ISOTONIC PROCEDURES FOR SELECTING POPULATIONS BETTER THAN A STANDARD FOR TWO-PARAMETER EXPONENTIAL DISTRIBUTIONS*

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ABSTRACT

Let $\pi_1,\ldots,\pi_k$ be k independent two-parameter exponential populations, where π_i has the associated density $f(x;\mu_i,\theta_i) = \theta_i^{-1}\exp\{-(x-\mu_i)/\theta_i\}$, $x \geq \mu_i$, $\mu_i \geq 0$, $\theta_i > 0$, $i = 1,\ldots,k$. The parameters μ_i are known as the threshold, or "guarantee time," parameters. The two-parameter exponential distribution is employed in reliability studies where the failure cannot occur before some particular guarantee time. It is assumed that μ_i, $i = 1,\ldots,k$, are unknown but their ordering is known. We assume without loss of generality that $\mu_1 \leq \ldots \leq \mu_k$. These k populations are compared in terms of their guaranteed lifetimes with a known standard μ_0 or the unknown guaranteed life of a control population π_0 with parameters μ_0 and θ_0. Any π_i, $i = 1,\ldots,k$, is better than the standard or the control if $\mu_i \geq \mu_0$. The goal is to select a subset (possibly empty) of the k populations so that all populations that are better than the standard (or the control) are included in the selected subset with a guaranteed minimum probability P*. Isotonic procedures are proposed and investigated assuming that $\theta_0 = \theta_1 = \ldots = \theta_k = \theta$. These procedures, naturally, are based on isotonic estimators of $\mu_1,\ldots,\mu_k$. All four cases arising out of μ_0 and θ being known or unknown are considered. The procedures are compared with some other proposed alternative procedures in terms of the expected number of inferior populations included in the selected subset. Tables of constants associated with these procedures are also given.

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Key words: Isotonic procedures, selection procedures, standard, negative exponential, guarantee time, subset selection, simple ordering prior

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1. Introduction

The problem of selecting populations better than a standard under an ordering prior has been considered by Gupta and Yang (1981) for the normal means problem and by Gupta and Huang (1982) for the binomial parameters problem. In this paper we consider the case of two-parameter exponential populations for which an interest lies in comparing location parameters (guarantee times).

A two-stage procedures for selecting the best of k such populations has been considered by Desu, Narula and Villarreal (1977) and Mukhopadhyay and Hamdy (1984). Mukhopadhyay (1984) also considered the sequential procedure for selecting the better exponential population. These three papers are based on "indifference zone approach". However our paper is based on "subset selection approach".

In Section 2, notations and definitions used in this paper are introduced. Isotonic selection procedures are considered in Section 3, according to the control parameter and the common scale parameter which may be known or unknown. In Section 4, some other procedures for this problem are also considered. Comparisons of these procedures based on the expected number of bad populations in the selected subset is investigated. Tables of associated constants for the proposed procedures are given in Table I through Table IV.

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2. Notations and Definitions

Let $E(\mu,\theta)$ denote the two-parameter exponential distribution with probability density function

$$(2.1) \qquad f(x;\ \mu,\theta) = \begin{cases} \theta^{-1}\exp\{-\theta^{-1}(x-\mu)\}, & \text{if } x > \mu \\ 0 \qquad\qquad\qquad\qquad , & \text{if } x \le \mu \end{cases}$$

where $-\infty < \mu < \infty$ and $\theta > 0$. The parameter μ is called the guarantee time and θ is the scale parameter which in this case is the standard deviation.

Suppose that $\pi_0,\pi_1,\ldots,\pi_k$ are (k+1) independent populations. It is assumed that the observations from π_i follow a $E(\mu_i,\theta)$ distribution, $i = 0,1,\ldots,k$. The guarantee time is the parameter of interest. It is assumed that $\mu_1 \le \mu_2 \le \cdots \le \mu_k$; however, the true values of these μ_i's are not known. We consider π_0 as a control (or standard). We say that population π_i is "good" if $\mu_i \ge \mu_0$. Our goal is to select a subset of these k populations so that all "good" populations are included in the selected subset.

Let $\Omega = \{\underline{\mu} = (\mu_0,\mu_1,\ldots,\mu_k)|-\infty < \mu_1 \le \mu_2 \le\cdots\le \mu_k<\infty,\ -\infty < \mu_0 < \infty\}$ be the parameter space. Let us denote the sets $a_i = \{i, i+1,\ldots,k\}$, $1 \le i \le k$ and $a_0 \equiv \phi$ (the empty set). If action a_i is taken, it means the subset $\{\pi_i,\pi_{i+1},\ldots,\pi_k\}$ of the k populations is selected. Since by our assumption μ_i's are ordered according to an ascending ordering prior, it is, therefore, appropriate to restrict our attention to the action space $\mathcal{A} = \{a_0,a_1,\ldots,a_k\}$. Let X_{ij}, $j = 1,2,\ldots,n$ be a random sample from population π_i, $i = 0,1,\ldots,k$. The sample space is denoted by $\mathcal{X} = \{\underline{x} = (x_{01},\ldots,x_{0n},x_{11},\ldots,x_{1n},\ldots,x_{k1},\ldots,x_{kn})|$ $\mu_i < x_{ij} < \infty$, $j = 1,2,\ldots,n, i=0,1,\ldots,k\}$.

<u>Definition 2.1</u>. A selection procedure δ is isotonic if it selects π_i with parameter μ_i and if $\mu_i < \mu_j$, then it also selects π_j. We will restrict our attention to isotonic selection procedure δ which satisfy the P*-condition:

$$\inf_{\underline{\mu}\in\Omega} P_{\underline{\mu}}(CS|\delta) \geq P^*,$$

where P* is a pre-assigned value, and a correct selection (CS) means the selection of any subset which contains all good populations.

Definition 2.2. A poset $(S, \lesssim)$ denotes a non-empty set S with a binary partial order $\lesssim$ defined on it.

Definition 2.3. A real-valued function f defined on a poset $(S, \lesssim)$ is called isotonic if f preserves the order on S, i.e. $x \lesssim y$, implies $f(x) \leq f(y)$.

Definition 2.4. Let g be a real-valued function and let W be a positive-valued function, both defined on a poset $(S, \lesssim)$. An isotonic function g* on S is called an isotonic regression of g with weight W if $\sum_{x\in S}[g(x)-g^*(x)]^2 W(x)$ attains its minimum values over set of all isotonic functions on S.

It is well-known (see Barlow, Bartholomew, Bremner and Brunk (1972)) that there exists one and only one isotonic regression of a given g with a given weight W defined on S.

