ANOTHER LOOK AT POISSON PROCESSES

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Technical Report #83-28

Department of Statistics
Purdue University

July 1984
(Revised July 1985)

*These investigations were supported in part by a David Ross Fellowship at Purdue University.

**These investigations were supported in part by the U.S. National Science Foundation Grant No. MCS-8102733.

Math Subject Classification
No. 60G55

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1. INTRODUCTION.

Poisson processes are generally characterized by the independence of increments (for arrival processes) or counts (for point processes). We show that the weaker condition of independence of [no arrivals] or [no points in sets] for non-overlapping intervals (disjoint sets) is sufficient to characterize Poisson processes among processes where events occur one at a time and no point has positive probability of being the site of an event. The reader may refer to an earlier work due to Kallenberg (1973), (1974), where the homogeneous Poisson processes are characterized under different conditions.

2. DEFINITIONS.

Let A be a random function on $[0,\infty)$ to the non-negative integers such that $A(0) = 0$ and A is right continuous nondecreasing with unit jumps. We shall refer to such a function as an arrival process. In order for an arrival process to be a Poisson process, the following definitions are most commonly made in literature (see, e.g., Çinlar (1975), Doob (1953), Hoel, Port and Stone (1972), Karlin and Taylor (1975), Parzen (1962), Prabhu (1965), Rényi (1970), and Ross (1983)).

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*These along with other results were reported in an invited presentation at the XIII International Conference on Stochastic Processes and Their Applications, held at Varansi, India, during Dec., 1983

**These investigations were supported in part by a David Ross Fellowship at Purdue University. Current Address: National Sun Yat-Sen University, Kaohsiung, Taiwan, R.O.C.

***These investigations were supported in part by the U.S. National Science Foundation Grant No. MCS-8102733.

<u>**DEFINITION 1**</u>. An arrival process $\{A(t), t \geq 0\}$ is said to be a homogeneous Poisson process if the following two conditions hold:

(i) for any $n > 1$ and $0 \leq t_0 < t_1 < \ldots < t_n < \infty$, the random variables $A(t_1) - A(t_0), A(t_2) - A(t_1), \ldots, A(t_n) - A(t_{n-1})$ are mutually independent;

(ii) for any $t, s \geq 0$, the distribution of $A(t+s) - A(s)$ is independent of s.

<u>DEFINITION 2</u>. An arrival process $\{A(t), t \geq 0\}$ is said to be a possibly non-homogeneous Poisson process if the following two conditions hold:

(i) for any $n > 1$ and $0 \leq t_0 < t_1 < \ldots < t_n < \infty$, the random variables $A(t_1) - A(t_0), A(t_2) - A(t_1), \ldots, A(t_n) - A(t_{n-1})$ are mutually independent;

(ii) $a(t) \equiv EA(t)$, $t \geq 0$, is continuous for all $t \geq 0$.

Another approach is that of point process (see Karlin and Taylor (1975), p. 31). The intuitive idea is that there is a complete separable metric space X and on X there is a random point set Q. If $X = [0,\infty)$, one way to get such a point process is to take the set Q of arrivals of an arrival process; although the space need not be $[0,\infty)$, this appraoch is almost general. We assume that there is a collection $\mathcal{C}$ of Borel subsets of X satisfying the following conditions:

(a) The collection $\mathcal{C}$ covers X and separates the points of X, and if $C \in \mathcal{C}$, $N(C)$ = number of elements of $C \cap Q$ is a (possibly infinite) random variable.

(b) For every $C_1, \ldots, C_m \in \mathcal{C}$, there exists a finite collection $\mathcal{D}$ of <u>disjoint</u> $D_i \in \mathcal{C}$, $i=1,2,\ldots,n$ such that each C_i is a union of a subfamily of $\mathcal{D}$.

(c) For every C with $P(N(C)=0)=0$, there exists a sequence $\{E_i\}$ of elements of $\mathcal{C}$ such that $C=\cup E_i$ and $P(N(E_i)=0) > 0$ for each i.

(d) For every $C \in \mathcal{C}$, there is a sequence $\{\pi_j\}$ of finite partitions of C in $\mathcal{C}$ such that

$$P(N(C) \neq \sup_j M(\pi_j)) = 0, \tag{1}$$

where $M(\pi)$ denotes the number of cells K of the partition π with $N(K)>0$.

