AN INCREASING DIFFUSION

by

Thomas S. Salisbury
Purdue University

Technical Report #84-18

Department of Statistics
Purdue University

June 1984

An Increasing Diffusion

by Thomas S. Salisbury

§1. Introduction

In [2], E. Çinlar and J. Jacod consider, among other things, the problem of whether every continuous strong Markov process of bounded variation is deterministic (a problem apparently also posed by S. Orey). They show that this question is equivalent to that of whether every strong Markov process satisfying an ODE $X_t' = F(X_t)$ is deterministic. At the time of writing [2], they thought they had a proof that this was indeed the case. They later found an error in this proof, but subsequently established the result in the case that (X_t) is one-dimensional. More formally, they can show the following

(1.1) Theorem. Let (X_t) be a real valued (time homogeneous) continuous Hunt process of bounded variation. Then X_t is a.s. a deterministic function of X_0.

We will show that this result is false in dimensions bigger than one. In fact, we will produce a non time homogeneous real valued continuous Hunt process that is not deterministic. It will arise as a deterministic function of a space-time version of a time changed Brownian motion.

I would like to thank Erhan Çinlar and Jean Jacod for advertising their problem. I would also like to thank Burgess Davis for many helpful conversations and invaluable suggestions.

§2. The Construction

Let $\Omega = [0,\infty) \times C([0,\infty),\mathbb{R})$. The canonical realization of a continuous space-time stochastic process is

$$(\tau_t, B_t)(s,\omega) = (s+t, \omega(t)).$$

Let $\mathcal{F}_t = \sigma(\tau_0, B_s;\ s \leq t)$, and set $\mathcal{F} = \mathcal{F}_\infty$. Let $P^{s,x}$ be the law on $(\Omega,\mathcal{F})$, of space-time Brownian motion started at (s,x). As usual, "a.s." means $P^{s,x}$-a.s., for every s,x.

A set of the form

$$[u,v] \times [a,b] \subset [0,\infty) \times \mathbb{R}$$

will be called a <u>box</u>. For each union G of finitely many disjoint boxes $[u_i,v_i] \times [a_i,b_i]$, we will define a process M_t^G as follows: It behaves like a Brownian motion till the first hit of G. If this occurs in the i'th box, we 'hold' M_t^G until $\tau_t = v_i$, and then resume Brownian behaviour until the next hit of G, etc... (see Figure (2.4)).

More formally, let $S^G(0) = A^G(0) = 0$, and define $S(n) = S^G(n)$ and $A(n) = A^G(n)$ inductively as follows:

$$S(n+1) = S(n) + \inf\{t > 0;\ (\tau_0 + A(S(n)) + t, B_{S(n)+t}) \in G\}$$

$$A(t) = A(S(n)) + t - S(n), \quad \text{for} \quad t \in (S(n), S(n+1))$$

$$A(S(n+1)) = v_i, \text{ if } (\tau_0 + A(S(n+1)-), B_{S(n+1)}) \in [u_i,v_i] \times [a_i,b_i].$$

Let $\hat{A}^G(t)$ be the (continuous) inverse of $A^G(t)$, and define

$$M^G(t) = B_{\hat{A}^G(t)}.$$

We will let G become dense in an appropriate manner, and show that these processes converge.

Let

$\Gamma(t,y,\rho) = \sup\{m;$ there are disjoint open subintervals $I(1)..I(m)$ of $[0,t]$, each of length $\geq \rho$, such that for each i there is an $s \in I(i)$ with $|B_s - y| \leq \rho\}$

$\Lambda(t,y,\zeta,\rho) =$ the number of upcrossings of $[y+p, y+\zeta]$ and downcrossings of $[y-\zeta, y-\rho]$ by $(B_s)_{s\in[0,t]}$.

(2.1) Lemma.

(a) For every $t \geq 0$, $E^{s,x}[\rho\Gamma(t,y,\rho)] \to 0$ as $\rho \downarrow 0$, uniformly in s, x and y.

(b) For every $t \geq 0$ and $\zeta > 0$, $E^{s,x}[\Lambda(t,y,\zeta,p)]$ remains bounded uniformly in s, x and y, as $p \downarrow 0$.

Proof: (b) follows from Doob's up and downcrossing bounds, via the translational invariance of Brownian motion, and the strong Markov property at $\inf\{s;\ B_s = y \pm \rho\}$. Similarly, by the strong Markov property at $\inf\{s;\ B_s = y \pm \rho\}$, we have that

$$E^{s,x}[\rho\Gamma(t,y,\rho)]$$

$$\leq \sup_{|z|\leq\rho} E^{0,z}[2\rho + \inf\{s;\ B_s = \pm\rho\}] + E^{0,0}[\rho\Gamma(t,\rho,\rho)]$$

$$\leq 2p + p^2 + E^{0,0}[\text{Lebesgue measure of a } \rho\text{-neighbourhood of } \{s \leq t;\ B_s \in [0,2\rho]\}].$$

As $\rho \downarrow 0$, the integrand decreases boundedly to the Lebesgue measure of $\{s \leq t;\ B_s = 0\}$, which is zero. □

If ϕ is nondecreasing and right continuous (for our purposes, actually continuous), denote its right continuous inverse by $\hat{\phi}$ (that is, $\hat{\phi}(t) = \inf\{s \geq 0;\ \phi(s) > t\}$). If G is the union of finitely many disjoint boxes, we let

$$A_\phi^G(t)(s,\omega) = A^G(t)(s,\omega\circ\phi),$$

and define $\hat{A}_\phi^G$ and S_ϕ^G similarly.