Let $Y_i = \min_{1\leq j\leq n} X_{ij}$, where $X_{ij} \sim E(\mu_i,\theta)$, $j = 1,2,\ldots,n$, $i = 0,1,\ldots,k$. Let $S = \{\mu_1,\ldots,\mu_k | \mu_1 \leq\cdots\leq \mu_k\}$. Consider the functions $g(\mu_i) = Y_i$ and $W(\mu_i) = n/\theta = w_i$, $i = 1,\ldots,k$. Then by the maximin formula, the isotonic regression of g with weight W is g*, where

$$g^*(\mu_i) = \max_{1\leq s\leq i} \min_{s\leq t\leq k} \{\frac{Y_s +\ldots+ Y_t}{t-s+1}\}.$$

The isotonic estimator of μ_i is denoted by $\hat{X}_{i:k}$, $i = 1,2,\ldots,k$, where

$$\hat{X}_{i:k} = \max_{1\leq s\leq i} \hat{\hat{X}}_{s:k} \tag{2.2}$$

and

$$\hat{\hat{X}}_{s:k} = \min\{Y_s, \frac{Y_s+Y_{s+1}}{2},\ldots, \frac{Y_s +\ldots+ Y_k}{k-s+1}\}. \tag{2.3}$$

It is known that the isotonic estimators $\hat{X}_{i:k}$, $i = 1,\ldots,k$ are also the maximum likelihood estimators of μ_i, $i = 1,2,\ldots,k$, for the two-parameter exponential distributions.

3. Isotonic Seletion Procedures

3.1. μ_0 and θ are known

Let us define

$$\Omega_i = \{\underline{\mu} \in \Omega | \mu_{k-i} < \mu_0 \le \mu_{k-i+1}\},\ i = 1,2,\ldots,k-1,$$

$$\Omega_k = \{\underline{\mu} \in \Omega | \mu_0 \le \mu_1\},$$

and

$$\Omega_0 = \{\underline{\mu} \in \Omega | \mu_k < \mu_0\}.$$

Then Ω_i are disjoint and $\Omega = \bigcup_{i=0}^{k} \Omega_i$. Furthermore,

$$\inf_{\underline{\mu}\in\Omega} P_{\underline{\mu}}(CS|\delta) = \min_{1\le i\le k} \inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta), \text{ for any } \delta,$$

and

$$\inf_{\underline{\mu}\in\Omega} P_{\underline{\mu}}(CS|\delta) \ge P^* \text{ iff } \inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta) \ge P^*,\ i = 1,2,\ldots,k.$$

If μ_0 is known, no samples are drawn from π_0 and $\underline{X} = (X_{11},\ldots,X_{1n},\ldots,X_{k1},\ldots,X_{kn})$. We propose a selection procedure $\delta_1^{(1)}$ as follows:

$$\text{(3.1)}\qquad \delta_1^{(1)}(\underline{X}) = a_{\varepsilon(\underline{X})}, \text{ where } \varepsilon(\underline{X}) = \min\{i | \hat{X}_{i:k} \ge \mu_0 + d_{i:k}^{(1)} \tfrac{\theta}{n}\},$$

here $\hat{X}_{i:k}$ is defined by (2.2) and $d_{i:k}^{(1)}$, $i = 1,2,\ldots,k$ are determined to satisfy the P*-condition.

<u>Lemma 3.1</u>. For any $\underline{\mu} \in \Omega_i$, $1 \le i \le k$, $P_{\underline{\mu}}(CS|\delta_1^{(1)})$ is increasing in μ_j, $1 \le j \le k$.

Proof. If $\underline{\mu} \in \Omega_i$, then

$$P_{\underline{\mu}}(CS|\delta_1^{(1)}) = P_{\underline{\mu}}\{\bigcup_{j=1}^{k-i+1} \{\hat{X}_{j:k} \ge \mu_0 + d_{j:k}^{(1)} \tfrac{\theta}{n}\}\}$$

$$= P_{\underline{\mu}}\{\bigcup_{j=1}^{k-i+1} \bigcup_{r=1}^{j} \{\hat{\hat{X}}_{r:k} \ge \mu_0 + d_{j:k}^{(1)} \tfrac{\theta}{n}\}\}$$

$$= E_{\underline{\mu}}\{I_A(\underline{X})\},$$

where $A = \bigcup_{j=1}^{k-i+1} \bigcup_{r=1}^{j} \{\hat{\hat{X}}_{r:k} \geq \mu_0 + d_{j:k}^{(1)} \frac{\theta}{n}\}$.

Since $I_A(\underline{x})$ is increasing in $(x_{j1},\ldots,x_{jn})$, $1 \leq j \leq k$, and the distribution of X_{ij} has stochastically increasing property, hence $E_{\underline{\mu}}\{I_A(\underline{X})\}$ is increasing in μ_j, $1 \leq j \leq k$. This completes the proof of the lemma.

It follows from Lemma 3.1 that $\inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(1)}) = P_{\underline{\mu}^*}(CS|\delta_1^{(1)})$, where $\underline{\mu}^* = (\mu_0,-\infty,\ldots,-\infty,\underbrace{\mu_0,\ldots,\mu_0}_{i \text{ terms}})$, and

$$P_{\underline{\mu}^*}(CS|\delta_1^{(1)}) = P_{\underline{\mu}^*}\{\hat{\hat{X}}_{k-i+1:k} \geq \mu_0 + d_{k-i+1:k}^{(1)} \frac{\theta}{n}\}$$

$$= P\{\hat{\hat{Z}}_{k-i+1:k} \geq d_{k-i+1:k}^{(1)}\},$$

where $Z_1,\ldots,Z_k$ are i.i.d. $E(0,1)$.

Now $\hat{\hat{Z}}_{k-i+1:k}$ has the same distribution as $\hat{\hat{Z}}_{1:i}$. If we let

$$(3.2) \qquad V_i = \hat{\hat{Z}}_{1:i} = \min_{1\leq r\leq i} \{\frac{1}{r} \sum_{j=1}^{r} Z_j\},$$

then we have

$$(3.3) \qquad \inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(1)}) = P\{V_i \geq d_{k-i+1:k}^{(1)}\}, \quad i = 1,2,\ldots,k,$$

and the following theorem follows.

Theorem 3.2. For given $P^*(0 < P^* < 1)$, if $d_{k-i+1:k}^{(1)}$ is the solution to the equation $P(V_i \geq x) = P^*$, where V_i is defined by (3.2). Then $\delta_1^{(1)}$ defined by (3.1) satisfies the P^*-condition.

Remarks:

(1) If $x \leq 0$, then $P(V_i \geq x) = 1$, hence we restrict our attention to $d_{i:k}^{(1)} > 0$, $i = 1,2,\ldots,k$.

