S_j) = 0, X_n = i_o, S_{2n} - S_n < -\varepsilon,$$

$$\max_{n<j\leq 2n}(S_j - S_n) = 0, X_{2n}=i_o \ldots, S_{Nn} - S_{Nn-n} < -\varepsilon,$$

$$\max_{nN-n<j\leq nN}(S_j - S_{nN-n}) = 0, X_{Nn} = i_o | X_o = i_o)$$

$$= [P(S_n < -\varepsilon, \max_{0\leq j\leq n}(S_j) = 0, X_n = i_o | X_o = i_o)]^N > \delta^N. \square$$

LEMMA 3.2. If $E_\pi U < 0$ and X_o has initial distribution $\underset{\sim}{\pi}$, then there exists an $\eta > 0$ such that for all $A > 0$, there is an N_A such that for $n > N_A$, we have

$$P(\max(\max_{0\leq j\leq n}(S_j), S_n + A) = 0) > \eta.$$

PROOF. First, choose i_o and n as in Lemma 3.1. Then, since

$n^{-1}S_n \to E_\pi U$, a.s., where $E_\pi U < 0$, for every $\gamma > 0$, there is a $B > 0$ such that

$$(3.6) \qquad P(\sup_{j\geq 1}(S_j) < B \mid X_o = i_o) > (1-\gamma).$$

From Corollary 3.1 and expression (3.6), we have for $m > n[\frac{B}{\varepsilon} + 1]$,

$$(3.7) \qquad P(\max(0,S_1,\ldots,S_m) = 0)$$

$$\geq \pi_{i_0} P(S_{n[\frac{B}{\varepsilon}+1]} < -B, \max_{0\leq j\leq n[\frac{B}{\varepsilon}+1]}(S_j) = 0, X_{n[\frac{B}{\varepsilon}+1]} = i_o,$$

$$\max_{n[\frac{B}{\varepsilon}+1]\leq i\leq m}(S_i - S_{n[\frac{B}{\varepsilon}+1]}) < B \mid X_o = i_o)$$

$$\geq \pi_{i_o} \delta^{[\frac{B}{\varepsilon}+1]}(1-\gamma).$$

Now let $2\eta = \pi_{i_0} \delta^{[\frac{B}{\varepsilon}+1]}(1-\gamma)$. Then, there is an $N_A > n[\frac{B}{\varepsilon} + 1]$, such that for $m > N_A$, we have

$$(3.8) \qquad \sum_{j\in \mathcal{S}} \pi_j P(S_m > -A \mid X_o = j) < \eta.$$

Finally using (3.7) and (3.8), it follows that

$$P(\max(\max(0,S_1,\ldots,S_m),S_m + A) = 0)$$

$$\geq P(\max(0,S_1,\ldots,S_m) = 0) - P(S_m > -A) > \eta,$$

which completes the proof.□

LEMMA 3.3. *If $E_\pi U < 0$, then for an arbitrary distribution of (X_0, Z_0), there exist a $\delta > 0$ and an N such that for $n > N$, we have $P(Z_n = 0) > \delta$.*

PROOF. Let $T = \inf\{n > 0: X_n = \bar{X}_n\}$. Then $T < \infty$, a.s., by Proposition 2.2. Now choose K_1, K_2, such that $P(T > K_1) < \eta/3$ and $P(Z_{K_1} > K_2) < \eta/3$ hold, where η is as in Lemma 3.2. Then for $n > K_1$, we have

$$(3.9)\qquad P(Z_n = 0) \geq P(Z_n = 0,\ T \leq K_1, Z_{K_1} \leq K_2)$$

$$= P(\max(S_n - S_{K_1} + Z_{K_1}, \max_{K_1+1 \leq i \leq n} (S_n - S_i)) = 0,\ T \leq K_1, Z_{K_1} \leq K_2)$$

$$\geq P(\max(K_2 + S_n - S_{K_1}, \max_{K_1+1 \leq i \leq n} (S_n - S_i)) = 0,\ T \leq K_1) - P(Z_{K_1} > K_2).$$

Now, from the definition of a dual and auxiliary process, we have on first using Proposition 2.2 and then Proposition 2.1,

$$(3.10)\qquad P(\max(K_2 + S_n - S_{K_1}, \max_{K_1+1 \leq i \leq n} (S_n - S_i)) = 0,\ T \leq K_1)$$

$$\geq P(\max(K_2 + \bar{S}_n - \bar{S}_{K_1}, \max_{K_1+1 \leq i \leq n} (\bar{S}_n - \bar{S}_i)) = 0) - P(T > K_1)$$

$$= P(\max(K_2 + \hat{S}_{n-K_1}, \max_{0 \leq i < n-K_1} (\hat{S}_i)) = 0) - P(T > K_1).$$

Thus, for all $n > N_{K_2} + K_1$, it follows from Lemma 3.2 that

$$P(Z_n = 0) \geq \eta - P(Z_{K_1} > K_2) - P(T > K_1) \geq \eta/3. \square$$

Now to establish that $\{(X_n, Z_n)\}$ is an ergodic Markov chain, we need to appeal to the lemmas in section 2. However, to use these we must define a ϕ-measure and the state space.

Let $S = \mathcal{I} \times [0,\infty)$ and $\mathfrak{F}$ be the ϕ-field generated by the sets $B = (\{i\} \times [0,x])$, $i \in \mathcal{I}$, $x \geq 0$. Let

(3.11) $$P^n((j,x),A) = P((X_n, Z_n) \in A \mid X_0 = j,\ Z_0 = x), \text{ for } A \in \mathfrak{F}.$$

Also, for i_0 of Lemma 3.1, define

(3.12) $$\phi(A) = \sum_{n=1}^{\infty} 2^{-n} P^n((i_0, 0), A)$$

We then have,

<u>LEMMA 3.4.</u> *If* $E_\pi U < 0$, *then the chain* $\{(X_n, Z_n)\}$ *is* ϕ-*irreducible for* ϕ *as defined in* (3.12).

