A SIMPLE FORM FOR INVERSE MOMENTS OF NON-CENTRAL CHI-SQUARE RANDOM VARIABLES AND THE RISK OF JAMES-STEIN ESTIMATORS

by

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Summary

By providing a simple form for the expected value of the inverse of a noncentral chi-square random variable, it is possible to give a simple form for the mean and risk of the James-Stein estimator. A theorem is also given for the evaluation of other inverse moments of the non-central chi-square random variable.

Section 1. Introduction

Let X be a k-dimensional normal random vector with mean θ and identity covariance matrix I_k. Under squared error loss the risk of an estimator $\tilde{\theta}(X)$ of θ is

$$R(\tilde{\theta},\theta) = E\left[(\tilde{\theta}(X)-\theta)^t(\tilde{\theta}(X)-\theta)\right] .$$

For $k \geq 3$, James and Stein [6] gave estimators of the form

$$\hat{\theta}_c(X) = \left(1 - \frac{c}{||X||^2}\right) X$$

(where c is an appropriately chosen constant) and $||X||^2 = X^tX$. They showed that $\hat{\theta}_c$ dominates $\hat{\theta}_0(X) \equiv X$, i.e.

$$R(\hat{\theta}_c,\theta) < R(\hat{\theta}_0,\theta) \equiv k \text{ for all } \theta.$$

This result has generated a great deal of interest and been extended in many directions. (See Berger [2], Brandwein and Strawderman [4], Efron and Morris [5]. In fact, results of Brandwein [3] extend this result from the normal distribution for X to all distributions for X which are spherically symmetric distributions about θ.) The best choice of c is (k-2) for $\hat{\theta}_c$ and the purpose of this paper is to present a simple form for the risk with c = (k-2) although the method would work for other values of c and other specifications of the model with their corresponding Stein-type estimators. A theorem which may be of independent interest is given in Section 3 which provides a simple form for $E\left[(\chi^2_{k,\lambda})^{-n}\right]$ where $\chi^2_{k,\lambda}$ is a noncentral chi-square random variable and $k > 2n$.

Section 2. A Simple Form for the Risk and Mean of the James-Stein Estimator.

Assume the k-dimensional vector X is normally distributed with mean vector θ and identity covariance matrix I_k with $k \geq 3$.

The James-Stein estimator $\hat{\theta}$ is defined as

$$\hat{\theta}(X) = \left(1 - \frac{(k-2)}{||X||^2}\right) X ,$$

with risk

$$R(\theta,\hat{\theta}) = E\left[||\hat{\theta}(X)-\theta||^2\right] = k-(k-2)^2 E\left[\left(\chi^2_{k,\lambda}\right)^{-1}\right] .$$

(See Judge and Bock [7], p. 171.) From the theorem for the evaluation of $E\left[\left(\chi^2_{k,\lambda}\right)^{-1}\right]$, we have for even k,

$$R(\theta,\hat{\theta}) = k - (k-2)\left(\frac{k}{2} - 1\right)! \left(\frac{-2}{\lambda}\right)^{\frac{k}{2}-1} \left[e^{-\frac{\lambda}{2}} - \sum_{\ell=0}^{\frac{k}{2}-2} \frac{\left(-\frac{\lambda}{2}\right)^{\ell}}{\ell!}\right] .$$

If k is odd, then

$$R(\theta,\hat{\theta}) = k - (k-2)\left(\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)(-2/\lambda)^{\frac{k-1}{2}-1} \left[2(\lambda/2)^{-\frac{1}{2}} \mathcal{D}(\lambda/2)^{\frac{1}{2}}\right.$$

$$\left. -I_{[5,\infty)}(k) \sum_{n=0}^{\frac{k-1}{2}-1} (-\lambda/2)^n \left(\frac{\Gamma(n+1+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)\right],$$

where

$$\mathcal{D}(y) \equiv e^{-y^2} \int_0^y e^{t^2}\, dt$$

is Dawson's integral.

Note that $\mathcal{D}(y)$ is nonnegative and that its maximum value is .5410442246... which occurs for y = .9241388730... (see page 298, Handbook [1].) For large y, $\mathcal{D}(y)$ is essentially $\frac{1}{2} y^{-1}$. Tables for $\mathcal{D}(y)$ are given in [1].

Here is a picture of $\mathcal{D}(y)$ from page 297 of Handbook [1].

