Volterra Equations Driven by Semimartingales

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SUMMARY

Existence and uniqueness of solutions is established for stochastic Volterra integral equations driven by right continuous semimartingales. This resolves (in the affirmative) a conjecture of M. Berger and V. Mizel.

Running Head: Volterra Integral Equations

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1. Introduction

Stochastic integral equations of Volterra type typically arise by modelling systems corrupted by noise (cf, e.g. [2]). The white noise case leads to Itô-integrals with Brownian differentials (e.g., [1], [2], [19]), and Berger and Mizel [2] conjectured that general right continuous semimartingale differentials can be handled. Partial results in this direction, using continuous McShane differentials, were obtained by Rao and Padgett [20]. In particular, consider the equation:

$$X_t = H_t + \sum_{i=1,r} \int_0^t f_i(t,s;\ X_u, u < s)\,dZ_s^i, \tag{1.1}$$

or more typically the linear equations:

$$X_t = H_t + \sum_{i=1,r} \int_0^t K_i(t,s) X_s\,dZ_s^i, \tag{1.2}$$

where Z^i are right continuous semimartingales. Several authors (e.g. [1], [2], [19]) have considered equations such as (1.1) and (1.2) when $Z_t^1 = t$ and $Z_t^2 = B_t$ (a Brownian motion). A recent article of Kolodii [12] considers semimartingale differentials, but the author's hypotheses eliminate discontinuous local martingales with paths of infinite variation on compacts, and thus the theorem for general semimartingale differentials is not established.

One cannot apply directly the standard techniques used to study stochastic differential equations driven by semimartingales, principally because if $Z = M + A$ is a decomposition of Z

(with M a local martingale), the Métivier-Pellaumail-Doob maximal inequalities cannot be applied to

$$\int_0^t f(t,s;\ X_u, u < s)\, dM_s,$$

because of the presence of "t" in the integrand.

In Theorem (4.2) we establish the existence and uniqueness of solutions of equations of the form (1.1) with arbitrary right continuous semimartingale differentials, thus resolving a conjecture (in the affirmative) of Berger and Mizel [2, p. 336]. These results include (as a special case) those of Rao and Padgett [20], since the semimartingale integral extends the McShane integral. We also show (Corollary (4.11)) that X is a semimartingale and we give its decomposition.

Our method combines the idea of Berger and Mizel of establishing a "transformation rule" (Theorem (3.3)) with the elegant results and methods of a recent article of A. S. Sznitman [21]. In proving the existence and uniqueness of solutions, we use the idea of "controlling" a semimartingale due to Métivier and Pellaumail, together with a type of Picard iteration method developed in [18].

Allowing stochastic Volterra kernels as in equation (1.2) leads one naturally to consider non-adapted (i.e., anticipating) integrands. One approach towards handling this might be to follow the idea of K. Itô [9], which has been systematically developed primarily by the French (cf, especially T. Jeulin [11]): that is, one may expand the underlying filtration in such a way that the semimartingale differential remains a semimartingale (although with possibly a different decomposition) and hence still can be used as a differential in the equation. Unfortunately these theories are still in their infancy; in Comment (3.6) we indicate how one might proceed in a particularly simple setting.

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2. Preliminaries

We assume the reader is familiar with the basic results of stochastic integration with respect to semimartingales as presented in any of [10], [16], [5], or [14]. For the reader's convenience we will recall in this section the basic notations and definitions we will use, as well as some of the more recent and less known results that we will need.

We will assume throughout that we have an underlying filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ satisfying the "usual hypotheses": that $\mathcal{F}$ is P-complete, and that $\mathcal{F}_0$ contains all P-null sets, and that $(\mathcal{F}_t)$ is right continuous. A right continuous, adapted process Z is called a <u>semimartingale</u> if Z has a decomposition $Z = M + A$, where M is a local martingale and A is a right continuous, adapted process with paths of finite variation on compacts. Such a decomposition is not unique.

