A Generalization of Autocorrelation and Partial Autocorrelation Functions Useful for Identification of ARMA(p,q) Processes

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1. Introduction.

For identification of the order of moving average process ~~or~~ autoregressive process the autocorrelation function ρ_k and the partial autocorrelation function ρ'_k have been successfully used (see Box and Jenkins, 1976). For general mixed ARMA(p,q) processes neither the autocorrelation function ρ_k nor the partial autocorrelation function ρ'_k show a simple behavior which leads to a simple and useful procedure of determining the orders p and q.

In this article we propose a new definition of generalized autocorrelation function $\rho(k,\ell)$ $(k > 0,\ \ell > 0)$ with the following properties.

(P1) $\quad \rho(1,\ell) = \rho_\ell, \qquad \ell \geq 1,$

(P2) $\quad \rho(k,1) = \rho'_k, \qquad k \geq 1,$

(P3) $\quad -1 \leq \rho(k,\ell) \leq 1,$

(P4) $\quad$ For an ARMA(p,q) process

$$\rho(k,\ell) = 0 \quad \text{if} \quad k > p \text{ and } \ell > q.$$

The first property (P1) says that when $\rho(k,\ell)$'s are arranged in a two dimensional array, then the first row coincides with autocorrelation coefficients. Similarly (P2) says that the first column coincides with partial autocorrelation coefficients. $\rho(k,\ell)$ is in fact a correlation coefficient between certain random variables and this is reflected in (P3). (P4) is the main property of the proposed procedure that makes it useful for the identification of ARMA(p,q) processes.

Recently several generalizations of autocorrelation and partial autocorrelation functions have been proposed for the purpose of identification of ARMA(p,q) processes. They include R- and S-arrays of Gray, Kelley, and McIntire (1978); "GPACF" (Generalized Partial AutoCorrelation Function) discussed in Woodward and Gray (1981) Tiao and Box (1981), and Jenkins and Alavi (1981); Corner Method of Beguin, Gourieroux, and Monfort (1980); ESACF (Extended Sample AutoCorrelation Function) of Tsay and Tiao (1984); SCAN (Smallest CANonical correlation) of Tsay and Tiao (1983). The basic ideas underlying these procedures are quite similar. Some differences between these procedures are discussed in Tsay and Tiao (1983) and Tsay and Tiao (1984). ESACF and SCAN of Tsay and Tiao seem to be more sophisticated among these procedures. The major advantage of ESACF and SCAN is that they can handle nonstationarity (unit roots) in autoregressive part directly. The procedure proposed below is restricted to stationary ARMA processes. On the other hand the proposed procedure has the following attractive properties: a) it contains autocorrelation and partial autocorrelation coefficients as special cases, b) simple asymptotic theory generalizing the usual asymptotic theory for autocorrelation and partial autocorrelation coefficients, c) quick computation by recursive formulae generalizing the Durbin-Levinson recursive method (Durbin (1960), Levinson (1946)). Because of these properties, the proposed procedure seems to be a natural generalization of autocorrelation and partial autocorrelation functions. It is also an elaboration of ideas in Section 5 of Bartlett and Diananda (1950).

In Section 2 the definition of the generalized autocoreelation function is given and then the properties (P1) - (P4) are proved. In Section 3 the asymptotic distribution of the sample generalized autocorrelation function is studied. Theorem 3.2 unifies and generalizes the well known results on the asymptotic distributions of sample autocorrelation coefficients r_k $(k > q)$ for MA(q) processes and the sample partial autocorrelation coefficients r_k' $(k > p)$ for AR(p) processes. In Section 4 recursive formulae for computing $\rho(k,\ell)$ are given. This recursive procedure is straightforward generalization of the well known recursive procedure of obtaining partial autocorrelation coefficients (Durbin (1960), Levinson (1946)). In Section 5 some simulation results are given to illustrate the behavior of the proposed procedure.

Throughout this article we mean by ARMA(p,q) process the time series x_t generated by

$$\sum_{i=0}^{p} \beta_i x_{t-i} = \sum_{j=0}^{q} \alpha_j v_{t-j},$$

or

$$\beta(L)x_t = \alpha(L)v_t ,$$

where $\beta_0 = \alpha_0 = 1$, L is the lag operator, $\beta(L) = 1+\beta_1 L+\ldots+\beta_p L^p$, $\alpha(L) = 1+\alpha_1 L+\ldots+\alpha_q L^q$, and v_t is the white noise term with $Ev_t^2 = \sigma_v^2$. Furthermore for regularity we assume that the roots of the polynomial equation $\beta(L) = 0$ lie outside the unit circle and v_t's are independently and identically distributed. We will not repeat these assumptions later.

2. Definition and Some Properties of the Generalized Autocorrelation Function.

Let $\sigma(k)$ denote the autocovariance function of a second-order stationary sequence. For $k > 0$ let $\underset{\sim}{\sigma}(\ell,k) = (\sigma(\ell), \sigma(\ell+1),\ldots,\sigma(\ell+k-1))'$, and $\underset{\sim}{\sigma}(\ell,-k) = (\sigma(\ell),\sigma(\ell-1),\ldots,\sigma(\ell-k+1))'$. Let $\underset{\sim}{\Sigma}(\ell,k)$ denote a $k\times k$ (nonsymmetric) Toeplitz matrix with $\sigma(\ell)$ on the main diagonal, i.e.,

$$\underset{\sim}{\Sigma}(\ell,k) = \begin{pmatrix} \sigma(\ell) & \sigma(\ell+1) & \cdots & \sigma(\ell+k-1) \\ \sigma(\ell-1) & \sigma(\ell) & \cdots & \sigma(\ell+k-2) \\ \vdots & \vdots & & \vdots \\ \sigma(\ell-k+1) & \sigma(\ell-k+2) & \cdots & \sigma(\ell) \end{pmatrix}. \tag{2.1}$$

Now we give the definition of a generalized autocorrelation function.

