ON AN OPTIMAL C(α)-TEST OF POISSON HYPOTHESIS* AGAINST COMPOUND POISSON ALTERNATIVES

(Dedicated to the Memory of Late Professor Jerzy Neyman)

by

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Technical Report #83-7

Purdue University
West Lafayette, Indiana

March 1983

*Presented as an invited paper at the Pacific Area Statistical Conference, held in December, 1982, at Tokyo, Japan.

**These investigations were supported in part by the U.S. National Science Foundation Grant No. MCS-8102733.

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1. INTRODUCTION. A nonnegative integer valued random varialbes (r.v.) X is said to have compound Poisson distribution if its probability generating function (p.g.f.) G(s) is given by

$$G(s) \equiv E(s^X) = \exp\{-\lambda(1-h(s))\}, \quad |s| \leq 1, \tag{1}$$

for some constant $\lambda > 0$, and a p.g.f. h(s) given by

$$h(s) = \sum_{k=0}^{\infty} r_k s^k, \quad |s| \leq 1, \tag{2}$$

where the nonnegative coefficients r_k's add up to one. These distributions as alternatives to Poisson distribution often arise in many live situations and include distributions such as negative binomial, Neyman type A distributions, to mention a few (see for instance Neyman [8], Feller [4], Neyman and Puri ([11], [12]) and Puri [13]). The purpose of the present work is to develop a test of the hypothesis that the underlying distribution is Poisson against the compound Poisson alter-

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*These investigations were supported in part by the U.S. National Science Foundation Grant No. MCS-8102733.

natives based on a sample $X_1, X_2, \ldots, X_n$. This same problem in the past formed the basis of Neyman's contagious distributions [8], but more recently it arose again in our work in the area of radiation biology (see Neyman and Puri [11], [12] and Puri [13]).

Again in (1) since the parameter λ is arbitrary, without loss of generality we may assume that $h(0) = r_0 = 0$. The Poisson hypothesis H_0 under test and composite alternative hypothesis H_1 of compound Poisson can now be equivalently stated as H_0: $\eta = 1$ versus H_1: $\eta > 1$, where $\eta = h'(1)$. This suggests parameterising our original problem by introducing a suitable dependence of the p.g.f. $h(\cdot)$ on a nonnegative parameter ξ (for instance, ξ could be taken to be $\eta-1$), so that if we rewrite the p.g.f. (2) as

$$(3) \qquad h_\xi(s) = \sum_{k=1}^{\infty} r_k(\xi) s^k, \quad |s| \leq 1,$$

with $r_k(\xi) \geq 0$, $\sum_{k=1}^{\infty} r_k(\xi) = 1$, we have for $k \geq 1$,

$$(4) \qquad \lim_{\xi \to 0} r_k(\xi) = \delta_{1k} \; ; \quad \lim_{\xi \to 0} h_\xi(s) \equiv h_0(s) \equiv s,$$

where δ_{1k} is the Kronecker delta. With this our hypotheses become

$$(5) \qquad H_0\colon \xi = 0; \quad H_1\colon \xi > 0,$$

where λ is the nuisance parameter. From this point on our approach in developing the desired test is based on Neyman's $C(\alpha)$-test theory (see

Neyman [9], Neyman and Scott [10], Bartoo and Puri [1] and Bühler and Puri [2]), which yields tests that are locally asymptotically optimal in a class of so called $C(\alpha)$-tests. In the next section we give a brief outline of this theory, which is then used in Section 3 in developing under appropriate conditions an optimal $C(\alpha)$-test for our problem. The test so obtained is applied in section 4 to data taken from Neyman [8]. The paper ends with some concluding remarks in section 5.

2 NEYMAN'S $C(\alpha)$-TEST THEORY.

Quite often in modeling real live situations, the distributions of the observable random variables turn out to be much more involved than the standard text book type distributions. Often they also involve many nuisance parameters θ's beside the parameter ξ under test. Also the estimators available for the nuisance parameters θ's may not be too good and in particular may be biased. Keeping these nonstandard situations in mind, Neyman [9] developed tests for testing the hypothesis say, H_0: $\xi = \xi_0$ against the alternative hypothesis say, H_1: $\xi > \xi_0$, in the presence of nuisance parameters. These tests are locally asymptotically most powerful in a class of so called $C(\alpha)$-tests.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (I.I.D.) r.v.'s with probability density function (p.d.f.) $p(x;\xi,\underset{\sim}{\theta})$ with respect to a σ-finite measure μ, which is independent of ξ and $\underset{\sim}{\theta}$, where $\xi \in [0,a)$ for some $a > 0$, and $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \Theta$, with Θ being an open set in R^r. Also we assume that the support of the distribution of X is independent of ξ and $\underset{\sim}{\theta}$. Let the null hypothesis for convenience be H_0: $\xi = 0$, which is to be tested against H_1: $\xi > 0$, in the presence of nuisance parameters $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \Theta$. We impose the following conditions on the p.d.f. $p(x;\xi,\underset{\sim}{\theta})$.