We will say that an increasing process A <u>controls</u> a semimartingale Z if for each bounded, predictable process H and for each stopping time T we have

$$(2.1)\quad E\{\sup_{t<T}(\int_0^t H_s dZ_s)^2\} \leqq E\{\tilde{A}_{T-}\int_0^{T-} H_s^2 dA_s\}$$

where $A_0 = 0$, $\tilde{A}_t = \max(1,A_t)$, $A_{T-} = \lim_{\substack{t\to T\\ t<T}} A_t$, and where A

is assumed to be increasing, right continuous and adapted. The following key result is due to M. Métivier and J. Pellaumail [14, or 8 and 13].

(2.2) THEOREM. *If* Z *is a semimartingale, then there exists a finite-valued increasing process* A *controlling* Z.

We will also need the existence of smooth versions of stochastic integrals when the integrand depends on a parameter. The following theorem is a trivial corollary of a general result due to Sznitman [21, Proposition 5 on p. 53].

(2.3) THEOREM. *Let* Z *be a semimartingale,* k *a nonnegative integer, and* $H(x,\cdot)$ *a predictable,* Z*-integrable (uniformly in* x*) process that is* $\mathscr{C}^k$ *in* x *such that the* k^{th} *derivative is Lipschitz continuous in* x. *Then there exists a version* $N(x,t)$ *of*

$$\frac{d^k}{dx^k}\int_0^t H(x,s)\,dZ_s$$

that is continuous in x *and cadlag in* t.

A process X is called *cadlag* if it is adapted and has paths which are right continuous with left limits. The space of functions mapping $\mathbb{R}_+$ to $\mathbb{R}$ which are right continuous and have left limits is denoted $\mathbb{D}$. Given a local martingale N and an adapted process A with right continuous paths of finite variation on compacts, we let

$$j(N,A) = [N,N]_\infty^{1/2} + \int_0^\infty |dA_s| ,$$

where $[N,N]$ is the quadratic variation process of the local martingale N.

For a semimartingale Z the $\mathscr{H}^p$ norm of Z is given by

$$\|Z\|_{\mathscr{H}^p} = \inf_{Z=N+A} \|j(N,A)\|_{L^p}$$

where the infimum is taken over all possible decompositions of Z. For a local martingale N one can easily see that

$$\|N\|_{\mathscr{H}^p} = \| [N,N]_\infty^{1/2} \|_{L^p}.$$

For a process X, we let

$$X^* = \sup_{s<\infty} X_s \ ; \ \|X\|_{\mathscr{S}^2} = \|X^*\|_{L^2}.$$

In general, our notation is that of P. A. Meyer as established in [5] or [16], to which we refer the reader for any unexplained notations or definitions.

3. The Transformation Rule

In this section a transformation rule for stochastic integrals is established, along with some related results. This extends the results of Berger and Mizel [1], and is closely related to the work of Sznitman [21], who considered more abstract situations.

(3.1) DEFINTIION. Let $\underset{\sim}{Z}_t = (Z_t^1, \cdots, Z_t^k)$ be a vector of $(\mathcal{F}_t)_{t \geqq 0}$-semimartingales, and let $\underset{\sim}{H}(x,s,\omega) = (H^1, \cdots, H^\ell)$ be a vector of parameterized processes: $H^j\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$.

$$\mathcal{G}_s^o = \mathcal{F}_s \vee \sigma\{H^j(x,u,\cdot);\ 0 \leqq u \leqq s;\ x \geqq u;\ 1 \leqq j \leqq \ell\}$$

$$\mathcal{G}_s = \bigcap_{u>s} \mathcal{G}_u^o. \quad \text{(Also write } \mathcal{G}^{\underset{\sim}{H}} = \mathcal{G}).$$

Then $\underset{\sim}{H}$ is $\underset{\sim}{Z}$-<u>acceptable</u> if (i) Z^i is a $\mathcal{G}$-semimartingale $(1 \leqq i \leqq k)$, and (ii) for each x, $(s,\omega) \to H^j(x,s,\omega)$ is $\mathcal{G}$-predictable for $s \leqq x$; and (iii) $(x,s,\omega) \to H^j(x,s,\omega)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_s$ measurable $(1 \leqq j \leqq \ell)$.