(2) It is clear that $d^{(1)}_{k-i+1:k} = d^{(1)}_{1:i}$ for all $1 \leq i \leq k$. Furthermore, $V_i \geq V_{i+1}$ implies $d^{(1)}_{1:i}$ is decreasing in i.

In order to find the $d^{(1)}_{i:k}$'s we need to find the joint distribution of $Z_1, Z_1+Z_2, \ldots,$ and $Z_1 + \ldots + Z_i$, $1 \leq i \leq k$. Theorem 3.3 gives an explicit form to find $d^{(1)}_{i:k}$'s.

Theorem 3.3. For $x > 0$, $P(V_i \geq x) = e^{-ix} \sum_{j=1}^{i} b_j x^{j-1}$, $1 \leq i \leq k$, where

(3.4) $$b_j = i^{(j-2)}(i-j+1)/(j-1)!.$$

Proof. Consider the transformation $U_1 = Z_1$, $U_2 = Z_1+Z_2, \ldots, U_i = Z_1 + \ldots + Z_i$, then $U_1, \ldots, U_i$ have joint pdf e^{-u_i}, $0 < u_1 < u_2 < \ldots < u_i < \infty$. Hence

$$P\{V_i \geq x\} = \int_{ix}^{\infty} e^{-u_i}\left(\frac{u_i^{i-1}}{(i-1)!} - \frac{u_i^{i-2}}{(1-2)!} x\right) du_i$$

$$= \frac{(ix)^{i-1}}{(i-1)!} e^{-ix} - (x-1)\int_{ix}^{\infty} e^{-u_i} \frac{u_i^{i-2}}{(i-2)!} du_i$$

$$= e^{-ix} \sum_{j=1}^{i} b_j x^{j-1},$$

where b_j is defined by (3.4).

From Theorem 3.3, for $1 \leq i \leq k$, $d^{(1)}_{k-i+1:k}$ is the solution to the equation

(3.5) $$e^{-ix} \sum_{j=1}^{i} b_j x^{j-1} = P^*, \text{ where } b_j \text{ is defined by (3.4).}$$

The values of $d^{(1)}_{1:i} (\equiv d^{(1)}_{k-i+1:k})$, for $k = 1(1)20$, and $P^* = 0.800(0.025)\ 0.975$ and 0.990 are tabulated in Table I.

3.2. μ_0 known, θ unknown

If the common value of θ is unknown, let $\hat{\theta} = \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - Y_i)/v$, where $v = k(n-1)$. Then $2v\hat{\theta}/\theta$ is distributed as chi-square with 2v degrees of freedom and is independent of $Y_1,\ldots,Y_k$ (see Epstein and Sobel (1954)). We propose a selection procedure $\delta_1^{(2)}$ by

(3.6) $\quad \delta_1^{(2)}(\underline{X}) = a_{\varepsilon(\underline{X})}$, where $\varepsilon(\underline{X}) = \min\{i \mid \hat{X}_{i:k} \geq \mu_0 + d_{i:k}^{(2)} \frac{2v\hat{\theta}}{n}\}$.

Analogous to the proof of Theorem 3.2, we have the following result.

Theorem 3.4. For given $P^*(0 < P^* < 1)$, if $d_{k-i+1:k}^{(2)}$ is the solution to the equation $P(V_i \geq \frac{2v\hat{\theta}}{\theta} x) = P^*$, where V_i is defined by (3.2) and $2v\hat{\theta}/\theta \sim \chi^2_{2v}$ are independent, then $\delta_1^{(2)}$ defined by (3.6) satisfies the P*-condition.

Theorem 3.5. For $x > 0$, $P(V_i \geq \frac{2v\hat{\theta}}{\theta} x) = \sum_{j=1}^{i} b_j (2x)^{j-1} \frac{\Gamma(v+j-1)}{\Gamma(v)(1+2ix)^{v+j-1}}$, where b_j is defined by (3.4).

Proof. The proof is straightforward.

Remarks:

(1) $d_{k-i+1:k}^{(2)}$ depends on $v = k(n-1)$ and $d_{k-i+1:k}^{(2)} \neq d_{1:i}^{(2)}$.

(2) $d_{k-i+1:k}^{(2)}$ is the solution to the equation

(3.7) $$\sum_{j=1}^{i} b_j (2x)^{j-1} \frac{\Gamma(v+j-1)}{\Gamma(v)(1+2ix)^{v+j-1}} = P^*, \ x > 0, \text{ where}$$

b_j is defined by (3.4).

The values of $d_{i:k}^{(2)}$ for $k = 2(1)6$, $P^* = 0.900(0.025)0.975$ and 0.990, with common sample size $n = 5(5)20$ are tabulated in Table II.

3.3. μ_0 unknown, θ known

In the case where μ_0 is unknown, we take additional observations X_{0j}, $j = 1,2,\ldots,n$ from π_0 and denote $\underline{X}$ by $(X_{01},\ldots,X_{0n},X_{11},\ldots,X_{1n},\ldots,X_{k1},\ldots,X_{kn})$. Let $Y_i = \min_{1\leq j\leq n} X_{ij}$, $i = 0,1,\ldots,k$, and $\hat{X}_{i:k}$ be defined as in (2.2). We propose a selection procedure $\delta_1^{(3)}$ by

(3.8) $\quad \delta_1^{(3)}(\underline{X}) = a_{\varepsilon(\underline{X})}$, where $\varepsilon(\underline{X}) = \min\{i \mid \hat{X}_{i:k} \geq Y_0 - d_{i:k}^{(3)} \frac{\theta}{n}\}$.

Theorem 3.6. For given P^* $(0 < P^* < 1)$, if $d_{k-i+1:k}^{(3)}$ is the solution to the equation

(3.9) $$\sum_{j=1}^{i} b_j e^{-x} \int_{-x}^{\infty} z^{j-1} e^{-(i+1)z} dz = P^*, \quad x \leq 0,$$

or the equation

(3.10) $$1 - e^{-x}\{1 - \sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j}\} = P^*, \; x > 0,$$

where $1 \leq i \leq k$ and b_j is defined by (3.4). Then $\delta_1^{(3)}$ defined by (3.8) satisfies the P^*-condition.