Let Ψ consist of all nondecreasing, continuous ϕ such that $\phi(0) = 0$ and $\hat{\phi}(t) - t$ is nondecreasing.

(2.2) Lemma. For $\phi \in \Psi$, and G as above,

$$\hat{A}^G(t) \geq \phi(\hat{A}_\phi^G(t)) \quad \text{for every } t \geq 0.$$

Proof: Write $A(t)$ and $A'(t)$ for $A^G(t)$ and $A_\phi^G(\hat{\phi}(t))$ respectively. Since ϕ is continuous, the inverse of A' is $\phi\circ\hat{A}_\phi^G$, so that it will suffice to show that $A(t) \leq A'(t)$ for every t. Let $t(n) = S^G(n)$ for $n \geq 0$. We will show by induction that $A(t) \leq A'(t)$ for every $t \in [t(n),t(n+1))$.

Note first that on any interval (s,r) not containing any time $t(n)$, we have that

$$\begin{aligned}A'(t) - A(t) &\geq (A'(s) + \hat{\phi}(t) - \hat{\phi}(s)) - (A(s) + t - s)\\ &\geq (A'(s) + t - s) - (A(s) + t - s)\\ &= A'(s) - A(s).\end{aligned}$$

This starts the induction off, as $\phi(0) = 0$ implies that $A(0) = A_\phi^G(0) \leq A'(0)$. Similarly, assuming that $A'(t) \geq A(t)$ on $[t(n), t(n+1))$, we need only show that $A'(t(n+1)) \geq A(t(n+1))$. If $A'(t(n+1)-) \geq A(t(n+1))$, there is nothing to show. Thus assume that $A'(t(n+1)-) < A(t(n+1))$, and let $[u,v] \times I$ be the box of G to which $(\tau_0 + A(t(n+1)-), B_{t(n+1)})$ belongs. Then

$$u \leq \tau_0 + A(t(n+1)-) \leq \tau_0 + A'(t(n+1)-) \quad \text{(by induction)}$$

$$< \tau_0 + A(t(n+1)) = v.$$

But in this case,

$$\tau_0 + A'(t(n+1)) \geq \tau_0 + A_\phi^G(\hat{\phi}(t(n+1)-)) = v = \tau_0 + A(t(n+1)),$$

showing the result. □

Now let $\Phi(\delta)$ consist of all nondecreasing continuous functions $\phi\colon [0,\infty) \to [0,\infty)$ such that $|\phi(t)-t| < \delta$ for every $t \geq 0$. In the following lemma, it is more or less clear that some δ will work. Specifying that δ and then verifying the result is a tedious task however, and the proof has been relegated to the next section; readers are advised to omit it!

(2.3) Lemma. For G as above, and $\varepsilon > 0$

$$P^{s,x}(\sup\{|\hat{A}^G(t) - \hat{A}_\phi^G(t)|;\ t \geq 0,\ \phi \in \Phi(\delta)\} \geq \varepsilon) \to 0$$

as $\delta \downarrow 0$, uniformly in s, x.

Proof: See §3.

Now let $\Sigma\varepsilon(n) < \infty$. Set $G(0) = \phi$, $\xi(0) = 1$, and define $G(n)$ and $H(n)$ inductively as follows;
Use Lemma (2.3) to find $\xi(n) \in (0,\xi(n-1)/2)$ so that

$$\sup_{s,x} P^{s,x}(\sup\{|\hat{A}^{G(n-1)}(t) - \hat{A}_{\phi}^{G(n-1)}(t)|;\ t \geq 0,\ \phi \in \Phi(4\xi(n))\} \geq \varepsilon(n))$$
$$< \varepsilon(n).$$

Then use that lemma again, to find $\delta(n) \in (0,\xi(n))$ so that

$$\sup_{s,x} P^{s,x}(\sup\{|\hat{A}^{G(n-1)}(t) - \hat{A}_{\phi}^{G(n-1)}(t)|;\ t \geq 0,\ \phi \in \Phi(\delta(n))\} \geq \xi(n))$$
$$< \varepsilon(n).$$

