On the Narrowest Tube of a Simple Symmetric Random Walk

by

E. Csáki and A. Földes*
Mathematical Institute of the Hungarian Academy of Sciences
H-1376 Budapest, Box 428, Hungary

Technical Report #83-4

March 1983

Purdue University
West Lafayette, IN

*This research was partly done while the second author was visiting the Department of Statistics at Purdue University. The authors would like to acknowledge the financial assistance of the Department of Statistics.

On the Narrowest Tube of a Simple Symmetric Random Walk

by

E. Csáki and A. Földes

Summary. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $P(X_1 = +1) = P(X_1 = -1) = 1/2$. Put $S_0 = 0$, $S_n = X_1 + \ldots + X_n$ $(n \geq 1)$. Our aim is to investigate the a.s. behavior of $U(a_N, N) = \min_{0 \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} |S_{j+i} - S_j|$ and $V(a_N, N) = \min_{0 \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} |S_{j+i}|$.

It is shown that for $a_N = [c \log N]$ both $U(a_N, N)$ and $V(a_N, N)$ are a.s. constant for large N, except for certain values of c, when U and V can take two values for large N.

1. INTRODUCTION

Let $\{W(t),\ t \geq 0\}$ be a standard Wiener process. Suppose that a_t is a non-decreasing function of t such that $0 < a_t \leq t$ and t/a_t is non-decreasing. By investigating the small values of the increments

$$(1.1) \qquad \zeta(t, a_T) = \sup_{0 \leq s \leq a_T} |W(t+s) - W(t)|,$$

Csörgő and Révész [5, Theorem 1.7.1] proved the following results.

THEOREM A

$$(1.2) \qquad \liminf_{T \to \infty} \gamma_T \inf_{0 \leq t \leq T-a_T} \zeta(t, a_T) \overset{a.s.}{=} 1,$$

<u>where</u>

(1.3) $$\gamma_T = \left(\frac{8(\log\frac{T}{a_T} + \log\log T)}{\pi^2 a_T}\right)^{1/2}$$

<u>If we also have</u>

(1.4) $$\frac{\log(T/a_T)}{\log\log T} \nearrow +\infty \text{ as } T \to \infty,$$

<u>then</u>

(1.5) $$\lim_{T\to\infty} \gamma_T \inf_{0\leq t\leq T-a_T} \zeta(t,a_T) \overset{a.s.}{=} 1$$

In Csáki and Földes [2] a similar result is given for

(1.6) $$\xi(t,a_T) = \sup_{0\leq s\leq a_T} |W(t+s)|,$$

i.e. for the Wiener process instead of the increments.

THEOREM B

(1.7) $$\liminf_{T\to\infty} \beta_T \inf_{0\leq t\leq T-a_T} \xi(t,a_T) \overset{a.s.}{=} 1,$$

<u>where</u>

(1.8) $$\beta_T = \left(\frac{4\log\frac{T}{a_T} + 8\log\log T}{\pi^2 a_T}\right)^{1/2}$$

<u>If we also have condition (1.4), then</u>

(1.9) $$\lim_{T\to\infty} \beta_T \inf_{0\le t\le T-a_T} \xi(t,a_T) \overset{a.s.}{=} 1$$

Similar problems can also be investigated for partial sums of i.i.d. random variables. Let $\{X_i, i \ge 1\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1. Put $\delta_0 = 0$, $S_n = X_1+\ldots+X_n$ $(n \ge 1)$. Suppose that $\{a_n, n \ge 1\}$ is a non-decreasing sequence of integers such that $0 < a_n \le n$ and n/a_n is non-decreasing. In the case $\lim_{n\to\infty} a_n/\log n = \infty$, referring to a small deviation result of Mogul'skii [7] the following theorem is formulated in Csörgő and Révész [5, Theorem 3.3.1.*]

THEOREM C

In the case when $\lim_{N\to\infty} a_N/\log N = \infty$ _we have_

(1.10) $$\liminf_{N\to\infty} \min_{0\le j\le N-a_N} \max_{0\le i\le a_N} \gamma_N |S_{j+i} - S_j| \overset{a.s.}{=} 1,$$

where γ_N _is given by (1.3). If the condition (1.4) is also satisfied, then_

(1.11) $$\lim_{N\to\infty} \min_{0\le j\le N-a_N} \max_{0\le i\le a_N} \gamma_N |S_{j+i} - S_j| \overset{a.s.}{=} 1.$$

It is not difficult to see that an analogue of Theorem B is also true. Our first result states

THEOREM 1. _In the case when_ $\lim_{N\to\infty} a_N/\log N = \infty$ _we have_

(1.12) $$\liminf_{N\to\infty} \min_{0\le j\le N-a_N} \max_{0\le i\le a_N} \beta_N |S_{j+i}| \overset{a.s.}{=} 1,$$

where β_N is given by (1.8). If the condition (1.4) is also satisfied, then

(1.13) $$\lim_{N\to\infty} \min_{0\le j\le N-a_N} \max_{0\le i\le a_N} \beta_N |S_{j+i}| \overset{a.s.}{=} 1.$$

Theorem 1 can be proved by the same way as Theorem B was proved in Csáki and Földes [2] using the above mentioned small deviation results of Mogulskii [7]. So we omit this proof.

The more interesting case is $a_N = [c \log N]$ and is yet unsolved. The following conjecture is formulated in Csörgő and Révész [5]:

CONJECTURE

(1.14) $$\lim_{N\to\infty} \min_{0\le j\le N-a_N} \max_{0\le i\le a_N} |S_{j+i} - S_j| \overset{a.s.}{=} \alpha(c),$$

where $\alpha(c)$ is a function which uniquely determines the distribution of X_i.

