IDENTIFIABILITY OF TIME SERIES MODELS WITH ERRORS IN VARIABLES*†

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ABSTRACT

Straightforward derivations are provided for some identifiability results for Time Series models with Errors in Variables.

Introduction

It is a pleasure to join in the Festschrift for Ted Hannan on the occasion of his 65th birthday. One of Ted's most impressive contributions to Time Series Analysis is his realization and exploitation of the role of multivariable polynomials in the identifiability problem for multivariate ARMA Time Series (see his 1969 article and his review article in 1979). Another area of Time Series that revolves around identifiability is the Error in Variables problem and Ted was an early worker here too (Hannan, 1963). This essay is concerned with such problems.

The Problem

Recently Maravall (1979) has provided a careful exposé of the identifiability of some Time Series models with Errors in Variables (EIV). The interesting results revealed by Maravall are that often identifiability is not a problem as it is in the traditional static case. He shows that the identifiability problem differs depending on whether or not the input or exogenous sequence is serially uncorrelated or serially correlated. In any case the results are expressed as simple counting rules.

However Maravall's argument is very long (developed over 120 pages) - something that he lamented in the preface. In this essay a simple direct development is given of the basic theorems of Maravall. The present discussion clearly reveals the origin of the identifiability conditions. Actually the results obtained include Maravall's in that the exogenous sequence need not be modelled by an ARMA process for some of the results.

Söderström (1980) has independently given some identifiability results for EIV problems but his model is different from the one used by Maravall. He also models the input sequence as an ARMA process though. Further, Söderström excluded the case that the input sequence is white noise (so carefully discussed by Maravall). Otherwise the two sets of results are more or less the same. The relationship will be discussed below. Recently Anderson and Deistler (1982) have extended Söderström's discussion but again excluded the case of white noise input.

Section 1 opens the discussion of the single input single output (SISO) case, pointing out how the white noise input case is different. In Section 2 the basic results of Maravall are derived in a direct and simple way. In Section 3 Söderström's results are reviewed and extended to cover the case of a white noise input.

1. SISO identifiability with EIV: first steps

We begin with a basic model for the observed output (y) and input (z) and unobserved output (Y) and input (Z) as follows

$$y_t = Y_{t-1}+\varepsilon_{yt} \tag{1a}$$

$$A_y(L)Y_t = B_y(L)Z_t+v_t \tag{1b}$$

$$z_t = Z_t + \varepsilon_{zt} \tag{1c}$$

$$A_v(L)v_t = B_v(L)\nu_{vt} \tag{1d}$$

where $LY_t = Y_{t-1}$ etc.; $A_w(L) = \Sigma_0^{p_w} a_{w_i} L^i$ etc.; the ε's and ν are white noises and uncorrelated among each other; $A_{y0} = 1 = A_{v0} = B_{v0}$. Also we denote $(a_{y1}\ldots a_{yp_y})'$ by $\underline{a}_y$ etc.

$$\text{all roots of } Z^p A_y(Z^{-1}) = 0 \quad \text{are} < 1 \text{ in modulus} \tag{1e}$$

$$Z_t \text{ is stationary} \tag{1f}$$

To begin a study of the identifiability of model 1 (ie equations 1) we naturally calculate the following covariances

$$E(y_0 y_\tau) = \gamma_y(\tau) = \gamma_Y(\tau) \qquad \tau \neq 0 \tag{2a}$$

$$E(z_0 z_\tau) = \gamma_z(\tau) = \gamma_Z(\tau) \qquad \tau \neq 0 \tag{2b}$$

$$E(y_0 z_{-\tau}) = \gamma_{yz}(\tau) = \gamma_{YZ}(\tau) \qquad \text{all } \tau \tag{2c}$$