<u>PROOF.</u> For every $A \in \mathfrak{F}$ with $\phi(A) > 0$, we need to establish that

$$\sum_{n=1}^{\infty} 2^{-n} P^n((j,x), A) > 0, \text{ for } j \in \mathcal{I},\ x \geq 0.$$

For every j and $x \geq 0$, there is an n_j and a B such that

$$P(X_{n_j} = i_o,\ Z_{n_j} \le B | X_o = j,\ Z_o = x) > 0.$$

We denote this positive probability by $\delta_j(x)$.

Then for $m = n_j + n[\frac{B}{\varepsilon} + 1]$ (where n,ε, are as in Corollary 3.1), we have

(3.13) $$P(X_m = i_o,\ Z_m = 0 | X_o = j,\ Z_o = x)$$

$$\ge P(X_m = i_o,\ \max(Z_{n_j} + S_m - S_{n_j},\ \max_{n_j \le i \le m} (S_m - S_i)) = 0,$$

$$Z_{n_j} \le B,\ X_{n_j} = i_o | X_o = j,\ Z_o = x)$$

$$\ge P(X_{n_j} = i_o,\ Z_{n_j} \le B | X_o = j,\ Z_o = x)$$

$$\cdot\ P(X_{m-n_j} = i_o, \max(B+S_{m-n_j},\ \max_{0 \le i \le m-n_j} (S_{m-n_j} - S_i)) = 0 | X_o = i_o)$$

$$= \delta_j(x) P(\hat{X}_{m-n_j} = i_o,\ \max(B+\hat{S}_{m-n_j},\ \max_{0 \le i \le m-n_j} (\hat{S}_i)) = 0 | \hat{X}_o = i_o)$$

$$\ge \delta_j(x)\delta^{[\frac{B}{\varepsilon} + 1]}.$$

From this it follows that $\sum_{n=1}^{\infty} 2^{-n}P^n((j,x),A) > 0$, for all j,x.□

Now that we have established that there is a ϕ-measure such that $\{(X_n,Z_n)\}$ is ϕ-irreducible, we have from Theorem 2.3 that there exists a subinvariant measure μ. As such, we can use the lemmas in section 2 to establish the ergodicity of $\{(X_n,Z_n)\}$.

THEOREM 3.5. If $E_\pi U < 0$, the Markov chain $\{(X_n,Z_n)\}$ is ergodic.

PROOF. In view of Lemma 2.4, we only need to show that for some (j,x) and some $A \in \mathfrak{F}_\mu$ (see (2.4) for definition of $\mathfrak{F}_u$), that

$\underline{\lim}(n^{-1} \sum_{i=1}^{n} P^i((j,x),A)) > 0.$

Applying Lemma 3.1 to the dual process, choose an i_o such that

$$(3.14) \qquad P(\max_{0 \le j \le n} (S_n - S_j) = 0,\ X_n = i_o | X_o = i_o)$$

$$= P(\max_{0 \le j \le n} (\hat{S}_j) = 0,\ \hat{X}_n = i_o | \hat{X}_o = i_o) > 0.$$

Let us take $(j,x) = (i_o,0)$ and $A = (\{i_o\} \cup \{1,2,\ldots,M\}) \times \{0\}$, where M is such that

$$\pi_{i_o} + \sum_{i=1}^{M} \pi_i > 1 - \delta/4,$$

for δ of Lemma 3.3.

Since $\phi((i_o,0)) > 0$ from (3.14), and $\mu >> \phi$ (see Theorem 2.3), we have that $\mu(A) > 0$. Also clearly by construction $\mu(A) < \infty$, so that $A \in \mathfrak{F}_\mu$.

Letting $B = \{i_o\} \cup \{1,\ldots,M\}$, we pick an N such that for $n > N$, we have for $i \in B$, (3.15) $$P(X_n = i | X_o = i_o) > \pi_i - \delta(4(M+1))^{-1}.$$

Let N_1 be as in Lemma 3.3, so that

$$P(Z_n = 0 | X_o = i_o, Z_o = 0) > \delta,$$

for $n > N_1$. Then for $n > \max(N, N_1)$, we have

(3.16) $$P(X_n \in B, Z_n = 0 | X_o = i_o, Z_o = 0)$$

$$> P(Z_n = 0 | X_o = i_o, Z_o = 0) - P(X_n \notin B | X_o = i_o, Z_o = 0)$$

$$> \delta - [1 - \sum_{i \in B} (\pi_i - \delta(4(M+1))^{-1})] > \delta/2.$$

Thus $\underline{\lim}(n^{-1} \sum_{i=1}^{n} P^i((i_o,0),A)) > \delta/2$, and in view of Lemma 2.4, this establishes that $\{X_n, Z_n\}$ is ergodic.□

4. LIMIT BEHAVIOR OF $\{(X(t),Z(t))\}$ VIA A RENEWAL EQUATION.

We define

(4.1) $$T^A_{j,x} = \inf\{t > T_1 : (X(t), Z(t)) \in A | X(0) = j, Z(0) = x\}.$$

Then it can be readily seen that for i_o of section 3,

$$(4.2) \qquad P(X(t) = j,\ Z(t) \le x \mid X(0) = i_0,\ Z(0) = 0)$$

$$= P(X(t) = j, Z(t) \le x,\ T_{i_0,0}^{(i_0,0)} > t \mid X(0) = i_0, Z(0) = 0)$$

$$+ \int_0^t P(X(t-\tau) = j,\ Z(t-\tau) \le x \mid X(0) = i_0, Z(0) = 0)\, dP(T_{i_0,0}^{(i_0,0)} \le \tau).$$

Equation (4.2) is therefore our basic renewal equation for the process $(X(t),Z(t))$.