As a small step towards the solution of this problem we investigate it for the simple symmetric random walk, i.e. when $P(X_1 = +1) = P(X_1 = -1) = 1/2$ and show that (1.14) is nearly true in this case. Define

(1.15) $$U(a,N) = \min_{0\le j\le N-a} \max_{0\le j\le a} |S_{j+i} - S_j|$$

and call it the narrowest tube for the increments. Since $U(a_N,N)$ is integer valued, (1.14) would mean that $U(A_N,N)$ takes only the particular value $\alpha(c)$ when N is large enough. In fact this is true for almost every c, but there are exceptional values of c when $U(a_N,N)$ can take 2 values even for large N.

We investigate also

$$(1.16)\qquad V(a,N) = \min_{0\le j\le N-a} \max_{0\le i\le a} |S_{j+i}|$$

and call it the narrowest tube around zero.

We formulate our main results.

THEOREM 2. <u>Assume that</u> $X_1, X_2, \ldots$ <u>are i.i.d. random variables with</u> $P(X_1 = +1) = P(X_i = -1) = 1/2$. <u>Let</u> $a_N = [c \log N]$, $c > 0$ <u>and define</u> $\alpha = \alpha(c) > 1$ <u>as the solution of the equation</u>

$$(1.17)\qquad \cos \frac{\pi}{2\alpha} = e^{-\frac{1}{c}} .$$

<u>If</u> $\alpha(c)$ <u>is not an integer, then for almost all</u> ω, <u>there exists an</u> $N_0 = N_0(c,\omega)$ <u>such that</u>

$$(1.18)\qquad U(a_N,N) = [\alpha(c)] \text{ if } N \ge N_0.$$

<u>If</u> $\alpha(c)$ <u>is an integer, then for almost all</u> ω <u>there exists an</u> $N_0 = N_0(c,\omega)$ <u>such that</u>

(1.19) $\alpha(c) - 1 \leq U(a_N, N) \leq \alpha(c)$ if $N \geq N_0$.

Moreover

(1.20) $P(U(a_N, N) = \alpha(c) - 1 \text{ i.o.}) = 1$

and

(1.21) $P(U(a_N, N) = \alpha(c) \text{ i.o.}) = 1$

THEOREM 3. *Assume that* $X_1, X_2, \ldots$ *are i.i.d. random variables with* $P(X_i = +1) = P(X_i = -1) = 1/2$. *Let* $a_N = [c \log N]$, $c > 0$ *and define* $\alpha^* = \alpha^*(c) > 1$ *as the solution of the equation*

(1.22) $$\cos \frac{\pi}{2\alpha^*} = e^{-\frac{1}{2c}}$$

If $\alpha^*(c)$ *is not an integer, then for almost all* ω *there exists an* $N_0^* = N_0^*(c, \omega)$ *such that*

(1.23) $V(a_N, N) = [\alpha^*(c)]$ if $N \geq N_0^*$.

If $\alpha^*(c)$ *is an integer, then for almost all* ω *there exists an* $N_0^* = N_0^*(c, \omega)$ *such that*

(1.24) $\alpha^*(c) - 1 \leq V(a_N, N) \leq \alpha^*(c)$ if $N \geq N_0$.

Moreover

(1.25) $P(V(a_N,N) = \alpha^*(c) - 1 \text{ i.o.}) = 1$

and

(1.26) $P(V(a_N,N) = \alpha^*(c) \text{ i.o.}) = 1$

Theorem 2 and Theorem 3 will be proved in Section 2 and Section 3, respectively.

The basic formula used in our proofs is due to Ellis [6] (see also Takács [9]):

LEMMA A. *Let* $a \geq 1$, $\alpha \geq 1$, $\beta \geq 1$, x *be integers such that* $-\beta < x < \alpha$. *Then*

$$P(-\beta < S_k < \alpha,\ k = i,\dots,a-1,\ S_a = x) = \tag{1.27}$$
$$\frac{2}{\alpha+\beta}\sum_{k=0}^{\alpha+\beta}\left(\cos\frac{k\pi}{\alpha+\beta}\right)^a \sin\frac{k\pi\alpha}{\alpha+\beta}\ \sin\frac{k\pi(\alpha-x)}{\alpha+\beta}$$

By using the formula

$$\sum_{x=-\beta}^{\alpha} \sin\frac{k\pi(\alpha-x)}{\alpha+\beta} = \frac{1+\cos\frac{k\pi}{\alpha+\beta}}{\sin\frac{k\pi}{\alpha+\beta}}\left(\frac{1-(-1)^k}{2}\right) \tag{1.28}$$

we obtain

$$P(-\beta < S_k < \alpha,\ k = 1,\dots,a) =$$

(1.29)

$$\frac{2}{\alpha+\beta} \sum_{k=1}^{\alpha+\beta-1} \left(\cos \frac{k\pi}{\alpha+\beta}\right)^a \sin \frac{k\pi\alpha}{\alpha+\beta} \frac{1 + \cos \frac{k\pi}{\alpha+\beta}}{\sin \frac{k\pi}{\alpha+\beta}} \left(\frac{1 - (-1)^k}{2}\right)$$

For large a the dominating terms of the above sums are for $k=1$ and $k=\alpha+\beta-1$ and it is easy to see that for large a the following inequalities hold:

COROLLARY.

(1.30) $$K_1\left(\cos \frac{\pi}{\alpha+\beta}\right)^a \leq P(-\beta < S_k < \alpha,\ k=1,\dots,a) \leq K_2\left(\cos \frac{\pi}{\alpha+\beta}\right)^a$$

with some constants K_1 and K_2, depending only on $\alpha+\beta$ and not on a. In our proofs K_1 and K_2 will denote the above constants, but K^*, K_1^*, etc. will denote further constants, whose values are not important for the proof and may change from time to time.