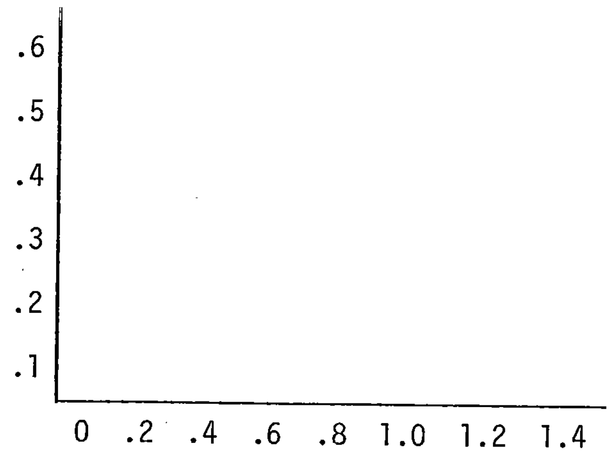

Figure 1

It has an inflection point at y = 1.5019752682.... where $\mathcal{D}(y) = .4276866160$. (See page 298, Handbook [1].)

In the case that k = 4,

$$R(\hat{\theta},\theta) = 4 - 4\lambda^{-1}\left(1-e^{-\frac{\lambda}{2}}\right).$$

If k = 6,

$$R(\hat{\theta},\theta) = 6 - 4\left(\left(\frac{\lambda}{2}\right)^2/2!\right)^{-1}\left(e^{-\frac{\lambda}{2}} - 1 + \frac{\lambda}{2}\right)$$

$$= 6 - 32\lambda^{-2}\left(e^{-\frac{\lambda}{2}} - 1 + \frac{\lambda}{2}\right).$$

If k = 3,

$$R(\hat{\theta},\theta) = 3 - \left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right).$$

If k = 5,

$$R(\hat{\theta},\theta) = 5 - 9\lambda^{-1}\left(1-2\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right)\right).$$

Note that $\hat{\theta}$ is a biased estimator of θ. The mean of $\hat{\theta}$ is

$$E[\hat{\theta}] = \left(1 - (k-2)\, E\left[\frac{1}{\chi^2_{K+2,\lambda}}\right]\right)\theta$$

For k even, we have

$$E[\hat{\theta}] = \theta\left(1 - ((k-2)/2)\ \left(\frac{k-2}{2}\right)!\ (-2/\lambda)^{\frac{k}{2}} \left[e^{-\frac{\lambda}{2}} - \sum_{\ell=0}^{\frac{k}{2}-1} \frac{\left(-\frac{\lambda}{2}\right)^{\ell}}{\ell!}\right]\right)$$

For k odd, we have

$$E[\hat{\theta}] = \theta\left(1 - ((k-2)/2)\left(\frac{\Gamma(\frac{k-2}{2})}{\Gamma(\frac{1}{2})}\right)(-2/\lambda)^{\frac{(k-1)}{2}} \left[2\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right)\right.\right.$$

$$\left.\left. - I_{[3,\infty)}(k) \sum_{n=0}^{\frac{(k-1)}{2}-1} \left(-\frac{\lambda}{2}\right)^{n} \left(\frac{\Gamma(n+1+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)\right]\right)$$

<u>Section 3</u>. <u>A Simple Form for the Expected Value of the Inverse Moments of a Noncentral Chi-square Random Variable</u>.

This corollary is a special case of the theorem that follows. The form is closed for the expected value of inverse moments of a noncentral chi-square random when the degrees of freedom are even.

<u>Corollary</u>: Let $\chi^2_{k,\lambda}$ have a noncentral chi-square distribution with noncentrality parameter λ. If k is an even integer greater than two, then

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-1}\right] = \tfrac{1}{2}\left(\frac{k}{2} - 2\right)!\ \left(\frac{-2}{\lambda}\right)^{\frac{k}{2}-1} \left[e^{-\frac{\lambda}{2}} - \sum_{\ell=0}^{\frac{k}{2}-2} \frac{\left(-\frac{\lambda}{2}\right)^{\ell}}{\ell!}\right].$$

If k is an odd integer greater than two, then

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-1}\right] = \tfrac{1}{2}\left(\frac{\Gamma(\frac{k-2}{2})}{\Gamma(\frac{1}{2})}\right)(-2/\lambda)^{\frac{(k-1)}{2}-1} \left[2\left(\frac{\lambda}{2}\right)^{-\frac{1}{2}} \mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right)\right.$$

$$\left. - I_{[5,\infty)}(k) \sum_{n=0}^{\frac{(k-1)}{2}-2} \left(-\frac{\lambda}{2}\right)^{n} \frac{\Gamma(n+1+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right],$$

where

$$\mathcal{D}(y) = e^{-y^2} \int_0^y e^{t^2} dt$$

is Dawson's integral.