(3.2) THEOREM. <u>Let</u> Z <u>be a semimartingale, and let</u> $H = H(x,s,\omega)$ <u>be</u> Z-<u>integrable and</u> Z-<u>acceptable</u>. <u>Suppose</u> $H_1(x,s,\omega) = \frac{\partial H}{\partial x}(x,s,\omega)$ <u>exists, is locally bounded (uniformly in</u> x), <u>and satisfies</u> $|H_1(x,s,\omega) - H_1(y,s,\omega)| \leqq K|x-y|$, <u>some</u> $K > 0$, <u>for each fixed</u> (s,ω) $(s \leqq x)$. <u>Then there exist processes</u> $N(x,t)$ <u>and</u> $L(x,t)$ <u>such that</u>:

(i) <u>for each fixed</u> ω, $x \to N(x,t,\omega)$ <u>is</u> $\mathscr{C}^1$ <u>and</u> $x \to L(x,t,\omega)$ <u>is continuous</u>.

(ii) <u>for each fixed</u> x, N <u>and</u> L <u>are versions, respectively of</u>:

$$N(x,t) = \int_0^t H(x,s)dZ_s$$
$$L(x,t) = \int_0^t H_1(x,s)dZ_s$$

(iii) $\frac{\partial N}{\partial x}(x,t) \equiv N_1(x,t)$ <u>is a semimartingale, and</u> $L(x,t)$ <u>is a version of</u> $N_1(x,t)$.

PROOF. Parts (i) and (ii) are immediate consequences of Theorem (2.3).

For part (iii), we first assume Z is an $\mathscr{H}^2$ martingale and that H_1 is bounded. Since $N(x,t) \in \mathscr{H}^2$, each x, we need to show only that $\lim_{h\to o} \frac{1}{h}\{N(x+h,t) - N(x,t)\} = L(x,t)$, with convergence being in $\mathscr{H}^2$. Note that:

$$\lim_{h\to o} \left\|\frac{1}{h}\{N(x+h,t) - N(x,t)\} - L(x,t)\right\|^2_{\mathscr{H}^2}$$
$$= \lim_{h\to o} E\{\int_0^t (\frac{1}{h}\{H(x+h,s) - H(x,s)\} - H_1(x,s))^2 d[Z,Z]_s\}$$
$$= \lim_{h\to o} E\{\int_0^t \{H_1(c(h),s) - H_1(x,s)\}^2 d[Z,Z]_s\}$$

where $[Z,Z]$ denotes the quadratic variation process of the $\mathscr{H}^2$ martingale Z, and where $c(h)$ is between x and $x+h$. Since $x \to H_1(x,s)$ is continuous and bounded, Lebesgue's dominated convergence theorem yields the result.

In the general case, by optional stopping we may assume H_1 is bounded; moreover one can always find a sequence of stopping times T^n increasing to ∞ a.s. such that $Z^{T^n} = M^n + A^n$ with $M^n \in \mathscr{H}^2$ and A an adapted process with paths of finite variation on compacts (cf, e.g., [5, p. 463]). The result holds for A^n by Lebesgue-Stieltjes integration theory, and it holds for M^n by the preceding; linearity gives the result for Z^{T^n}, and hence for Z. □

(3.3) THEOREM (Transformation Rule). Let Z be a semimartingale and let $H = H(x,s,\omega)$ be Z-integrable and also satisfy the hypotheses of Theorem (3.2). Let $Y_t = \int_0^t H(t,s)\,dZ_s$. Then Y is a $\mathscr{G}^H$ semimartingale, and if $Z = M + A$ is a $\mathscr{G}^H$-decomposition of Z, then

(3.4)
$$Y_t = \int_0^t H(s,s)\,dM_s + \left\{\int_0^t H(s,s)\,dA_s + \int_0^t \left(\int_0^s H_1(s,u)\,dZ_u\right) ds\right\}$$

is a decomposition of Y.