Clearly $N(C) \geq M(\pi)$ for every partition π of C. Note that condition (c) implies the nonexistence of a point in X which is almost sure to be in Q. Also here we are not requiring that $\mathcal{C}$ be a semi-ring; however, in view of condition (b), the collection of sets closed under finite unions of the sets of $\mathcal{C}$ forms a ring which we shall denote by $\mathcal{R}$. We introduce the following definitions:

<u>DEFINITION 3</u>. A point process Q defined on a space X (along with a collection $\mathcal{C}$) as above is said to be a possibly nonhomogeneous spacial Poisson process provided that for any $n \geq 2$ and disjoint sets $C_1,\ldots,C_n \in \mathcal{C}$, the random variables $N(C_1),\ldots,N(C_n)$ are mutually independent Poisson random variables.

Note that for such a point process Q, the sets $C \in \mathcal{C}$ with $EN(C)<\varepsilon$ form a covering of X for every $\varepsilon>0$.

<u>DEFINITION 4</u>. Let λ be a nonnegative finitely additive set function defined on $\mathcal{C}$. Then a point process Q defined on a space X as above is said to be a λ-homogeneous spacial Poisson process if it satisfies definition 3 and if for any disjoint sets $A_1,\ldots,A_m$ and disjoint sets $B_1,\ldots,B_n$ all in $\mathcal{C}$,

$$\sum_i \lambda(A_i) \geq \sum_j \lambda(B_j) \Rightarrow P(\sum_i N(A_i)=0) \leq P(\sum_j N(B_j)=0). \tag{2}$$

Note that it is possible for $\lambda(B) \neq \lambda(C)$ whenever $B \neq C$, so some condition of the type (2) is needed.

3. RESULTS. Let Q be a point process defined on a space X (together with the collection $\mathbb{C}$) as in section 2 and let $Z(C)$ be the event $C \cap Q \neq \phi$ (empty set) or equivalently that $N(C)>0$. Then we have

THEOREM. *If the events $Z(C_i)$ are independent for disjoint C_i's $\in \mathbb{C}$ and if for every $\varepsilon>0$ and every $A \in \mathbb{C}$ with $P(Z(A))<1$, there exists a finite covering of A by sets $\{C_i\}$ in $\mathbb{C}$, each with $P(Z(C_i))<\varepsilon$, then Q is a Poisson point process. If in addition $P(Z(C))$ satisfies the condition (2) for some nonnegative finitely additive function λ on $\mathbb{C}$, then Q is λ-homogeneous and $E(N(C))$ is proportional to $\lambda(C)$.*

Before we prove the theorem, we need the following lemma.

LEMMA 1. *Let Y be a Poisson random variable with mean $\theta>0$. Let $Y'=0$ if $Y=0$ and $Y'=1$ if $Y\geq 1$. Then*

$$0 < P(Y \neq Y') < E(Y-Y') < \theta^2/2. \tag{3}$$

Proof.

$$P(Y \neq Y') = \sum_{k=2}^{\infty} P(Y=k)$$

$$< \sum_{k=2}^{\infty} (k-1)P(Y=k)$$

$$= E(Y-Y') = \theta-(1-\exp(-\theta)) < \theta^2/2. \qquad \square$$

Proof of the Theorem. Here all the sets will be considered in $\mathbb{C}$ unless stated otherwise. The idea of the proof is that if a set C is divided into "small" sets B_i's, the number of points in C is certainly at least the number of B_i's with $N(B_i)\neq 0$; since points do not coincide it will be unlikely that any $N(B_i)>1$. We modify this approach slightly. For each C we define $\eta(C) = -\ell n\,[1-P(Z(C))]$. Note the independence of the Z's for disjoint sets in $\mathbb{C}$ is equivalent to the additivity of η on $\mathbb{C}$. Let A_i, $i=1,2,\ldots, I$, be disjoint elements of $\mathbb{C}$. Since $\eta(A)=0$ implies $N(A)=0$ a.s., without loss of generality we assume that $\eta(A_i)>0$, $i=1,2,\ldots, I$. We consider first the case when $\eta(A_i)<\infty$, for all i. We shall prove that $N(A_i)$, $i=1,2,\ldots, I$, are mutually independent having Poisson distributions with means $\eta(A_i)$, $i=1,2,\ldots, I$. Note that for each A_i, there exists a sequence $\{\pi_{ij}\}$ of finite partitions of A_i such that