Under some appropriate conditions, the basic renewal theorem (see Karlin and Taylor [9]), will now yield the desired asymptotic behavior of $\{(X(t),Z(t))\}$ using (4.2). As a first step, in satisfying the conditions needed for the basic renewal theorem, we prove below that $E(T_{i_0,0}^{(i_0,0)}) < \infty$.

THEOREM 4.1. *If* $E_\pi U < 0$, *and* $\beta < \infty$, *then* $ET_{i_0,0}^{(i_0,0)} < \infty$, *for* i_0 *of Lemma 3.1 and* β *as in* (1.9).

PROOF. Note that $\beta = \sum \pi_i m_i$, where $m_i = \int_0 t\, dB_i(t)$, and $B_i(t) = P(T_n - T_{n-1} \le t \mid X_{n-1} = i)$.

Again the observation that $\{(X_n, Z_n, T_n)\}$ is a semi-Markov process on a general state space allows us to apply the results of section 2. In the notation used there, we define for every $i \in \mathcal{J}$ and $x \ge 0$, a non-negative variable Y_{ix} with

$$(4.3) \qquad P(Y_{ix} \le t) = P(T_n - T_{n-1} \le t \mid X_{n-1} = i, Z_{n-1} = x) = P(T_n - T_{n-1} \le t \mid X_{n-1} = i).$$

Here we have used the fact that $\{T_n - T_{n-1}\}$ and $\{Z_n\}$ are conditionally independent given $\{X_n\}$. Thus $Y_{ix} = Y_{iy}$, for $x,y \ge 0$, so that we label the common value as Y_i, for each i.

Now Theorem 3.5 guarantees the existence of an invariant probability positive measure π for $\{(X_n,Z_n)\}$, since $\{(X_n,Z_n)\}$ is ergodic. Letting $A = (i_0,0)$ and applying theorem 2.6, we have

$$(4.4)\qquad \pi(i_0,0)ET_{i_0,0}^{(i_0,0)} \leq \int_{\mathcal{J} \times [0,\infty)} EY_{(x,y)}\pi(dx,dy) = \sum_{i\in\mathcal{J}} EY_i\pi(i,[0,\infty))=\sum\pi_i m_i < \infty.$$

Finally, since $\phi(i_0,0) > 0$ implies $\pi(i_0,0) > 0$, it follows that $ET_{i_0,0}^{(i_0,0)} \leq \beta(\pi(i_0,0))^{-1} < \infty.\square$

We now establish the main result of this chapter, which tells us that in the subcritical case that $\{(X(t),Z(t))\}$ converges in distribution.

THEOREM 4.2. IF $E\pi U < 0$, and $\beta < \infty$, then as $t \to \infty$,

$$P(X(t) = i,Z(t) \leq x|X(0) = i_0,Z(0) = 0) \to P(X = i,Z \leq x),$$

for every continuity point of $P(Z \leq x)$, and for some random variables X and Z.

PROOF. From (4.2), it follows from the basic renewal theorem that if $T_{i_0,0}^{(i_0,0)}$ is non-arithmetic and $P(X(t) = i,Z(t) \leq x, T_{i_0,0}^{(i_0,0)} > t|X(0) = i_0, Z(0) = 0)$ is directly

Riemann integrable, then as $t \to \infty$

(4.5) $$P(X(t) = i, Z(t) \leq x | X(0) = i_0, Z(0) = 0)$$

$$\to (ET_{i_0,0}^{(i_0,0)})^{-1} \int_0^\infty P(X(t) = i, Z(t) \leq x, T_{i_0,0}^{(i_0,0)} > t | X_0 = i_0, Z(0) = 0)$$

$$\cdot \, dP(T_{i_0,0}^{(i_0,0)} \leq t).$$

From Theorem 4.1, it follows that $ET_{i_0,0}^{(i_0,0)} = \int_0^\infty P(T_{i_0,0}^{(i_0,0)} > t)dt$ is finite. Also, since

(4.6) $$P(X(t) = i, Z(t) \leq x, T_{i_0,0}^{(i_0,0)} > t | X(0) = i_0, Z(0) = 0) \leq P(T_{i_0,0}^{(i_0,0)} > t),$$

the desired direct Riemann integrability is equivalent to Riemann integrability over a finite interval, since $ET_{i_0,0}^{(i_0,0)} < \infty$. Now, by elementary arguments (see Royden [18]), it can be established that if a function is right continuous, it will have a countable number of discontinuities, which yields its Riemann integrability over a finite interval. As such, it suffices to establish that

$P(X(t)=i, Z(t) \leq x, T_{i_0,0}^{(i_0,0)} > t | X(0) = i_0, Z(0) = 0)$ is right continuous for t. To this end, we note that for $t > s$,

$$(4.7)\qquad P(X(t) = i, Z(t) \leq x, T_{i_0,0}^{(i_0,0)} > t \mid X(0) = i_0, Z(0) = 0)$$

$$- P(X(s) = i, Z(s) \leq x, T_{i_0,0}^{(i_0,0)} > s \mid X(0) = i_0, Z(0) = 0)$$

$$\leq P(M(t) - M(s) > 0 \mid X(0) = i_0, Z(0) = 0).$$

Thus, it is enough to show that

$$(4.8)\qquad \lim_{\varepsilon \downarrow 0} P(M(t+\varepsilon) - M(t) = 0 \mid X(0) = i_0, Z(0) = 0) = 1.$$