2. THE TUBE FOR THE INCREMENTS

In this Section we prove Theorem 2, based on the following 4 lemmas.

LEMMA 2.1. <u>If</u> $\alpha \geq 1$ <u>is an integer and</u>

(2.1) $$\cos \frac{\pi}{2\alpha} < e^{-\frac{1}{c}} .$$

then for almost all ω *there exists an* $N_0 = N_0(c,\omega)$ *such that*

$$U(a_N,N) \geq \alpha \quad \textit{for} \quad N > N_0, \tag{2.2}$$

where $U(a,N)$ *is defined by (1.15) and* $a_N = [c \log N]$.

LEMMA 2.2. If $\alpha \geq 2$ *is an integer and*

$$\cos \frac{\pi}{2\alpha} > e^{-\frac{1}{c}}, \tag{2.3}$$

then for almost all ω *there exists an* $N_0^* = N_0^*(c,\omega)$ *such that*

$$U(a_N,N) < \alpha \quad \textit{for} \quad N > N_0^*, \tag{2.4}$$

where $U(a,N)$ *is defined by (1.15) and* $a_N = [c \log N]$.

LEMMA 2.3. *Let the events* A_j^a $(j=0,1,\ldots)$ be defined by

$$A_j^a = \{ \max_{0 \leq i \leq a} |S_{j+i} - S_j| < \alpha(c)\}, \tag{2.5}$$

where the solution $\alpha(c)$ *of (1.17) is an integer. Then*

$$P(A_{N-a_N}^{a_N} \text{ i.o.}) = 1, \tag{2.6}$$

where $a_N = [c \log N]$.

LEMMA 2.4. *The following inequality holds true for* $\alpha \geq 1$, $k \geq 1$ *and* a *large enough:*

$$(2.7)\quad \begin{aligned} &P(U(a,(k+1)a - 1) \geq \alpha) \\ &\geq (1 - K_2\, a(\cos \tfrac{\pi}{2\alpha})^a)^k - K_2\, a(\cos \tfrac{\pi}{2\alpha})^a , \end{aligned}$$

where $U(a,N)$ *is defined by (1.15) and* K_2 *is the constant of (1.30).*

PROOF OF LEMMA 2.1.

From the inequality (2.1) it follows that there exists an integer $\rho > 0$ such that

$$(2.8)\qquad \cos \frac{\pi}{2\alpha} < e^{-\frac{1+\rho}{c\rho}} .$$

Define $N_k = k^\rho$. Then by (1.30),

$$\begin{aligned} P(U(a_{N_k}, N_{k+1}) < \alpha) &\leq N_{k+1} P(\max_{0 \leq j \leq a_{N_k}} |S_j| < \alpha) \\ &\leq K_2 N_{k+1} (\cos \tfrac{\pi}{2\alpha})^{a_{N_k}} \leq K^*(k+1)^\rho (\cos \tfrac{\pi}{2\alpha})^{c\,\rho \log k} \\ &\leq K^* k^{-\beta}, \end{aligned}$$

where, from (2.8)

$$(2.9) \qquad \beta = \rho + \rho c \log \cos \frac{\pi}{2\alpha} < -1.$$

Hence

$$(2.10) \qquad \sum_k P(U(a_{N_k}, N_{k+1}) < \alpha) < \infty$$

and Lemma 2.1 follows from Borel-Cantelli lemma and from the simple inequality

$$(2.11) \qquad U(a_{N_k}, N_{k+1}) \le U(a_N, N) \text{ if } N_k \le N < N_{k+1}.$$

□

PROOF OF LEMMA 2.2.

Since

$$(2.12) \qquad U(a_N, N) \le \inf_{0 \le j \le \frac{N}{a_N} - 1} \ \sup_{0 \le i \le a_N} |S_{i+ja_N} - S_{ja_N}|$$

and $\sup_{0 \le i \le a_N} |S_{i+ja_N} - S_{ja_N}|$ are independent for $j=0,1,2,\ldots$, we obtain from (1.30)

$$P(U(a_N,N) \geq \alpha) \leq (P(\sup_{0\leq i\leq a_N} |S_i| \geq \alpha))^{[\frac{N}{a_N}]} \leq$$

$$(1 - K_1(\cos \frac{\pi}{2\alpha})^{a_N})^{[\frac{N}{a_N}]} \leq e^{-K_1(\frac{N}{a_N} - 1)(\cos \frac{\pi}{2\alpha})^{c \log N}}$$

$$\leq \exp\left\{-K_1(\frac{N}{c \log N} - 1)N^{c \log \cos \frac{\pi}{2\alpha}}\right\},$$

if N is large enough. But from (2.3), $c \log \cos (\pi/(2\alpha)) > -1$, and hence

$$\sum_{N=1}^{\infty} P(U(a_N,N) \geq \alpha) < \infty.$$

So Lemma 2.2 follows from Borel-Cantelli lemma.

□

PROOF OF LEMMA 2.3.

From (1.17) and (1.30) we obtain

$$(2.13) \qquad P(A_{N-a_N}^{a_N}) \geq K_1(\cos \frac{\pi}{2\alpha})^{a_N} \geq \frac{K_1^*}{N}$$

By choosing $N_k = [(c + 1)k \log k]$, it can be easily seen that

$$N_k < N_{k+1} - a_{N_{k+1}}$$

holds for large enough k, hence the events $A_{N_k-a_{N_k}}^{a_{N_k}}$ are independent for $k \geq k_0$ and since

$$\sum_k P(A_{N_k - a_{N_k}}^{a_{N_k}}) = \infty,$$

(2.6) follows from Borel-Cantelli lemma.