(i) $\pi_{ij} \subset \mathbb{C}$,

(ii) $\pi_{i,j+1}$ refines π_{ij},

(iii) $\sup\limits_{K\in\pi_{ij}} \eta(K) \to 0$, as $j \to \infty$,

(iv) $P(N(A_i) \neq \sup\limits_{j} M(\pi_{ij})) = 0$,

where

$$M(\pi_{ij}) \equiv \sum_{K\in\pi_{ij}} Y(K), \tag{4}$$

and Y(K) are independent Bernoulli random variables with $P(Y(K)=1) = (1-\exp[-\eta(K)])$. Here (ii) follows from (b), (iii) follows from condition assumed in the theorem and condition (b), and (iv) follows from condition (d). Now for each $K \in \pi_{ij}$ define $W(K)=0$ when $Y(K)=0$ and when $Y(K)\geq 1$, let $W(K) = r$ with probability

$$\frac{1}{r!}\,[\eta(K)]^r \exp[-\eta(K)]\;(1-\exp[-\eta(K)])^{-1}, \qquad r = 1,2,\ldots\,, \tag{5}$$

so that $W(K)$, $K\in\pi_{ij}$ are mutually independent Poisson random variables with means $\eta(K)$. Now let

$$L_i(\pi_{ij}) = \sum_{K\in\pi_{ij}} W(K). \tag{6}$$

Then L_i has a Poisson distribution with mean $\eta(A_i)$. Moreover since the A_i's are disjoint, the L_i's are mutually independent. Also using the lemma we have

$$P(Y(K) \neq W(K)) \leq \frac{1}{2}\,[\eta(K)]^2, \tag{7}$$

so that

$$\begin{aligned}\sum_{K\in\pi_{ij}} P(Y(K) \neq W(K)) &\leq \frac{1}{2} \sum_{K\in\pi_{ij}} [\eta(K)]^2 \\ &\leq \frac{1}{2}\,[\sup_{K\in\pi_{ij}} \eta(K)] \sum_{K\in\pi_{ij}} \eta(K) \\ &= \frac{1}{2}\,\eta(A_i) \sup_{K\in\pi_{ij}} \eta(K),\end{aligned}$$

which tends to zero as $j\to\infty$ in view of (iii). Thus $P(M(\pi_{ij}) \neq L_i(\pi_{ij}))$ tends to zero as $j\to\infty$. Also, in view of (iv), $P(N(A_i) \neq M(\pi_{ij}))$ tends to zero as $j\to\infty$, so that it follows that $P(N(A_i) \neq L_i(\pi_{ij}))$ tends to zero. Consequently $N(A_i)$, $i=1,2,\ldots, I$, are mutually independent Poisson random variables with means $\eta(A_i)$, $i=1,2,\ldots, I$. Consider now the case with $\eta(A_i)=\infty$ for some i or equivalently with $P(N(A_i)=0)=0$. Since in view of conditions (b) and (c) there exists a nondecreasing sequence of $\{E_{ij}\}_{j=1}^{\infty}$ such that each E_{ij} can be expressed as a finite union of disjoint sets in $\mathcal{C}$, $A_i = \cup E_{ij}$ with $\eta(E_{ij})\to\infty$; since as $j\to\infty$, $N(E_{ij})$ are Poisson random variables with means $\eta(E_{ij})<\infty$ for all j, it is easy to see that $P(N(A_i)=\infty)=1$. In either case we can say that $N(A_i)$, $i=1,2,\ldots, I$, are independent Poisson random variables with means $\eta(A_i)$, $i = 1,2,\ldots, I$. Finally if condition (2) of definition 4 is satisfied for some nonnegative finitely additive function λ on $\mathcal{C}$, it follows that the process Q is λ-homogeneous. The only thing that we still need to prove subject to (2) is that $EN(C) = \eta(C)$ is proportional to $\lambda(C)$. In this context we note that since both η and λ are finitely additive, their definitions can be extended to the ring $\mathcal{R}$. Also for the last part of the theorem we find it convenient to work in terms of these extended versions of η and λ defined on $\mathcal{R}$. The assumed condition (2) in terms of their extended versions can now be rewritten as

$$\lambda(A) \geq \lambda(B) \Rightarrow \eta(A) \geq \eta(B), \; \forall\, A,\, B \in \mathcal{R}, \tag{8}$$

or equivalently

$$\eta(A) < \eta(B) \Rightarrow \lambda(A) < \lambda(B), \; \forall\, A,\, B \in \mathcal{R}. \tag{9}$$

Also the condition of the theorem satisfied by $\eta(A)$ for all $A \in \mathcal{C}$ with