PROOF. Establishing (3.4) will, of course, show that Y is a semimartingale. We have

$$Y_t = \int_0^t H(t,s)\,dZ_s$$
$$= \int_0^t \{H(t,s) - H(s,s)\}\,dZ_s + \int_0^t H(s,s)\,dZ_s,$$

and $$H(t,s) - H(s,s) = \int_s^t H_1(u,s)\,du = \int_0^t 1_{(s\leqq u)} H_1(u,s)\,du,$$

hence by the Fubini theorem for stochastic integration (cf, e.g., Jacod [10, p. 181]) we have:

$$\int_0^t \{H(t,s) - H(s,s)\}dZ_s = \int_0^t \{\int_0^t 1_{(s\leqq u)} H_1(u,s)\,du\}dZ_s = \int_0^t \{\int_0^u H_1(u,s)\,dZ_s\}du,$$

and the theorem follows. □

(3.5) COMMENT. A sufficient condition for H to be Z-integrable (one of the hypotheses of Theorem (3.3)) is that H be locally bounded, uniformly in x. With a little extra regularity in the second variable this is easily verified in practice: let

$$T(n,x) = \inf\{s > 0: |H(x,s)| \geqq n\}$$

$$R_n = \sup_{x\leqq n} T(n,x).$$

Since $x \to H(x,s)$ is assumed continuous, one has $R_n = \sup_{\substack{x\leqq n\\ x\in Q}} T(n,x)$ as well, so that R_n is a countable supremum of stopping times and hence also a stopping time.

If one assumes $s \to H(x,s)$ is left continuous, then $|H(x,s \wedge R_n)| \leqq n$. If $s \to H(x,s)$ is right continuous, then the stopping times R_n are predictable (cf, eg, [4, p. 74]), and letting $R_{n,k}$ be an announcing sequence for R_n, we have $|H(x,s \wedge R_{n,k})| \leqq n$, each k. Letting k_0 be so large that $P(R_{n,k_0} < R_n - \frac{1}{2^n}) < \frac{1}{2^n}$, we can choose a sequence of stopping times $Q_n = R_{n,k_0}$ for each n such that $\lim Q_n = \infty$ a.s. and $|H(x,s \wedge Q_n)| \leqq n$ a.s.

(3.6) COMMENT. Recall that the filtration $\mathcal{G}^H$ is given in (3.1) as follows:

$$\mathcal{G}_s^o = \mathcal{F}_s \vee \sigma\{H^i(x,u,\cdot);\ 0 \leqq u \leqq s;\ x \geqq u,\ 1 \leqq i \leqq \ell\}$$

$$\mathcal{G}_s = \bigcap_{u>s} \mathcal{G}_u^o = \mathcal{G}^{\underset{\sim}{H}}.$$

The question of when an $\mathcal{F}$-semimartingale remains a $\mathcal{G}$-semimartingale is a difficult one. If $Z = M + A$ is an $\mathcal{F}$-decomposition of Z, then A is obviously also $\mathcal{G}$-adapted with right continuous paths of finite variation on compacts, and hence Z is a $\mathcal{G}$-semimartingale as soon as the $\mathcal{F}$-local martingale M is one. That all $\mathcal{F}$-local martingales be $\mathcal{G}$-semimartingales is called "hypothesis H'" in the literature. General - but not very practical for our situation - conditions for it to hold have been obtained (cf, eg, [11], [17]), especially in the important special case when Z is a Brownian motion ([9], [11], [22]).