Proof. If $\underline{\mu} \in \Omega_i$, $P_{\underline{\mu}}(CS|\delta_1^{(3)})$ is increasing in μ_j, $1 \leq j \leq k$ and is decreasing in μ_0. Hence

$$\inf_{\underline{\mu} \in \Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(3)}) = \inf_{\mu_0} P_{\underline{\mu}^*}(CS|\delta_1^{(3)}),$$

where $\underline{\mu}^* = (\mu_0, -\infty, \ldots, -\infty, \underbrace{\mu_0, \ldots, \mu_0}_{i \text{ terms}})$,

and is independent of μ_0. Therefore

$$\inf_{\underline{\mu} \in \Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(3)}) = P\{V_i \geq Z_0 - d_{k-i+1:k}^{(3)}\}$$

$$= \begin{cases} \sum_{j=1}^{i} b_j e^{-d_{k-i+1:k}^{(3)}} \int_{-d_{k-i+1:k}^{(3)}}^{\infty} z^{j-1} e^{-(i+1)z} dz, & \text{if } d_{k-i+1:k}^{(3)} \leq 0 \\ 1 - e^{-d_{k-i+1:k}^{(3)}}\{1 - \sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j}\}, & \text{if } d_{k-i+1:k}^{(3)} > 0. \end{cases}$$

Remarks:

(1) If $d^{(3)}_{k-i+1:k} \leq 0$, then $P(V_i \geq Z_0 - d^{(3)}_{k-i+1:k}) \leq P(V_i \geq Z_0) \leq P(V_1 \geq Z_0) = 1/2$. Hence, for $P^* > 1/2$, there is no solution in the case when $d^{(3)}_{k-i+1:k} \leq 0$. We should restrict attention to $d^{(3)}_{k-i+1:k} > 0$ and use the equation (3.10) or

$$d^{(3)}_{k-i+1:k} = -\ln((1-P^*)/(1-\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j})).$$

The values of $d^{(3)}_{1:k}$, for $k = 1(1)20$, and $P^* = 0.800(0.025)0.975$ and 0.990 are tabulated in Table III.

(2) $d^{(3)}_{k-i+1:k} = d^{(3)}_{1:i}$ is increasing in i, $1 \leq i \leq k$.

(3) If $P^* > 1/2$, then $0 < (1-P^*)/(1-\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j}) < 1$ and hence $d^{(3)}_{k-i+1:k} > 0$.

(4) $\int_x^\infty z^{j-1} e^{-(i+1)z} dz = \frac{\Gamma(j)}{(i+1)^j} \sum_{\ell=0}^{j-1} \frac{((i+1)x)^\ell}{\ell!} e^{-(i+1)x}, \quad x > 0.$

If $d^{(3)}_{k-i+1:k} \leq 0$, then $-d^{(3)}_{k-i+1:k}$ is the solution to the equation

$$(3.11) \qquad (\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j} \sum_{\ell=0}^{j-1} \frac{((i+1)x)^\ell}{\ell!}) e^{-ix} = P^*, \qquad x \geq 0.$$

3.4. μ_0 unknown, θ unknown

We define a selection procedure $\delta_1^{(4)}$ by

$$(3.12) \qquad \delta_1^{(4)}(\underline{X}) = a_{\varepsilon(\underline{X})}, \text{ where } \varepsilon(\underline{X}) = \min\{i \mid \hat{X}_{i:k} \geq Y_0 - d^{(4)}_{i:k} \frac{2v_1\hat{\theta}_1}{n}\},$$

where $v_1 = (k+1)(n-1)$, $\hat{\theta}_1 = \sum_{i=0}^{k} \sum_{j=1}^{n} (X_{ij} - Y_i)/v_1$.

Theorem 3.7. For given $P^*(0 < P^* < 1)$, if $d^{(4)}_{k-i+1:k}$ is the solution to the equation

$$(3.13)\qquad \sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j} \sum_{\ell=0}^{j-1} \frac{(i+1)^{\ell}(-2x)^{\ell}\Gamma(v_1+\ell)}{\ell!\Gamma(v_1)(1-2ix)^{v_1+\ell}} = P^*, \quad x \leq 0$$

or the equation

$$(3.14)\qquad 1-\left(1-\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j}\right)(1+2x)^{-v_1} = P^*, \quad x > 0,$$

where b_j is defined by (3.4). Then $\delta_1^{(4)}$ defined by (3.12) satisfies the P^*-condition.

Proof. $\inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(4)}) = P(V_i \geq Z_0 - d_{k-i+1:k}^{(4)} \frac{2v_1\hat{\theta}_1}{\theta})$

$$= \begin{cases} \sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j} \sum_{\ell=0}^{j-1} \frac{(i+1)^{\ell}(-2d_{k-i+1:k}^{(4)})^{\ell}\Gamma(v_1+\ell)}{\ell!\Gamma(v_1)(1-2id_{k-i+1:k}^{(4)})^{v_1+\ell}}, & \text{if } d_{k-i+1:k}^{(4)} \leq 0 \\ 1-\left(1-\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j}\right)(1+2d_{k-i+1:k}^{(4)})^{-v_1}, & \text{if } d_{k-i+1:k}^{(4)} > 0. \end{cases}$$

<u>Remark</u>: If $d_{k-i+1:k}^{(4)} \leq 0$, then $\inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_1^{(4)}) \leq 1/2$. Hence, if $P^* > 1/2$, $d_{k-i+1:k}^{(4)}$ is the solution to the equation (3.14) or

$$d_{k-i+1:k}^{(4)} = \{\{(1-P^*)/(1-\sum_{j=1}^{i} b_j \frac{\Gamma(j)}{(i+1)^j})\}^{-1/v_1}-1\}/2.$$

The values of $d_{i:k}^{(4)}$ for $k = 2(1)6$, $P^* = 0.900(0.025)0.975$ and 0.990, with common sample size $n = 5(5)20$ are tabulated in Table IV.

4. <u>Some Other Proposed Selection Procedures</u>

4.1. <u>μ_0 and θ known</u>

(1) In Section 3.1, if we take $d = d_{1:k}^{(1)}$ and define a selection procedure $\delta_2^{(1)}$ by

$$(4.1)\qquad \delta_2^{(1)}: \text{ Select } \pi_i \text{ iff } \hat{X}_{i:k} \geq \mu_0 + d\,\frac{\theta}{n}, \quad i = 1,2,\ldots,k.$$

Since $d_{1:k}^{(1)} = \min_{1\le i\le k} d_{1:i}^{(1)}$, it is easy to see that $\inf_{\underline{\mu}\in\Omega} P_{\underline{\mu}}(CS|\delta_2^{(1)}) \ge P^*$.

Furthermore, $\hat{X}_{i:k} \ge \hat{X}_{j:k}$ for $i > j$ implies $\delta_2^{(1)}$ is an isotonic selection procedure.

(2) Let $\tilde{X}_j = \max\{Y_1,\ldots,Y_j\}$, $1 \le j \le k$ and define a selection procedure $\delta_3^{(1)}$ by

(4.2) $\quad \delta_3^{(1)}(\underline{X}) = a_{\varepsilon(\underline{X})}$, where $\varepsilon(\underline{X}) = \min\{i|\tilde{X}_i \ge \mu_0 + d_i \frac{\theta}{n}\}$.