$\eta(A)\to\infty$, and the condition (b) on $\mathbb{C}$ imply (*) given below for the extended version of η on $\mathcal{R}$.

$$\left.\begin{array}{l}\text{For every } \varepsilon>0 \text{ and every } A \in \mathcal{R} \text{ with } \eta(A)<\infty, \text{ there exists a finite}\\ \text{partition of } A \text{ in } \mathcal{R}, \text{ say } \{D_i\} \text{ with } A=\cup D_i \text{ and } \eta(D_i)<\varepsilon \text{ for every } D_i.\end{array}\right\} \quad (*)$$

The proportionality of η and λ now follows from lemma 3, which in turn needs lemma 2 given below. □

<u>LEMMA 2</u>. <u>Let two nonnegative finitely additive set functions</u> η <u>and</u> λ <u>be defined on sets of a ring</u> $\mathcal{R}$ <u>and satisfy the condition</u> (8). <u>Let</u> η <u>satisfy the conditions</u> (*) <u>and</u> (c) <u>given above</u>. <u>Also to avoid trivial cases we assume that there exists a set</u> $B \in \mathcal{R}$ <u>such that</u> $0<\eta(B)<\infty$. <u>Then we have</u>

(i) $\lambda(C) = 0 \Leftrightarrow \eta(C) = 0$,

(ii) λ <u>too satisfies the condition</u> (*), <u>and</u>

(iii) <u>for sets</u> A_1 <u>and</u> $A_2 \in \mathcal{R}$

$$\lambda(A_1) < \lambda(A_2) < \infty \Rightarrow \eta(A_1) < \eta(A_2). \quad (10)$$

<u>Proof</u>. (i) That $\lambda(C) = 0 \Rightarrow \eta(C) = 0$ follows by using (8) with $A = \phi$ and $B=C$. Conversely let $\eta(C) = 0$. Since $0<\eta(B)<\infty$ so that $\eta(C)<\eta(B)$, using (8) we have $\lambda(B)>0$. Again in view of (*), for an arbitrary n, there exists a finite partition of B in $\mathcal{R}$, say $\{D_i\}$ such that $\cup D_i=B$ and $\eta(D_i)<[\eta(B)/n]$ for all i. Evidently the $\#\{D_i\text{'s}: \eta(D_i)>0\}>n$. However, since for such D_i's, $0=\eta(C)<\eta(D_i)$, we have $\lambda(C)<\lambda(D_i)$ so that

$$\{i: \eta(D_i)>0\} \subset \{i: \lambda(D_i)>0\}. \quad (11)$$

Consequently since

$$\#\{D_i\text{'s}: \eta(D_i)>0\}\ \lambda(C) < \sum_{D_i:\ \eta(D_i)>0} \lambda(D_i) \leq \sum_i \lambda(D_i) = \lambda(B), \tag{12}$$

we have $\lambda(C) < \lambda(B)/n$ for all $n \geq 1$, so that $\lambda(C) = 0$.

(ii) Let $C \in \mathcal{R}$ be a set with $\eta(C)$ and $\lambda(C)$ both finite. Since when $\lambda(C)=0$ (*) is trivially satisfied, in view of (i) we let both $\eta(C)$ and $\lambda(C)$ be positive. Now for every n, using (*), there exists a finite partition of C in $\mathcal{R}$, say $\{D_i\}$ with $C=\cup D_i$ and $\eta(D_i) \leq [\eta(C)/n]$ for all i. Let D^* be a D_i with $\eta(D^*) = \min\{\eta(D_i): \eta(D_i)>0\}$. Then $0<\eta(D^*) \leq \eta(C)/n$. Since $\eta(D_i)=0 \Leftrightarrow \lambda(D_i)=0$, we also have $\lambda(D^*)>0$; in fact we can choose D^* to be such that $\lambda(D^*) = \min\{\lambda(D_i): \lambda(D_i)>0\}$, so that $0<\lambda(D^*) \leq \lambda(C)/n$. Now take a new finite partition of C in $\mathcal{R}$ say $\{\tilde{D}_\ell\}$ such that $\eta(\tilde{D}_\ell)<\eta(D^*)$. From (*) it follows that