For this it is easily established that

$$(4.9)\qquad P(M(t+\varepsilon) - M(t) = 0 \mid X(0) = i_0)$$

$$= P(Y_{i_0} > t + \varepsilon) + \sum_{j \in \mathscr{S}} \int_0^t dR_{i_0,j}(\tau) P(Y_j > t + \varepsilon - \tau),$$

where

$$(4.10)\qquad R_{i_0,j}(t) = \sum_{n=1}^{\infty} P(M_j(t) \geq n \mid X(0) = i_0),$$

and

$$(4.11)\qquad M_j(t) = \sum_{i=1}^{M(t)} I(X_i = j).$$

From (4.9) we immediately have

$$\lim_{\varepsilon \downarrow 0} P(M(t+\varepsilon) - M(t) = 0 | X(0) = i_0)$$

$$= P(Y_{i_0} > t) + \sum_{j \in \mathcal{J}} \int_0^t dR_{i_0,j}(\tau) P(Y_j > t - \tau) = 1.$$

Thus, the desired right-continuity is established, which completes the proof of the theorem for the case when the distribution of $T_{i_0,0}^{(i_0,0)}$ is non-arithmetic. On the other hand, if $T_{i_0,0}^{(i_0,0)}$ is arithmetic, then by a simple argument, the behavior of $\{(X_n,T_n)\}$ is equivalent to $\{\tilde{X}_n\}$, an appropriately defined Markov chain, and the result again follows from previous arguments.□

<u>THEOREM 4.3</u>. <u>If</u> $E_\pi U < 0$ <u>and</u> $\beta < \infty$, <u>then as</u> $t \to \infty$

$$P(X(t) = i, Z(t) \leq x) \to P(X = i, Z \leq x),$$

<u>for every continuity point</u> x <u>of</u> $P(Z \leq x)$, <u>where the random variables</u> X <u>and</u> Z <u>are defined as in Theorem</u> 4.2.

<u>PROOF</u>. We first need to establish that $P(T_{X(0),Z(0)}^{(i_0,0)} < \infty) = 1$. For this, it follows by applying Lemma 2.5 to the process $\{(X_n,Z_n)\}$, that there exists a ϕ-null set $N(i_0,0)$ such that

$$(4.12)\qquad P\Big(\bigcup_{n=1}^{\infty} ((X_n,Z_n) = (i_o,0))\,|\,X_o = j,\ Z_o = x\Big) < 1$$

for $(j,x) \in N(i_o,0)$. Clearly, $P(T^{(i_o,0)}_{X(0),Z(0)} < \infty) = 1$ for all arbitrary initial distributions if and only if $N(i_o,0) = \emptyset$. In order to establish that $N(i_o,0) = \emptyset$, we note that since $\phi(i_o,0) > 0$ (which implies that $(i_o,0) \notin N(i_o,0)$), using a rather straightforward argument, we have that for all $(j,x) \notin N(i_o,0)$

$$(4.13)\qquad P((X_n,Z_n) = (i_o,0),\ \text{i.o.}\,|\,X_o = j,\ Z_o = x) = 1.$$

Now, for every j, there must be a y_j with $(j,y_j) \notin N(i_o,0)$. Using the facts that for every z,

$$(4.14)\qquad P((X_n,Z_n) = (i_o,0),\ \text{i.o.}\,|\,X_o = j,\ Z_o = z)$$

$$= P((X_n,\ \max[z + S_n,\ \max_{1\le j\le n} (S_n - S_j)]) = (i_o,0),\ \text{i.o.}\,|\,X_o = j),$$

and that $S_n \to -\infty$, a.s., we have

$$(4.15)\qquad P((X_n,\max[z + S_n,\ \max_{1\le j\le n} (S_n - S_j)])$$

$$\ne (X_n,\max[y_j + S_n,\ \max_{1\le j\le n} (S_n - S_j)]),\ \text{i.o.}\,|\,X_o = j)$$

$$\le P(S_n > \min(-z,-y_j),\ \text{i.o.}\,|\,X_o = j) = 0.$$

Consequently from (4.15), we have

$$(4.16) \qquad P((X_n,Z_n) = (i_o,0),\ \text{i.o.}|X_o = j,\ Z_o = z)$$

$$= P((X_n,Z_n) = (i_o,0),\ \text{i.o.}|X_o = j,\ Z_o = y_j) = 1.$$

Thus for all j, $j \times [0,\infty) \notin N(i_o,0)$, so that $N(i_o,0) = \emptyset$, and hence $P(T_{X(0),Z(0)}^{(i_o,0)} < \infty) = 1$, for arbitrary initial distributions.

As such, for all ε, there exist K_1 and K_2 such that $P(T_{X(0),Z(0)}^{(i_o,0)} \le K_1) > 1 - \varepsilon$, and for $t > K_2$

$$(4.17) \qquad |P(X(t) = j,Z(t) \le x|X(0) = i_o,Z(0) = 0) - P(X = j,\ Z \le x)| < \varepsilon,$$

for every continuity point x of $P(Z \le x)$. From this it follows that for $t < K_1 + K_2$, we have

$$(4.18) \qquad P(X(t) = j,Z(t) \le x,\ T_{X(0),Z(0)}^{(i_o,0)} \le K_1) \le P(X(t) = j,Z(t) \le x)$$

$$\le P(X(t)=j,Z(t) \le x,\ T_{X(0),Z(0)}^{(i_o,0)} \le K_1) + \varepsilon,$$

where

$$(4.19) \qquad P(X(t) = j,Z(t) \le x,\ T_{X(0),Z(0)}^{(i_o,0)} \le K_1)$$

$$= \int_0^{K_1} P(X(t-\tau) = j,Z(t-\tau) \le x|X(0) = i_o,Z(0) = 0)dP(T_{X(0),Z(0)}^{(i_o,0)} \le \tau).$$

