On the Stochastic Integrals of Gaussian Processes and Local Times

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ABSTRACT

Let X_t, $0 \leq t \leq T$ be a Gaussian process and let $L(x)$ be the local time of X_t (occupation time density with respect to the clock $d\langle X,X\rangle_t$) up to time T at x. Then

$$\int_0^T g(X_t)dX_t = \int_{X_0}^{X_T} g(y)dy - \frac{1}{2}\int_{-\infty}^{\infty} L(x)dg(x)$$

under certain conditions.

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1. Introduction. The direct connection between the stochastic integrals of a process X_t and its local time $L_t(x)$ is the Tanaka's formula

$$(1.1)\qquad (X_t-x)^+ = (X_0-x)^+ + \int_0^t 1_{[x,\infty)}(X_s)dX_s + \frac{1}{2}L_t(x).$$

The local time $L_t(x)$ satisfies

$$(1.2)\qquad L_t(x) = \lim_{n\to\infty} n\int_0^t 1_{[x,x+\frac{1}{n})}(X_s)\, d\langle X,X\rangle_s$$

where $\langle X,X\rangle_s$ is the quadratic variation of X_t over $[0,s]$. That is

$$(1.3)\qquad \langle X,X\rangle_s = \lim \sum_i (X_{s_{i+1}} - X_{s_i})^2$$

along any sequence of partitions $\{s_0=0<s_1<\cdots<s_n=s\}$ whose mesh goes to 0 sufficiently fast. The meaning of (1.2) is that $L_t(x)$ is the occupation time density relative to the clock $d\langle X,X\rangle_s$ up to time t at x. In the case that X_t is a standard Wiener process (see [6, p. 101]), then $\langle X,X\rangle_s = s$. If X_t is a continuous semimartingale, then $\langle X,X\rangle_s = \langle X^c,X^c\rangle_s$ where X^c is the local martingale part of X in the Doob-Meyer decomposition. If X_t is not continuous, then the above result needs some modification (see [7, p. 32]). The purpose of this note is to show that the above result is also good for a Gaussian process whose covariance function satisfies certain conditions.

2. Integration. Let X_t, $0\le t\le T$, be a zero mean Gaussian process. We need to define the stochastic integral of the type $\int_0^T g(X_t)dX_t$, where g is a Borel function. The stochastic integral of a Gaussian process was defined in [4] through the tensor product of some Hilbert space. A Riemann-Stieltjes type of integration was given

in [5]. We shall use the latter.

Let the covariance function $R(s,t)$ of X_t satisfy the following conditions:

(2.1) $R(s,t)$ is continuous in (s,t) and has continuous first partial derivatives in $\{(s,t): 0 \leq s \leq t \leq T\}$.

(2.2) There exists a constant $C>0$ such that, for $s \neq t$, $|R(s+ds,t+dt) - R(s+ds,t) - R(s,t+dt) + R(s,t)| \leq C|dsdt|$.

Then the quadratic variation $\langle X,X\rangle_t$ exists and equals to (see [1])

$$\langle X,X\rangle_t = \int_0^t f(s)ds \tag{2.3}$$

where $f(t)$ is a nonnegative continuous function of t which is obtained by

$$f(t) = \lim_{h\to 0+} \frac{1}{h}\{2R(t,t) - R(t,t-h) - R(t,t+h)\}. \tag{2.4}$$

Let bC^1 denote the space of all differentiable functions on R which have bounded derivatives. If $g \in bC^1$, then see([5, Thm 3.1, p. 623]).

$$\int_0^T g(X_t)dX_t = \lim \sum_i g(X_{t_i}) (X_{t_{i+1}} - X_{t_i}) \tag{2.5}$$

exists along any sequence of partitions $\{t_0=0<t_1<\cdots<t_n=T\}$ whose mesh goes to 0. Furthermore, it is equal to

$$\int_0^T g(x_t)dX_t = \int_{X_0}^{X_T} g(y)dy - \frac{1}{2} \int_0^T g'(x_t)d\langle X,X\rangle_t \tag{2.6}$$

where $\langle X,X\rangle_t$ is given by (2.3). However, if $g \notin bC^1$, then, unlike the martingales case which have orthogonal increments, it is difficult to show the existence of (2.5) in general. We shall seek an alternative definition through (2.6).

Let Φ denote the space of all infinitely differentiable probability density functions with compact support. For each Borel function h and $\varphi \in \Phi$, let

(2.7) $$(h*\varphi)_m (x) = m\int_{-\infty}^{\infty} h(x-y)\varphi(my)dy = m\int_{-\infty}^{\infty} h(y)\varphi(mx-my)dy, \quad m \geq 1.$$

Definition 2.1. Let g be a Borel function which is integrable over any finite interval. If there exists a r.v. Y_g such that

(2.8) $$Y_g = \lim_{m\to\infty} \int_0^T (g*\varphi)_m' (X_t) f(t)dt$$

in some sense for all $\varphi \in \Phi$, then we define

(2.9) $$\int_0^T g(X_t)dX_t = \int_{X_0}^{X_T} g(y)dy - \frac{1}{2} Y_g.$$

Remark 22. It is clear from (2.7) and (2.8) that if $g \in bC^1$, then $Y_g = \int_0^T g'(X_t) f(t)dt$. Hence (2.9) becomes (2.6). Therefore, Definition 2.1 is an extension of (2.5).