Definition. Let $k \geq 0$, $\ell \geq 0$. A generalized autocorrelation function $\rho(\cdot,\cdot)$ is defined as

$$\begin{aligned} \rho(k+1,\ell+1) = {} & [\sigma(k+\ell+1) - \underset{\sim}{\sigma}(\ell+1,k)'\underset{\sim}{\Sigma}(\ell,k)^{-1}\underset{\sim}{\sigma}(k+\ell,-k)] \\ & /[\sigma(0)-2\underset{\sim}{\sigma}(1,k)'\underset{\sim}{\Sigma}(\ell,k)'^{-1}\underset{\sim}{\sigma}(\ell+1,k) \\ & +\underset{\sim}{\sigma}(\ell+1,k)'\underset{\sim}{\Sigma}(\ell,k)^{-1}\underset{\sim}{\Sigma}(0,k)\underset{\sim}{\Sigma}(\ell,k)'^{-1}\underset{\sim}{\sigma}(\ell+1,k)] \\ & \quad \text{if } |\underset{\sim}{\Sigma}(\ell,k)| \neq 0, \\ = {} & 0 \quad \text{if } |\underset{\sim}{\Sigma}(\ell,k)| = 0. \end{aligned} \tag{2.2}$$

When $k = 0$ terms involving $\underset{\sim}{\Sigma}(\ell,0)$ are taken to be zero since they are zero-dimensional. Hence $\rho(1,\ell+1)=\sigma(\ell+1)/\sigma(0)=\rho_{\ell+1}$. It is also easy to check that when $\ell=0$ then $\rho(k+1,1)$ coincides with the partial autocorrelation function ρ'_{k+1}. The properties (P1), (P2) are verified.

We now discuss the motivation for the above definition and then prove the properties (P3) and (P4). For a set of arbitrary numbers $b_1,\ldots,b_k$, ($b_0\equiv1$), let

$$Y = Y_t = \sum_{i=0}^{k} b_i x_{t-i} = b(L)x_t, \tag{2.3}$$

where L is the lag operator and $b(L) = 1+b_1L+\ldots+b_kL^k$. Let $P = L^{-1}$ be the forward shift operator and let

$$Z = Z_{t-\ell-k-1} = b(P)x_{t-\ell-k-1} = \sum_{i=0}^{k} b_i x_{t-\ell-k-1+i}. \tag{2.4}$$

Note that Z is defined in terms of the reversed process, x_{-t}. The time reversibility of the covariance structure of stationary sequence is essential in considering Z. Now consider the correlation coefficient between Y and Z: Cor(Y,Z). By the time reversibility of the process we have

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(Z) \\ &= \underset{\sim}{b}'\underset{\sim}{\Sigma}(0,k+1)\underset{\sim}{b} \\ &= \sigma(0)+2\underset{\sim}{b}'_{(1)}\underset{\sim}{\sigma}(1,k)+\underset{\sim}{b}'_{(1)}\underset{\sim}{\Sigma}(0,k)\underset{\sim}{b}_{(1)} \\ &= \sum_{i,j=0}^{k} b_i b_j \sigma(i-j), \end{aligned} \tag{2.5}$$

where $\underset{\sim}{b} = (1,b_1,\ldots,b_k)'$ and $\underset{\sim}{b}_{(1)} = (b_1,\ldots,b_k)'$. Note that Var(Y) can be written as $b(P)b(L)\sigma(0)$ where P, L are now operating on the argument i of $\sigma(i)$. (See the notation H_t in Bartlett and Diananda, 1950). The covariance between Y and Z is given by

$$\begin{aligned} \mathrm{Cov}(Y,Z) &= \underset{\sim}{b}'\underset{\sim}{\Sigma}(\ell+1,k+1)\tilde{\underset{\sim}{b}} \\ &= \sigma(\ell+k+1)+2\underset{\sim}{b}'_{(1)}\underset{\sim}{\sigma}(\ell+k,-k)+\underset{\sim}{b}'_{(1)}\underset{\sim}{\Sigma}(\ell,k)\tilde{\underset{\sim}{b}}_{(1)} \\ &= \sum_{i,j=0}^{k} b_i b_j \sigma(\ell+k+1-i-j) \\ &= b(L)^2\sigma(\ell+k+1), \end{aligned} \tag{2.6}$$

where $\tilde{\underset{\sim}{b}} = (b_k, b_{k-1}, \ldots, b_0)'$ and $\tilde{\underset{\sim}{b}}_{(1)} = (b_k, \ldots, b_1)'$. In general we will place a $\sim$ over a vector to denote the vector with its elements in the reversed order, for example, $\underset{\sim}{\sigma}(\ell+k,-k) = \tilde{\underset{\sim}{\sigma}}(\ell+1,k)$. Combining (2.5) and (2.6) we have Cor(Y,Z) = $b(L)^2\sigma(k+\ell+1)/b(L)b(P)\sigma(0)$.

Now we choose a particular set of $(b_1,\ldots,b_k)' = \underset{\sim}{b}_{(1)}$ defined by

$$\underset{\sim}{\Sigma}(\ell,k)'\underset{\sim}{b}_{(1)} = -\underset{\sim}{\sigma}(\ell+1,k), \tag{2.7}$$

or $\underset{\sim}{b}_{(1)} = -\underset{\sim}{\Sigma}(\ell,k)'^{-1}\underset{\sim}{\sigma}(\ell+1,k)$ provided that $\underset{\sim}{\Sigma}(\ell,k)$ is nonsingular. With this choice of $\underset{\sim}{b}_{(1)}$, Cor(Y,Z) reduces to the right hand side of (2.2) and this proves the property (P3). Note that (2.7) can also be expressed as

$$\underset{\sim}{\Sigma}(\ell,k)\tilde{\underset{\sim}{b}}_{(1)} = -\underset{\sim}{\sigma}(\ell+k,-k). \tag{2.8}$$