Now use Lemma (2.1) to find $\rho(n) \in (0,2^{-n})$ so that

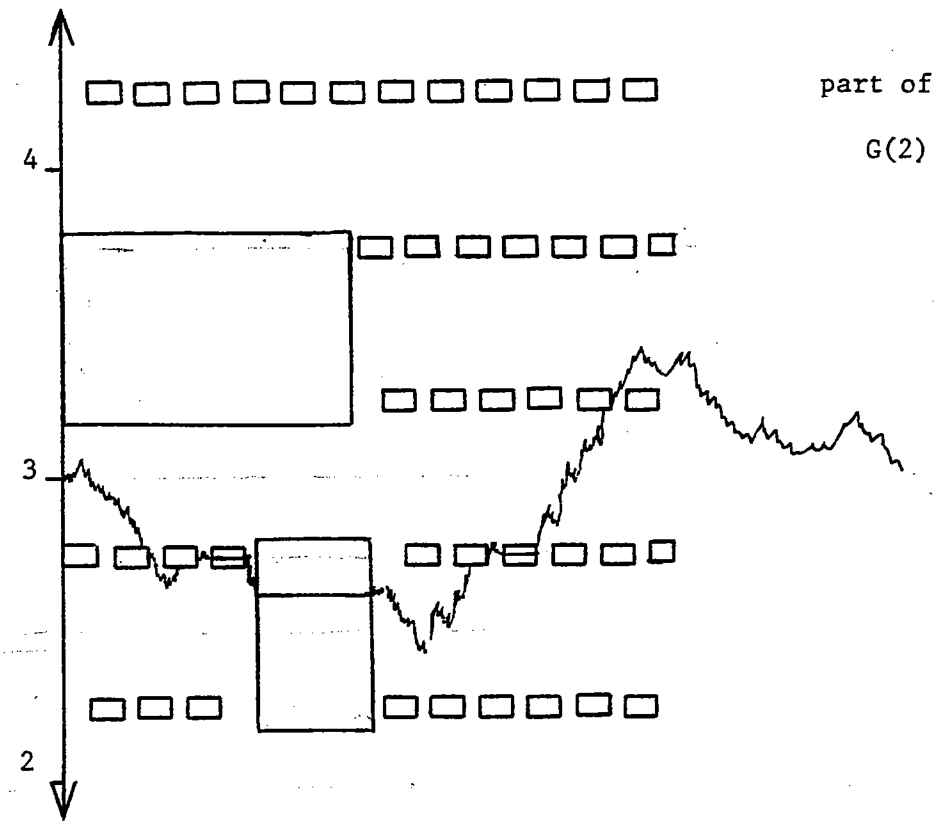
$$\sup_{s,x,y} P^{s,x}(3\rho(n)[\Gamma(n,y,\rho(n)) + \Lambda(n,y,2^{-n},\rho(n)) + 1]$$
$$\geq \frac{\delta(n)}{2^{2n+1}}) < \frac{\varepsilon(n)}{2^{2n+1}}.$$

Let $H(n)$ be the union of those boxes

$$([\rho(n)(4k + (-1)^j + 1),\ \rho(n)(4k + (-1)^j + 4)] \cap [0,n])$$
$$\times [\frac{2j+1}{2^n} - \rho(n),\ \frac{2j+1}{2^n} + \rho(n)]$$

not intersecting $G(n-1)$, for which $k \geq 0$ and $|j| < 2^{2n}$.
Let $G(n) = G(n-1) \cup H(n)$.

(2.4)



Recall (cf. [1]) that a continuous $\mathbb{R}^2$-valued strong Markov process is a <u>Hunt process</u> if its semigroup maps Borel functions to Borel functions.

(2.5) <u>Theorem</u>

(a) $\hat{A}^{G(n)}$ decreases to a function $\hat{A}$ as $n \to \infty$. Moreover, the convergence is a.s. uniform.

(b) $\hat{A}(t)$ is continuous in t and $\to \infty$ as $t \to \infty$, a.s..

(c) $\hat{A}(r)$ is an $(\mathcal{F}_{t+})$ stopping time for each $r \geq 0$. Let $\mathcal{G}_t = \mathcal{F}_{\hat{A}(t)+}$. If T is a $(\mathcal{G}_{t+})$ stopping time then $\hat{A}(T)$ is an $(\mathcal{F}_{t+})$ stopping time, and $\mathcal{G}_{T+} \subset \mathcal{F}_{\hat{A}(T)+}$.

(d) Let $M_t = B_{\hat{A}(t)}$. Then for all $(\mathscr{G}_{t+})$ stopping times T we have that

$$M_{T+t} = M_t(\tau_T, B_{\hat{A}(T)+\cdot}) \quad \text{for every } t, \text{ a.s..}$$

(e) $((\tau_t, M_t), \mathscr{G}_{t+}, P^{s,x})$ is a continuous Hunt process.

[Note: the monotone convergence of (a) is not strong enough. It will be crucial that $\hat{A}$ is not constant, and for that we need uniform convergence].