(C_1) The derivatives

$$\varphi_{\theta_j}(x;\underset{\sim}{\theta}) = \left.\frac{\partial \ell n\, p(x;\xi,\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\xi=0}, \quad j=1,2,\ldots,r, \tag{6}$$

and

$$\varphi_{\xi}(x;\underset{\sim}{\theta}) = \left.\frac{\partial \ell n\, p(x;\xi,\underset{\sim}{\theta})}{\partial \xi}\right|_{\xi=0}, \tag{7}$$

exist and are all Cramér functions (see Neyman [9] for their definition).

(C_2) Under H_0, $\varphi_{\xi}(X;\underset{\sim}{\theta})$ is not expressible as a linear function of $\varphi_{\theta_j}(X;\underset{\sim}{\theta})$, $j=1,2,\ldots,r$, with probability one.

It may be remarked here that the functions (6) and (7) besides having first two moments under H_0, satisfy few other "regularity conditions". These regularity conditions are similar to the ones imposed by Cramér [3] in his treatment on consistency of maximum likelihood estimates. Consequently functions satisfying these regularity conditions are referred to by Neyman [9] as Cramér functions. We shall not spell out these conditions in detail here in defining these conditions; instead we refer the reader to Neyman [9] for their definition.

Let $g(x;\underset{\sim}{\theta})$ be a measurable Cramér function, which we center around its expectation to yield

$$f(x;\underset{\sim}{\theta}) = \{g(x;\underset{\sim}{\theta}) - E_0[g(X;\underset{\sim}{\theta})]\}, \tag{8}$$

where the zero subscripts in E_0 here and in σ_0 below indicate that the expectation and the variance σ_0^2 are obtained under H_0. Furthermore, let

$$g^*(x;\underset{\sim}{\theta}) = f(x;\underset{\sim}{\theta}) - \sum_{j=1}^{r} b_j(\underset{\sim}{\theta})\, \varphi_{\theta_j}(x;\underset{\sim}{\theta}), \tag{9}$$

where b_j's are the first order partial regression coefficients of $f(X;\underset{\sim}{\theta})$ on $\varphi_{\theta}(X;\underset{\sim}{\theta})$'s, computed under H_0. Finally let $S(\alpha) \subset R$ be a measurable set with an almost everywhere continuous indicator function such that

(10) $$\frac{1}{\sqrt{2\pi}} \int_{S(\alpha)} \exp[-\tfrac{1}{2}t^2]dt = \alpha.$$

A typical member of Neyman's class of $C(\alpha)$-tests is now defined for each pair $g(x;\underset{\sim}{\theta})$, a Cramér function and a set $S(\alpha)$, by rejecting H_0 whenever $Z_n(\hat{\underset{\sim}{\theta}}_n) \in S(\alpha)$, where

(11) $$Z_n(\hat{\underset{\sim}{\theta}}_n) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \frac{g^*(X_i;\hat{\underset{\sim}{\theta}}_n)}{\sigma_0(g^*;\hat{\underset{\sim}{\theta}}_n)} ,$$

(12) $$\sigma_0^2(g^*,\underset{\sim}{\theta}) = \mathrm{Var}_0(g^*(X_i;\underset{\sim}{\theta})),$$

and $\hat{\underset{\sim}{\theta}}_n$ is a so called locally root n consistent estimator for $\underset{\sim}{\theta}$, defined by Neyman [9] to be such that for every $j=1,2,\ldots,r$, and for some constants $A_j \neq 0$, the random quantities $|\hat{\theta}_{jn}-\theta_j-A_j\xi|\sqrt{n}$ remain bounded in probability, as $n\to\infty$, for all ξ and $\underset{\sim}{\theta}$. Again based on a class Γ of local alternatives $\{\xi_n\}$ such that $\sqrt{n}\ \xi_n$ remains bounded as $n\to\infty$, Neyman considers a local asymptotic optimality criterion (see Neyman [9] for details) and obtains a test which is optimal in that sense within the class $C(\alpha)$ of tests. This optimal test corresponds to rejecting H_0, whenever $Z_n^*(\hat{\underset{\sim}{\theta}}_n) > z_{1-\alpha}$, where