Using (4.17), we obtain

$$(4.20)\qquad [P(X = j,Z \le x) - \varepsilon]P(T_{X(0),Z(0)}^{(i_0,0)} \le K_1)$$

$$\le P(X(t) = j,Z(t) \le x,\ T_{X(0),Z(0)}^{(i_0,0)} \le K_1)$$

$$\le [P(X = j,Z \le x) + \varepsilon]P(T_{X(0),Z(0)}^{(i_0,0)} \le K_1).$$

Finally, using (4.18) and (4.20), we obtain

$$(4.21)\qquad P(X = j,Z \le x) - 2\varepsilon \le P(X(t) = j,Z(t) \le x) \le P(X = j,\ Z \le x) + 2\varepsilon,$$

and from this the theorem follows.□

This completes the analysis of the asymptotic behavior of $\{(X(t),Z(t)\}$ in the subcritical case. For the same case the following section deals with the joint asymptotic behavior of the storage level at time t and the cumulative amount of the output demands from the storage which were unmet during (0,t].

5. SOME FURTHER RESULTS.

In this section, along with the level in the storage at time t in the subcritical case, we examine the behavior of a random variable L(t) which represents the cumulative amount of demands from the system which were unmet during time (0,t] as given in (1.6) and (1.7). From these we also note that

(5.1) $\quad L(t) = L_{M(t)} = Z(t) - S_{M(t)} - Z(0).$

To establish the joint behavior of L(t) and Z(t), we need a theorem which provides conditions for the asymptotic independence of the two variables. To this end, we define a new variable $Z_\tau(t)$ by

(5.2) $$Z_\tau(t) = \max_{M(t-\tau)\le j\le M(t)} (S_{M(t)} - S_j).$$

Note that since $Z(t) = \max(Z(0) + S_{M(t)}, \max_{1\le j\le M(t)} (S_{M(t)} - S_j))$, we have that for $0\le\tau_1\le\tau_2\le t$,

(5.3) $$Z_{\tau_1}(t) \le Z_{\tau_2}(t) \le Z(t).$$

We now prove the following lemma, which establishes a useful property of $Z_\tau(t)$.

LEMMA 5.1. If $E_\pi U < 0$ and $\beta < \infty$, then for every $\varepsilon > 0$, there exist T_1 and T_2 with $T_1 > T_2$, such that $P(Z(t) \ne Z_\tau(t)) < \varepsilon$, for all $t > T_1$ and $\tau > T_2$.

PROOF. Consider the state i_0 used in the previous section, and let

(5.4) $$N_{(i_0,0)}(t) = \sup\{1 \le n \le M(t): \ (X_n,Z_n) = (i_0,0)\}$$

and $T_t = T_{N_{(i_o,0)}}(t)$. Since $ET^{(i_o,0)}_{i_o,0} < \infty$ (from section 4), it follows from renewal theory (see Karlin and Taylor [9]) that for all initial dsitributions of $(X(0),Z(0))$, as $t \to \infty$,

$$(5.5)\qquad P(t - T_t > z) \to (ET^{(i_o,0)}_{i_o,0})^{-1} \int_z^{\infty} P(T^{(i_o,0)}_{i_o,0} > x)dx.$$

Thus we can choose a T_2 satisfying $(ET^{(i_o,0)}_{i_o,0})^{-1} \int_{T_2}^{\infty} P(T^{(i_o,0)}_{i_o,0} > x)dx < \frac{\varepsilon}{2}$, and a T_1 such that for $t > T_1$,

$$(5.6)\qquad P(t - T_t > T_2) \le (ET^{(i_o,0)}_{i_o,0})^{-1} \int_{T_2}^{\infty} P(T^{(i_o,0)}_{i_o,0} > x)dx + \frac{\varepsilon}{2} < \varepsilon.$$

Note that $t - T_t \le T_2$ implies there exists an x, satisfying $t - T_2 \le x \le t$, such that $Z(x) = 0$. And since

$$(5.7)\qquad Z(t) = (Z(x) + S_{M(t)} - S_{M(x)}, \max_{M(x)<j\le M(t)} (S_{M(t)} - S_j))$$

$$= \max_{M(x)\le j\le M(t)} (S_{M(t)} - S_j) \le \max_{M(t-T_2)\le j\le M(t)} (S_{M(t)} - S_j)) = Z_{T_2}(t),$$

we have that $t - T_t \le T_2$ implies $Z(t) = Z_\tau(t)$, for $\tau \ge T_2$. As such, for $t > T_1$, $\tau > T_2$,

$$(5.8)\qquad P(Z(t) \ne Z_\tau(t)) \le P(t - T_t > T_2) < \varepsilon,$$

which completes the proof.□

Another useful property of $Z_\tau(t)$ is that $Z_\tau(t)$ "ignores" the inputs and outputs which occurred before $t-\tau$, in the sense that they do not enter into the formula for $Z_\tau(t)$. Because of this, $Z_\tau(t)$ is conditionally independent of anything that has happened in time $[0,t-\tau)$, given $X_{M(t-\tau)+1}$ and $T_{M(t-\tau)+1}$. Thus, we can establish the following theorem.