We now give an example of how one might apply this theory. Let

$$\mathcal{H}_s^o = \mathcal{F}_s \vee \sigma\{H^i(x,u,\cdot);\ 0 \leqq u \leqq s;\ x \geqq 0;\ 1 \leqq i \leqq \ell\}$$

$$\mathcal{H}_s = \bigcap_{u>s} \mathcal{H}_u^o.$$

Then $\mathcal{F}_s \subseteqq \mathcal{G}_s \subseteqq \mathcal{H}_s$, and it is well known that if Z is an $\mathcal{H}$-semimartingale, then it is also a $\mathcal{G}$-semimartingale. The following is a trivial consequence of a result of J. Jacod and P. A. Meyer [11, p. 26].

(3.7) THEOREM. *Let* $(A_i)_{i\in I}$ *be a partition of* Ω *such that* $A_i \in \mathcal{F}$, $P(A_i) > 0$, $(i \in I)$, *and* $\sigma\{A_i;\ i \in I\} = \mathcal{J}$. *If* $\mathcal{H}_s = \bigcap_{u>s} (\mathcal{F}_u \vee \mathcal{J})$, *each* s, *then each* $\mathcal{F}$-*semimartingale is an* $\mathcal{H}$-*semimartingale*.

This indicates a way to adjoin all the "extra" information of the Volterra coefficient *initially* in such a way that the differential remains a semimartingale.

Clearly, however, this is one of the least interesting possibilities, and it would be interesting to find a framework such that one could have more general random Volterra coefficients.

4. Existence and Uniqueness of Solutions

We begin this section by describing what functions will be allowed as coefficients.

(4.1) CONDITION. Let $f\colon \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{D} \to \mathbb{R}$ be such that

(i) f is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{D})$-measurable;

(ii) If $\mathcal{G}_s^f = \bigcap_{u>s} \mathcal{F}_u \vee \sigma\{f(t,v,H);\ 0 \leqq v \leqq u;\ t \geqq u;\ H \in \mathbb{D}\}$, then for any cadlag, $\mathcal{G}^f$-adapted process Y, the process $J_s(\omega) = f(\omega,t,s,Y.(\omega))$, $(s \leqq t)$, is $\mathcal{G}^f$-predictable.

(iii) f has the following Lipschitz property (X, Y $\mathcal{G}^f$-adapted and cadlag):

$$\sup_{u<s} |f(u,u,X) - f(u,u,Y)| \leqq K \sup_{u<s} |X_u - Y_u|,$$

(iv) f_1 exists and has a Lipschitz property:

$$\sup_{\substack{u<s \\ t\geqq s}} |f_1(t,u,X.) - f_1(t,u,Y.)| \leqq K \sup_{u<s} |X_u - Y_u|,$$

where X, Y are $\mathcal{G}^f$-adapted and cadlag.

In the above, f_1 denotes the partial derivative of f in the first variable.

(4.2) DEFINITION. Let $\underset{\sim}{Z} = (Z^1,\cdots,Z^k)$ be a vector of $(\Omega, \mathcal{F},\mathcal{F}_t,P)$ semimartingales, and suppose that $\underset{\sim}{f} = (f^1,\cdots,f^k)$ satisfy Condition (4.1). If $\underset{\sim}{Z}$ remains a semimartingale for the filtration $\mathcal{G}^{\underset{\sim}{f}}$, then we say that $\underset{\sim}{f}$ is a $\underset{\sim}{Z}$-compatible coefficient.

(4.3) THEOREM. *Given semimartingales* $(Z^1,\cdots,Z^k) = \underset{\sim}{Z}$, *and a* $\underset{\sim}{Z}$-*compatible coefficient* $\underset{\sim}{f} = (f^1,\cdots,f^k)$, *we let* H *be a* $\mathcal{G}^{\underset{\sim}{f}}$-*adapted cadlag process. Then there exists a solution, and only one, of:*

$$X_t = H_t + \sum_{i=1,k} \int_0^t f^i(t,s,X.)_s dZ_s^i.$$

Before formally beginning the proof, we state and prove two lemmas we will need.