□

Since

(2.14) $$U(a_N, N) \le \max_{0 \le i \le a_N} |S_{N-a_N+i} - S_{N-a_N}|,$$

Lemma 2.3 gives also

(2.15) $$P(U(a_N, N) \le \alpha(c) - 1 \quad \text{i.o.}) = 1.$$

PROOF OF LEMMA 2.4.

Let the events $A_j = A_j^a$ $(j=0,1,\ldots)$ be defined by (2.5) and assume that a is large enough so that (1.30) holds true. Then

(2.16) $$\{U(a,(k+1)a - 1 \ge \alpha\} = \bar{A}_0 \bar{A}_1 \ldots \bar{A}_{ka-1}$$

Introduce the following notations:

(2.17) $$p_k = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{ka-1}), \quad k = 1,2,\ldots$$

(2.18) $$D_k = \bar{A}_{(k-1)a} \ldots \bar{A}_{ka-1}, \quad k = 1,2,\ldots$$

For $k = 1$, we have

$$p_1 = P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{a-1}) \geq 1 - aP(A_0) \geq 1 - 2aP(A_0)$$

and (2.7) follows from (1.30).

Furthermore

$$p_1 = P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{a-1}) = P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{a-1}D_2) + P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{a-1}\bar{D}_2)$$

$$\leq p_2 + P(\bar{D}_2).$$

Hence from $P(\bar{D}_2) \leq \sum_{i=a}^{2a-1} P(A_i) = aP(A_0)$,

$$p_2 \geq p_1 - P(\bar{D}_2) \geq 1 - 2aP(A_0) \geq (1 - aP(A_0))^2 - aP(A_0)$$

and (2.7) for $k = 2$ follows again from (1.30).

For $k \geq 3$ we have

$$p_{k-1} = P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{(k-1)a-1}) =$$

$$= P(\bar{A}_0\bar{A}_1\ldots\bar{A}_{(k-1)a-1}\, D_k) + P(\bar{A}_0\ldots\bar{A}_{(k-1)a-1}\, \bar{D}_k)$$

$$\leq p_k + P(\bar{D}_k)p_{k-2}$$

Using this inequality with k-1 replaced by k-2,

$$p_{k-1} \leq p_k + P(\bar{D}_k)(p_{k-1} + P(\bar{D}_{k-1})p_{k-3})$$

$$\leq p_k + p_{k-1}P(\bar{D}_k) + P(\bar{D}_k)P(\bar{D}_{k-1}).$$

Since

$$P(\bar{D}_k) = P(\bar{D}_{k-1}) \leq aP(A_0),$$

we obtain

$$(2.19) \qquad p_k \geq (1 - aP(A_0))p_{k-1} - a^2P^2(A_0)$$

and by induction it is easy to see that

$$(2.20) \qquad p_k \geq (1 - aP(A_0))^k - aP(A_0)$$

Applying (1.30) again, we get (2.7).

$\square$

Now we are ready to prove Theorem 2. Lemma 2.1 and Lemma 2.2 clearly imply (1.18) and (1.19), while Lemma 2.1 and Lemma 2.3 imply (1.20). It remains to prove (1.21).

By putting $a = a_N = [c \log N]$, $k = [N/a_N]$, $\alpha = \alpha(c)$ into the inequality (2.7), one can easily see that

$$(2.21) \qquad \liminf_{N\to\infty} P(U([c \log N], N) \geq \alpha(c)) > 0,$$

where $\alpha(c)$, the solution of (1.17) is an integer. Consequently

$$(2.22) \qquad P(U([c \log N], N) \geq \alpha(c) \text{ i.o.}) > 0.$$

We can not claim however that the above probability is equal to 1, since

$$(2.23) \qquad \{U([c \log N], N) \geq \alpha(c) \text{ i.o.})\}$$

is not a tail event. A small modification of the argument given above, however gives the desired result. Consider

$$(2.24) \qquad \{ \min_{a_N+1\leq j\leq N-a_N} \max_{0\leq i\leq a_N} |S_{j+i} - S_j| \geq \alpha(c) \text{ i.o.}\}$$

which is already a tail event since $a_N \to \infty$ as $N \to \infty$, therefore its probability is either 0 or 1. But similarly to the above argument one can verify also that the probability of (2.24) is bounded away from 0 and hence

$$(2.25) \qquad P\left(\min_{a_N+1 \le j \le N-a_N} \max_{0 \le i \le a_N} |S_{j+i} - S_j| \ge \alpha(c) \text{ i.o.}\right) = 1.$$

On the other hand it follows from Theorem C that for almost all ω,

$$(2.26) \qquad \min_{0 \le j \le a} \max_{0 \le i \le a} |S_{j+i} - S_j| \ge \alpha(c)$$

if a is large enough, therefore we have also

$$(2.27) \qquad P(U([c \log N], N) \ge \alpha(c) \text{ i.o.}) = 1$$

which together with Lemma 2.2 yields (1.21). This completes the proof of Theorem 2.

3. THE TUBE AROUND ZERO

In this Section we prove our Theorem 3, based on the following 4 lemmas.

LEMMA 3.1. _If_ $\alpha \ge 1$ _is an integer and_

$$(3.1) \qquad \cos \frac{\pi}{2\alpha} < e^{-\frac{1}{2c}}$$

then for almost ω _there exists_ an $N_0 = N_0(c,\omega)$ _such that_

(3.2) $V(a_N,N) \geq \alpha$ *for* $N \geq N_0$

where $V(a,N)$ *is defined by (1.16) and* $a_N = [c \log N]$.