Then, for any i, $1 \le i \le k$

$$\inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_3^{(1)}) = P(Z_{k-i+1} \ge d_{k-i+1}) = e^{-d_{k-i+1}}.$$

If $d_{k-i+1} = -\ell n\, P^*$ for all i, then $\delta_3^{(1)}$ satisfies the P^*-condition.

<u>Remark</u>: $\delta_3^{(1)}$ is equivalent to:

$$\delta_3^{(1)}: \text{ Select } \pi_i \text{ iff } \tilde{X}_i \ge \mu_0 - \ell n\, P^* \frac{\theta}{n}, \quad i = 1,2,\ldots,k.$$

(3) Gupta and Sobel (1958) proposed a selection procedure without assuming any ordering prior. If we define a similar selection procedure $\delta_4^{(1)}$ by

(4.3) $\quad \delta_4^{(1)}$: Select π_i iff $Y_i \ge \mu_0 + d\,\frac{\theta}{n}$, $\quad i = 1,2,\ldots,k$,

then

$$\inf_{\underline{\mu}\in\Omega_i} P_{\underline{\mu}}(CS|\delta_4^{(1)}) = e^{-id}, \quad \text{if } \ d > 0.$$

Hence $\quad \inf_{\underline{\mu}\in\Omega} P_{\underline{\mu}}(CS|\delta_4^{(1)}) = e^{-kd}$ and $d = -\frac{1}{k}\,\ell n\, P^*$.

Note that the selection procedure $\delta_4^{(1)}$ is not isotonic.

4.2. <u>μ_0 known, θ unknown</u>

(1) Similar to Section 3.2, we can define a selection procedure $\delta_2^{(2)}$ by

(4.4) $\delta_2^{(2)}$: Select π_i iff $\hat{X}_{i:k} \geq \mu_0 + d\,\frac{2v\hat{\theta}}{n}$, $i = 1,2,\ldots,k$,

where $d = d_{1:k}^{(2)}$.

(2) We define a selection procedure $\delta_3^{(2)}$ by

(4.5) $\delta_3^{(2)}$: Select π_i iff $\tilde{X}_i \geq \mu_0 + d\,\frac{2v\hat{\theta}}{n}$, $i = 1,2,\ldots,k$,

where $d = ((P^*)^{-1/v}-1)/2$.

(3) We define a selection procedure $\delta_4^{(2)}$ by

(4.6) $\delta_4^{(2)}$: Select π_i iff $Y_i \geq \mu_0 + d\,\frac{2v\hat{\theta}}{n}$, $i = 1,2,\ldots,k$,

where $d = ((P^*)^{-1/v}-1)/2k$. Then $\delta_i^{(2)}$, $i = 2,3,4$ satisfy the P*-condition.

4.3. <u>μ_0 unknown, θ known</u>

(1) If we define $\delta_2^{(3)}$ by

(4.7) $\delta_2^{(3)}$: Select π_i iff $\hat{X}_{i:k} \geq Y_0 - d\,\frac{\theta}{n}$, $i = 1,2,\ldots,k$,

where $d = d_{1:k}^{(3)}$.

(2) If we define $\delta_3^{(3)}$ by

(4.8) $\delta_3^{(3)}$: Select π_i iff $\tilde{X}_i \geq Y_0 - d\,\frac{\theta}{n}$, $i = 1,2,\ldots,k$,

where

$$d = \begin{cases} \ln 2P^* , & \text{if } P^* \leq 1/2 \\ -\ln 2(1-P^*), & \text{if } P^* > 1/2. \end{cases}$$

(3) If we define $\delta_4^{(3)}$ by

(4.9) $\delta_4^{(3)}$: Select π_i iff $Y_i \geq Y_0 - d\,\frac{\theta}{n}$, $i = 1,2,\ldots,k$,

where

$$d = \begin{cases} \frac{1}{k} \ln (k+1)P^* , & \text{if } P^* \le 1/(k+1) \\ -\ln \frac{k+1}{k} (1-P^*), & \text{if } P^* > 1/(k+1). \end{cases}$$

Then $\delta_i^{(3)}$, i = 2,3,4 satisfy the P*-condition.

4.4. <u>μ_0 unknown, θ unknown</u>

(1) We define $\delta_2^{(4)}$ by

(4.10) $\delta_2^{(4)}$: Select π_i iff $\hat{X}_{i:k} \ge Y_0 - d \frac{2v_1\hat{\theta}_1}{n}$, $i = 1,2,\ldots,k$,

where $d = d_{1:k}^{(4)}$.

(2) We define $\delta_3^{(4)}$ by

(4.11) $\delta_3^{(4)}$: Select π_i iff $\tilde{X}_i \ge Y_0 - d \frac{2v_1\hat{\theta}_1}{n}$, $i = 1,2,\ldots,k$,

where

$$d = \begin{cases} \{1-(2P^*)^{-1/v_1}\}/2 , & \text{if } P^* \le 1/2 \\ \{\{2(1-P^*)\}^{-1/v_1}-1\}/2, & \text{if } P^* > 1/2. \end{cases}$$

(3) We define $\delta_4^{(4)}$ by

(4.12) $\delta_4^{(4)}$: Select π_i iff $Y_i \ge Y_0 - d \frac{2v_1\hat{\theta}_1}{n}$, $i = 1,2,\ldots,k$,

where

$$d = \begin{cases} \{1-((k+1)P^*)^{-1/v_1}\}/2k , & \text{if } P^* \le 1/(k+1) \\ \{\{(\frac{k+1}{k})(1-P^*)\}^{-1/v_1}-1\}/2, & \text{if } P^* > 1/(k+1). \end{cases}$$

Then $\delta_i^{(4)}$, i = 2,3,4 satisfy the P*-condition.

5. Expected Number (Size) of Bad Populations in the Selected Subset

In this section, we assume that μ_0 and θ are known. Let $E(S'|\delta)$ denote the expected number of bad populations in the selected subset when the selection procedure δ is used. For the procedure satisfying the P*-condition, usually we want the procedure with small expected number of bad populations in the selected subset. For procedure $\delta_1^{(1)}$ we have the following theorem:

Theorem 5.1. For any j, $0 \leq j \leq k$, $\sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_1^{(1)})$
$= \sum_{r=1}^{j} P\{\bigcup_{h=1}^{r}\{Z_{h:j} \geq d_{h:k}^{(1)}\}\}$, where $\hat{\hat{Z}}_{h:j}$ is defined as in (2.3) and $Z_1,\ldots,Z_k$ are i.i.d. $E(0,1)$.