$$\lambda(\tilde{D}_\ell) < \lambda(D^*) \leq \lambda(C)/n,\ \forall\ \ell,$$

which proves (*) for λ. Next consider the case where $C \in \mathcal{R}$, $\eta(C)=\infty$ but $\lambda(C)<\infty$. In view of condition (c), there is a sequence $C_k \nearrow C$ with $C_k \in \mathcal{R}$, $0<\eta(C_k)<\infty$ and $\eta(C_k) \nearrow \infty$, as $k\to\infty$. Likewise $\lambda(C_k) \nearrow \lambda(C)$. Choose k_0 large enough such that $\lambda(C \setminus C_{k_0})<\varepsilon$. Now treat the set C_{k_0} as above since both $\lambda(C_{k_0})$ and $\eta(C_{k_0})$ are finite. The finite partition of C_{k_0} so obtained together with the set $(C\setminus C_{k_0}) \in \mathcal{R}$, yields the desired partition of the set C.

(iii) Let $A_1, A_2 \in \mathcal{R}$, $\lambda(A_1) < \lambda(A_2) < \infty$. With $\varepsilon=\lambda(A_2)-\lambda(A_1)$, in view of (*) there exists a finite partition of A_2 in $\mathcal{R}$, say $\{D_i\}$, such that $A_2=\cup D_i$,

$\lambda(D_i)<\varepsilon$ for all i. Without loss of generality, let D_1 be such that $\lambda(D_1)>0$ and hence $\eta(D_1)>0$. Consequently $\lambda(A_2\setminus D_1) > \lambda(A_1)$ so that using (8) we have $\eta(A_2\setminus D_1) \geq \eta(A_1)$. Hence

$$\eta(A_2) = \eta(A_2\setminus D_1) + \eta(D_1) > \eta(A_2\setminus D_1) \geq \eta(A_1). \qquad \square$$

<u>LEMMA 3</u>. <u>Let</u> η <u>and</u> λ <u>satisfy the conditions of lemma 2</u>. <u>Define</u> $\psi(x)=y$ <u>if for some</u> $A \in \mathfrak{R}$, $\lambda(A)=x$ <u>and</u> $\eta(A)=y$.

(i) ψ <u>is uniformly continuous over a dense set in</u> $[0,\alpha)$ <u>and hence can be extended uniquely to a strictly monotone</u> ψ <u>with its domain</u> $= [0,\alpha)$, <u>where</u> $\alpha = \sup_{A\in\mathfrak{R}} \lambda(A)$.

(ii) <u>The extended</u> ψ <u>satisfies</u>

$$\psi(x_1+x_2) = \psi(x_1) + \psi(x_2), \quad 0 \leq x_1,\ x_2 < \infty,\ x_1 + x_2 < \alpha, \tag{13}$$

<u>so that</u> $\psi(x) = \beta x$, <u>for some</u> $\beta>0$ <u>and hence</u> η <u>and</u> λ <u>are proportional</u>.

<u>Proof</u>. (i) Clearly the domain of the function ψ, say $\mathcal{D}(\psi)$, is dense in $[0,\alpha)$. It is sufficient to prove that for every $\varepsilon>0$, there exists a $\delta>0$ such that if $0\leq x_1<x_2<x_1+\delta$, $x_1, x_2 \in \mathcal{D}(\psi)$, we have $\psi(x_2)-\psi(x_1)<\varepsilon$. In view of condition (*), there exists $A \in \mathfrak{R}$ with $0<\eta(A)<\varepsilon$, for a given $\varepsilon>0$. Take $\delta = \frac{1}{2}\lambda(A)$. Now for $0 < x_1 < x_2 < \delta$, $x_1, x_2 \in \mathcal{D}(\psi)$, let E_1 and $E_2 \in \mathfrak{R}$ be such that $x_1 = \lambda(E_1)$ and $x_2 = \lambda(E_2)$. Then

$$0 < \lambda(E_1) < \lambda(E_2) < \delta < \lambda(A),$$

from which, using lemma 2 (iii), follows

$$0 < \eta(E_1) < \eta(E_2) < \eta(A) < \varepsilon,$$

so that $\eta(E_2) - \eta(E_1) = \psi(x_2) - \psi(x_1) < \varepsilon$. Again for the case with $0<x_1<x_2<x_1+\delta$, $x_2 > \delta$, x_1, $x_2 \in \mathcal{D}(\psi)$, as before let $x_1 = \lambda(E_1)$ and $x_2 = \lambda(E_2)$ for some E_1, $E_2 \in \mathfrak{R}$. Then using lemma 2 (ii), there exists a finite partition $\{D_i\}$ of sets in $\mathfrak{R}$ such that $E_2 = \cup D_i$ and $\lambda(D_i)<\delta$, for all i. Thus there exists a set C, which is union of some of the D_i's such that $\delta < \lambda(C) < 2\delta = \lambda(A)$ holds, so that

$$\lambda(E_2\backslash C) = \lambda(E_2) - \lambda(C) = x_2 - \lambda(C) < x_2 - \delta < x_1 = \lambda(E_1). \tag{14}$$