Proof: Let $\nu(n)$ be the total time that $(\tau_t, M_t^{G(n)})$ spends in $H(n)$. Fix $n \geq 1$ for now, and let

$$\hat{\phi}(t) = t + \sum_{r \leq t} (A^{G(n)}(r) - A^{G(n)}(r-))1_{\{(\tau_0 + A^{G(n)}(r-), B_r) \in H(n)\}}.$$

Let ϕ be the (continuous) inverse of $\hat{\phi}$. Then $\phi \in \Psi$ and

$$A^{G(n)}(t) = A_\phi^{G(n-1)}(\hat{\phi}(t)),$$

so that also

$$\hat{A}^{G(n)}(t) = \phi(\hat{A}_\phi^{G(n-1)}(t)).$$

The monotone convergence part of (a) now follows by Lemma (2.2).

Now suppose that $[u,v] \times [y-\rho(n), y+\rho(n)]$ is a box of $H(n)$ of length $3\rho(n)$ (this is a restriction only if $v = n$), which is hit by $(\tau_t, M_t^{G(n)})$. Let

$$I = (\hat{A}^{G(n)}(u), \hat{A}^{G(n)}(v) + \rho(n)).$$

If $B_s \in (y-2^{-n}, y+2^{-n})$ for $s \in I$, then $|I| > \rho(n)$.

If not, then $(B_s)_{s \in I}$ makes an upcrossing of $[y+\rho(n), y+2^{-n}]$, or a downcrossing of $[y-2^{-n}, y-\rho(n)]$. Thus

$$\#\{\text{boxes of } H(n) \text{ hit by } (\tau_t, M_t^{G(n)})\}$$

$$\leq \sum_{|j|<2^{2n}} [N(n, \frac{2j+1}{2^n}, \rho(n)) + L(n, \frac{2j+1}{2^n}, \frac{1}{2^n}, \rho(n)) + 1],$$

and hence

$$P^{s,x}(\nu(n) \geq \delta(n)) \leq \sum_{|j|<2^{2n}} \frac{\varepsilon(n)}{2^{2n+1}} < \varepsilon(n).$$

If $\nu(n) < \delta(n)$ then $\phi \in \Phi(\delta(n))$. Therefore

(2.6) $$P^{s,x}(\sup_t |\hat{A}^{G(n-1)}(t) - \hat{A}^{G(n)}(t)| \geq 2\xi(n))$$

$$\leq P^{s,x}(\sup_t |\hat{A}^{G(n-1)}(t) - \hat{A}_\phi^{G(n-1)}(t)| \geq \xi(n)) < 2\varepsilon(n),$$

so that the remainder of (a) follows from the easy half of the Borel-Cantelli Lemma. Part (b) follows in turn, as each $\hat{A}^{G(n)}$ is eventually linear with unit slope.

To show (c), let A be the right continuous inverse of $\hat{A}$. Then $A(t-) = \lim A^{G(n)}(t-)$ for every t. By construction, each $A^{G(n)}(t)$ is adapted to $(\mathcal{F}_t)$, hence so is $A(t-)$. Thus $A(t)$ is adapted to $(\mathcal{F}_{t+})$, and

$$\{\hat{A}(r) \leq t\} = \{r \leq A(t)\} \in \mathcal{F}_{t+},$$

so that $\hat{A}(r)$ is an $(\mathcal{F}_{t+})$ stopping time. We may therefore define $\mathcal{G}_t = \mathcal{F}_{\hat{A}(t)+}$. Then also

$$\{A(t) < s\} = \{t < \hat{A}(s)\} \in \mathcal{F}_{\hat{A}(s)+} = \mathcal{G}_s,$$

so that $A(t)$ is a $(\mathcal{G}_{s+})$-stopping time.

Let T be a $(\mathcal{G}_{s+})$-stopping time. For $B \in \mathcal{G}_{T+}$, we have that

$$B \cap \{\hat{A}(T) \leq t < \hat{A}(s)\}$$

$$= B \cap \{T \leq A(t) < s\} \in \mathcal{G}_s = \mathcal{F}_{\hat{A}(s)+},$$

so that for each r also

$$B \cap \{\hat{A}(t) \leq t < \hat{A}(s) \leq r\} \in \mathcal{F}_{r+}.$$

Taking the union over $s \in \mathbb{Q}$, we see that for each $r > t$,

$$B \cap \{\hat{A}(T) \leq t\} \in \mathcal{F}_{r+}.$$

Thus $\hat{A}(T)$ is an $(\mathcal{F}_{t+})$ stopping time, and $\mathcal{G}_{T+} \subset \mathcal{F}_{\hat{A}(T)+}$.

Turning to (d), let T be a $(\mathcal{G}_{t+})$ stopping time. Then $\tau_T \in \mathcal{F}_{\hat{A}(T)+}$, so that by the strong Markov property of (B_t) at $\hat{A}(T)$,

$$P^{s,x}(\sup\{|\hat{A}^{G(n)}(t)(\tau_T, B_{\hat{A}(T)+\cdot}) - \hat{A}_\phi^{G(n)}(t)(\tau_T, B_{\hat{A}(T)+\cdot})|;$$

$$t \geq 0,\ \phi \in \Phi(4\xi(n+1))\} \geq \varepsilon(n+1))$$

$$= E^{s,x}[P^{\tau_T, B_{\hat{A}(T)}}(\sup\{|\hat{A}^{G(n)}(t) - \hat{A}_\phi^{G(n)}(t)|;\ t \geq 0,$$