Note: (1) Recall that

$$\Gamma(m+\tfrac{1}{2}) \;/\; \Gamma(\tfrac{1}{2})$$

$$= (m-\tfrac{1}{2}) \;\; (m-1-\tfrac{1}{2}) \; \ldots . (\tfrac{1}{2})$$

a product of m terms.

(2) The values of $\mathcal{D}(y)$ are nonnegative and the maximum value is less than .542. For large y, $\mathcal{D}(y)$ is approximately $\tfrac{1}{2}y^{-1}$.

The expression is very simple for even degrees of freedom. In particular,

For k = 4, $E\left[\left(\chi^2_{4,\lambda}\right)^{-1}\right]$

$$= \lambda^{-1}\left(1-e^{-\frac{\lambda}{2}}\right).$$

For k = 6, $E\left[\left(\chi^2_{6,\lambda}\right)^{-1}\right]$

$$= 2\lambda^{-2} \; (e^{-\frac{\lambda}{2}} - 1 + \lambda/2).$$

If k = 3, then

$$E\left[\left(\chi^2_{3,\lambda}\right)^{-1}\right] \;=\; (\tfrac{\lambda}{2})^{\frac{1}{2}} \;\; \mathcal{D}\left((\tfrac{\lambda}{2})^{\frac{1}{2}}\right) \; .$$

For large values of the argument $\mathcal{D}(y)$ is essentially $\frac{1}{2} y^{-1}$. For instance if $\left(\frac{\lambda}{2}\right)^{-1}$ is .005 (ie. $\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \approx 14$), then $\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathcal{D}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} = .501259494$

Thus $E\left[\left(\chi^2_{3,\lambda}\right)^{-1}\right] \approx \frac{1}{2}\left(\frac{\lambda}{2}\right)^{-1} = \lambda^{-1}$ for large λ.

The corollary is a special case of the following theorem.

<u>Theorem</u>: Let k and n be nonnegative integers and assume k is greater than 2n. If k is an even integer then

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-n}\right] =$$

$$= \frac{2^{-1}\left(-\frac{\lambda}{2}\right)^{-\left(\frac{k}{2}-n\right)}}{(n-1)!} \sum_{\ell=0}^{n-1} \left(\frac{\lambda}{2}\right)^{-\ell} \binom{n-1}{\ell} \left(\frac{k}{2}-1-n+\ell\right)! \left\{ e^{-\frac{\lambda}{2}} - \sum_{t=0}^{\left(\frac{k}{2}-1-n+\ell\right)} \frac{\left(-\frac{\lambda}{2}\right)^t}{t!} \right\} .$$

If k is an odd integer,

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-n}\right] =$$

$$= \frac{2^{-1}\left(-\frac{\lambda}{2}\right)^{-\left(\frac{(k-1)}{2}-n\right)}}{(n-1)!} \sum_{\ell=0}^{n-1} \left(\frac{\lambda}{2}\right)^{-\ell} \binom{n-1}{\ell} \left(\frac{\Gamma\left(\frac{k}{2}-n+\ell\right)}{\Gamma\left(\frac{1}{2}\right)}\right)$$

$$\left\{ 2\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathcal{D}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} - I_{[1,\infty)}\left(\frac{(k-1)}{2}-n+\ell\right) \sum_{t=0}^{\frac{(k-1)}{2}-1-n+\ell} \left(-\frac{\lambda}{2}\right)^t \Big/ \left(\frac{\Gamma\left(t+1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right) \right\}$$

Proof:

Let k be an integer greater than 2n. Then

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-n}\right] = e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \; E\left[\left(\chi^2_{k+2j}\right)^{-n}\right]$$

$$= e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \; \frac{2^{-n}\Gamma(\frac{k}{2}-n+j)}{\Gamma(\frac{k}{2}+j)}$$

$$= \tfrac{1}{2}\, e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \left\{\frac{1}{\prod_{\ell=0}^{n-1}\left(\frac{k}{2}-n+j+\ell\right)}\right\}$$

$$= \tfrac{1}{2}\, e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \left\{\frac{1}{(n-1)!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \frac{(-1)^\ell}{\left(\frac{k}{2}-n+j+\ell\right)}\right\}$$

$$= \frac{2^{-1} e^{-\frac{\lambda}{2}}}{(n-1)!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^\ell \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \int_0^1 x^{\frac{k-2}{2}-n+j+\ell}\, dx$$

$$= \frac{2^{-1} e^{-\frac{\lambda}{2}}}{(n-1)!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^\ell \int_0^1 \left(\sum_{j=0}^{\infty} \frac{\left(\frac{x}{2}\right)^j}{j!}\right) x^{\frac{k-2}{2}-n+\ell}\, dx$$

$$= \frac{2^{-1}}{(n-1)!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^\ell \int_0^1 e^{(x-1)\left(\frac{\lambda}{2}\right)} x^{\frac{k-2}{2}-n+\ell}\, dx$$

If k is even, then k/2 is an integer and lemma 2 of the Appendix implies that

$$E\left[\left(\chi^2_{k,\lambda}\right)^{-n}\right] =$$

$$= \frac{2^{-1}}{(n-1)!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^{\ell} \left(\frac{k}{2} - 1 - n + \ell\right)! \left(-\frac{\lambda}{2}\right)^{-\left(\frac{k}{2} - n + \ell\right)}$$

$$\left\{ e^{-\frac{\lambda}{2}} - \sum_{t=0}^{\left(\frac{k}{2} - 1 - n + \ell\right)} \frac{\left(-\frac{\lambda}{2}\right)^t}{t!} \right\}$$

$$= \frac{2^{-1}(-1)^{n-\frac{k}{2}} \left(\frac{\lambda}{2}\right)^{-\left(\frac{k}{2} - 1\right)}}{(n-1)!} \sum_{s=0}^{n-1} \binom{n-1}{s} \left(\frac{\lambda}{2}\right)^s \left(\frac{k}{2} - 2 - s\right)!$$

$$\left\{ e^{-\frac{\lambda}{2}} - \sum_{t=0}^{\left(\frac{k}{2} - 2 - s\right)} \frac{\left(-\frac{\lambda}{2}\right)^t}{t!} \right\} .$$

If k is an odd integer setting $a = \frac{\lambda}{2}$ and $j = \frac{(k-1)}{2} - n + \ell$ in lemma 1 of the Appendix gives

$$E\left[\left({}^{2}_{k,\lambda}\right)^{-n}\right]$$

$$= \frac{2^{-1}}{(n-1)!} \sum_{\ell=0}^{n-1} (-1)^{\ell} \binom{n-1}{\ell} \frac{\Gamma\left(\frac{k}{2} - n + \ell\right)}{\Gamma\left(\frac{1}{2}\right)} \left(-\frac{\lambda}{2}\right)^{-\left(\frac{(k-1)}{2} - n + \ell\right)}$$

$$\left\{ \left(\frac{\lambda}{2}\right)^{-\frac{1}{2}} 2\mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right) - I_{[1,\infty)}\left(\frac{(k-1)}{2} - n + \ell\right) \sum_{t=0}^{\left(\left(\frac{k-1}{2}\right) - 1 - n - \ell\right)} \left(-\frac{\lambda}{2}\right)^t \Big/ \left(\frac{\Gamma\left(t + 1 + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right) \right\}$$

$$= \frac{2^{-1}(-1)^{n-\frac{(k-1)}{2}} \left(\frac{\lambda}{2}\right)^{-\left(\frac{(k-1)}{2} - 1\right)}}{(n-1)!} \sum_{s=0}^{n-1} \binom{n-1}{s} \left(\frac{\lambda}{2}\right)^{s} \frac{\Gamma(\frac{k}{2} - 1 - s)}{\Gamma(\frac{1}{2})}$$

$$\left\{ \left(\frac{\lambda}{2}\right)^{-\frac{1}{2}} 2\mathcal{D}\left(\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\right) - I_{[0, \frac{(k-1)}{2} - 2)}(s)\right.$$

$$\left. \sum_{t=0}^{\left(\frac{(k-1)}{2} - 2 - s\right)} \left(-\frac{\lambda}{2}\right)^{t} \Bigg/ \left(\frac{\Gamma(t + 1 + \frac{1}{2})}{\Gamma(\frac{1}{2})}\right) \right\}$$

where $\mathcal{D}(y) \equiv e^{-y^2} \int_0^y e^{t^2}\, dt$ is the Dawson integral.

qed.