THEOREM 5.2. *Let* $E_\pi U < 0$, *and* $\beta < \infty$. *Also let* $Y(t)$ *and* $Y'(t)$ *be two processes such that,*

(1) $P(Y(t) \le x) \to P(Y \le x)$ *for all continuity points of* $P(Y \le x)$ *as* $t \to \infty$,

(2) $Y'(t-\tau) - Y(t) \overset{P}{\to} 0$, *as* $t \to \infty$, *and*

(3) $Y'(t-\tau)$ and $Z_\tau(t)$ *are conditionally independent given* $X_{M(t-\tau)+1}$ *and* $T_{M(t-\tau)+1}$.

Then as $t \to \infty$, $P(Z(t) \le x, Y(t) \le y) \to P(Z \le x)P(Y \le y)$, *for all continuity points* x *and* y *of the distributions of* Z *and* Y, *respectively, where* Z *is as described in Theorem* 4.2.

REMARK. Since $Y'(t-\tau) - Y(t) \overset{P}{\to} 0$, we also have $P(Y'(t) \le y) \to P(Y \le y)$, as $t \to \infty$, for all continuity points y of $P(Y \le y)$. Thus in order to obtain our result, (1) could be replaced by:

(1') $P(Y'(t) \le y) \to P(Y \le x)$ *for all continuity points* x *of* $P(Y \le x)$, *as* $t \to \infty$.

PROOF OF THEOREM 5.2. In view of Lemma 5.1, for every $\varepsilon > 0$, there exist T_1 and T_2 such that for $t > T_1$, and $\tau > T_2$,

$$(5.9) \qquad P(Z(t) \neq Z_\tau(t)) < \frac{\varepsilon}{2}.$$

Also, for $\tau > T_2$, there exists a $T_\tau > T_1$ such that for $t > T_\tau$,

$$(5.10) \qquad P(|Y(t) - Y'(t-\tau)| > \varepsilon) < \frac{\varepsilon}{2}.$$

Thus it follows that for $t > T_\tau$,

$$(5.11) \qquad P(Z_\tau(t) \leq x,\ Y'(t-\tau) \leq y - \varepsilon) - \varepsilon \leq P(Z(t) \leq x,\ Y(t) \leq y)$$

$$\leq P(Z_\tau(t) \leq x,\ Y'(t-\tau) \leq y + \varepsilon) + \varepsilon.$$

Now defining $T_t^+ = T_{M(t)+1}$, $X_t^+ = X_{M(t)+1}$ and $P_t^+(i,x) = P(X_t^+ = i, T_t^+ \leq x)$, we have

$$(5.12) \qquad P(Z_\tau(t) \leq x,\ Y'(t-\tau) \leq y)$$

$$= \sum_{i \in \mathcal{J}} \int_{t-\tau}^{\infty} P(Z_\tau(t) \leq x,\ Y'(t-\tau) \leq y \,|\, X_{t-\tau}^+ = i, T_{t-\tau}^+ = z) P_{t-\tau}^+(i,dz)$$

$$= \sum_{i \in \mathcal{J}} \int_{t-\tau}^{\infty} P(Z_\tau(t) \leq x \,|\, X_{t-\tau}^+ = i, T_{t-\tau}^+ = z) P(Y'(t-\tau) \leq y \,|\, X_{t-\tau}^+ = i, T_{t-\tau}^+ = z) P_{t-\tau}^+(i,dz).$$

Here the second equality follows from the fact that $Y'(t-\tau)$ and $Z_\tau(t)$ are conditionally independent given $X^+_{t-\tau}$ and $T^+_{t-\tau}$.
Now by (Çinlar [5]), we have for any τ

(5.13) $$P(X^+_{t-\tau}=k,\ T^+_{t-\tau} - (t-\tau) > y) \to (\Sigma\pi_j m_j)^{-1} \int_y^\infty \Sigma\pi_j(p_{jk}-A_{jk}(u))du,$$

as $t \to \infty$, where p_{jk}, $A_{jk}(u)$, and m_j are as defined in (1.1) and (1.4). Since $\beta=\Sigma\pi_j m_j < \infty$, for every $\varepsilon > 0$, we can pick a finite set $A \subset \mathcal{J}$ such that

(5.14) $$\sum_{k\in A}[(\Sigma\pi_j m_j)^{-1} \int_0^\infty \sum_{j\in\mathcal{J}} \pi_j(p_{jk} - A_{jk}(u))du] > 1 - \varepsilon.$$

Consequently, for every $\varepsilon > 0$, there exists a K_1 such that for $t > K_1$,

(5.15) $$P(X^+_t \in A) > 1 - 2\varepsilon.$$

Also, for every $\varepsilon > 0$, we can choose a $z > 0$ such that

$$\sum_{k\in A}[(\Sigma\pi_j m_j)^{-1} \int_z^\infty \sum_{j\in\mathcal{J}} \pi_j(p_{jk} - A_{jk}(u))du] < \varepsilon.$$

Thus it follows from (5.13) that for every $\varepsilon > 0$, there exist z and $K_2 > K_1$ such that for $t > K_2$,

$$(5.16)\qquad P(X_t^+ \in A,\ T_t^+ - t > z) < 2\varepsilon,$$

so that using (5.15) and (5.16), we have for $t > K_2$

$$(5.17)\qquad P((X_t^+ \notin A) \cup (X_t^+ \in A,\ T_t^+ - t > z)) < 4\varepsilon.$$

Also, from Theorem 4.3 it follows that for every $\varepsilon > 0$, there exists an L such that for $k \in A$ and $t > L$,

$$(5.18)\qquad \left|\int_0^\infty P(Z(t) \le x | X(0) = k, Z(0) = y) dP((U(k))^+ \le y) - P(Z \le x)\right| < \varepsilon.$$