Proof. For any j, $0 \leq j \leq k$, if $\underline{\mu} \in \Omega_{k-j}$, we have

$$E_{\underline{\mu}}(S'|\delta_1^{(1)}) = \sum_{r=1}^{j} P_{\underline{\mu}}(\pi_r \text{ is selected}|\delta_1^{(1)})$$

$$= \sum_{r=1}^{j} P_{\underline{\mu}}\{\bigcup_{h=1}^{r}\{\hat{\hat{X}}_{h:k} \geq \mu_0 + d_{h:k}^{(1)}\frac{\theta}{n}\}\}.$$

Using the property similar to Lemma 3.1, we have

$$\sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_1^{(1)}) = \sum_{r=1}^{j} P_{\underline{\mu}^{**}}\{\bigcup_{h=1}^{r}\{\hat{\hat{X}}_{h:k} \geq \mu_0 + d_{h:k}^{(1)}\frac{\theta}{n}\}\}.$$

$$= \sum_{r=1}^{j} P\{\bigcup_{h=1}^{r}\{\hat{\hat{Z}}_{h:j} \geq d_{h:k}^{(1)}\}\},$$

where $\underline{\mu}^{**} = (\underbrace{\mu_0,\mu_0,\ldots,\mu_0}_{j \text{ terms}},\infty,\ldots,\infty)$.

For procedure $\delta_2^{(1)}$, it is easy to show that

$$\sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_2^{(1)}) = \sum_{r=1}^{j} P\{\bigcup_{h=1}^{r}\{\hat{\hat{Z}}_{h:j} \geq d_{1:k}^{(1)}\}\} \text{ and}$$

$$\sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_1^{(1)}) \le \sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_2^{(1)}), \text{ for } 0 \le j \le k.$$

Hence $\delta_1^{(1)}$ is uniformly better than $\delta_2^{(1)}$. Furthermore,

$$\sup_{\underline{\mu}\in\Omega} E(S'|\delta_2^{(1)}) = \sup_{\underline{\mu}\in\Omega_0} E_{\underline{\mu}}(S'|\delta_2^{(1)}), \text{ since}$$

$$\sum_{r=1}^{j+1} P\{\bigcup_{h=1}^{r}\{\hat{\hat{Z}}_{h:j+1} \ge d\}\} \ge \sum_{r=2}^{j+1} P\{\bigcup_{h=2}^{r}\{\hat{\hat{Z}}_{h:j+1} \ge d\}\}$$

$$= \sum_{r=1}^{j} P\{\bigcup_{h=1}^{r}\{\hat{\hat{Z}}_{h:j} \ge d\}\}.$$

For procedure $\delta_3^{(1)}$, we have the following theorem:

<u>Theorem 5.2</u>. For any j, $0 \le j \le k$, $\sup_{\underline{\mu}\in\Omega_{k-j}} E(S'|\delta_3^{(1)}) = j-q(1-q^j)/P^*$ and

$\sup_{\underline{\mu}\in\Omega} E(S'|\delta_3^{(1)}) = k-q(1-q^k)/P^*$, where $q = 1-P^*$.

Proof. $$\sup_{\underline{\mu}\in\Omega_{k-j}} E(S'|\delta_3^{(1)}) = \sup_{\underline{\mu}\in\Omega_{k-j}} \sum_{\gamma=1}^{j} P_{\underline{\mu}}\{\max_{1\le s\le r} Y_s \ge \mu_0+d\,\frac{\theta}{n}\}$$

$$= \sum_{r=1}^{j} \{1-P\{\max_{1\le s\le r} Z_s < d\}\} = j-q(1-q^j)/P^*,$$

and $\sup_{\underline{\mu}\in\Omega_{k-j}} E(S'|\delta_3^{(1)})$ is increasing in j.

In order to compare the procedures $\delta_1^{(1)}$ and $\delta_3^{(1)}$, we need the following lemma:

<u>Lemma 5.3</u>. For $i = 1,\ldots,k$, let $A_i = \{\hat{\hat{Z}}_{i:k} \ge d_{i:k}^{(1)}\}$, then

$$P\{\bigcup_{i=1}^{j} A_i \cap A_{j+1}\} > P(\bigcup_{i=1}^{j} A_i)P^* \text{ for all } j,\ 1 \le j \le k-1,\ k \ge 2.$$

Proof. If $\hat{\hat{z}}_{j+1:k} \geq d^{(1)}_{j+1:k}$ and $\hat{\hat{z}}_{i:j} \geq d^{(1)}_{i:k}$ for some i, $1 \leq i \leq j$, then

$$\frac{z_i + \ldots + z_\ell}{\ell-i+1} = \frac{(j-i+1)(z_i+\ldots+z_j)/(j-i+1)+(\ell-j)(z_{j+1}+\ldots+z_\ell)/(\ell-j)}{\ell-i+1}$$

$$\geq \frac{(j-i+1)d^{(1)}_{i:k} + (\ell-j)d^{(1)}_{i:k}}{\ell-i+1} = d^{(1)}_{i:k}, \quad \text{for } j+1 \leq \ell \leq k.$$

Hence

$$P(\bigcup_{i=1}^{j} A_i \cap A_{j+1}) = P(\bigcup_{i=1}^{j} \{\hat{\hat{Z}}_{i:j} \geq d^{(1)}_{i:k}\} \cap A_{j+1})$$

$$= P(\bigcup_{i=1}^{j} \{\hat{\hat{Z}}_{i:j} \geq d^{(1)}_{i:k}\})P(A_{j+1})$$

$$> P(\bigcup_{i=1}^{j} A_i)P^*.$$

<u>Theorem 5.4</u>. For all $k \geq 2$, $\sup_{\underline{\mu}\in\Omega_0} E_{\underline{\mu}}(S'|\delta_1^{(1)}) \leq \sup_{\underline{\mu}\in\Omega_0} E_{\underline{\mu}}(S'|\delta_3^{(1)})$.

Proof. By Lemma 5.3 and the induction principle, we have

$$P(\bigcup_{i=1}^{j} A_i) \leq 1-(1-P^*)^j \text{ for all } j,\ 1 \leq j \leq k.$$

Hence

$$\sup_{\underline{\mu}\in\Omega_0} E(S'|\delta_1^{(1)}) = \sum_{r=1}^{k} P\{\bigcup_{h=1}^{r} \{\hat{\hat{Z}}_{h:k} \geq d^{(1)}_{h:k}\}\}$$

$$\leq \sum_{r=1}^{k} \{1-(1-P^*)^r\} = \sup_{\underline{\mu}\in\Omega_0} E(S'|\delta_3^{(1)}).$$

<u>Remark</u>: Theorem 5.4 tells us that procedure $\delta_1^{(1)}$ is better than $\delta_3^{(1)}$ in the sense that in Ω_0 it tends to select smaller number of bad populations, however, procedure $\delta_1^{(1)}$ is not uniformly better than $\delta_3^{(1)}$.