$$\phi \in \Phi(4\xi(n+1))\} \geq \varepsilon(n+1))] < \varepsilon(n+1).$$

Thus by the Borel-Cantelli Lemma and (2.6), we have that a.s. there is an n_0 such that for $n \geq n_0$,

$$|\hat{A}^{G(n)}(t) - \hat{A}^{G(n+1)}(t)| < 2\xi(n+1) \text{ for every } t, \text{ and}$$

$$\sup\{|\hat{A}^{G(n)}(t)(\tau_T, B_{\hat{A}(T)+\cdot}) - \hat{A}_\phi^{G(n)}(t)(\tau_T, B_{\hat{A}(T)+\cdot})|;$$

$$t \geq 0,\ \phi \in \Phi(4\xi(n+1))\} < \varepsilon(n+1).$$

Let $n \geq n_0$. Then

$$|\hat{A}^{G(n)}(T) - \hat{A}(T)| \leq \sum_{k \geq n} |\hat{A}^{G(k)}(T) - \hat{A}^{G(k+1)}(T)|$$

$$< 2\sum_{k \geq n} \xi(k+1) < 2\sum_{k \geq n} 2^{n-k}\xi(n+1) = 4\xi(n+1).$$

By (a) we have that $\hat{A}(T) \leq \hat{A}^{G(n)}(T)$, so

$$\phi(t) = t + \hat{A}^{G(n)}(T) - \hat{A}(T) \in \Phi(4\xi(n+1)).$$

We have that $\phi(\hat{A}(T)+t) = \hat{A}^{G(n)}(T) + t$, and by construction,

$$\hat{A}^{G(n)}(t)(\tau_T, B_{\hat{A}^{G(n)}(T)+\cdot}) = \hat{A}^{G(n)}(T+t) - \hat{A}^{G(n)}(T) \text{ for every } t.$$

Thus

$$|\hat{A}^{G(n)}(t)(\tau_T, B_{\hat{A}(T)+\cdot}) - \hat{A}^{G(n)}(T+t) + \hat{A}^{G(n)}(T)| < \varepsilon(n+1).$$

Letting $n \to \infty$, we see that

$$\hat{A}(t)(\tau_T, B_{\hat{A}(T)+\cdot}) = \hat{A}(T+t) - \hat{A}(T) \text{ for every } t, \text{ a.s.},$$

showing (d).

Finally, let T be a $(\mathcal{G}_{t+})$ stopping time, $Z \in \mathcal{G}_{T+}$ bounded, and f bounded and measurable on path space. Then

$$E^{s,x}[Zf((\tau_{T+\cdot},M_{T+\cdot}))]$$

$$= E^{s,x}[Zf((\tau_\cdot,M_\cdot)(\tau_T,B_{\hat{A}(T)+\cdot}))]$$

$$= E^{s,x}[ZE^{\tau_T,M_T}[f((\tau_\cdot,M_\cdot))]].$$

This shows the strong Markov property. Because of their Brownian heritage, the transition function for each of the $(\tau_t,M_t^{G(n)})$ can (given sufficient time) be found more or less explicitly. Thus each of these is a continuous Hunt process, hence so is (τ_t,M_t). □

We will need the following fact later:

(2.7) Lemma. Let $(s,y) \in [u,v] \times [a,b] \subset G(n)$. Then

$$P^{s,y}(M_t = y \quad \text{for} \quad t \in [0,v-s]) = 1.$$

Proof. By construction,

$$\hat{A}^{G(k)}(v-s) = 0, \quad P^{s,y} - \text{a.s.}$$

for each $k \geq n$. Thus also $\hat{A}(v-s) = 0$, $P^{s,y}$ - a.s., showing the result. □

Now enumerate the boxes of $G = \cup G(n)$ as $[t_n-x_n,t_n+x_n] \times [a_n,b_n]$. Choose $y_n > 0$ so that

$$\beta = \sum_{n=1}^{\infty} y_n/x_n < 1.$$

Let

$$f(y) = \begin{cases} 0, & y \le 0 \\ y, & 0 \le y \le 1 \\ 1, & y \ge 1 \end{cases},$$

$$g_m(t,y) = 1 + t - \sum_{n=1}^{m} y_n f\left(\frac{y-a_n}{b_n-a_n}\right)\left(\left[1 - \left|\frac{t-t_n}{x_n}\right|\right] \vee 0\right).$$

Each g_m is continuous on $[0,\infty) \times \mathbb{R}$. Since $|g_{m+1} - g_m| \le y_m$, and

$$\sum y_m \le \sum \frac{y_m}{x_m} < \infty$$

we see that the g_m converge uniformly to a continuous function g.