It remains to show (P4). Note that (2.7) is the set of k Yule-Walker equations for the following ARMA(k,ℓ) process:

$$Y_t = \beta(L)x_t = \alpha(L)v_t,$$

where $\beta(L) = 1 + \beta_1 L + \ldots + \beta_k L^k$, $\alpha(L) = 1 + \alpha_1 L + \ldots + \alpha_\ell L^\ell$, and v_t's are the white noise terms. If $\underset{\sim}{\Sigma}(\ell,k)$ is nonsingular then (2.7) gives the values of the parameters $\beta_1,\ldots,\beta_k$. Now the smallest time index in $\alpha(L)v_t$ is $t - \ell$ in $\alpha_\ell v_{t-\ell}$ and the largest time index in $Z_{t-\ell-k-1}$ is $t - \ell - 1$ in $\beta_k x_{t-\ell-1}$. Therefore $Y = Y_t$ and $Z = Z_{t-\ell-k-1}$ are uncorrelated and $\rho(k+1,\ell+1) = 0$. Note that if $k \geq p$, $\ell \geq q$ then an ARMA(p,q) process is a special case of ARMA(k,ℓ) processes having $\beta_{p+1} = \ldots = \beta_k = 0$ and $\alpha_{q+1} = \ldots = \alpha_\ell = 0$. This proves the property

(P4) with the provision that $\rho(k+1,\ell+1) = 0$ if $|\underset{\sim}{\Sigma}(\ell,k)| = 0$. Now all properties (P1)-(P4) have been verified.

We have mathematically verified all properties (P1)-(P4), but the motivation for defining $\rho(k+1,\ell+1) = 0$ when $|\underset{\sim}{\Sigma}(\ell,k)| = 0$ has to be discussed more carefully for further developments in the next section. An important question is whether $\underset{\sim}{\Sigma}(\ell,k)$ is nonsingular or singular when $k \geq p$, $\ell \geq q$ for an ARMA(p,q) process. For an $n\times n$ matrix $\underset{\sim}{A}$ let $N(\underset{\sim}{A})$ denote the null space of $\underset{\sim}{A}$ and let $\nu(\underset{\sim}{A})$ be the nullity of $\underset{\sim}{A}$, namely

$$\nu(\underset{\sim}{A}) = \dim(N(\underset{\sim}{A})) = n - \text{rank}\ (\underset{\sim}{A}).$$

By a nondegenerate ARMA(p,q) process we mean the process $\beta(L)x_t = \alpha(L)v_t$, where $\deg \beta(L) = p$, $\deg \alpha(L) = q$, $\beta_p \neq 0$, and $\alpha_q \neq 0$.

Proposition 2.1. Consider a nondegenerate ARMA(p,q) process. If $k \geq p$, $\ell \geq q$, then

$$\nu(\underset{\sim}{\Sigma}(\ell,k)) = \min(k-p,\ell-q). \tag{2.9}$$

Furthermore if $\nu = \nu(\underset{\sim}{\Sigma}(\ell,k)) > 0$, then a basis of $N(\underset{\sim}{\Sigma}(\ell,k)')$ is given by $\underset{\sim}{\eta}_1 = (\beta_0,\beta_1,\ldots,\beta_p,0,\ldots,0)'$, $\underset{\sim}{\eta}_2 = (0,\beta_0,\ldots,\beta_p,0,\ldots,0)',\ldots,$ $\underset{\sim}{\eta}_\nu = (0,\ldots,0,\beta_0,\ldots,\beta_p,0,\ldots,0)'$, where $\beta_0 \equiv 1$ and $\underset{\sim}{\eta}_\nu$ has first $\nu-1$ zero elements. Similarly a basis of $N(\underset{\sim}{\Sigma}(\ell,k))$ is given by $\underset{\sim}{\eta}_1,\ldots,\underset{\sim}{\eta}_\nu$ with their elements in the reversed order.

The proof of this proposition will be given in Appendix. Now consider (2.7). It has a specific solution $\underset{\sim}{\eta}^* = (\beta_1,\ldots,\beta_p,0,\ldots,0)'$. A general solution of (2.7) is then $\underset{\sim}{b}_{(1)} = (b_1,\ldots,b_k)' = \underset{\sim}{\eta}^* + \sum_{i=1}^{\nu} c_i\underset{\sim}{\eta}_i$ where $c_1,\ldots,c_\nu$

are arbitrary constants. With these b_i's consider Y in (2.3) and Z in (2.4). We claim that Cov(Y,Z) = 0 and Var(Y) = Var(Z) is bounded away from zero, so that Cor(Y,Z) = 0. By Proposition 2.1 we have $\underset{\sim}{\Sigma}(\ell,k)\tilde{\underset{\sim}{\eta}}_i = \underset{\sim}{\eta}_i'\underset{\sim}{\Sigma}(\ell,k) = \underset{\sim}{0}$, $i = 1,\ldots,\nu$. Also by (2.8) we have $\underset{\sim}{\eta}_i'\underset{\sim}{\sigma}(\ell+k,-k) = 0$, $i = 1,\ldots,\nu$. Using these results in (2.6) we obtain

$$\mathrm{Cov}(Y,Z) = \sum_{i,j=0}^{p} \beta_i\beta_j\sigma(\ell+k+1-i-j).$$

But this is zero by the same argument given above to prove the property (P4). Furthermore $Y = \sum_{i=0}^{k} b_i x_{t-i} = \beta(L)x_t + \sum_{i=1}^{\nu} c_i\beta(L)x_{t-i} = \alpha(L)v_t + \sum_{i=1}^{\nu} c_i\alpha(L)v_{t-i}$. Hence $\mathrm{Var}(Y) \geq \sigma_v^2$ for all $c_1,\ldots,c_\nu$. This proves the above claim. In terms of generalized inverses we have proved the following:

<u>Proposition 2.2.</u> For an ARMA(p,q) process the first expression of (2.2) is zero for all (k,ℓ) such that $k \geq p$ and $\ell \geq q$, if $\underset{\sim}{\Sigma}(\ell,k)^{-1}$ is taken as any generalized inverse.

This proposition shows that it is logically consistent to define $\rho(k+1,\ell+1) = 0$ if $|\underset{\sim}{\Sigma}(\ell,k)| = 0$. We will see in the next section that these considerations are essential for the discussion of sample generalized autocorrelation coefficients.

3. Asymptotic Distribution of Sample Generalized Autocorrelation Coefficients.

In this section we discuss distributional properties of sample generalized autocorrelation coefficients. In practical applications sample autocovariances, $\hat{\sigma}(\ell)$, $\ell = 0,1,\ldots$, are computed from observed time series. These sample autocovariances can now be substituted into (2.2). We call the resulting quantity sample generalized autocorrelation coefficient and denote it by $r(k+1,\ell+1)$. These sample generalized autocorrelation coefficients can be arranged in a two-dimensional table. If there is a pair (p,q) such that $r(k,\ell)$'s are close to zero for all pairs (k,ℓ) such that $k > p$ and $\ell > q$, then this indicates that the observed time series comes from an ARMA(p,q) process.