The step that now follows naturally is to seek the parameters $\underline{a}_y$, $\underline{b}_y$ by taking cross covariances in (1b) with $Z_{t+\tau}$ to find

$$A_y(L)\gamma_{YZ}(-\tau) = B_y(L)\gamma_Z(-\tau) \qquad \text{all } \tau. \tag{3a}$$

Now a problem is immediately apparent. Suppose Z_t is a white noise then clearly for $-\tau > q_y$

$$\gamma_{\tilde{Y}Z}(-\tau) = A_y(L)\gamma_{YZ}(-\tau) = 0$$

and we can find $\underline{a}_y$ by assembling and solving these equations in matrix form. However we have trouble in finding $\underline{b}_y$ for the only equations apparently available are for $-\tau = 0,+1,\ldots,q_y$, namely

$$\gamma_{\tilde{Y}Z}(-\tau) = b_{y\tau}\gamma_Z(0) \tag{4}$$

so that only $B_{y\tau} = b_{y\tau}\gamma_Z(0)$ can be found. So we are missing a scale factor.

For this reason it is clear (as Maravall found) that we need two types of result. One when Z_t is white the other when Z_t is not.

Before continuing the discussion an important simplification with regard to the identifiability of ARMA models is made. In the Appendix it is shown that the p+q+1 parameters $\underline{a},\underline{b},\sigma^2$ of an ARMA (p,q) model such as (1d) are equivalent to the r+1 = p+q+1 autocovariances $\gamma_0,\gamma,\ldots,\gamma_r$ (i.e. each set may be obtained from the other). Thus identifiability of an ARMA model is established if these autocovariances are found. This equivalence is more than just a theoretical point. The author has recently discussed algorithms for constructing the exact likelihood in both the scalar and multivariate cases parameterised by autocovariances. See Solo (1982), Solo (1983a).

Now let us observe from (2b) that $\gamma_Z(1)\ldots\gamma_2(r)$ are available for any r. So that the whole covariance sequence of Z_t is available once $\gamma_Z(0)$ is found. Then if appropriate, an ARMA model may be constructed for Z_t (as observed in the Appendix this includes the orders p_Z, q_Z). For these reasons we are henceforth only concerned with the identifiability of $\gamma_Z(0)$.

So far as v_t is concerned we similarly need only show how $\gamma_v(0)$, $\gamma_v(1)\ldots\gamma_v(r_v)$ may be found. It will still prove convenient though to state results in terms of $\underline{a}_v, \underline{b}_v, \sigma^2_{\nu v}$.

The following results will be established in the next section. For other results see Section 3.

<u>Result 1C</u>. If Z_t is serially correlated then

$[\underline{a}_y : \underline{b}_y]$, $[\gamma_Z(0), \sigma^2_{\varepsilon Z}]$, $\underline{a}_v$ are always identified.

<u>Result 1C'</u>. If Z_t is serially correlated then

$[\underline{b}_v, \sigma^2_{\nu v}]$, $\sigma^2_{\varepsilon y}$ are also identified iff

$$p_y \geq \max(0, q_v - p_v + 1)$$

<u>Result 1W</u>. If Z_t is a white noise $\underline{a}_y$, $\underline{a}_v$ are always identified.

<u>Result 1W'</u>. If Z_t is a white noise then

$[\underline{b}_y, \sigma^2_{\varepsilon y}]$, $[\gamma_Z(0), \sigma^2_{\varepsilon Z}]$, $[\underline{b}_v, \sigma^2_{\nu v}]$ are also identified iff

$$\min(p_y, q_y) \geq \max(0, 1+q_v-p_v)$$

$$\max(p_y, q_y) \geq 1+\max(0, 1+q_v-p_v)$$

<u>Remark 1</u>. Note in the above results that p_y, q_y, p_v can be obtained from the covariances. However in Results 1C', 1W', q_v must be known.

Remark 2. For our result 1W' cf Maravall's Theorem 4; while for 1C' cf his Theorem 7. Note though we do not require as does Maravall that Z_t be ARMA. Actually Maravall considered multiple but independent inputs and so joined Theorems 4, 7 into Theorem 8 (see p. 121 and the Table p. 122). These results are easily established by straightforward extensions of the discussion given below.