LEMMA 3.2. *If* $\alpha \geq 2$ *is an integer and*

(3.3) $$\cos \frac{\pi}{2\alpha} > e^{-\frac{1}{2c}}$$

then for almost all ω *there exists an* $N_0^* = N_0^*(c,\omega)$ *such that*

(3.4) $V(a_N,N) < \alpha$ for $N \geq N_0^*$

where $V(a,N)$ is defined by (1.16) and $a_N = [c \log N]$.

LEMMA 3.3. *Let the events* $A_j^a (j = 0,1,\ldots)$ *be defined by*

(3.5) $$A_j^a = \{ \max_{0\leq i\leq a} |S_{j+i}| < \alpha^*(c)\}$$

where the solution $\alpha^*(c)$ *of (1.22) is an integer. Then*

(3.6) $$P(A_{N-a_N}^{a_N} \text{ i.o.}) = 1,$$

where $a_N = [c \log N]$.

LEMMA 3.4. *The following inequality holds true for* $\alpha \geq 1$, $n > a$ *and* a *large enough:*

$$P(\tilde{V}(a,n+a) \geq \alpha) \geq \tag{3.7}$$
$$\geq \prod_{j=a+1}^{n} \left(1 - \frac{K^*}{\sqrt{j}} \left(\cos \frac{\pi}{2\alpha}\right)^{a}\right)$$

with some constant K^* *(which may depend on* α, *but not on* a *and* n), *where*

$$\tilde{V}(a,n+a) = \min_{a<j\leq n} \max_{0\leq i\leq a} |S_{j+i}| \tag{3.8}$$

PROOF OF LEMMA 3.1.

Define

$$A_j = A_j^a = \{-\alpha < S_{j+i} < \alpha, \ i = 0,1,\ldots,a\} \quad (j \geq 0). \tag{3.9}$$

Then

$$P(A_j) = \sum_{z=-\alpha+1}^{\alpha-1} P(A_j | S_j = z))(S_j = z). \tag{3.10}$$

From Stirling's formula we obtain

$$P(S_j = z) \leq \frac{K_0}{\sqrt{j}} \qquad \text{for } -\alpha < z < \alpha, \quad j > 0 \tag{3.11}$$

with certain constant K_0, depending only on α. Hence

$$(3.12) \qquad P(A_j) \leq \frac{K_0}{\sqrt{j}} \sum_{z=-\alpha+1}^{\alpha-1} P(A_j | S_j = z).$$

Applying (1.30), we get

$$(3.13) \qquad P(A_j) \leq \frac{K^*}{\sqrt{j}} \left(\cos \frac{\pi}{2\alpha}\right)^{a}.$$

Therefore

$$P(V(a_N,N) < \alpha) \leq \sum_{j=0}^{N-a_N} P(A_j) \leq$$

$$(3.14) \qquad \leq K^*\left(\cos \frac{\pi}{2\alpha}\right)^{a_N}\left(1 + \sum_{j=1}^{N-a_N} \frac{1}{\sqrt{j}}\right) \leq$$

$$\leq K^*\sqrt{N-a_N} \left(\cos \frac{\pi}{2\alpha}\right)^{a_N}.$$

Considering a subsequence $N_k = k^\rho$ $k = 1,2,\ldots$ with integer $\rho > 0$, we clearly have for $N_k \leq N < N_{k+1}$

$$(3.15) \qquad V(a_N,N) \geq V(a_{N_k},N_{k+1})$$

hence it is enough to prove that under the conditions of our Lemma 3.1 for almost all ω there exists $k^* = k_0^*(c,\omega)$ such that

$$(3.16) \qquad V(a_{N_k},N_{k+1}) \geq \alpha \quad \text{for } k \geq k_0^*.$$

From (3.14) we obtain

$$\sum_{k=1}^{\infty} P(V(a_{N_k}, N_{k+1}) < \alpha) \le \sum_{k=1}^{\infty} K^*(\alpha)\sqrt{N_{k+1}} \left(\cos \frac{\pi}{2\alpha}\right)^{a_{N_k}} \le$$

$$(3.17) \qquad \le \sum_{k=1}^{\infty} K^*(\alpha)\sqrt{N_{k+1}} \left(\cos \frac{\pi}{2\alpha}\right)^{c \log N_k} \le$$

$$\le \sum_{k=1}^{\infty} K^*(\alpha)\ (k+1)^{\rho/2} \quad k^{\rho\ c \log \cos \frac{\pi}{2\alpha}}$$

The last number (3.17) clearly converges if

$$\rho/2 + \rho\ c \log \cos \frac{\pi}{2\alpha} < -1$$

which is equivalent to

$$(3.18) \qquad \cos \frac{\pi}{2\alpha} < e^{-\frac{2+\rho}{2\rho c}} .$$

If (3.1) holds then there exists a big enough integer ρ which satisfies (3.18) implying the convergence of (3.17) and hence our lemma.

□

PROOF OF LEMMA 3.2.

To prove Lemma 3.2. we need the following result. Let

(3.19) $\quad T_n = \min\{j: \ 0 \leq j,\ S_{n+j} = 0\}$

We define a sequence of stopping times as follows. Let $\underline{a}$ be a positive integer, and let

$$\eta_0(a) = 0,\ \xi_k(a) = \eta_{k-1}(a) + a, \quad \eta_k(a) = T_{\xi_k(a)} \quad k=1,2.$$

Denote $\quad \alpha_k(a) = \eta_k(a) = \xi_k(a)$

Then clearly $\{\alpha_k(a)\}$ $k=1,2,\ldots$ is a sequence of independent identically distributed random variables having the same distribution as T_a.
Let ν_N be the largest integer for which

(3.10) $$\sum_{i=1}^{\nu_N} \alpha_i(a_N) + (\nu_N + 1)a_N \leq N$$

For ν_N we proved the following result (see Csáki and Földes [3] and also for more general recurrent random walk in Csáki and Földes [4]).