In order to compare the procedures $\delta_3^{(1)}$ and $\delta_4^{(1)}$, we need the following lemma.

Lemma 5.5. $k(1-(P^*)^{1/k}) \uparrow -\ell n\, P^*$, $0 < P^* < 1$.

Proof. Let $f(k) = k(1-(P^*)^{1/k})$, then

$$f'(k) = 1-(P^*)^{1/k} + \frac{1}{k}(\ell n\, P^*)(P^*)^{1/k}.$$

$f'(k) > 0$ iff $-\frac{1}{k}\ell n\, P^* > \ell n(1-\frac{\ell n\, P^*}{k})$.

The result follows since $-\frac{1}{k}\ell n\, P^* > 0$ and $\lim_{k\to\infty} k(1-(P^*)^{1/k}) = -\ell n\, P^*$.

Theorem 5.6. If $k \geq 2$ and $P^* > 1/2$, then $\sup_{\underline{\mu}\in\Omega} E(S'|\delta_3^{(1)}) < \sup_{\underline{\mu}\in\Omega} E(S'|\delta_4^{(1)})$.

Proof. It is easy to show that $\sup_{\underline{\mu}\in\Omega_{k-j}} E_{\underline{\mu}}(S'|\delta_4^{(1)}) = j(P^*)^{1/k}$ and hence

$\sup_{\underline{\mu}\in\Omega} E_{\underline{\mu}}(S'|\delta_4^{(1)}) = k(P^*)^{1/k}$.

$$\sup_{\underline{\mu}\in\Omega} E_{\underline{\mu}}(S'|\delta_3^{(1)}) < \sup_{\underline{\mu}\in\Omega} E_{\underline{\mu}}(S'|\delta_4^{(1)}) \quad \text{iff}$$

$$k(1-(P^*)^{1/k}) < (1-P^*)(1-(1-P^*)^k)/P^*.$$

If $P^* > 1/2$, then $-\ell n\, P^* < (1-P^*)(2-P^*)$. By Lemma 5.5, we have

$k(1-(P^*)^{1/k}) < -\ell n\, P^*$.

$\sup_{\underline{\mu}\in\Omega} E_{\underline{\mu}}(S'|\delta_3^{(1)}) < \sup_{\underline{\mu}\in\Omega} E_{\underline{\mu}}(S'|\delta_4^{(1)})$, since

$$(1-P^*)(2-P^*) = (1-P^*)(1-(1-P^*))/P^*$$

$$< (1-P^*)(1-(1-P^*)^k)/P^*.$$

Remark: Theorem 5.6 tells us that procedure $\delta_3^{(1)}$ is uniformly better than procedure $\delta_4^{(1)}$.

Table I

Table of $d_{1:k}^{(1)}$ values associated with procedure $\delta_1^{(1)}$.

$k \backslash d_{1:k}^{(1)}$	P*								
	0.990	0.975	0.950	0.925	0.900	0.875	0.850	0.825	0.800
1	.0100	.0253	.0512	.0779	.1053	.1335	.1625	.1923	.2231
2	-	.0250	.0500	.0752	.1006	.1261	.1520	.1781	.2046
3	-	-	-	.0750	.1000	.1252	.1504	.1757	.2012
4	-	-	-	-	-	.1250	.1501	.1752	.2004
5	-	-	-	-	-	-	.1500	.1751	.2001
6	-	-	-	-	-	-	-	.1750	.2000
7-20*	-	-	-	-	-	-	-	-	-

The "-" in Table I means that the value is the same as the preceding one in the same column.

* For k = 7(1)20, values of $d_{1:k}^{(1)}$ are the same for any given P* in the above table.

Table II

Table of $d_{i:k}^{(2)}$ values associated with procedure $\delta_1^{(2)}$.

		n = 5					n = 10				
k	i \ P*	0.990	0.975	0.950	0.925	0.900	0.990	0.975	0.950	0.925	0.900
2	1	.0006	.0015	.0031	.0047	.0063	.0002	.0006	.0013	.0020	.0027
	2	-	-	.0032	.0048	.0066	-	.0007	.0014	.0021	.0029
3	1	.0004	.0010	.0020	.0031	.0041	.0002	.0004	.0009	.0013	.0018
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	.0021	.0032	.0044	-	-	-	.0014	.0019
4	1	.0003	.0007	.0015	.0023	.0031	.0001	.0003	.0006	.0010	.0013
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-
	4	-	-	.0016	.0024	.0033	-	-	.0007	-	.0014
5	1	.0002	.0006	.0012	.0018	.0025	.0001	.0002	.0005	.0008	.0011
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-
	4	-	-	-	-	-	-	-	-	-	-
	5	-	-	-	.0019	.0026	-	-	-	-	-
6	1	.0002	.0005	.0010	.0015	.0020	.0001	.0002	.0004	.0006	.0009
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-
	4	-	-	-	-	-	-	-	-	-	-
	5	-	-	-	-	-	-	-	-	-	-
	6	-	-	-	.0016	.0021	-	-	-	.0007	-

The "-" in Table II means that the value is the same as the preceding one in the same column.

Table II (continued)

Table of $d_{i:k}^{(2)}$ values associated with procedure $\delta_1^{(2)}$.

		n = 15					n = 20				
k	i \ P*	0.990	0.975	0.950	0.925	0.900	0.990	0.975	0.950	0.925	0.900
2	1	.0001	.0004	.0008	.0013	.0017	.0001	.0003	.0006	.0009	.0013
	2	-	-	.0009	-	.0018	-	-	-	.0010	-
3	1	.0001	.0002	.0005	.0008	.0011	.0001	.0003	.0004	.0006	.0008
	2	-	-	-	-	-	-	-	-	-	-
	3	-	.0003	.0006	.0009	.0012	-	-	-	-	.0009
4	1	.0001	.0002	.0004	.0006	.0008	.0001	.0001	.0003	.0004	.0006
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-
	4	-	-	-	-	.0009	-	-	-	.0005	-
5	1	.0001	.0001	.0003	.0005	.0007	.0001	.0001	.0002	.0003	.0005
	2	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-
	4	-	-	-	-	-	-	-	-	-	-
	5	-	-	-	-	-	-	-	-	.0004	-

The "-" in Table II means that the value is the same as the preceding one in the same column.