Remark 3. The case of correlated inputs is much harder (Maravall only found sufficient conditions) and will be discussed elsewhere.

2. SISO identifiability in EIV models: Details.

First we treat the correlated case.

Result 1C.

Assemble equations (3a) for $\tau = 1,2,\ldots,(r_y+1)$ in matrix form and solve (via (2b), (2c)) for $(\underline{a}_y, \underline{b}_y)$. This is straightforward unless Z_t is MA with $q_Z < q_y$ or ARMA with $p_Z < q_y$ for then the last rows in the matrix are null or linearly dependent on previous ones. We deal with this in a moment.

Next put $\tau = 0$ in (3a) to yield an equation for $\gamma_Z(0)$. In case $b_{y0} = 0$ put $\tau = 1$ etc. Finally from $\gamma_z(0) = \gamma_Z(0) + \sigma^2_{\varepsilon Z}$ we obtain $\sigma^2_{\varepsilon Z}$.

To cover the MA or ARMA case we have only to be a little more judicious. First reduce the ARMA case to the MA. We can (cf Appendix) find $A_Z(L)$ with $A_Z(L)\gamma_Z(-\tau) = 0$ for $\tau > q_Z$. Thus introducing $\tilde{Z}_t = A_Z(L)Z_t$ we find (3a) becomes

$$A_y(L)\gamma_{Y\tilde{Z}}(-\tau) = B_y(L)\gamma_{\tilde{Z}}(-\tau) \tag{3b}$$

and $\tilde{Z}_t$ is MA(q_Z). We find for $\tau > q_Z$

$$A_y(L)\gamma_{Y\tilde{Z}}(-\tau) = 0$$

which gives (via 2c) a set of equations for $\underline{a}_y$. So introduce $\tilde{Y}_t = A_y(L)Y_t$ then (3b) becomes

$$\gamma_{\tilde{Y}\tilde{Z}}(-\tau) = B_y(L)\gamma_{\tilde{Z}}(-\tau). \tag{3c}$$

This yields a set of $q_y + 2q_Z+1$ equations for the q_y+2 unknowns $\underline{b}_y$, $\gamma_Z(0)$. By writing out an example or two the reader can see that these equations have a quadrilateril shape. So by solving from the top down and bottom up a set of linear equations is obtained for $\underline{b}_y$, $\gamma_Z(0)$. The argument continues as above for σ^2_Z.

We have $\underline{a}_v$ left to find. Take covariances in (1b) to see

$$\begin{aligned}\gamma_v(\tau) &= E\{v_0[A_y(L)Y_{-\tau}-B_y(L)Z_{-\tau}]\}\\ &= E(v_0A_y(L)Y_{-\tau})\\ &= \Sigma_0^{p_y}\ \Sigma_0^{p_y}a_{yi}a_{yj}\gamma_Y(i-j-\tau)-\Sigma_0^{q_y}\ \Sigma_0^{q_y}b_{yi}a_{yj}\gamma_{YZ}(i-j-\tau).\end{aligned} \tag{5a}$$

This set of equations involves $\gamma_{YZ}(\cdot)$ which is identified and $\gamma_Y(\cdot)$ which is identified except for $\gamma_Y(0)$. However for $\tau > p_y$, $\gamma_Y(0)$ does not appear so we can calculate $\gamma_v(p_y+q_v)\ldots\gamma_v(p_y+r_v)$ and hence (cf Appendix A) determine the autoregressive (AR) parameters $\underline{a}_v$. Result 1C is thus established.

Result 1C'.