(13) $$Z_n^*(\hat{\underset{\sim}{\theta}}_n) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \left\{ \frac{\varphi_{\xi}(X_i;\hat{\underset{\sim}{\theta}}_n) - \sum_{j=1}^{r} b_j^0(\hat{\underset{\sim}{\theta}}_n)\,\varphi_{\theta_j}(X_i;\hat{\underset{\sim}{\theta}}_n)}{\sigma_0(\hat{\underset{\sim}{\theta}}_n)} \right\},$$

$z_{1-\alpha}$ is the upper α-point of the standard normal distribution and

(14) $$\sigma_0^2(\underset{\sim}{\theta}) = \mathrm{Var}_0(\varphi_\xi(X;\underset{\sim}{\theta}) - \sum_{j=1}^{r} b_j^0 \varphi_{\theta_j}(X;\underset{\sim}{\theta})).$$

Also to arrive at (13) we have taken the function g* in (9) as

(15) $$g^*(x;\underset{\sim}{\theta}) = \varphi_\xi(x;\underset{\sim}{\theta}) - \sum_{j=1}^{r} b_j^0(\underset{\sim}{\theta}) \varphi_{\theta_j}(x;\underset{\sim}{\theta}),$$

where as before b_j^0's are the first order partial regression coefficients of $\varphi_\xi(X;\underset{\sim}{\theta})$ on $\varphi_{\theta_j}(X;\underset{\sim}{\theta})$, computed under H_0. Finally for the local alternatives Γ described above, the asymptotic power of the above optimal test is given by

(16) $$1 - \Phi(z_{1-\alpha} - \sigma_0(\underset{\sim}{\theta})\xi_n\sqrt{n}),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The above theory of Neyman [9] was generalized by Bartoo and Puri [1] and Bühler and Puri [2] for the cases where X_i's are mutually independent but are not necessarily identically distributed and also where the hypothesis under test involves more than one parameters. The reader may refer to these papers for the corresponding optimal $C(\alpha)$ tests and other relevant details covering these cases.

3 AN OPTIMAL $C(\alpha)$-TEST FOR POISSON HYPOTHESIS.

We now apply the $C(\alpha)$-test theory of the preceding section to the problem introduced in section 1, namely of testing a Poisson hypothesis against compound Poisson alternatives. For our random variable X of

(1) with $h(\cdot)$ replaced by $h_\xi(\cdot)$ of (3), let

$$(17) \qquad p_m(\xi,\lambda) = P(X = m|\xi,\lambda),\ m=0,1,2,\ldots \quad .$$

Now it is well known (see Katti [5]) that these probabilities are related recursively to each other and to $r_k(\xi)$'s of (3) as follows:

$$(18) \qquad \begin{cases} p_0(\xi,\lambda) = \exp(-\lambda) \\ p_{m+1}(\xi,\lambda) = \frac{\lambda}{m+1} \sum_{k=0}^{m} (m-k+1)p_k(\xi,\lambda)r_{m-k+1}(\xi),\ m \geq 0. \end{cases}$$

Also in view of (4) we have

$$(19) \qquad p_m(0,\lambda) = \frac{\lambda^m}{m!} \exp(-\lambda),\ m \geq 0.$$

Besides assuming that the probabilities $r_k(\xi)$'s satisfy (4), we further add the following assumptions (A_1) - (A_3) on these.

(A_1) For $k \geq 1$, $r_k(\xi)$'s are twice differentiable with respect to ξ for $\xi \geq 0$ (the derivatives at $\xi = 0$ are to be considered right hand derivatives) and that these differentiations are valid under the summation sign of $\sum_{k=1}^{\infty} r_k(\xi) \equiv 1$. Also we assume that $\sum_{j=1}^{\infty} jr_j'(0)$ is finite, with $r_1'(0) < 0$.

Note that in view of (4) $r_1'(0) \leq 0$ holds in any case. What we are requiring here is that $r_1'(0)$ be strictly negative (see also lemma 2(i)). Also in view of (18) the assumption (A_1) implies that for $m \geq 0$, $p_m(\xi,\lambda)$

are twice differentiable with respect to $\xi \geq 0$ and also for $\lambda > 0$.