Fix s for the moment. Each $g_m(s,\cdot)$ is nondecreasing and moreover is strictly increasing on any interval (a_n,b_n) with $n \le m$ and $s \in (t_n - x_n, t_n + x_n)$. By construction, the union of such intervals becomes dense in $\mathbb{R}$ as $m \to \infty$, so that in fact each $g(s,\cdot)$ is strictly increasing, hence one to one. Let $h(s,\cdot)$ be its inverse. Then h is continuous. For $(s,x) \in [0,\infty) \times \mathbb{R}$, define

$$Q^{s,x} = P^{s,h(s,x)}.$$

(2.8) Theorem. Let $X_t = g(\tau_t, M_t)$. Then (X_t) is continuous and nondecreasing a.s., yet

$$((\tau_t, X_t), \mathcal{G}_{t+}, Q^{s,x})$$

is a nondeterministic Hunt process.

Proof: Since $\hat{A}(t) \leq t$ for every t, (M_t) is a $Q^{s,x}$-martingale for each s, x. It is nonconstant by (b) of Theorem (1.5), hence is nondeterministic. Therefore since each $g(r,\cdot)$ is one to one, (τ_t, X_t) is both nondeterministic and strong Markov. It is continuous since g and (τ_t, M_t) are. It is therefore Hunt, since g and h are Borel. Thus all that remains to be shown is that (X_t) is a.s. nondecreasing. It suffices to show that each $g_m(\tau_t, M_t)$ is.

By definition each $g_m(\cdot, y)$ is absolutely continuous with 'derivative'

$$1 - \sum_{n=1}^{m} \frac{y_n}{x_n} f\left(\frac{y-a_n}{b_n-a_n}\right) \operatorname{sign}(t_n - \cdot)$$

$$\geq 1 - \sum_{n=1}^{m} \frac{y_n}{x_n} > 1 - \beta > 0.$$

Thus each $g_m(\cdot, y)$ is increasing. Moreover, $[0,\infty) \times \mathbb{R}$ may be decomposed into finitely many boxes (now allowing infinite sides) which either are subsets of some $G(n)$, or on which $g_m(t,y)$ does not depend on y. It is clear that $g_m(\tau_t, M_t)$ increases while (τ_t, M_t) remains in any rectangle of the latter type, and by Lemma (2.7) it also increases on rectangles of the former type. □

§3. Proof of Lemma (2.3)

① Definitions and outline of proof:

Let $m \geq 1$ be the number of boxes of G, and write

$$G = \bigcup_{i=1}^{m} [u_i, v_i] \times [a_i, b_i].$$

For convenience, let $I_i = [a_i, b_i]$. Choose $\gamma \in (0,1)$ so that $|a_i - b_j| > \gamma$ whenever $I_i \cap I_j = \phi$, and $|u_i - v_j| > \gamma$ otherwise (thus all boxes have length at least γ in the time direction, and are at least distance γ apart).

Given $\lambda \in (0,1)$, choose $\eta \in (0, (\gamma/5) \wedge \varepsilon)$ so that

$$P^{o,o}(|B_t| < \gamma \text{ for } t \in [0, 2\eta]) > 1 - \frac{\lambda}{4} \text{ and}$$

$P^{o,o}((t, B_t) \text{ hits } D) < \lambda/8m^2$, for every set D of the form $[s, s+4\eta] \times \{y\}$ not intersecting the $\gamma/2$-neighbourhood of $(0,0)$.

Then choose $\delta \in (0, \eta/3)$ so that

$$P^{o,o}(B_t = 0 \text{ for some } t \in (2\delta, \eta-\delta)) > 1 - \frac{\lambda}{4}.$$

Write $A(t) = A^G(t)$. Let

$$T(1) = \inf\{t > 0;\ (\tau_t, B_t) \in \bigcup_i [u_i - \delta, v_i + \delta] \times I_i\},$$

$$J(1) = \begin{cases} 0, & \text{if } T(1) = \infty \\ i, & \text{if } (\tau_{T(1)}, B_{T(1)}) \in [u_i - \delta, v_i + \delta] \times I_i, \end{cases}$$

and define $T(n)$, $J(n)$ inductively for $n \geq 2$ by

$$T(n) = \inf\{t > T(n-1);\ (\tau_0 + A(t-), B_t) \in \bigcup_{i \in \{1..m\} \setminus \{J(1)..J(n-1)\}} [u_i - 2\eta, v_i + 2\eta] \times I_i\}$$

$$J(n) = \begin{cases} 0, & \text{if } T(n) = \infty \\ i, & \text{if } (\tau_0 + A(T(n)-), B_{T(n)}) \in [u_i - 2\eta, v_i + 2\eta] \times I_i . \end{cases}$$