(4.4) LEMMA. *Let* $H = H(x,t,\omega)$ *be a bounded, parameterized process satisfying the hypotheses of Theorem* (3.2), *and let* f *satisfy condition* (4.1). *Let* Z *be a semimartingale. Then there exists an increasing process* L *controlling* Z *in the following sense: for any stopping time* T *bounded a.s. by a constant* a:

(i)
$$E\{\sup_{t<T}(\int_0^t H(t,s)dZ_s)^2\}$$
$$\leqq 2E\{\tilde{L}_{T-}\int_0^{T-} H(u,u)^2 dL_u\}$$
$$+ 8a \int_0^a E\{\tilde{L}_{T-}\int_0^{T-} H_1(s,u)^2 dL_u\}ds;$$

where $\tilde{L}_t = \max(1,L_t)$.

(ii) <u>for</u> X <u>and</u> Y <u>two processes with paths in</u> $\mathbb{D}$:

$$E\{\sup_{t<T}(\int_0^t f(t,s,X.)_s - f(t,s,Y.)_s dZ_s)^2\}$$

$$\leqq (2+8a^2)E\{\tilde{L}_{T-}\int_0^{T-}(X.-Y.)_u^{*2}dL_u\}.$$

PROOF. First we prove (i). The existence of an L controlling Z is assured by Theorem (2.2), a result of M. Métivier and J. Pellaumail. Thus we have that

$$E\{\sup_{s<t}(\int_0^t H(s,s)dZ_s)^2\}$$

$$\leqq E\{\tilde{L}_{T-}\int_0^{T-}H(s,s)^2 dL_s\}.$$

Using $(a+b)^2 \leqq 2a^2+2b^2$, we need only show there exists an L controlling $\int_0^t\{\int_0^s H_1(s,u)dZ_u\}ds$. Let $Z = M+A$ be a decomposition where M is locally square-integrable and A is of finite variation. We have:

$$E\{\sup_{t<T}(\int_0^t|\int_0^s H_1(s,u)dZ_u|ds)^2\}$$

(4.5) $$\leqq E\{T\int_0^T(\int_0^{s\wedge T-}H_1(s,u)dZ_u)^2ds\}$$

$$\leqq a\int_0^a E\{(\int_0^{s\wedge T-}H_1(s,u)dZ_u)^2ds\}.$$

Let $N_t^s = \int_0^{s\wedge t} H_1(s,u)\,dM_u$, which of course is a local martingale for each fixed s; and let $B_t^s = \int_0^{s\wedge t} H_1(s,u)\,dA_u$; then (4.5) becomes

$$\leqq 2a\int_0^a E(N_{T-}^s)^2 + E(B_{T-}^s)^2 ds$$

$$\leqq 8a\int_0^a \{E([N^s,N^s]_{T-} + \langle N^s,N^s\rangle_{T-})$$

$$(4.6) \qquad + E(|A|_{T-}\int_0^{s\wedge T-} H_1(s,u)^2|dA_s|)\}ds$$

$$\leqq 8a\int_0^a (E\{\int_0^{T-} H_1(s,u)^2 d([M,M]_u + \langle M,M\rangle_u)\}$$

$$+ E\{|A|_{T-}\int_0^{T-} H_1(s,u)^2|dA_s|\})ds.$$

Let $C_u = [M,M]_u + \langle M,M\rangle_u$, and let $D_u = \sqrt{2C_u}$, so that $dC_u = \frac{1}{2}(D_u + D_{u-})dD_u$. Then (4.6) yields:

$$\leqq 8a\int_0^a E\{D_{T-}\int_0^{T-} H_1(s,u)^2 dD_u +$$

$$|A|_{T-}\int_0^{T-} H_1(s,u)^2|dA_s|\}ds.$$

Finally, set $L_u = (|A|_u + D_u)$, and we are done.