3. Formulas. Let us now derive some relations between the stochastic integrals of X_t and the local time $L_t(x)$. First we note that conditions (2.1) and (2.2) imply that

(3.1) $$E|X_t-X_s|^2 = f(t) \ |t-s| + o(t-s)$$

for small $|t-s|$. Hence, $E|X_t-X_s|^4 \geq C(t-s)^2$, $0 \leq t, s \leq T$, for some constant C. It is well known that this inequality implies that the process X_t is equivalent to a process whose sample paths are continuous with probability 1. Therefore, we can assume that X_t has continuous sample paths with probability 1. From (3.1), we see that

$$\int_0^T \int_0^T \left\{E|X_t-X_s|^2\right\}^{-\frac{1}{2}} f(t)\, f(s)\, dtds \quad .$$

Then (see [2, Lemma 5.1, p. 276]) the nonnegative monotone increasing process $v(x) = \int_0^T 1_{(-\infty,x)}(X_s)f(s)ds$ is absolutely continuous w.r.t. the Lebesque measure with probability 1. Its Radon-Nikodym derivative $L(x)$ $(=L_T(x))$ exists a.s. and belongs to $\mathscr{L}^2(R)$. $L(x)$ is the local time of X_t up to time T at x.

Lemma 3.1. Let g be a continuous monotone function. If $\int_{-\infty}^{\infty} L(x)dg(x)$ exists, then $\int_0^T g(X_t)dX_t$ exists. Furthermore,

$$\int_0^T g(X_t)dX_t = \int_{X_0}^{X_T} g(y)dy - \frac{1}{2}\int_{-\infty}^{\infty} L(x)dg(x) \quad \text{a.s.} \tag{3.2}$$

Proof. Let $\varphi \in \Phi$. Then $(g*\varphi)'_m(x)$ is a continuous function. Hence (see [2, (1.2), p. 270])

$$\int_0^T (g*\varphi)'_m\,(X_t)f(t)dt = \int_{-\infty}^{\infty} (g*\varphi)'_m\,(x)\; L(x)dx \tag{3.3}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m^2 g(y)\;\; \varphi'(mx-my)dy\; L(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m\varphi\,(mx-mg)\; L(x)\;\, dg(y)dx = I_m.$$

Since X_t is continuous, $L(x)$ vanishes outside the range of X_t, and hence $L(x)$ vanishes outside some compact set. This fact together with the fact that φ has compact support implies that the integrand in the last integral of (3.3) has compact support in R^2. Therefore, Fubini's Theorem applies. We have

$$I_m = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} m\varphi(mx-my)\ L(x)dx \right\} dg(y)$$

(3.4)

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} m\varphi(mx)\ L(x+y)dx \right\} dg(y)$$

Let the support of φ be contained in $[-K,K]$. Then the integral in the brace of the last term of (3.4) is an average of L, which belongs to $L^2(R)$, over $[y-\frac{K}{m}, y+\frac{K}{m}]$. Therefore it converges to L(y) for a.e. y. This fact plus the existence of $\int_{-\infty}^{\infty} L(y)\ dg(y)$ and the continuity of g imply that

(3.5) $$Y_g = \lim_{m\to\infty} I_m = \int_{-\infty}^{\infty} L(y)dg(y)$$

exists for all $\varphi \in \Phi$. Lemma 3.1 then follows from the Definition 2.1, (2.9) and (3.5).

If g is not continuous, we see from the above proof that we need also the condition

(3.6) $$L(x) = \lim_{m\to\infty} m \int_{-\infty}^{\infty} L(x+y)\ \varphi(my)dy,$$

for $\varphi \in \Phi$ and x where $g(x+) \neq g(x-)$, to get (3.5). Therefore, Lemma 3.1 will hold if the additional condition (3.6) is satisfied. We can apply the same argument to g which is of bounded variation over finite intervals.

<u>Theorem 3.2.</u> Let g be a real function which is of bounded variation over any finite interval. If $\int_{-\infty}^{\infty} L(x)dg(x)$ exists and the condition (3.6) is satisfied, then $\int_0^T g(X_t)dX_t$ exists and the identify (3.2) holds.

By letting $g(y) = 1_{[x,\infty)}(y)$, and noticing that (3.6) is satisfied for a.e. $x \in R$, we obtain the following Tanak's formula.

Corollary 3.3. For a.e. $x \in R$, and a.s.

$$(X_t - x)^+ = (X_0 - x)^+ + \int_0^T 1_{[x,\infty)}(X_t)dX_t + \frac{1}{2} L(x).$$

Remark 3.4. There are conditions (see [3,pp. 52-53]) such that $L(x)$ is continuously differentiable in x. Then

$$\int_{-\infty}^{\infty} L(x)dg(x) = \int_{-\infty}^{\infty} g(x)d\,L(x). \tag{3.7}$$

However, these known conditions are incompatible to the condition (2.1). It remains the possibility to find a condition such that $L(x)$ is absolutely continuous w.r.t. the Lebesque measure. Then (3.7) can be discussed for Borel measurable function g. And so can

$$\int_0^T g(X_t)dX_t = \int_{X_0}^{X_T} g(y)dy - \frac{1}{2}\int_{-\infty}^{\infty} g(x)d\,L(x).$$

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