Let

$$C'(1) = \{T(1) < \infty,\ \tau_{T(1)} \in [u_{J(1)} - \delta, v_{J(1)} - \eta],$$

$$(\tau_t, B_t) \text{ hits } (\tau_{T(1)} + 2\delta, \tau_{T(1)} + \eta - \delta) \times I_{J(1)}, \text{ and}$$

$$|B_t - B_{T(1)}| < \gamma \text{ for } t \in [T(1), T(1) + 2\eta]\},$$

$$C''(1) = \{T(1) < \infty,\ \tau_{T(1)} \in (v_{J(1)} - \eta, v_{J(1)} + \delta], \text{ and}$$

$$|B_t - B_{T(1)}| < \gamma \text{ for } t \in [T(1), T(1) + 2\eta]\},$$

$$C(1) = \{T(1) = \infty\} \cup C'(1) \cup C''(1).$$

For $n \geq 2$, let

$$C'(n) = \{T(n) < \infty \text{ and } \tau_0 + A(T(n)-) \in [u_{J(n)} + 2\eta, v_{J(n)} - 2\eta]\},$$

$$C''(n) = \{T(n) < \infty \text{ and } B_t \in (a_{J(n)}, b_{J(n)}) \text{ for } t \in [T(n), T(n) + 4\eta]\},$$

$$y(n) = \begin{cases} 1 & \text{on } C'(n) \\ 2 & \text{on } C''(n) \\ 0 & \text{elsewhere}, \end{cases}$$

$$C(n) = \{T(n) = \infty\} \cup C'(n) \cup C''(n),$$

$$C = C(1) \cap .. \cap C(m).$$

Let $S = S^G$, and

$$L = 1_{C''(1)},\ K = 1_{C''(1)\cap\{(\tau_t, B_t) \text{ hits } [u_{J(1)}, v_{J(1)}] \times I_{J(1)}\}}.$$

A straightforward induction shows that the following conditions hold:

(3.1) If $K=1$ then $S(1)=T(1)$ and $\tau_0 + A(S(1)) = v_{J(1)} < \tau_0 + A(S(1)-) + \eta$.

(3.2) $S(1) \in [T(1), T(1)+\eta)$ and $(\tau_0 + A(S(1)), B_{S(1)}) \in \{v_{J(1)}\} \times I_{J(1)}$ on $C'(1)$.

(3.3) $S(K-L+n) = T(n)$ on $C \cap C'(n)$ for $n \geq 2$.

(3.4) $S(K-L+n) = T(n) + 2\eta < T(n+1)$ and

$$(\tau_0 + A(S(K-L+n)-), B_{S(K-L+n)}) \in \{u_{J(n)}\} \times (a_{J(n)}, b_{J(n)})$$

on $C \cap C''(n)$, for $n \geq 2$.

(3.5) $S(K-L+n) = \infty$ on $C \cap \{T(n) = \infty\}$, for $n \geq 1$.

We will show from this (see ② below) that $P^{s,x}(C) > 1 - \lambda$ for every s and x.

Now let $\phi \in \Phi(\delta)$, and write

$$A' = A^G_\phi,\ S' = S^G_\phi,$$

$$K' = 1_{C''(1)\cap\{(\tau_t, B_{\phi(t)}) \text{ hits } [u_{J(1)}, v_{J(1)}] \times I_{J(1)}\}}.$$

Conditions analogous to (3.1)-(3.5) hold for these objects as well, but we will state matters slightly differently; we will show that the following conditions hold on C (for $n = 1..m$):

(3.6) If $K' = 1$ then $T(1) \leq \phi(S'(1)) < T(2)$, $\tau_0 + A'(S'(1)) = v_{J(1)}$, and $\tau_0 + A'(S'(1)-) \in (\tau_0 + A(T(1)-) - \delta, v_{J(1)}]$

(3.7) $T(L+n) \leq \phi(S'(K'+n))$, and if $T(L+n) < \infty$ then also $\phi(S'(K'+n)) < T(L+n+1)$.

(3.8) If $T(L+n) < \infty$ then $(\tau_0 + A'(S'(K'+n)), B_{\phi(S'(K'+n))}) \in \{v_{J(L+n)}\} \times I_{J(L+n)}$.

(3.9) If $T(L+n) < \infty$ then $|S(K+n) - S'(K'+n)| < \eta$.

The induction step will be shown in ③, and the induction started off (and (3.6) shown) in ④ and ⑤.

Lastly, we will show (in ⑥) that from these conditions it follows that

(3.10) $\sup_t |\hat{A}(t) - \hat{A}'(t)| < \varepsilon$ on C,

completing the proof of the lemma.

② Proof that $P^{s,x}(C) > 1 - \lambda$:

Fix s and x. For $i \neq j$, each component of $[u_j - 2\eta, u_j + 2\eta) \times \{a_j, b_j\}$ or $(v_j - 2\eta, v_j + 2\eta] \times \{a_j, b_j\}$ is of distance at least

$$\gamma - 2\eta > \delta + \frac{\gamma}{2}$$

from $\{v_i\} \times I_i$. By (3.1) we have that

$$A(T(1)) \in [v_{J(1)}, v_{J(1)} + \delta] \quad \text{on} \quad C''(1).$$

By (3.2)-(3.5), we may therefore apply the strong Markov property at either $T(1)$ or $S(n)$, and obtain that for $n = 1..m-1$,

$$P^{s,x}(C(n) \setminus C(n+1))$$

$$\leq \sum_{i=1}^{m} P^{s,x}(J(n)=i) \sup_{\substack{r\in[v_i,v_i+\delta) \\ y\in I_i}} P^{r,y}((\tau_t,B_t)$$