PROOF of (ii): Applying part (i), we have:

$$E\{\sup_{t<T}(\int_0^t f(t,s,X.)_s - f(t,s,Y.)_s dZ_s)^2\}$$

$$\leqq 2E\{\tilde{L}_{T-}\int_0^{T-} (X. - Y.)_u^2 dL_u\}$$

$$+ 8a\int_0^a E\{\tilde{L}_{T-}\int_0^{T-} (X. - Y.)_u^{*2} dL_u\}ds$$

using the Lipschitz hypothesis on f. Since the second integral above has an integrand not depending on s, the result follows.

□

(4.7) LEMMA. *Let* X^n *be adapted, cadlag processes. Let* L_u *be a strictly increasing, right continuous, adapted process such that* $L_u \geqq u$ *a.s., each* $u \geqq 0$. *Suppose that for any bounded stopping time* R ($R \leqq a$ *a.s., say) we have:*

$$E\{[(X_\cdot^{n+1} - X_\cdot^n)_{R-}^*]^2\}$$
$$\leqq C(a)E\{\int_0^{R-} (X_\cdot^n - X_\cdot^{n-1})_s^{*2} dL_s\}.$$

Then $(X^n)_{n \geqq 1}$ *converge locally in* $\mathcal{S}^2$ *to a unique cadlag, adapted process* X.

PROOF. Define $\tau_t = \inf\{s > 0: L_s > t\}$, the right continuous inverse of L. Since L is adapted, τ_t is a stopping time for each t. Moreover $L_t \geqq t$ implies $\tau_t \leqq t$ a.s. and hence is bounded for each t. We fix a t_0, set $a = t_0$, and consider $t \leqq t_0$, writing C for $C(t_0) = C(a)$. Then

$$E\{[(X_{\cdot}^{n+1} - X_{\cdot}^{n})^{*}_{\tau_{t-}}]^2\}$$

$$\leqq CE\{\int_0^{\tau_{t-}} [(X^n - X^{n-1})^{*2}_s] dL_s\}$$

$$\leqq CE\{\int_0^a 1_{[0,\tau_t[}(s)(X^n - X^{n-1})^{*2}_s dL_s\}$$

$$= CE\{\int_0^\infty 1_{[0,\tau_t[}(s) 1_{[0,a]}(s)(X^n - X^{n-1})^{*2}_s dL_s\}$$

$$\leqq CE\{\int_0^{L_\infty} 1_{[0,\tau_t[}(\tau_s) 1_{[0,a]}(\tau_s)(X^n - X^{n-1})^{*2}_{\tau_s} ds\}$$

$$\leqq CE\{\int_0^{L_\infty} 1_{[0,t[}(s) 1_{(\tau_s \leq a)}(X^n - X^{n-1})^{*2}_{\tau_s} ds\}$$

$$\leqq CE\{\int_0^t (X^n - X^{n-1})^{*2}_{\tau_s \wedge a} 1_{(s<t)} ds\}$$

$$\leqq C\int_0^t E\{(X^n - X^{n-1})^{*2}_{\tau_{s-}}\} ds.$$

Now set $\alpha_{n+1}(t) = E\{(X^{n+1} - X^n)^{*2}_{\tau_{t-}}\}$. Then we have established, for $t \leqq t_0 = a$, that

$$\alpha_{n+1}(t) \leqq C\int_0^t \alpha_n(s) ds.$$

Iteration yields $\alpha_{n+1}(t) \leqq \frac{M}{n!}$ for a finite constant M. Since $\frac{M}{n!}$ is the general term of a convergent series, we are done. □