Clearly we need to assess the sample variability of $r(k,\ell)$'s for judging whether they are significantly different from zero or not. Suppose that an observed time series x_t is a nondegenerate ARMA(p,q) process. The following question arises: Does $r(k,\ell)$ tend to be small for (k,ℓ) where $k > p$ and $\ell > q$? The difficulty here is that as a sample quantity the matrix $\hat{\Sigma}(\ell,k)$ is nonsingular with probability one so that the first expression of (2.2) is always used although the population matrix $\Sigma(\ell,k)$ is singular. Using Proposition 2.1 we can show that $r(k,\ell)$ tends to be small indeed.

<u>Theorem 3.1</u>. Let $r(k+1,\ell+1)$ be the sample generalized autocorrelation coefficient obtained from an ARMA(p,q) process of length T. Then for $k \geq p$, $\ell \geq q$, $r(k+1,\ell+1) = O_p(T^{-1/2})$.

<u>Proof</u>. We first note that the denominator of $r(k+1,\ell+1)$ stays bounded away from zero in probability. This can be shown as follows. As in (2.5) the denominator is of the form $b'\hat{\Sigma}(0,k+1)b$ where the first element of b is equal to 1. Hence $||b|| \geq 1$ and this implies that the denominator is bounded from below by the smallest characteristic root of $\hat{\Sigma}(0,k+1)$. However the smallest characteristic root of $\hat{\Sigma}(0,k+1)$ converges in probability to the smallest characteristic root of $\Sigma(0,k+1)$ which is positive because $\Sigma(0,k+1)$ is positive definite. This shows that the denominator is bounded away from zero in probability. Therefore it suffices to show that the numerator is of the order $O_p(T^{-1/2})$. Note that

$$\begin{aligned}\text{Num} &= \hat{\sigma}(k+\ell+1)-\hat{\sigma}(\ell+1,k)'\hat{\Sigma}(\ell,k)^{-1}\hat{\sigma}(k+\ell,-k)\\ &= (-1)^k\det\hat{\Sigma}(\ell+1,k+1)/\det\hat{\Sigma}(\ell,k).\end{aligned}$$

Consider

$$\Pr\{|\mathrm{Num}|<cT^{-1/2}\}=\Pr\{T^{(\nu+1)/2}|\det\hat{\underset{\sim}{\Sigma}}(\ell+1,k+1)|\leq cT^{\nu/2}|\det\hat{\underset{\sim}{\Sigma}}(\ell,k)|\},$$

where $\nu = \min(k-p,\ell-q)$.

Let the singular value decomposition of $\underset{\sim}{\Sigma}(\ell+1,k+1)$ and $\underset{\sim}{\Sigma}(\ell,k)$ be

$$\underset{\sim}{\Sigma}(\ell+1,k+1) = \underset{\sim}{Q}_1\underset{\sim}{D}\underset{\sim}{Q}_2,$$
$$\underset{\sim}{\Sigma}(\ell,k) = \underset{\sim}{R}_1\underset{\sim}{E}\underset{\sim}{R}_2,$$

where $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$, $\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$ are orthogonal matrices and $\underset{\sim}{D}$ and $\underset{\sim}{E}$ are diagonal matrices with diagonal elements $d_{ii} = \delta_i$, $i=1,\ldots,k+1$, and $e_{ii} = \varepsilon_i$, $i=1,\ldots,k$, respectively. Here we specify

$$\delta_1 \geq \ldots \geq \delta_{k-\nu} > 0 = \delta_{k-\nu+1} = \ldots = \delta_k = \delta_{k+1},$$
$$\varepsilon_1 \geq \ldots \geq \varepsilon_{k-\nu} > 0 = \varepsilon_{k-\nu+1} = \ldots = \varepsilon_k.$$

Let $\hat{\underset{\sim}{D}} = (\hat{d}_{ij}) = \underset{\sim}{Q}_1' \hat{\underset{\sim}{\Sigma}}(\ell+1,k+1)\underset{\sim}{Q}_2'$ and $\hat{\underset{\sim}{E}} = (\hat{e}_{ij}) = \underset{\sim}{R}_1' \hat{\underset{\sim}{\Sigma}}(\ell,k)\underset{\sim}{R}_2'$. From the joint asymptotic normality of sample covariances (see Sections 8.3 and 8.4 of Anderson, 1971, for example) it follows that $\hat{d}_{ij} = d_{ij} + O_p(T^{-1/2})$, $\hat{e}_{ij} = e_{ij}+O_p(T^{-1/2})$. hence

$$|\det\hat{\underset{\sim}{\Sigma}}(\ell+1,k+1)| = |\det\hat{\underset{\sim}{D}}| = O_p(T^{-(\nu+1)/2}),$$
$$|\det\hat{\underset{\sim}{\Sigma}}(\ell,k)| = |\det\hat{\underset{\sim}{E}}| = O_p(T^{-\nu/2}).$$

Therefore

$$\lim_{c\to\infty} \lim_{T\to\infty} \Pr\{|\text{Num}| < cT^{-1/2}\} = 1.$$

This completes the proof. □

Note that when $k > p$ or $\ell > q$ the asymptotic distribution of the numerator of $r(k+1,\ell+1)$ involves a ratio of normal random variables. Thus we expect to see a Cauchy-like behavior for the numerator of $r(k+1,\ell+1)$.

When $k = p$ or $\ell = q$, $T^{1/2}r(k+1,\ell+1)$ has an asymptotic normal distribution with mean zero. The following result unifies and generalizes the well known results on autocorrelation coefficients of MA(q) processes and on partial autocorrelation coefficients of AR(p) processes.