(A_2) These differentiations of $p_m(\xi,\lambda)$ are permitted under the summation sign of $\sum_{m=0}^{\infty} p_m(\xi,\lambda) \equiv 1$.

Define

(20) $$\varphi_\xi(m) = \frac{\partial \ln p_m(\xi,\lambda)}{\partial \xi}\Big|_{\xi=0}, \quad \varphi_\lambda(m) = \frac{\partial \ln p_m(\xi,\lambda)}{\partial \lambda}\Big|_{\xi=0}.$$

(A_3) We assume that

(21) $$E_{0,\lambda}[\varphi_\xi(X)]^2 < \infty,$$

and that the functions φ_ξ and φ_λ are Cramér functions (see Neyman [9] for their definition).

The reader may find an expression for (21) in lemma 3(iv) below. Also the following lemma gives the needed expressions for the functions φ_ξ and φ_λ.

LEMMA 1. Using (18) we obtain

(22) $$\begin{cases} \varphi_\xi(m) = \sum_{j=1}^{m} \frac{m!}{(m-j)!} \lambda^{-(j-1)} r_j'(0); \; m \geq 1, \\ \varphi_\xi(0) = 0 \end{cases}$$

(23) $$\varphi_\lambda(m) = (\frac{m}{\lambda} - 1), \; m \geq 0.$$

PROOF. Proof for φ_λ being similar, we outline briefly the derivation for φ_ξ. Let

(24) $$p_m'(0,\lambda) = \frac{\partial p_m(\xi,\lambda)}{\partial \xi}\bigg|_{\xi=0}, \quad m \geq 1.$$

Using (18) and (4), it can be easily seen that

(25) $$p_m'(0,\lambda) = \frac{\lambda}{m}\,(A_m(\lambda) + p_{m-1}'(0,\lambda)), \quad m \geq 1,$$

where $p_0'(0,\lambda) = 0$ and for $m \geq 1$,

(26) $$A_m(\lambda) = \sum_{k=0}^{m-1} (m-k)p_k(0,\lambda)r_{m-k}'(0).$$

Solving (25) recursively and after some algebraic simplification, we obtain

(27) $$p_m'(0,\lambda) = p_m(0,\lambda)\sum_{j=1}^{m} j\,\lambda^{-(j-1)} \cdot r_j'(0)\Big[\sum_{k=j}^{m} \frac{(k-1)!}{(k-j)!}\Big],$$

which easily leads to (22) after using the combinatorial identity

(28) $$\sum_{k=j}^{m} \binom{k-1}{j-1} = \binom{m}{j}; \quad 1 \leq j \leq m.$$ □

The following lemma is needed in the sequel.

<u>LEMMA</u> 2. (i) <u>Subject to</u> (4) <u>and</u> (A_1) <u>we have</u> $r_1'(0) < 0$ <u>and</u> $r_j'(0) \geq 0$ <u>for</u> $j \geq 2$, <u>with</u>

(29) $$-r_1'(0) = \sum_{j=2}^{\infty} r_j'(0).$$

(ii) $$E_{0,\lambda}[\varphi_\xi(X)]^2 > 0.$$

(iii) <u>Whatever be</u> $\lambda > 0$, <u>there does not exist a constant</u> c <u>such that</u>

(30) $$P_{0,\lambda}(\varphi_\xi(X) - c\,\varphi_\lambda(X) = 0) = 1.$$

PROOF. (i) easily follows from (4) and (A_1) and the fact that $\sum_{j=1}^{\infty} r_j(\xi) = 1$. Proof of (ii) follows by contradiction; for if $E_{0,\lambda}[\varphi_\xi(X)]^2 = 0$, this would mean that $P_{0,\lambda}(\varphi_\xi(X) = 0) = 1$ or equivalently $\varphi_\xi(m) = 0$, for $m \geq 1$. This, in turn, using (22) recursively, implies that $r_j'(0) = 0$, $\forall\, j \geq 1$, which contradicts part (i) and in particular $r_1'(0) < 0$. Finally the proof of (iii) follows from essentially a similar argument. □

Lemma 2(ii) together with the assumption (A_3) implies that $Var_{0,\lambda}(\varphi_\xi)$ is positive and finite. Lemma 2(iii) guarantees condition C_2 of the previous section, which in turn guarantees the positivity of the variance σ_0^2 of (14) and hence the existence of an optimal $C(\alpha)$-test for the present case.

The next lemma gives some further expressions needed for the construction of the optimal $C(\alpha)$-test.