Now to find the other parameters we first reduce v_t to the MA case. Multiply through (1b) by $A_v(L)$ to see

$$A_y'(L)Y_t = B_y'(L)Z_t + v_t' \tag{6}$$

where $A_y'(L) = A_y(L)A_v(L)$; $p_y' = \deg(A_y'(L)) = p_y + p_v$ etc.; v_t' is a MA(q_v) process. Now return to equation (5a) (with v, p_y, q_y replaced respectively by $v_y' p_y', q_y'$): call it then (5a').

We need the autocovariance sequence of v_t'. From equation (6) it is clearly available (via (2a)) once we know $\gamma_Y(0)$.

Now according to assumption $\gamma_{v'}(\tau) = 0 \quad \tau > q_v$. In order to determine $\gamma_Y(0)$ from this assumption we must have it appear in (5a') for a lag $\tau > q_v$. However it only appears when $\tau \le p_y'$. We can thus use our assumption to find $\gamma_Y(0)$ if and only if $p_y' > q_v$ i.e. $p_y \ge \max(0, 1+q_v-p_v)$. Then we find $\gamma_{v'}(\tau)$ by taking covariances in (6) while $\sigma^2_{\varepsilon y} = \gamma_y(0)-\gamma_Y(0)$. Result 1C' is thus established.

Now we turn to the white noise case.

Results 1W, 1W'.

Previous argument has established 1W. For 1W' recall that from (3a) we can only find $b_{y\tau}\gamma_Z(0)$ $-\tau = 0,1,\ldots,q_y$ and $\gamma_Z(0)$ is still missing. It is equation (5a) that can rescue us again.

If we substitute (4) into (5a) we see

$$\gamma_v(\tau) = \Sigma_0^{p_y} \Sigma_0^{p_y} a_{yi}a_{yj}\gamma_Y(i-j-\tau) - \gamma_Z^{-1}(0)\Sigma_0^{q_y}\gamma_{\tilde{Y}Z}(-i)\gamma_{\tilde{Y}Z}(i-\tau). \tag{5b}$$

Once more we reduce consideration to the case where v_t is MA. Observe that for $\tau > p_y$, $\gamma_Y(0)$ does not appear in (5b) while (in view of (4)) if $\tau > q_y$, $\gamma_Z(0)$ does not appear. Thus for $\tau > \max(p_y, q_y)$ we can directly calculate $\gamma_v(\tau)$ and so find the AR parameters $\underline{a}_v$. We then multiply through (1b) by $A_v(L)$ as before to obtain an equation like (b). We can then return to (5b) (now called (5b')) with v, p_y, q_y replaced by v', p_y', q_y' as before.

According to assumption $\gamma_{v'}(\tau) = 0$ for $\tau > q_v$. In order to determine $\gamma_Y(0)$, $\gamma_Z(0)$ from this assumption we must have them appear in (5b') for two lags $\tau > q_v$. Now $\gamma_Y(0)$ only appears when $\tau \le p_y'$ and $\gamma_Z(0)$ only appears when $\tau \le q_y'$. We then have two possibilities

(i) $\gamma_Y(0)$ appears in the lag q_v equation (which requires clearly $p_y' > q_v$) and $\gamma_Z(0)$ appears in the lag(q_v+1) equation (which requires $q_y' > q_v+1$)

(ii) vice versa

We can summarize this by saying we need

$$\min(p_y, q_y) \geq \max(0, 1+q_v-p_v)$$

$$\max(p_y, q_y) \geq 1+\max(0, 1+q_v-p_v)$$

Again we find $\sigma^2_{\varepsilon y} = \gamma_y(0) - \gamma_Y(0)$.

Result 1W' is thus established. (Since $\gamma_Y(0)$ and $\gamma_Z(0)$ are both available (5b) yields the remaining unknown covariances $\gamma_v(\tau)$).