LEMMA 3.3. <u>If</u> $a_N < \frac{1}{2}N$ <u>then there exists a small enough</u> C_1 <u>and a big enough</u> C_2 <u>such that for any</u> $0 < \varepsilon < 1$

(3.11) $$P(\nu_N < C_1(\frac{N}{a_N})^{\frac{1-\varepsilon}{2}}) \leq C_2(\frac{a_N}{N})^{\frac{\varepsilon}{2}} .$$

Now we are ready to prove Lemma 3.2. From the inequality $V(a_N,N) \le \inf_{0\le k\le \nu_N} \sup_{0\le i\le a_N} |S_{n_k}(a_N) + i|$ we obtain the following estimation. Let A_j be defined by (3.9). Then

$$(3.22)\qquad P(V(a_N,N) \ge \alpha) = P(\bigcap_{j=0}^{N-a_N} \overline{A_j}) \le$$

$$P(\bigcap_{k=0}^{\nu_N} \overline{A_{n_k}(a_N)}) \le$$

$$P(\bigcap_{k=0}^{\nu_N} \overline{A_{n_k}}(a_N),\ \nu_N < C_1(\frac{N}{a_N})^{\frac{1-\varepsilon}{2}}) +$$

$$P(\bigcap_{k=0}^{\nu_N} \overline{A_{n_k}}(a_N),\ \nu_N \ge C_1(\frac{N}{a_N})^{\frac{1-\varepsilon}{2}}) \le$$

$$C_2(\frac{a_N}{N})^{\frac{\varepsilon}{2}} + (1 - P(A_0))^{C_1(\frac{N}{a_N})^{\frac{1-\varepsilon}{2}}}$$

by LEMMA 3.3.

Consequently applying (1.30) we have for $a_N = [c \log N]$

$$(3.23)\qquad \begin{aligned} P(V(a_N,N) \ge \alpha) &\le C_2(\frac{c \log N}{N})^{\frac{\varepsilon}{2}} + \\ &+ \left(1 - K_1(\cos \frac{\pi}{2\alpha})^{[c \log N]}\right)^{C_1(\frac{N}{a_N})^{\frac{1-\varepsilon}{2}}} \le \end{aligned}$$

$$\leq C_2\left(\frac{c \log N}{N}\right)^{\frac{\varepsilon}{2}} + \exp\left\{-K_1^*\left(\cos\frac{\pi}{2\alpha}\right)^{c \log N}\left(\frac{N}{c \log N}\right)^{\frac{1-\varepsilon}{2}}\right\}$$

Choosing a sequence $N_k = k^\rho$ $k = 1,2,\ldots$ similarly to the proof of Lemma 3.1. it is enough to show that

$$\sum_{k=1}^{\infty} P(V(a_{N_{k+1}}, N_k) \geq \alpha) < +\infty. \tag{3.24}$$

$$\sum_{i=1}^{\infty} P(V(a_{N_{k+1}}, N_k) \geq \alpha) \leq \sum_{k=1}^{\infty}\left(C_2\left(\frac{c\ \rho \log (k+1)}{k^\rho}\right)^{\frac{\varepsilon}{2}} + \right.$$

$$\left. + \exp\left\{-K_1^*\left(\cos\frac{\pi}{2\alpha}\right)^{c\ \rho \log (k+1)}\left(\frac{k^\rho}{c\ \rho \log (k+1)}\right)^{\frac{1-\varepsilon}{2}}\right\}\right)$$

$$= \sum_{k=1}^{\infty}\left(C_2\left(\frac{c\ \rho \log (k+1)}{k^\rho}\right)^{\frac{\varepsilon}{2}} + \exp\left\{-K_1^*(k+1)^{c\ \rho \log \cos\frac{\pi}{2\alpha}} \frac{k^{\frac{\rho(1-\varepsilon)}{2}}}{(c\ \rho \log (k+1))^{\frac{1-\varepsilon}{2}}}\right\}\right)$$

which is clearly convergent if

$$c\ \rho \log \cos\frac{\pi}{2\alpha} + \rho\ \frac{1-\varepsilon}{2} > 0 \tag{3.25}$$

and

$$\frac{\rho\varepsilon}{2} > 1 \tag{3.25}$$

Under the condition of our Lemma we may choose a small enough $\varepsilon > 0$ such that

$$(3.27) \qquad \cos \frac{\pi}{2\alpha} > e^{-\frac{1-\varepsilon}{2c}}$$

and to this ε we might choose an integer $\rho > 0$ such that (3.26) should hold. For this choice of ρ and ε (3.25) is valid and this proves (3.24) implying the lemma.

□

PROOF OF LEMMA 3.3.

From (1.30), (3.10) and the local central limit theorem one can obtain that

$$(3.28) \qquad P(A_j^a) \geq \frac{K_1^*}{\sqrt{j}} \left(\cos \frac{\pi}{2\alpha}\right)^a$$

if a and j are large enough, where A_j^a is defined by (3.9). Hence from (1.22),

$$(3.29) \qquad P(A_{N-a_N}^{a_N}) \geq \frac{K_1^*}{N}$$

and we may choose a subsequence $N_k = [(c+1)k \log k]$ such that

$$(3.30) \qquad N_k < N_{k+1} - a_{N_{k+1}}$$

for k large and

$$(3.31) \qquad \sum_k P(A_{N_k-a_{N_k}}^{a_{N_k}}) = \infty.$$

But this is not enough, because in this case the events in the brackets are not independent, even for large k. We apply the following version of Borel-Cantelli lemma due to Erdös and Rényi (see Rényi [8]):