PROOF of Theorem (4.2): For notational simplicity we consider only the case $k = 1$. By the hypotheses on the coefficient f, using optional stopping if necessary, we may assume without loss of generality that f and f_1 are bounded. By Stricker's theorem and its refinements ([5], no. 63 bis, pp. 271, 2) we can assume, by changing to an equivalent probability law Q if necessary, that $H \in \mathcal{S}^2$ and $Z \in \mathcal{H}^2$, and hence Z has a decomposition $Z = M + A$ with $M \in \mathcal{H}^2$ and $\int_0^\infty |dA_s| \in L^2(dQ)$. The solution X obtained under Q will also be a solution under P due to the invariance of the stochastic calculus under changes to equivalent laws ([5, p. 338]). Thus henceforth we assume $Z \in \mathcal{H}^2$.

Set: $X_t^o = H_t$;

$$X_t^{n+1} = H_t + \int_0^t f(t,s,X_\cdot^n)_s dZ_s. \tag{4.10}$$

Then each $X^n \in \mathcal{S}^2$, and the controlling process L of Lemma (4.4) can be taken such that $E(L_t) < \infty$ for $t < \infty$. Moreover, by replacing L_t with $L_t + t$, L can be taken to be strictly increasing and such that $L_t \geqq t$ a.s.

We now apply Lemma 4.4 (ii), and Lemma (4.7), and the existence follows trivially.

As for uniqueness, if X and Y are two solutions, then

$$X - Y = \int_0^t f(t,s,X.) - f(t,s,Y.)dZ_s$$

(taking $k = 1$ for notational simplicity). Thus by Lemma (4.4) there exists a controlling process L such that

$$E\{\sup_{t<T}(\int_0^t f(t,s,X.)_s - f(t,s,Y.)_s dZ_s)^2\}$$
$$\leqq C(a)E\{\int_0^{T-} (X. - Y.)_u^{*2} dL_u\}.$$

Letting $\tau_t = \inf\{s > 0: L_s > t\}$ and proceeding as in the proof of Lemma (4.7), we have

$$E\{(X-Y)^{*2}_{\tau_{t-}}\} = \alpha(t) \leqq C\int_0^t \alpha(s)ds.$$

Gronwall's lemma then implies that $\alpha = 0$ if it is finite. We can ensure α is finite by changing to an equivalent probability measure, if necessary, so that X and Y are both in $\mathcal{S}^2$. Also, without loss of generality we can assume that $L_t \geqq t$ a.s., and also that $L_t < \infty$ a.s., each t. Thus $\lim_{t\to\infty} \tau_t = \infty$ a.s. Uniqueness then follows. □

(4.11) COROLLARY. With the same hypotheses of Theorem (4.2), the solution X of equation (4.3) is a semimartingale if H is, and if $H = N + C$, $Z^i = M^i + A^i$ are decompositions of the semimartingales H, Z^i, then a decomposition of X is given by:

$$\begin{aligned} X_t = &\{N_t + \sum_{i=1,k} \int_0^t f^i(s,s,X.)_s dM^i_s\} \\ &+ \{C_t + \sum_{i=1,k} \int_0^t f^i(s,s,X.)_s dA^i_s \\ &+ \int_0^t \sum_{i=1,k} (\int_0^s f^i_1(s,u,X.)_u dZ^i_u) ds\}. \end{aligned}$$

PROOF. For notational simplicity, let $k = 1$. Let X^n be the n^{th} iterate process as defined in equation (4.10) in the proof of Theorem (4.2). Then X^n is a semimartingale by induction, and hence by the transformation rule (Theorem (3.3)), we have that a decomposition of X^n is:

$$X_t^n = \{N_t + \int_0^t f(s,s,X_\cdot^{n-1})_s dM_s\}$$

$$+ \{C_t + \int_0^t f(s,s,X_\cdot^{n-1})_s dA_s$$

$$+ \int_0^t (\int_0^s f_1(s,u,X_\cdot^{n-1})_u dZ_u) ds\}.$$

The result now follows by standard localisation and limit arguments. □

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