<u>Theorem 3.2</u>. Consider a nondegenerate ARMA(p,q) process: $\beta(L)x_t = \alpha(L)v_t$. Let $Y_t = \beta(L)x_t = \alpha(L)v_t$ and let ρ_i^Y be the autocorrelation coefficients of Y_t. For (k,ℓ) such that $k=p$ or $\ell=q$, $T^{1/2}r(k+1,\ell+1)$ are asymptotically jointly normally distributed with mean zero. The asymptotic covariance between $T^{1/2}r(k+1,\ell+1)$ and $T^{1/2}r(k'+1,\ell'+1)$ is given by

$$\sum_{i=-q}^{q} \rho_i^Y \rho_{i-(k-k')-(\ell-\ell')}^Y. \tag{3.1}$$

For the proof the argument in Section 5.6 of Anderson (1971) can be generalized in a straightforward way:

<u>Proof</u>. The denominator of $r(k+1,\ell+1)$ converges in probability to $\text{VAR}(Y) = \text{VAR}(Z) = \sigma_v^2\sum_{j=0}^{q}\alpha_j^2$ for all k, ℓ such that $k = p$ or $\ell = q$. Hence we only need to consider the numerator.

Let

$$T^{-1/2}h_{k,\ell}=(1/T)\sum_{t=1}^{T}Y_tZ_{t-k-\ell-1}$$

$$=(1/T)\sum_{t=1}^{T}\beta(L)x_t\beta(P)x_{t-k-\ell-1}$$

$$=(1/T)\sum_{t=1}^{T}\alpha(L)v_t\beta(P)x_{t-k-\ell-1}.$$

The right hand side is the same as the numerator of $r(k+1,\ell+1)$ except for the adjustment of endpoints and that $\hat\beta_1,\ldots,\hat\beta_p(,\ldots,\hat\beta_k)$ are replaced by $\beta_1,\ldots,\beta_p$ $(,0,\ldots,0)$. $h_{k,\ell}$ corresponds to h_j in formula (27), Section 5.6 of Anderson (1971). With the same argument as in Anderson (1971) $h_{k,\ell}$'s can be shown to have an asymptotic joint normal distribution with mean zero. Their asymptotic covariances are given by $\lim_T E(h_{k,\ell}h_{k',\ell'})$. We evaluate this expected value first. Consider

$$E(Y_tZ_{t-k-\ell-1}Y_sZ_{s-k'-\ell'-1})$$

$$=E\{\alpha(L)v_t\beta(P)x_{t-k-\ell-1}\ \alpha(L)v_s\beta(P)x_{s-k'-\ell'-1}\}.$$

Note that the only nonzero contribution to the expectation comes from the term where the same v_{t-i}'s are taken from both $\alpha(L)v_t$ and $\alpha(L)v_s$. This term contributes

$$\alpha_i\alpha_{i-t+s}E(v_{t-i}^2)E(\beta(P)x_{t-k-\ell-1}\beta(P)x_{s-k'-\ell'-1})$$

to the expectation. Let $\gamma_i^Y = E(Y_tY_{t-i}) = \sigma_v^2\sum_{j=0}^{q-|i|}\alpha_j\alpha_{j+|i|}$ denote the autocorvariances of Y_t. By the time reversibility of the process the second expectation is the same as $\gamma^Y_{t-k-\ell-s+k'+\ell'} = E(\beta(L)x_{t-k-\ell}\beta(L)x_{s-k'-\ell'})$.

Now summing the terms involving v_{t-i}'s we obtain

$$E(Y_t Z_{t-k-\ell-1}\, Y_s Z_{s-k'-\ell'-1}) = \gamma^Y_{t-s}\gamma^Y_{t-k-\ell-s+k'+\ell'} .$$

Hence

$$E(h_{k,\ell}h_{k',\ell'}) = (1/T)\sum_{t,s=1}^{T} \gamma^Y_{t-s}\gamma^Y_{t-k-\ell-s+k'+\ell'}$$

$$\to \sum_i \gamma^Y_i\gamma^Y_{i-(k-k')-(\ell-\ell')} .$$

Note that the last summation is only finite. Now using $VAR(Y_t) = VAR(Z_t) = \gamma^Y_0$ and $\rho^Y_i = \gamma^Y_i/\gamma^Y_0$ we have

$$\lim_T E(h_{k,\ell}h_{k',\ell'})/[VAR(Y)VAR(Z)] = \sum_i \rho^Y_i\rho^Y_{i-(k-k')-(\ell-\ell')}$$

and this gives the asymptotic covariance.

It remains to verify that the difference between $h_{k,\ell}$ and $T^{1/2}$ × (numerator of $r(k+1,\ell+1)$ converges to zero in probability. Let $\hat{h}_{k,\ell}$ denote $T^{-1/2}$ times the numerator of $r(k+1,\ell+1)$. Then (except for the adjustments of end points) we have

$$\hat{h}_{k,\ell}-h_{k,\ell} = T^{1/2}\sum_{i,j=0}^{k}(\hat{\beta}_i\hat{\beta}_j-\beta_i\beta_j)\frac{1}{T}\sum_{t=1}^{T}x_{t-i}x_{t-k-\ell-1+j}$$

$$= \sum_{i,j=0}^{k} T^{1/2}(\hat{\beta}_i-\beta_i)\hat{\beta}_j\frac{1}{T}\, a_{i,k+\ell+1-j} \qquad (3.2)$$

$$+ \sum_{i,j=0}^{k} \beta_i T^{1/2}(\hat{\beta}_j-\beta_j)\frac{1}{T}\, a_{i,k+\ell+1-j},$$

where $a_{i,j} = \sum_{t=1}^{T} x_{t-i}x_{t-j}$. Now $\hat{\beta}_j \to \beta_j$ and $(1/T)a_{i,k+\ell+1-j} \to \sigma(k+\ell+1-j-i)$ in probability. Considering $i = k$, $j = p$ in the first term and $i = p$, $j = k$ in the second term on the extreme right hand side and noting $k+\ell+1-(k+p) = \ell+1-p \geq q+1-p$ we see that indices fall in the range where the Yule-Walker equations hold. Hence we have

$$\sum_j \hat{\beta}_j \frac{1}{T} a_{i,k+\ell+1-j} \to 0, \quad i = 1,\ldots,k,$$

$$\sum_i \beta_i \frac{1}{T} a_{i,k+\ell+1-j} \to 0, \quad j = 1,\ldots,k,$$

in probability. Therefore the two terms on the extreme right hand side of (3.2) converge to zero in probability and this completes the proof. □