If $\sum_k P(B_k) = \infty$ and

$$(3.32) \qquad \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \sum_{\ell=1}^{n} P(B_k B_\ell)}{\left(\sum_{k=1}^{n} P(B_k)\right)^2} \leq 1,$$

then $P(B_i \text{ i.o.}) = 1$. To verify (3.32) for $B_k = A_{N_k-a_{N_k}}^{a_{N_k}}$, consider two events $A_{j_1}^{a_1}$ and $A_{j_2}^{a_2}$, where $j_1 + a_1 < j_2$. Then the probability of joint occurrence of these two events can be written as

$$(3.33) \qquad \begin{aligned} &P(A_{j_1}^{a_1} A_{j_2}^{a_2}) = \\ &= \sum_{x_1=-\alpha+1}^{\alpha-1} \sum_{x_2=-\alpha+1}^{\alpha-1} P(A_{j_1}^{a_1} | S_{j_1+a_1} = x_1) P(S_{j_1+a_1} = x_1) \times \\ &\times P(A_{j_2}^{a_2} | S_{j_2} = x_2) P(S_{j_2} = x_2 | S_{j_1+a_1} = x_1), \end{aligned}$$

while

$$P(A_{j_1}^{a_1})P(A_{j_2}^{a_2}) =$$

(3.34)

$$= \sum_{x_1=-\alpha+1}^{\alpha-1} \sum_{x_2=-\alpha+1}^{\alpha-1} P(A_{j_1}^{a_1} | S_{j_1+a_1} = x_1) P(S_{j_1+a_1} = x_1) \times$$

$$\times P(A_{j_2}^{a_2} | S_{j_2} = x_2) P(S_{j_2} = x_2).$$

From the local limit theorem (or from Stirling's formula),

(3.35) $$P(S_{j_2} = x_2) \sim \sqrt{\frac{2}{\pi j_2}} \quad \text{for} \quad -\alpha < x_2 < \alpha$$

and

(3.36) $$P(S_{j_2} = x_2 | S_{j_1+a_1} = x_1) \sim \frac{\sqrt{2}}{\sqrt{\pi(j_2-j_1-a_1)}}$$

$$\text{for } -\alpha < x_1 < \alpha$$
$$-\alpha < x_2 < \alpha,$$

provided j_2 and $j_2-j_1-a_1$ are large enough and j_2 and $_2$ have the same parity. Hence it can be seen that for all $\varepsilon > 0$,

(3.37) $$P(A_{j_1}^{a_1} A_{j_2}^{a_2}) \leq (1 + \varepsilon) \sqrt{\frac{j_2}{j_2-j_1-a_1}} \; P(A_{j_1}^{a_1})P(A_{j_2}^{a_2})$$

provided $j_2-j_1-a_1$ is large enough. Since $N_{k+1} - a_{N_{k+1}} - N_k \to \infty$ as $k \to \infty$, we have for large enough k and $k < \ell$,

$$(3.38)\qquad P(B_k B_\ell) \le (1+\varepsilon)\sqrt{\frac{N_\ell - a_{N_\ell}}{N_\ell - a_{N_\ell} - N_k}}\; P(B_k)P(B_\ell).$$

Now from (3.38) one can verify (3.32) similarly to Csáki-Csörgő-Földes-Révész [1, Lemma 3.4.]

□

PROOF OF LEMMA 3.4.

Assume that a is large enough, so that the inequality (1.30) holds true. Let A_j be defined by (3.9). We show that

$$(3.39)\qquad P(\bar{A}_{a+1}\ldots\bar{A}_n) \ge P(\bar{A}_{a+1})\ldots P(\bar{A}_n),$$

provided a is large enough and $n > a$.

We start from the following identity:

$$P(\bar{A}_{a+1}\ldots\bar{A}_n) = P(\bar{A}_{a+1}\ldots\bar{A}_{n-1}) - P(A_n) + \sum_{k=a+1}^{n-1} P(\bar{A}_{a+1}\ldots\bar{A}_{k-1}A_k A_n).$$
(3.40)

The next step is to show that

(3.41) $$P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_kA_n) \geq P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_k)\, P(A_n)$$

We distinguish three cases.

Case (i). $n-a \leq k \leq n-1$.

In this case we have

(3.42) $$\begin{aligned} &P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_kA_n) = \\ &\sum_{x=-\alpha+1}^{\alpha-1} P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_kA_n \mid S_{k+a} = x)P(S_{k+a} = x) = \\ &= \sum_{x=-\alpha+1}^{\alpha-1} P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_k \mid S_{k+a}=x)P(\max_{k+a\leq i\leq n+a}|S_i|<\alpha \mid S_{k+a}=x)P(S_{k+a}=x) \end{aligned}$$

But from (1.30),

(3.43) $$P(\max_{k+a\leq i\leq n+a}|S_i| < \alpha \mid S_{k+a} = x) \geq K_1\left(\cos\frac{\pi}{2\alpha}\right)^{n-k}$$

for $-\alpha < x < \alpha$, $P(S_{k+a} = x) > 0$, and from (3.13),

(3.44) $$P(A_n) \leq \frac{K^*}{\sqrt{n}}\left(\cos\frac{\pi}{2\alpha}\right)^{a}.$$

Since $n-k \leq a$, we have for large enough n,

(3.45) $$P(\max_{k+a\leq i\leq n+a}|S_i| < \alpha \mid S_{k+a} = x) \geq P(A_n)$$

and this together with (3.42) yields (3.41).

Case (ii). $n-a-\sqrt{a} \leq k < n-a$.