Let us now choose $\tau > L+z$. Then for $t > K_2 + \tau$, from (5.12) and (5.17), we have,

$$(5.19)\qquad \sum_{i \in A} \int_{t-\tau}^{t-\tau+z} P(Z_\tau(t) \le x | X_{t-\tau}^+ = i, T_{t-\tau}^+ = u) P(Y'(t-\tau) \le y | X_{t-\tau}^+ = i, T_{t-\tau}^+ = u) \cdot P_{t-\tau}^+(i,du)$$

$$\le P(Z_\tau(t) \le x,\ Y'(t-\tau) \le y)$$

$$\le \sum_{i \in A} \int_{t-\tau}^{t-\tau+z} P(Z_\tau(t) \le x | X_{t-\tau}^+ = i,\ T_{t-\tau}^+ = u) P(Y'(t-\tau) \le y | X_{t-\tau}^+ = i, T_{t-\tau}^+ = u)$$

$$\cdot P_{t-\tau}^+(i,du) + 4\varepsilon.$$

Also, by definition of $Z_\tau(t)$, we have

$$(5.20)\qquad P(Z_\tau(t) \le x \mid X^+_{t-\tau} = i,\ T^+_{t-\tau} = u)$$

$$= \int_0^\infty P(Z(t-u) \le x \mid X(0)=i, Z(0)=z)\, dP((U(i))^+ \le z).$$

Since for $t-\tau \le u \le t - \tau + z$, we have $t-u \ge \tau - z > L$, it follows from (5.18) that

$$(5.21)\qquad \Big| \sum_{i\in A} \int_{t-\tau}^{t-\tau+z} P(Z_\tau(t) \le x \mid X^+_{t-\tau}=i, T^+_{t-\tau}=u) P(Y'(t-\tau) \le y \mid X^+_{t-\tau}=i, T^+_{t-\tau}=u)$$

$$\cdot\ P^+_{t-\tau}(i,du)$$

$$- \sum_{i\in A} \int_{t-\tau}^{t-\tau+z} P(Z \le x) P(Y'(t-\tau) \le y \mid X^+_{t-\tau}=i, T^+_{t-\tau}=u) P^+_{t-\tau}(i,du) \Big| \le \varepsilon,$$

so that from (5.19) we have

$$(5.22)\qquad P(Z \le x)\Big(\sum_{i\in A} \int_{t-\tau}^{t-\tau+z} P(Y'(t-\tau) \le y \mid X^+_{t-\tau}=i, T^+_{t-\tau}=u) P^+_{t-\tau}(i,du)\Big) - \varepsilon$$

$$\le P(Z_\tau(t) \le y,\ Y'(t-\tau) \le x)$$

$$\le P(Z \le x)\Big(\sum_{i\in A} \int_{t-\tau}^{t-\tau+z} P(Y'(t-\tau) \le y \mid X^+_{t-\tau}=i, T^+_{t-\tau}=u) P^+_{t-\tau}(i,du)\Big) + 5\varepsilon.$$

From (5.17), we have for all x, and $t > K_2+\tau$,

$$(5.23)\qquad P(Z \le x)P(Y'(t-\tau) \le y) - 5\varepsilon \le P(Z_\tau(t) \le x,\ Y'(t-\tau) \le y)$$

$$\le P(Z \le x)P(Y'(t-\tau) \le y) + 5\varepsilon.$$

Thus, to combine (5.11) and (5.23), we need to choose $\tau > \max(L+z, T_2)$, and $T > \max(T_\tau, K_2+\tau)$, for T_2, T_τ as in (5.9) and (5.10). With these it follows, using (5.11) and (5.23), that for $t > T$,

$$(5.24)\qquad P(Z \le x)P(Y(t) \le y - 2\varepsilon) - 7\varepsilon \le P(Z \le x)P(Y'(t-\tau) \le y - \varepsilon) - 6\varepsilon$$

$$\le P(Z_\tau(t) \le x,\ Y'(t-\tau) \le y - \varepsilon) - \varepsilon \le P(Z(t) \le x,\ Y(t) \le y)$$

$$\le P(Z_\tau(t) \le x,\ Y'(t-\tau) \le y + \varepsilon) + \varepsilon \le P(Z \le x)P(Y'(t-\tau) \le y + \varepsilon) + 6\varepsilon$$

$$\le P(Z \le x)P(Y(t) \le y + 2\varepsilon) + 7\varepsilon,$$

which completes the proof of the theorem.□

Before establishing the main theorem of this section, we need to define

$$(5.25)\qquad t_1 = \inf\{n > 0:\ X_n = i_0\}.$$

To establish the joint asymptotic behavior of Z(t) and L(t), we use Theorem 5.2 to establish the asymptotic independence of the two random variables, which yields the following theorem.

THEOREM 5.3. If $E_\pi U \leq 0$, $\beta < \infty$, and

$$\sigma^2_{i_0} = E[(\sum_{1}^{t_1}(U_i(X_i)) + T_{t_1}(E_\pi U))^2 | X_0 = i_0] < \infty,$$

then for arbitrary distributions of X(0) and Z(0), as $t \to \infty$, we have for all continuity points x of $P(Z \leq x)$,

$$P(Z(t) \leq x,\ L(t) + t\beta^{-1}(E_\pi U) \leq y(t\pi_{i_0}\beta^{-1}\sigma^2_{i_0})^{\frac{1}{2}}) \to P(Z \leq x)\phi(y).$$

PROOF. To appeal to Theorem 5.2, we define

$$(5.26) \qquad Y(t) = (t\pi_{i_0}\beta^{-1}\sigma^2_{i_0})^{-\frac{1}{2}}(L(t) + t(E_\pi U)\beta^{-1}),$$

$$(5.27) \qquad Y'(t) = ((t+\tau)\pi_{i_0}\beta^{-1}\sigma^2_{i_0})^{-\frac{1}{2}}(-S_{M(t)} + (t+\tau)(E_\pi U)\beta^{-1}).$$