LEMMA 3. (i) $Var_{0,\lambda}(\varphi_\lambda(X)) = \lambda^{-1}$,

(ii) $Cov_{0,\lambda}(\varphi_\xi(X), \varphi_\lambda(X)) = \sum_{j=1}^{\infty} j\, r_j'(0)$,

(iii) $b^0(\lambda) = \lambda \sum_{j=1}^{\infty} j\, r_j'(0)$,

(iv) $Var_{0,\lambda}(\varphi_\xi(X)) = V_0(\lambda)$,

where

(31) $$V_0(\lambda) = \sum_{k=1}^{\infty} r_k'(0) \sum_{\ell=1}^{\infty} r_\ell'(0) B_{\ell,k}(\lambda),$$

with

(32) $$B_{\ell,k}(\lambda) = \sum_{r=0}^{\min(\ell,k)} \lambda^{-r} \cdot \binom{\min(\ell,k)}{r} \frac{[\max(\ell,k)]!}{[\max(\ell,k)-r]!} \quad .$$

Proof of the above lemma is omitted as it follows from lengthy but rather straightforward standard calculations using the expressions (22) and (23). Unfortunately however in this generality, the expression for the $Var_{0,\lambda}(\varphi_\xi)$ could not be further simplified, although it could be represented in other alternative forms. Finally the following theorem gives the desired optimal $C(\alpha)$-test, which can be easily established using the theory of section 2 (in particular (13)) and the lammas 1-3.

THEOREM 1. Subject to the assumptions (A_1) - (A_3), an optimal $C(\alpha)$-test for testing H_0: $\xi = 0$ against H_1: $\xi > 0$ is to reject H_0 whenever

$\tilde{Z}_n(\hat{\lambda}) > z_{1-\alpha}$, where

(33) $$\tilde{Z}_n(\hat{\lambda}) = [n\sigma_0^2(\hat{\lambda})]^{-\frac{1}{2}} \cdot \sum_{i=1}^{n} \tilde{g}(X_i,\hat{\lambda}),$$

(34) $$\tilde{g}(X,\lambda) = \varphi_\xi(X) - b^0(\lambda)\varphi_\lambda(X),$$

(35) $$\sigma_0^2(\lambda) = Var_0[\tilde{g}(X,\lambda)]$$

$$= V_0(\lambda) - \lambda[\sum_{j=1}^{\infty} j\, r_j'(0)]^2,$$

$z_{1-\alpha}$ is the upper α-point of the standard normal distribution and $\hat{\lambda}$ stands for a locally root n consistent estimator of the nuisance parameter λ.

The next theorem deals with an important special case of theorem 1, where the test statistic (33) simplifies considerably.

THEOREM 2. Subject to the conditions of theorem 1, if moreover $r_k'(0) = 0$ for $k \geq 3$, then the test statistic (33) reduces to

$$\tilde{Z}_n(\hat{\lambda}) = \hat{\lambda}^{-1}(2n)^{-\frac{1}{2}} \sum_{i=1}^{n} [(X_i-\hat{\lambda})^2 - X_i]. \tag{36}$$

Furthermore if the sample mean $\bar{X}$ is a locally root n consistent estimator of λ, then taking $\hat{\lambda} = \bar{X}$, the statistic (36) further reduces to

$$\tilde{Z}_n(\hat{\lambda}) = (\frac{n}{2})^{\frac{1}{2}} [\frac{S^2}{\bar{X}} - 1], \tag{37}$$

where

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2. \tag{38}$$

The condition $r_k'(0) = 0$ for $k \geq 3$ in theorem 2 holds for several well known compound Poisson alternatives such as negative binomial distributions, Neyman type A distributions to mention a few (see also remark (a) of section 5). Again it is interesting to note that the test statistic (37) is the classical dispersion coefficient. Even more interestingly the optimal $C(\alpha)$-test based on (36) for the case when $r_k'(0) = 0$ for $k \geq 3$, coincides with the corresponding optimal $C(\alpha)$-test for Poisson hypothesis against certain mixtures of Poisson as alternatives, obtained by Klonecki [6] (see also LeCam and Traxler [7]). This is not surprising, since the class of compound Poisson distributions overlaps with the class of mixtures of Poisson distributions (see Puri and Goldie [14]). For instance, the negative binomial distributions belong to both these classes. Thus the $C(\alpha)$-test based on (36) is optimal against a much larger class of alternatives than the ones considered here. This property has been referred to as the 'robustness of optimality' property by Neyman (see [1] and [6]).