In this case we have

$$(3.46)\quad \begin{aligned} &P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_kA_n) = \\ &= \sum_{x=-\alpha+1}^{\alpha-1} P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_kA_n \mid S_{k+a} = x)P(S_{k+a} = x) \\ &= \sum_{x=-\alpha+1}^{\alpha-1} P(\bar{A}_{a+1}\cdots\bar{A}_{k-1}A_k \mid S_{k+a} = x)P(A_n \mid S_{k+a} = x)P(S_{k+a} = x). \end{aligned}$$

But from (1.30),

$$(3.47)\quad \begin{aligned} P(A_n \mid S_{k+a} = x) &= \sum_{y=-\alpha+1}^{\alpha-1} P(A_n \mid S_n=y)P(S_n=y \mid S_{k+a} = x) \\ &\geq K_1\left(\cos\frac{\pi}{2\alpha}\right)^a \sum_{y=-\alpha+1}^{\alpha-1} P(S_n = y \mid S_{k+a} = x) \\ &= K_1\left(\cos\frac{\pi}{2\alpha}\right)^a P(|S_n| < \alpha \mid S_{k+a} = x) \end{aligned}$$

From the local limit theorem

$$(3.48)\quad P(|S_n| < \alpha \mid S_{k+a} = x) \geq \frac{K_1^*}{\sqrt{n-k-a}} \geq \frac{K_1^*}{\sqrt[4]{a}}$$

for $-\alpha < x < \alpha$ and $P(S_{k+a} = x) > 0$, hence by (3.44) for large enough a,

$$(3.49) \qquad P(A_n | S_{k+a} = x) \geq P(A_n)$$

and (3.46) yields (3.41).

Case (iii). $a+1 \leq k < n-a-\sqrt{a}$.

Compare

$$(3.50) \qquad P(A_n) = \sum_{y=-\alpha+1}^{\alpha-1} P(A_n | S_n = y) P(S_n = y)$$

and

$$(3.51) \qquad P(A_n | S_{k+a} = x) = \sum_{y=-\alpha+1}^{\alpha-1} P(A_n | S_n = y) P(S_n = y | S_{k+a} = x)$$

Then either both $P(S_n=y)$ and $P(S_n=y|S_{k+a}=x)$ are zero or by the local limit theorem (or Stirling's formula)

$$(3.52) \qquad P(S_n = y) \sim \sqrt{\frac{2}{\pi n}} \quad (n \to \infty)$$

and

$$(3.53) \qquad P(S_n = y | S_{k+a} = x) \sim \frac{\sqrt{2}}{\sqrt{\pi(n-k-a)}} , \quad (n-k-a \to \infty)$$

Therefore

$$(3.54) \qquad P(S_n = y) \leq P(S_n = y | S_{k+a} = x)$$

for $-\alpha < x < \alpha$, $-\alpha < y < \alpha$ and a large enough, which gives also (see (3.50) and (3.51))

$$(3.55) \qquad P(A_n) \leq P(A_n | S_{k+a} = x)$$

implying (3.41) as in case (ii).

(3.40), (3.41) and a simple induction argument yield (3.39) and this together with (3.13) proves (3.7).

□

Now we cam complete the proof of Theorem 3 similarly to the proof of Theorem 2. Lemma 3.1 and Lemma 3.2 imply (1.23) and (1.24), while Lemma 3.1 and Lemma 3.3 imply (1.25). So we have to prove (1.26).

By putting $a = a_N = [c \log N]$, $n = N-a_N$ into (3.7) one can easily see that

$$(3.56) \qquad \liminf_{N\to\infty} P(\tilde{V}(a_N,N) \geq \alpha^*(c)) > 0$$

where $\alpha^*(c)$, the solution of (1.22) is an integer. Consequently

$$(3.57) \qquad P(\tilde{V}(a_N,N) \geq \alpha^*(c) \text{ i.o.}) > 0.$$

But the event

$$(3.58) \qquad \{\tilde{V}(a_N,N) \geq \alpha^*(c) \text{ i.o.}\}$$

is a tail event (for S_n), therefore its probability is either 0 or 1. Moreover

(3.59) $$\min_{0\leq j\leq a} \max_{0\leq i\leq a} |S_{j+i}| \geq \alpha^*(c) \quad \text{a.s.}$$

for large enough a follows from Theorem 1. Hence we have also

(3.60) $$P(V([c \log N], N) \geq \alpha^*(c) \text{ i.o.}) = 1$$

and this with Lemma 3.2 implies (1.26). The proof of Theorem 3 is complete.

REFERENCES

1. Csáki, E., Csörgő, M., Földes, A. Révész, P.: How big are the increments of the local time of a Wiener process? Ann. Probability, to appear.

2. Csáki, E., Földes, A.: On the narrowest tube of a Wiener process. Coll. Math. Soc. J. Bolyai 36. Limit theorems in probability and statistics, Veszprém (Hungary), 1982 (ed. P. Révész), to appear.

3. Csáki, E., Földes, A.: How big are the increments of the local time of a simple symmetric random walk? Coll. Math. Soc. J. Bolyai 36. Limit theorems in probability and statistics, Veszprém (Hungary), 1982 (ed. P. Révész), to appear.

4. Csáki, E., Földes, A.: How big are the increments of the local time of a recurrent random walk? Z. Wahrscheinlichkeitstheorie verv. Geb., to appear.

5. Csörgő, M., Révész, P.: Strong approximations in Probability and Statistics. New York: Academic Press 1981.

6. Ellis, R.E.: On the solution of equations in finite differences. Cambridge Mathematical Journal 4, 182 (1844).

7. Mogul'skii, A.A.: Small deviations in a space of trajectories. Theor. Probability Appl. 19, 726-736 (1974).

8. Rényi, A.: Probability theory. Amsterdam: North Holland 1970.

9. Takács, L.: Remarks on random walk problems. Publ. Math. Inst. Hung. Acad. Sci. 2, 175-182 (1958).