3. The Transfer Function Model

The transfer function model used by Söderström is as follows

$$Y^{\bullet}_t = T(L)Z_t \tag{7a}$$

$$y_t = Y^{\bullet}_t + v^{\bullet}_t \tag{7b}$$

$$z_t = Z_t + \varepsilon_{Zt} \tag{1c}$$

$$A^{\bullet}_v(L)v^{\bullet}_t = B_v(L)\nu^{\bullet}_{st} \tag{7c}$$

where $\nu^{\bullet}$ is a white noise; $T(L) = A^{-1}_y(L)B_y(L)$.

$$Z_t \text{ is stationary} \tag{1f}$$

$$v^{\bullet}_t \text{ is stationary.} \tag{7d}$$

Let us first observe the connexion with the ARMAX model used by Maravall. From (1a) we see

$$Y_t = Y^{\bullet}_t + A^{-1}_y(L)v_t.$$

Thus we can replace (1a), (1b) by (7a), (7b) where

$$v^{\bullet}_t = A^{-1}_y(L)v_t + \varepsilon_{yt}.$$

Thus Maravall's scheme is included in Söderström's.

As before we learn from (7) that

$$\gamma_{Y\cdot Z}(\tau) = T(L)\gamma_Z(\tau) \qquad \text{all } \tau \tag{8a}$$

$$\gamma_{yz}(\tau) = \gamma_{Y\cdot Z}(\tau) \qquad \text{all } \tau \tag{8b}$$

$$\gamma_z(\tau) = \gamma_Z(\tau) \qquad \tau \neq 0. \tag{8c}$$

Now in solving (8) for $\underline{a}_y$, $\underline{b}_y$ we meet exactly the same trouble as before if Z_t is a white noise. This point was not discussed by Söderström or Anderson and Deistler.

Note that if Z_t is serially correlated we can as before obtain $[\underline{a}_y,\underline{b}_y]$, $[\sigma^2_{\varepsilon Z},\gamma_Z(0)]$, $\underline{a}^{\bullet}_v$. From (7a) we then get $\gamma_{Y^{\bullet}}(\tau)$ for all τ and then from (7b) $\gamma_{v^{\bullet}}(\tau) = \gamma_y(\tau)-\gamma_{Y^{\bullet}}(\tau)$. Thus we have (cf Söderström (1980).

Result 2C. If Z_t is serially correlated then $[\underline{a}_y,\underline{b}_y]$, $[\sigma^2_{\varepsilon Z},\gamma_Z(0)]$, $[\underline{a}^{\bullet}_v,\underline{b}^{\bullet}_v,\sigma^2_{\nu v}]$ are identified i.e. all the parameters are identified.

Now Söderström's results are extended by proving

Result 2W. If Z_t is white noise then

$[\underline{a}_y,\underline{b}_y]$, $[\sigma^2_{\varepsilon Z},\gamma_Z(0)]$, $[\underline{a}^{\bullet}_v,\underline{b}^{\bullet}_v,\sigma^2_{\nu v}]$ are identified iff

$$p_y-q_y < p^{\bullet}_v-q^{\bullet}_v.$$

Since Z_t is a white noise we have as before that $\tilde{b}_{y\tau} = b_{y\tau}\gamma_Z(0)$ are identified. We are led then to write (7b) as

$$y'_t = Y'_t + v'_t \tag{9a}$$

$$\Rightarrow \quad \gamma'_t(\tau) = \gamma'_Y(\tau) + \gamma'_v(\tau) \tag{9b}$$

where

$$y'_t = A(L)y_t = A^{\bullet}_v(L)A_y(L)y_t \tag{10a}$$

$$Y'_t = \gamma_Z^{-1}(0)A^{\bullet}_v(L)\tilde{B}_y(L)Y_t \tag{10b}$$

$$v'_t = A_y(L)B^{\bullet}_v(L)\nu_{vt} \tag{10c}$$

so that the dashed quantities are MA processes. Now y_t is an ARMA process in this case (since Z_t is white) so that we can (as in the Appendix) identify A(L) and hence (since $A_y(L)$ is identified) $A_v^{\bullet}(L)$. Thus $\gamma_y'(\tau)$ can be found from $\gamma_y(\tau)$.