4 AN ILLUSTRATIVE EXAMPLE.

As an illustration we apply the above test based on (37) for testing the goodness of fit of a Poisson hypothesis against compound Poisson alternatives with $r_k'(0) = 0$ for $k \geq 3$ (and also against the mixture of Poisson alternatives considered by Klonecki [6]), to a set of data considered by Neyman in his classical paper on contagious distributions (see [8]). Neyman of course used there the classical chi-square test of goodness of fit of Poisson hypothesis. The data given below are taken from Neyman [8], but they go back to 'Student' [15], where he observed the distribution of yeast cells in 400 squares of haemacytometer.

# of cells	0	1	2	3	4	5	Total
observed frequency	213	128	37	18	3	1	400

Here with n=400, we obtain $\hat{\lambda} = \bar{X} = 0.68250$; $S^2 = 0.81169$ with $\tilde{Z}_n(\hat{\lambda}) = (\frac{n}{2})^{1/2} \cdot [\frac{S^2}{\bar{X}} - 1] = 2.6770$, which is highly significant with P-value = 0.0037, compared with the P-value > 0.02, obtained by Neyman while using the classical chi-square test of goodness of fit. The test (37) was also applied to another set of data on distribution of European cornborers considered by Neyman [8]. We wish to remark here that whenever either the compound Poisson distribution or the mixture of Poisson (see Klonecki [6]) are suspected as alternatives to Poisson hypothesis, the much simpler large sample test based on (37) is highly recommended (based on the above optimality considerations) instead of the classical chi-square test of goodness of fit for Poisson hypothesis.

5 A FEW CONCLUDING REMARKS.

(a) It is interesting to note that subject to appropriate minor modifications of assumptions (A_1) - (A_3), the above results of lemmas 1-3 and theorems 1-2 remain valid even when the p.g.f. (3) and hence the probabilities r_k's are allowed to depend upon the nuisance parameter λ besides ξ, as long as we make the additional assumption that the quantities

$$(39) \qquad \left. \frac{\partial r_k(\xi,\lambda)}{\partial\lambda} \right|_{\xi=0}$$

exist and are all zero, for $k \geq 1$. Consider the special case of negative binomial distribution which falls under this more general set up. Here

$$(40) \qquad E(s^X) = p^{\alpha}(1-ps)^{-\alpha}; \; |s| \leq 1, \; 0 < p < 1, \; \alpha > 0.$$

Reparameterising this with $\xi = \alpha^{-1}$ and $\lambda = -\alpha \ln p$, this can be rewritten as

$$(41) \qquad E(s^X) = \exp[-\lambda(1 - h_{\xi,\lambda}(s))],$$

where

$$(42) \qquad h_{\xi,\lambda}(s) = \sum_{k=1}^{\infty} r_k(\xi,\lambda)s^k,$$

with

$$(43) \qquad r_k(\xi,\lambda) = (\lambda\xi k)^{-1}(1 - \exp[-\lambda\xi])^k, \; k \geq 1.$$

Evidently $\lim_{\xi \downarrow 0} \xi_k(\xi,\lambda) = \delta_{1k}$. Also it can be easily shown that

$$(44) \qquad \left.\frac{\partial r_k(\xi,\lambda)}{\partial \lambda}\right|_{\xi=0} = 0, \text{ for } k \geq 1,$$

and that

$$(45) \qquad \left.\frac{\partial r_k(\xi,\lambda)}{\partial \xi}\right|_{\xi=0} = \begin{cases} -\lambda/2, & \text{for } k = 1 \\ \lambda/2, & \text{for } k = 2 \\ 0, & \text{for } k \geq 3. \end{cases}$$

It follows that the test based on (37) remains an optimal $C(\alpha)$-test for this case.

(b) The situation where the present problem arose (see Neyman and Puri [11], [12] and Puri [13]) was in the area of radiation biology, where for a possibly compound Poisson process, one can only observe the total counts of events over varying intervals of times $(0,t_i]$, $i=1,2,\ldots,n$, and not their actual times of occurrences. This means that X_i's are although mutually independent but are not necessarily identically distributed. For such situations similar optimal $C(\alpha)$-tests have been obtained using the generalizations of Neyman's theory of $C(\alpha)$-tests by Bartoo and Puri [1] and Bühler and Puri [2], and will be reported elsewhere.

(c) In closing we remark that the question of testing Poisson hypothesis is raised and answered here within the classical α-level testing hypothesis frame-work. The study of the same question from a decision theoretic and a Bayesian point of view will be dealt with elsewhere.

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