Appendix

Lemma 1. Assume j is a nonnegative integer. Then

$$\int_0^1 e^{(x-1)a} x^{j-\frac{1}{2}} dx$$

$$= \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} (-a)^{-j} \left\{ a^{-\frac{1}{2}} 2\mathscr{D}(a^{\frac{1}{2}}) - I_{[1,\infty)}(j) \sum_{m=0}^{j-1} (-a)^m / \left(\frac{\Gamma(m+1+\frac{1}{2})}{\Gamma(\frac{1}{2})} \right) \right\}$$

where

$$\mathscr{D}(y) \equiv e^{-y^2} \int_0^y e^{t^2} dt$$

is the Dawson integral.

Proof:

Suppose $j \geq 1$. Then using integration by parts,

$$\int_0^1 e^{(x-1)a} x^{j-\frac{1}{2}} dx$$

$$= \left[e^{(x-1)a} x^{j-\frac{1}{2}} a^{-1} \right]_{x=0}^{x=1} - \int_0^1 a^{-1} (j-\tfrac{1}{2}) x^{j-1-\frac{1}{2}} e^{(x-1)a} dx$$

$$= a^{-1} - a^{-1}(j-\tfrac{1}{2}) \int_0^1 e^{(x-1)a} x^{j-1-\frac{1}{2}} dx.$$

Repeated applications of integration by parts imply that

$$\int_0^1 e^{(x-1)a} x^{j-\frac{1}{2}}\, dx = \sum_{k=0}^{j-1} a^{-(k+1)} \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\frac{1}{2}-k)} (-1)^k +$$

$$+ (-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} a^{-j} \int_0^1 e^{(x-1)a}\, x^{-\frac{1}{2}}\, dx$$

Now for $j = 0$,

$$\int_0^1 e^{(x-1)a} x^{j-\frac{1}{2}}\, dx = \int_0^1 e^{(x-1)a} x^{-\frac{1}{2}}\, dx$$

The integral $\int_0^1 e^{(x-1)a} x^{-\frac{1}{2}}\, dx$ may be written as

$$= 2a^{-\frac{1}{2}} e^{-a} \int_0^{a^{\frac{1}{2}}} e^{t^2} dt$$

using the transformation $t = a^{\frac{1}{2}} x^{\frac{1}{2}}$.

This is $2a^{-\frac{1}{2}} \mathscr{D}(a^{\frac{1}{2}})$ where

$$\mathscr{D}(y) \equiv e^{-y^2} \int_0^y e^{t^2} dt$$

is the Dawson integral.

Thus

$$\int_0^1 e^{(x-1)a} x^{j-\frac{1}{2}}\, dx$$

$$= I_{[1,\infty)}(j) \sum_{k=0}^{j-1} a^{-(k+1)} \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\frac{1}{2}-k)} (-1)^k +$$

$$+ (-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} a^{-j-\frac{1}{2}} 2\mathscr{D}(a^{\frac{1}{2}})$$

$$= \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} (-a)^{-j} \left[a^{-\frac{1}{2}} 2\mathscr{D}(a^{\frac{1}{2}}) \right.$$

$$\left. - I_{[1,\infty)}(j) \sum_{k=0}^{j-1} a^{-(k+1-j)} \frac{(-1)^{k+1-j}}{\left(\frac{\Gamma\left(\frac{1}{2}+1+(j-1-k)\right)}{\Gamma(\frac{1}{2})}\right)} \right]$$

$$= \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})} (-a)^{j} \left[a^{-\frac{1}{2}} 2\mathscr{D}(a^{\frac{1}{2}}) - I_{[1,\infty)}(j) \right.$$

$$\left. \sum_{m=0}^{j-1} \frac{(-a)^{-m}}{\left(\frac{\Gamma(m+1+\frac{1}{2}))}{\Gamma(\frac{1}{2})}\right)} \right] \quad \text{qed.}$$

The proof of the following is given on p.188 of Judge and Bock [7].

<u>Lemma 2</u>

$$\int_0^1 y^m e^{-(y-1)c} dy = m!c^{-(m+1)}\{e^c - \sum_{n=0}^{m} c^n/n!\}.$$

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