Puri and Woolford [17] have shown that as $t \to \infty$, $P(Y'(t) \leq y) \to \phi(y)$. Thus in view of the remark following Theorem 5.2, and of the fact that $Y'(t-\tau)$ and $Z_\tau(t)$ are by construction conditionally independent given $X^+_{t-\tau}$, $T^+_{t-\tau}$, we only need to show that $Y(t) - Y'(t-\tau) \xrightarrow{p} 0$ in order to complete the proof of our theorem. For this since $L(t) = Z(t) - S_M(t) - Z(0)$, it is easy to see that $Y(t) - Y'(t-\tau) \xrightarrow{p} 0$ is equivalent to

(5.28) $$t^{-\frac{1}{2}}(S_{M(t-\tau)} - S_{M(t)}) \overset{p}{\to} 0.$$

For $N_{i_o}(t)=N_{(i_o,[0,\infty))}(t)$, as in (5.4), from (Puri and Woolford [17]), it follows that if $\beta < \infty$, then $t^{-\frac{1}{2}}(S_{N_{i_o}(t)} - S_{M(t)}) \overset{p}{\to} 0$, as $t \to \infty$.

Also, by Lemma 5.4, established below, we have $t^{-\frac{1}{2}}(S_{N_{i_o}(t)} - S_{N_{i_o}(t-\tau)}) \overset{p}{\to} 0$, as $t \to \infty$. Finally using the representation

(5.29) $$t^{-\frac{1}{2}}(S_{M(t-\tau)}-S_{M(t)})=t^{-\frac{1}{2}}(S_{M(t-\tau)}-S_{N_{i_o}(t-\tau)})+t^{-\frac{1}{2}}(S_{N_{i_o}(t-\tau)}-S_{N_{i_o}(t)})$$
$$+ t^{-\frac{1}{2}}(S_{N_{i_o}(t)}-S_{M(t)})$$

it follows that $t^{-\frac{1}{2}}(S_{M(t-\tau)}-S_{M(t)}) \overset{p}{\to} 0$, and hence the proof.□

LEMMA 5.4. *If* $\beta < \infty$, *then* $t^{-\frac{1}{2}}(S_{N_{i_o}(t)}-S_{N_{i_o}(t-\tau)}) \overset{p}{\to} 0$, *as* $t \to \infty$.

PROOF. Remembering that $t_1=\inf\{n\geq 1: X_n=i_o\}$, it is clear that if $X(0)=i_o$, we have the renewal equation

(5.30) $$P(S_{N_{i_o}(t)} - S_{N_{i_o}(t+\tau)} \leq x | X(0) = i_o)$$

$$= P(-S_{N_{i_o}}(t+\tau) \leq x,\ T_{t_1} > t \mid X(0) = i_o)$$

$$+ \int_0^t P(S_{N_{i_o}}(t-y) - S_{N_{i_o}}(t+\tau-y) \leq x \mid X(0) = i_o)\, dP(T_{t_1} \leq y \mid X(0) = i_o).$$

Since $E(T_{t_1} | X(0)=i_o) = \pi_{i_o}^{-1} \beta < \infty$, by an argument identical to that for Theorem 4.2, we can show that $P(-S_{N_{i_o}}(t+\tau) \leq x,\ T_{t_1} > t | X(0)=i_o)$ is directly Riemann integrable. Consequently
$P(S_{N_{i_o}}(t) - S_{N_{i_o}}(t+\tau) \leq x | X(0) = i_o) \to P(W \leq x)$, as $t \to \infty$, if T_{t_1} is non-arithmatic. Once again, if T_{t_1} is arithmetic, we can show that $\{(X_n, T_n)\}$ has behavior equivalent to a Markov chain $\{\tilde{X}_n\}$, and a straightforward argument yields that
$P(S_{N_{i_o}}(t) - S_{N_{i_o}}(t+\tau) \leq x | X(0) = i_o) \to P(W \leq x)$, as $t \to \infty$.

To establish that $P(S_{N_{i_o}}(t) - S_{N_{i_o}}(t+\tau) \leq x) \to P(W \leq x)$ as $t \to \infty$, for an arbitrary distribution of $X(0)$, an argument based on the waiting time for the first visit to i_o, very similar to the one used in Theorem 4.3, yields the desired result. Thus, we have shown that

$t^{-\frac{1}{2}}(S_{N_{i_o}(t-\tau)} - S_{N_{i_o}(t)}) \xrightarrow{P} 0$ as $t \to \infty$, and the proof is complete.□

REMARK. The above lemma is valid as long as $S_n = \sum_{i=1}^{n} Y_i(X_i)$ and, for every $j \in \mathcal{J}$, $\{Y_n(j), n=1,2,\ldots\}$ is an i.i.d. sequence, such that $\{Y_n(j)\}$ is independent of $\{Y_n(i)\}$, for $j \neq i$.

6. CONCLUDING REMARKS.

The approach adopted here of using conditions such as those of Tweedie [21] to establish ergodicity followed by the use of a renewal equation argument appear to be more generally applicable to models defined on semi-Markov processes. However, the creation of the measure ϕ and the selection of an appropriate 'recurrent point' appear to be the major problems, and must be tackled with due considerations of the model at hand. It should be pointed out that under certain conditions, if no recurrent point is available, a special (recurrent) set may suffice for the establishment of the needed renewal equation in order to follow through the present approach (see Athreya, McDonald and Ney [2] and Nummelin [13]).

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