Consequently,

$$\begin{aligned}\psi(x_1) > \psi(\lambda(E_2\backslash C)) &= \eta(E_2\backslash C)\\ &= \eta(E_2) - \eta(C)\\ &> \psi(x_2) - \psi(\lambda(A))\\ &= \psi(x_2) - \eta(A)\\ &> \psi(x_2) - \varepsilon.\end{aligned}$$

Here the strict inequalities follow from (8) and lemma 2 (iii). The same reasoning applies to justify the strict monotonocity of the extended ψ to the domain $[0,\alpha)$.

(ii) Let $\varepsilon>0$ and δ be as defined in the proof of part (i) above. From part (i) it follows that for $0 \leq x_1, x_2 < \infty$, $x_1 + x_2 < \alpha$, there exists $C \in \mathfrak{R}$, such that

$$|\lambda(C) - (x_1+x_2)| < \delta/2. \tag{15}$$

Then in view of lemma 2 (ii), there exists a finite partition $\{E_i\}$ of sets in $\mathfrak{R}$ with $C = \cup E_i$ and $\lambda(E_i) < \delta/2$ for all i. Consequently there exist disjoint sets C_1 and C_2, both unions of different E_i's such that $C = C_1 \cup C_2$ and

$$|\lambda(C_1) - x_1| < \delta/2, \qquad |\lambda(C_2) - x_2| < \delta. \tag{16}$$

Using part (i) above, (15), (16) and the fact that $\eta(C) = \eta(C_1) + \eta(C_2)$, it follows that $|\eta(C) - \psi(x_1+x_2)|<\varepsilon$, $|\eta(C_1) - \psi(x_1)|<\varepsilon$ and $|\eta(C_2) - \psi(x_2)|<\varepsilon$, so that

$$|\psi(x_1+x_2) - \psi(x_1) - \psi(x_2)| < 3\varepsilon. \tag{17}$$

The number ε being arbitrary, we have proved (13) and the lemma. $\square$

We observe that the usual methods of extending measures to a σ-field can be carried out also in this case, so that we have

<u>COROLLARY 1</u>. <u>Under the assumptions of the theorem and the added assumption</u>:

$$\left.\begin{array}{l} P(N(C) = 0) = \prod_{i=1}^{\infty} P(N(A_i) = 0) \\ \\ \underline{\text{whenever}}\ A_i,\ C \in \mathcal{C},\ C = \bigcap_{i=1}^{\infty} A_i,\ \ A_i\ \underline{\text{disjoint}}. \end{array}\right\}, \qquad (**)$$

$N(B_i)$ are independent Poisson random variables if B_i's are disjoint elements of the σ-field generated by $\mathcal{C}$.

Here the condition (**) is merely the σ-additivity of η.

COROLLARY 2. *If* $\{A(t), t \geq 0\}$ *is an arrival process on* $[0,\infty)$ *such that*

(i) *If* $0 \leq t_0 < t_1 < \ldots < t_n$ *the events* $[A(t_i) = A(t_{i-1})]$ *are independent, and*

(ii) *for each* $t \geq 0$ *and* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $P(A(t+\delta) = A((t-\delta) \vee 0)) > 1 - \varepsilon$,

then A *is a Poisson arrival process. If* (i) *holds and* $P(A(t+s) = A(t))$ *is independent of* t, A *is a homogeneous Poisson arrival process.*

NOTE. After this paper was written our attention was drawn to a recent paper due to Brown (1984) appeared in *The American Mathematical Monthly*, Vol. 91, pp. 116-123, which deals with the same topic as ours. However, the main point of our paper is that we only assume independence property for the *emptiness* of disjoint sets rather than for the numbers of points in them, and we do not even assume this for all measurable sets, but only for "sufficiently many" of them. Also the special case of arrival processes with $X = [0,\infty)$ was already dealt with in a Ph.D. dissertation of Huang (1983).

ACKNOWLEDGEMENT: The authors are grateful to the referee for insisting that they provide some of the missing details which led them to remove some gaps in an earlier version of the paper.

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