Table III

Table of $d^{(3)}_{1:k}$ values associated with procedure $\delta^{(3)}_1$.

k | p*	0.990	.975	.950	.925	.900	.875	.850	.825	.800
1	3.912	2.996	2.303	1.897	1.609	1.386	1.204	1.050	0.916
2	4.017	3.101	2.408	2.002	1.715	1.492	1.309	1.155	1.022
3	4.057	3.141	2.448	2.042	1.755	1.531	1.349	1.195	1.061
4	4.078	3.162	2.469	2.063	1.776	1.552	1.370	1.216	1.082
5	4.091	3.175	2.482	2.076	1.789	1.565	1.383	1.229	1.095
6	4.100	3.184	2.491	2.085	1.797	1.574	1.392	1.238	1.104
7	4.106	3.190	2.497	2.092	1.804	1.581	1.398	1.244	1.111
8	4.111	3.195	2.502	2.096	1.809	1.586	1.403	1.249	1.116
9	4.115	3.199	2.506	2.100	1.813	1.589	1.407	1.253	1.119
10	4.118	3.202	2.509	2.103	1.816	1.592	1.410	1.256	1.122
11	4.121	3.204	2.511	2.106	1.818	1.595	1.413	1.258	1.125
12	4.123	3.206	2.513	2.108	1.820	1.597	1.415	1.261	1.127
13	4.125	3.208	2.515	2.110	1.822	1.599	1.417	1.262	1.129
14	4.126	3.210	2.517	2.111	1.824	1.600	1.418	1.264	1.130
15	4.127	3.211	2.518	2.113	1.825	1.602	1.419	1.265	1.132
16	4.129	3.212	2.519	2.114	1.826	1.603	1.421	1.266	1.133
17	4.130	3.213	2.520	2.115	1.827	1.604	1.422	1.267	1.134
18	4.131	3.214	2.521	2.116	1.828	1.605	1.423	1.268	1.135
19	4.131	3.215	2.522	2.116	1.829	1.606	1.423	1.269	1.136
20	4.132	3.216	2.523	2.117	1.830	1.606	1.424	1.270	1.136

Table IV

Table of $d_{i:k}^{(4)}$ values associated with procedure $\delta_1^{(4)}$.

		n= 5					n= 10				
k	i p*	0.990	0.975	0.950	0.925	0.900	0.990	0.975	0.950	0.925	0.900
2	1	0.199	0.147	0.111	0.091	0.077	0.080	0.061	0.047	0.038	0.033
	2	0.193	0.142	0.106	0.086	0.072	0.078	0.059	0.045	0.036	0.031
3	1	0.144	0.108	0.083	0.068	0.058	0.060	0.046	0.035	0.029	0.025
	2	0.143	0.107	0.081	0.067	0.057	0.059	0.045	0.035	0.029	0.024
	3	0.138	0.103	0.077	0.063	0.053	0.057	0.043	0.033	0.027	0.023
4	1	0.113	0.086	0.066	0.054	0.046	0.047	0.036	0.028	0.023	0.020
	2	0.112	0.085	0.065	0.054	0.046	0.047	0.036	0.028	0.023	0.020
	3	0.111	0.084	0.064	0.053	0.045	0.047	0.036	0.027	0.023	0.019
	4	0.108	0.081	0.061	0.050	0.042	0.045	0.034	0.026	0.022	0.018
5	1	0.093	0.071	0.054	0.045	0.039	0.039	0.030	0.024	0.020	0.017
	2	0.093	0.070	0.054	0.045	0.038	0.039	0.030	0.023	0.019	0.017
	3	0.092	0.070	0.054	0.044	0.038	0.039	0.030	0.023	0.019	0.017
	4	0.091	0.069	0.053	0.044	0.037	0.039	0.030	0.023	0.019	0.016
	5	0.089	0.066	0.050	0.041	0.035	0.038	0.029	0.022	0.018	0.015
6	1	0.079	0.060	0.047	0.039	0.033	0.034	0.026	0.020	0.017	0.014
	2	0.079	0.060	0.046	0.038	0.033	0.034	0.026	0.020	0.017	0.014
	3	0.078	0.060	0.046	0.038	0.033	0.033	0.026	0.020	0.017	0.014
	4	0.078	0.059	0.046	0.038	0.032	0.033	0.026	0.020	0.016	0.014
	5	0.077	0.059	0.045	0.037	0.032	0.033	0.025	0.019	0.016	0.014
	6	0.075	0.056	0.043	0.035	0.030	0.032	0.024	0.019	0.015	0.013

		n= 15					n= 20				
k	i p*	0.990	0.975	0.950	0.925	0.900	0.990	0.975	0.950	0.925	0.900
2	1	0.050	0.038	0.030	0.024	0.021	0.037	0.028	0.022	0.018	0.015
	2	0.049	0.037	0.028	0.023	0.020	0.036	0.027	0.021	0.017	0.014
3	1	0.038	0.029	0.022	0.019	0.016	0.027	0.021	0.016	0.014	0.012
	2	0.037	0.028	0.022	0.018	0.016	0.027	0.021	0.016	0.013	0.011
	3	0.036	0.027	0.021	0.017	0.015	0.026	0.020	0.015	0.013	0.011
4	1	0.030	0.023	0.018	0.015	0.013	0.022	0.017	0.013	0.011	0.009
	2	0.030	0.023	0.018	0.015	0.013	0.022	0.017	0.013	0.011	0.009
	3	0.030	0.023	0.017	0.015	0.012	0.022	0.017	0.013	0.011	0.009
	4	0.029	0.022	0.017	0.014	0.012	0.021	0.016	0.012	0.010	0.009
5	1	0.025	0.019	0.015	0.013	0.011	0.018	0.014	0.011	0.009	0.008
	2	0.025	0.019	0.015	0.012	0.011	0.018	0.014	0.011	0.009	0.008
	3	0.025	0.019	0.015	0.012	0.011	0.018	0.014	0.011	0.009	0.008
	4	0.024	0.019	0.015	0.012	0.010	0.018	0.014	0.011	0.009	0.008
	5	0.024	0.018	0.014	0.011	0.010	0.017	0.013	0.010	0.008	0.007
6	1	0.021	0.017	0.013	0.011	0.009	0.016	0.012	0.009	0.008	0.007
	2	0.021	0.016	0.013	0.011	0.009	0.016	0.012	0.009	0.008	0.007
	3	0.021	0.016	0.013	0.011	0.009	0.016	0.012	0.009	0.008	0.007
	4	0.021	0.016	0.013	0.011	0.009	0.015	0.012	0.009	0.008	0.007
	5	0.021	0.016	0.012	0.010	0.009	0.015	0.012	0.009	0.008	0.006
	6	0.020	0.016	0.012	0.010	0.008	0.015	0.011	0.009	0.007	0.006

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The problem of selecting populations, from two-parameter exponential populations, which are better than a standard under an ordering prior is investigated. If the negative exponential distribution is the model for lifetime, then the problem is to select all those populations for which the guarantee lifetimes are larger than that of a standard. Comparisons of these procedures based on the expected number of bad populations in the selected subset is investigated. Tables of associated constants for the proposed procedures are given in Table I through Table IV.

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