For a nondegenerate ARMA(p,q) process $\gamma_i^Y = \mathrm{Cov}(Y_t, Y_{t-i})$ can be consistently estimated by

$$\begin{aligned}\hat{\gamma}_i^Y &= \hat{\underset{\sim}{\beta}}'\hat{\underset{\sim}{\Sigma}}(i,p+1)\hat{\underset{\sim}{\beta}} \\ &= \hat{\sigma}(i) - \hat{\underset{\sim}{\sigma}}(q+1,p)'\hat{\underset{\sim}{\Sigma}}(q,p)^{-1}\hat{\underset{\sim}{\sigma}}(i-1,-p) \\ &\quad - \hat{\underset{\sim}{\sigma}}(i+1,p)'\hat{\underset{\sim}{\Sigma}}(q,p)'^{-1}\hat{\underset{\sim}{\sigma}}(q+1,p) \\ &\quad + \hat{\underset{\sim}{\sigma}}(q+1,p)'\hat{\underset{\sim}{\Sigma}}(q,p)^{-1}\hat{\underset{\sim}{\Sigma}}(i,p)\hat{\underset{\sim}{\Sigma}}(q,p)'^{-1}\hat{\underset{\sim}{\sigma}}(q+1,p).\end{aligned} \tag{3.3}$$

Let $r_i^Y = \hat{\gamma}_i^Y/\hat{\gamma}_0^Y$, $i = 1,\ldots,q$. Then as a corollary to Theorem 3.2 we have

<u>Corollary 3.1</u>. Suppose that x_t is a nondegenerate ARMA(p,q) process. Then $T^{1/2}r(p+1,q+1)/[1+2\sum_{i=1}^{q}(r_i^Y)^2]^{1/2}$ has an asymptotic standard normal distribution.

Corollary 3.1 can be conveniently used as a test statistic for the null hypothesis that an observed process is a nondegenerate ARMA(p,q) process.

4. Recursive Formulae.

One advantage of our generalized autocorrelation function is that it can be quickly computed by recursive formulae presented below. It is a straightforward generalization of the well known recursive procedure for partial autocorrelation coefficients (Durbin (1960), Levinson (1946)). Starting from $(q+1)$-st autocorrelation coefficient $\rho_{q+1}=\rho(1,q+1)$, the formula gives a recursive relation for obtaining $\rho(p+1,q+1)$ for successive values of p. For q=0 it reduces to Durbin-Levinson procedure.

In this section the distinction between population quantities and sample quantities is not important. Hence we omit $\wedge$ although the recursive formulae are used with sample quantities in practice. Also we use p,q as general running indices rather than true orders of AR and MA parts.

Let $q \geq 0$ be fixed in the following discussion. We define p dimensional vectors $\underset{\sim}{b}(p)$, $\underset{\sim}{c}(p)$, $\underset{\sim}{d}(i;p)$, $\underset{\sim}{e}(i;p)$ $(i=0,\ldots,q)$ as follows:

<u>Definition</u>:

$$\begin{aligned}
\underset{\sim}{\Sigma}(q,p)'\underset{\sim}{b}(p) &= -\underset{\sim}{\sigma}(q+1,p) \\
\underset{\sim}{\Sigma}(q,p)'\underset{\sim}{c}(p) &= -\underset{\sim}{\sigma}(q-p,p) \\
\underset{\sim}{d}(i;p) &= \underset{\sim}{\Sigma}(i,p)\underset{\sim}{b}(p), \quad i = 0,\ldots,q, \\
\underset{\sim}{e}(i;p) &= \underset{\sim}{\Sigma}(i,p)\underset{\sim}{c}(p), \quad i = 0,\ldots,q.
\end{aligned} \tag{4.1}$$

Using these definitions, γ_i in (3.3) can be expressed as

$$\begin{aligned}
\gamma_i = {} & \sigma(i)+\underset{\sim}{b}(p)'\underset{\sim}{\sigma}(i-1,-p)+\underset{\sim}{\sigma}(i+1,p)'b(p) \\
& + \underset{\sim}{d}(i;p)'\underset{\sim}{b}(p), \quad i = 0,\ldots,q.
\end{aligned}$$

Then from (2.2) we have

$$\rho(p+1,q+1) = [\sigma(p+q+1)+\underset{\sim}{\sigma}(p+q,-p)'\underset{\sim}{b}(p)]/\gamma_0 \ .$$

Now we give the initial condition and updating formulae for $\underset{\sim}{b}(p)$, $\underset{\sim}{c}(p)$, $\underset{\sim}{d}(i;p)$, $\underset{\sim}{e}(i;p)$.

<u>Initial condition</u>.

$$\underset{\sim}{0} = \underset{\sim}{b}(0) = \underset{\sim}{c}(0) = \underset{\sim}{d}(i;0) = \underset{\sim}{e}(i;0),$$
$$i = 0,\ldots,q.$$

Actually these are zero-dimensional and the condition is trivial.

<u>Updating</u>. Let

$$\underset{\sim}{b}(p+1) = \begin{pmatrix} \underset{\sim}{b}^{(1)}(p+1) \\ b_{p+1}(p+1) \end{pmatrix}, \qquad \underset{\sim}{d}(i;p+1) = \begin{pmatrix} \underset{\sim}{d}^{(1)}(i;p+1) \\ d_{p+1}(i;p+1) \end{pmatrix}, \quad i = 0,\ldots,q,$$

where $b_{p+1}(p+1)$, $d_{p+1}(i;p+1)$ are scalars and $\underset{\sim}{b}^{(1)}(p+1)$, $\underset{\sim}{d}^{(1)}(i;p+1)$ are p dimensional vectors. Let

$$\underset{\sim}{c}(p+1) = \begin{pmatrix} c_1(p+1) \\ \underset{\sim}{c}^{(2)}(p+1) \end{pmatrix}, \qquad \underset{\sim}{e}(i;p+1) = \begin{pmatrix} e_1(i;p+1) \\ \underset{\sim}{e}^{(2)}(i;p+1) \end{pmatrix}, \quad i = 0,\ldots,q,$$