The idea is of course to use (9b) to get an equation for $\gamma_Z(0)$. For this we need at least one lag for which $\gamma_Y'(\tau) \neq 0$ yet $\gamma_v'(\tau) = 0$. Now (10b) shows us that $\gamma_Z(0)$ appears in (9b) for $\tau \leq p_{v\cdot} + q_y$. However from (10c) we see $\gamma_v'(\tau)$ vanishes for $\tau > p_y + q_v$. We clearly will have an equation for $\gamma_Z(0)$ iff

$$p_{v\cdot} + q_y > p_y + q_v.$$

Then since $\gamma_Z(0)$ is identified, so is $B_y(L)$ and hence via (7a) so is $\gamma_Y^{\bullet}(\tau)$. Thus $\gamma_v(\tau)$ is found from (7b). Finally $\sigma^2_{\varepsilon Z} = \gamma_z(0) - \gamma_Z(0)$. Result 2W is thus verified.

Remark. If Z_t is a white noise and $\gamma_Z(0)$ cannot be identified we still have valuable information. Indeed T(L) is known to within a scale factor $(\gamma_Z(0))$. Further we can bound $\gamma_Z(0)$ since it is $\leq \gamma_z(0)$. Of course independent knowledge of $\sigma^2_{\varepsilon Z}$ resolves the problem completely. In any case all the phase information (ie lagging) is available in T(L): this can of course be extremely useful.

Appendix. Parameterization of ARMA models by autocovariances

It is shown here how the parameters $\underline{a}$, $\underline{b}$, σ^2 of and ARMA (p,q) model determine and are determined by the r+1 = p+q+1 autocovariances $\gamma(0),\gamma(1),\ldots,\gamma(r)$. Obtaining the γ's from $\underline{a}$, $\underline{b}$, σ^2 is simply the question of generating the autocovariances of an ARMA process. The best algorithm for this is due to Hwang (1978) and independently Wilson (1979). Thus only the reverse transformation is considered here.

Consider the ARMA model $A(L)\omega_t = B(L)\nu_t$ and take cross covariances with $\omega_{t-\tau}$ for $\tau > q$ to see

$$A(L)\gamma_\tau = 0 \qquad \tau > q$$

and $\gamma_\tau \equiv \gamma_\omega(\tau)$. We can write this in matrix form as

$$\underline{H}_p(a_1 \ldots q_p)' = (\gamma_{q+1} \cdots \gamma_{r+1})'$$

where $\underline{H}_p$ is the Hankel matrix (i.e. one whose cross diagonal entries are equal)

$$\underline{H}_p = \begin{bmatrix} \gamma_q & \gamma_{q-1} & \cdots & \gamma_{q-p+1} \\ \gamma_{q-1} & & & \\ \vdots & & & \\ \gamma_{q-p+1} & \cdots\cdots & & \gamma_{q-2p+2} \end{bmatrix}.$$

From this set of equations $\underline{a}$ is clearly obtained. Actually p can also be found since it is defined by

p is the smallest index $\ni$ Rank $H_{p+i} = R \quad \forall i$

(see eg Solo, 1983a).

There are several ways to proceed now (cf Solo, 1983b) but a simple one is to reduce consideration to the MA case. Since by definition $u_t = A(L)\omega_t$ is a MA(q) process we can use $\gamma_0,\gamma_1,\ldots,\gamma_r$ to determine $\gamma_u(0)\ldots\gamma_u(q)$ from the equations.

$$\gamma_u(\tau) = \Sigma_0^p \Sigma_0^p a_i a_j \gamma_{i-j+\tau} \; ; \quad a_0 = 1.$$

Then from the (q+1) MA covariances $\gamma_u(\cdot)$ we can determine σ^2, $\underline{b}$ by spectral factorization i.e. say the iterative procedure of Wilson (1969).

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