where $c_1(p+1)$, $e_1(i;p+1)$ are scalars and $\underset{\sim}{c}^{(2)}(p+1)$ and $\underset{\sim}{e}^{(2)}(i;p+1)$ are p dimensional vectors. Then

$$b_{p+1}(p+1) = -\frac{\sigma(q+p+1) + \underset{\sim}{\sigma}(q+p,-p)'\underset{\sim}{b}(p)}{\sigma(q) + \underset{\sim}{\sigma}(q+p,-p)'\underset{\sim}{c}(p)} \ ,$$

$$\underset{\sim}{b}^{(1)}(p+1) = \underset{\sim}{b}(p) + b_{p+1}(p+1)\underset{\sim}{c}(p) \ ,$$

$$c_1(p+1) = -\frac{\sigma(q-p-1) + \underset{\sim}{g}(q-1,-p)'\underset{\sim}{c}(p)}{\sigma(q) + \underset{\sim}{g}(q-1,-p)'\underset{\sim}{b}(p)},$$

$$\underset{\sim}{c}^{(2)}(p+1) = \underset{\sim}{c}(p) + c_1(p+1)\underset{\sim}{b}(p),$$

$$d_{p+1}(i'p+1) = b_{p+1}(p+1)\sigma(i) + \underset{\sim}{g}(i-p,p)'\underset{\sim}{b}^{(1)}(p+1),$$

$$\underset{\sim}{d}^{(1)}(i;p+1) = \underset{\sim}{d}(i;p) + b_{p+1}(p+1)\,[\underset{\sim}{e}(i;p) + \underset{\sim}{g}(i+p,-p)],$$

$$e_1(i;p+1) = c_1(p+1)\sigma(i) + \underset{\sim}{g}(i+1,p)'\,\underset{\sim}{c}^{(2)}(p+1),$$

$$\underset{\sim}{e}^{(2)}(i;p+1) = \underset{\sim}{e}(i;p) + c_1(p+1)\,[\underset{\sim}{d}(i;p) + \underset{\sim}{g}(i-1,-p)].$$

These formulae can be readily verified by writing (4.1) for p+1 in appropriately partitioned form. When $q = 0$ $\underset{\sim}{c}(p)$ equals $\underset{\sim}{b}(p)$ with its elements in the reversed order. $\underset{\sim}{d}(0;p)$ reduces to $-\underset{\sim}{g}(1,p)$ and $\underset{\sim}{e}(0;p)$ reduces to $-\tilde{\underset{\sim}{g}}(1,p)$. Hence these need not be calculated separately. However for general $q > 0$, these quantities form essential working values and have to be stored in memory for each updating step.

5. Simulation Results.

Here we present some simulation results. An ARMA(1,1) process ($\beta(L) = 1-.5L$, $\alpha(L) = 1+L$) and ARMA (3,3) process ($\beta(L) = (1-.5L)(1+.5L^2)$, $\alpha(L) = (1+.5L)(1+L^2)$) of length 200 were generated 100 times. For each pair (p,q) the null hypothesis H_0: $\rho(p+1, q+1) = 0$ was tested using Corollary 3.1 with asymptotic significance level .1. The entries of the following tables show how many times the null hypothesis was rejected out of 100 trials. In Table 1 the asymptotic expected number of (2,2) element is 10 (in comparison to 8 which was actually observed). In Table 2 the same holds for (4,4) element. The pattern implied by the property (P4) of $\rho(k,\ell)$ in Section 1 can be clearly seen in these tables.

Table 1

$$x_t - .5x_{t-1} = v_t + v_{t-1}$$

number of rejections out of 100 trials

p\q							
100	100	49	19	13	12	10	9
100	8	8	3	5	6	4	7
100	6	6	3	2	2	0	0
99	5	4	3	2	0	1	1
80	12	2	0	0	1	1	0
65	6	4	0	1	0	1	1
50	9	2	1	1	1	0	0
37	3	2	2	0	0	0	0

Table 2

$$x_t - .5x_{t-1} + .5x_{t-2} - .25x_{t-3} = v_t + .5v_{t-1} + v_{t-2} + .5v_{t-3}$$

number of rejections out of 100 trials

p\q							
100	100	58	12	11	9	6	8
99	87	37	8	13	11	5	7
36	10	65	7	14	2	2	0
11	54	0	8	0	0	3	1
79	3	1	2	0	1	1	0
26	41	0	4	0	0	0	0
52	3	5	1	1	1	0	1
11	34	2	4	1	0	0	0

Appendix.

Here we give a proof of Proposition 2.1. We consider several cases.

Case 1: $k = p$.

For the case $k = p$, $\ell = q$, Anderson (1971, p.240) shows that $\underset{\sim}{\Sigma}(q,p)'$ is nonsingular. For the case $\ell > q$, Anderson's argument can be modified in a straightforward way to show that $\underset{\sim}{\Sigma}(\ell,p)'$ is nonsingular. The essential condition for the proof of this case is that $\beta_p \neq 0$.

Case 2: $k > p, \ell = q$.

We shall show that $\underset{\sim}{\Sigma}(q,k)'$ is nonsingular in this case. The essential condition for this case will be the asumption $\alpha_q \neq 0$. We argue by induction. First we show that $\underset{\sim}{\Sigma}(q,p+1)'$ is nonsingular. Consider $\underset{\sim}{\Sigma}(q,p+1)'\underset{\sim}{\xi} = \underset{\sim}{0}$. We want to show that this implies $\underset{\sim}{\xi} = \underset{\sim}{0}$. In the partitioned form this can be written as

$$\begin{pmatrix} \sigma(q) & \underset{\sim}{\sigma}(q-1,-p)' \\ \underset{\sim}{\sigma}(q+1,p) & \underset{\sim}{\Sigma}(q,p)' \end{pmatrix} \begin{pmatrix} \xi_1 \\ \underset{\sim}{\xi}_{(2)} \end{pmatrix} = \underset{\sim}{0}, \qquad \text{(A1)}$$

where ξ_1 is a scalar and $\underset{\sim}{\xi}_{(2)}$ is a p-dimensional column vector. From the second set of equations we have

$$\underset{\sim}{\xi}_{(2)} = -\xi_1 \underset{\sim}{\Sigma}(q,p)'^{-1} \underset{\sim}{\sigma}(q+1,p) = \xi_1 \underset{\sim}{\beta}_{(1)}, \qquad \text{(A2)}$$

where $\underset{\sim}{\beta}_{(1)} = (\beta_1, \ldots, \beta_p)'$. Then the first equation recudes to

$$0 = \xi_1\left(\sum_{i=0}^{p} \sigma(q-i)\beta_i\right).$$

But

$$\begin{aligned}\sum_{i=0}^{p} \sigma(q-i)\beta_i &= E\{[\beta(L)x_t]x_{t-q}\}\\ &= E\{[\alpha(L)v_t]x_{t-q}\}\\ &= \alpha_q E(v_{t-q}x_{t-q}) = \alpha_q\sigma_v^2.\end{aligned}$$

Hence $0 = \xi_1\alpha_q\sigma_v^2$. This implies $\xi_1 = 0$ and consequently $\underset{\sim}{\xi}_{(2)} = \underset{\sim}{0}$ and $\underset{\sim}{\xi} = \underset{\sim}{0}$. This proves that $\underset{\sim}{\Sigma}(q,p+1)'$ is nonsingular.

For induction suppose that $\underset{\sim}{\Sigma}(q,j)'$ is nonsingular. We want to show that $\underset{\sim}{\Sigma}(q,j+1)'$ is nonsingular as well. The argument is the same as for the case $j = p$. The differences are that p is replaced by j in (A1) and $\underset{\sim}{\beta}_{(1)}$ is replaced by $(\beta_1,\ldots,\beta_p,0,\ldots,)'$ in (A2).

This proves that $|\underset{\sim}{\Sigma}(q,k)'| \neq 0$ for all $k > p$.

Case 3. $k > p$, $\ell > q$, and $k - p \leq \ell - q$.

We shall show that $\nu(\underset{\sim}{\Sigma}(\ell,k)') = k-p$ and $\underset{\sim}{\eta}_1,\ldots,\underset{\sim}{\eta}_{k-p}$ given in Proposition 2.1 form a basis of $N(\underset{\sim}{\Sigma}(\ell,k)')$.

The upper left $p\times p$ corner of $\underset{\sim}{\Sigma}(\ell,k)'$ is $\underset{\sim}{\Sigma}(\ell,p)'$. As we have already shown $\underset{\sim}{\Sigma}(\ell,p)'$ is nonsingular. Therefore rank $(\underset{\sim}{\Sigma}(\ell,k)') \geq p$ or $\nu(\underset{\sim}{\Sigma}(\ell,k)') \leq k-p$. Now we want to show that $\nu(\underset{\sim}{\Sigma}(\ell,k)') \geq k-p$. In order to show this it suffices to check that $\underset{\sim}{\eta}_1,\ldots,\underset{\sim}{\eta}_{k-p}$ given in Proposition 2.1 belong to $N(\underset{\sim}{\Sigma}(\ell,k)')$ because they are clearly linearly independent and then $\nu(\underset{\sim}{\Sigma}(\ell,k)') = \dim N(\underset{\sim}{\Sigma}(\ell,k)') \geq k-p$. Note that in this case $\underset{\sim}{\eta}_{k-p} = (0,\ldots,0,\beta_0,\ldots,\beta_p)$ and the element at the upper right corner of $\underset{\sim}{\Sigma}(\ell,k)'$ is $\sigma(\ell-k+1)$. Now $k-p \leq \ell - q$ implies $\ell-k+1 \geq q+1-p$.

This implies that indices fall in the range where the Yule-Walker equations hold. Then $\Sigma(\ell,k)'\eta_i$, $i = 1,\ldots,k-p$ simply reduce to the Yule-Walker equations and therefore $0 = \Sigma(\ell,k)'\eta_i$, $i=1,\ldots,k-p$. This proves that $\nu(\Sigma(\ell,k)') = k-p$. It has been also shown that $\eta_1,\ldots,\eta_{k-p}$ form a basis of $N(\Sigma(\ell,k)')$.

Case 4. $k > p$, $\ell > q$, and $k - p > \ell - q$.

Let $r = q-\ell+k$. Then $r > p$. The element at the upper right corner of $\Sigma(\ell,k)'$ is $\sigma(\ell-k+1)$. Now $(\ell-k+1)+r-1=q$. We see that the upper right $r\times r$ corner of $\Sigma(\ell,k)'$ is $\Sigma(q,r)'$. We have already shown that $\Sigma(q,r)'$ is nonsingular. Therefore rank $(\Sigma(\ell,k)') \geq$ rank $(\Sigma(q,r)') = r$ or $\nu(\Sigma(\ell,k)') \leq \ell-q$. Now we want to show that $\nu(\Sigma(\ell,k)') \geq \ell-q$. As in the previous case it suffices to show that $\eta_1,\ldots,\eta_{\ell-q}$ belong to $N(\Sigma(\ell,k)')$. In this case $\eta_{\ell-q}$ has additional $k-(\nu-1)-(p+1) = k-p-(\ell-q)$ zeros at the end and the last nonzero element of $\eta_{\ell-q}$ is the $(\ell-q+p)$-th $(:\ell-q+p=\nu-1+p+1)$ element. Therefore the last nonzero term in $g(\ell,-k)'\eta_{\ell-q}$ is $\beta_p\sigma(\ell-(\ell-q+p)+1) = \beta_p\sigma(q+1-p)$. We see that again the indices fall in the range where the Yule Walker equations hold and we obtain $0 = \Sigma(\ell,k)'\eta_i$, $i = 1,\ldots,\ell-q$. This implies that $\nu(\Sigma(\ell,k)') = \ell-q$ and $\eta_1,\ldots,\eta_{\ell-q}$ form a basis of $N(\Sigma(\ell,k)')$.

All cases have been examined and this completes the proof. $\square$

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A new definition of generalized autocorrelation function is proposed. It has properties useful for identification of ARMA processes. The advantages of the proposed procedure include simple asymptotic theory and quick recursive computation. These are natural generalizations of corresponding results for autocorrelation and partial autocorrelation functions.

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