$$\text{hits} \bigcup_{j\neq i} ([u_j-2\eta,u_j+2\eta)\cup(v_j-2\eta,v_j+2\eta]) \times \{a_j,b_j\})$$

$$< 4(m-1) \frac{\lambda}{8m^2} < \frac{\lambda}{2m}.$$

Similarly,

$$P^{s,x}(\Omega\setminus C(1)) \leq \sum_{i=1}^{m} P^{s,x}(J(1)=i) \sup_{\substack{r\in[u_i-\delta,v_i+\delta] \\ y\in I_i}}$$

$$[P^{r,y}(|B_t-y| \geq \gamma \text{ for some } t\in[0,2\eta]$$

$$+ P^{r,y}(B_t \neq y \text{ for any } t\in(2\delta,\eta-\delta))]$$

$$< \frac{\lambda}{2}.$$

Thus

$$P^{s,x}(C) > 1 - \frac{(m-1)\lambda}{2m} - \frac{\lambda}{2} > 1-\lambda.$$

③ Proof of (3.7)-(3.9); induction step:

Let $k \geq 1$, and suppose that (3.6) holds, as do (3.7)-(3.9) for $n = 1..k$. Then

$$S'(K'+k+1) = \inf\{t > S'(K'+k); (v_{J(L+k)}+t-S'(K'+k), B_{\phi(t)}) \in G\}.$$

ϕ cannot be constant on $[S'(K'+k), S'(K'+k+1)]$ (if it were, then $[\tau_0+A'(S'(K'+k)), \tau_0+A'(S'(K'+k))+S'(K'+k+1)-S'(K'+k)]\times\{B_{\phi(S'(K'+k))}\}$ would stretch from one box of G to another, hence would be of length at least γ, whereas

$$S'(K'+k+1) - S'(K'+k)$$

$$= S'(K'+k+1)-\phi(S'(K'+k+1))+\phi(S'(K'+k))-S'(K'+k) < 2\delta < \gamma).$$

Thus

$$\phi(S'(K'+k+1)) = \inf\{r > \phi(S'(K'+k)); (v_{J(L+k)}+t-S'(K'+k), B_r)$$
$$\in G \text{ for some } t \text{ with } \phi(t) = r\}.$$

By induction, if $\phi(t) = r$, then

$$(3.11) \qquad |(t-S'(K'+k))-(r-S(K+k))| < \delta + \eta < 2\eta.$$

Since also $T(L+k) \leq \phi(S'(K'+k))$, and $(\tau_0+A'(t-), B_{\phi(t)})$ $\notin [u_{J(n)}, v_{J(n)}] \times I_{J(n)}$ for any $n \leq L+k$ and $t > S'(K'+k)$, we have that $T(L+k+1) \leq \phi(S'(K'+k+1))$.
On $C \cap \{Y(L+k+1) = 1\}$ we have by (3.11) that

$$\phi(S'(K'+k+1)) = T(L+k+1), \text{ and}$$

$$\tau_0 + A'(S'(K'+k+1)-) \in [u_{J(L+k+1)}, v_{J(L+k+1)}],$$

showing (3.7) and (3.8). (3.9) follows by (3.3).
On $C \cap \{Y(L+k+1) = 2\}$, we have by definition that

$$B_{\phi(t)} \in (a_{J(L+k+1)}, b_{J(L+k+1)})$$
$$\text{for } t \in [\hat{\phi}(T(L+k+1)-), \hat{\phi}(T(L+k+1)+4\eta)].$$

Also,

$$
\begin{aligned}
&|\tau_0 + A'(\hat{\phi}(T(L+k+1)-)-) - (u_{J(L+k+1)} - 2\eta)| \\
(3.12) \qquad &= |v_{J(L+k)} + \hat{\phi}(T(L+k+1)-) - S'(K'+k) \\
&\quad - (v_{J(L+k)} + T(L+k+1) - S(K+k)| \\
&< \delta + \eta < 2\eta - 2\delta,
\end{aligned}
$$

so that

$$
\begin{aligned}
&u_{J(L+k+1)} - 4\eta < \tau_0 + A'(\hat{\phi}(T(L+k+1)-)-) < u_{J(L+k+1)} \\
&\quad < \tau_0 + A'(\hat{\phi}(T(L+k+1)-)-) + \hat{\phi}(T(L+k+1)+4\eta) - \hat{\phi}(T(L+k+1)-).
\end{aligned}
$$

Since $4\eta < \gamma$, we conclude that

$$
(\tau_0 + A'(t-), B_{\phi(t)})_{t \in [\hat{\phi}(T(L+k+1)-), \hat{\phi}(T(L+k+1)+4\eta)]}
$$

hits no box of G other than $[u_{J(L+k+1)}, v_{J(L+k+1)}] \times I_{J(L+k+1)}$, and that it hits that box in the set

$$
\{u_{J(L+k+1)}\} \times I_{J(L+k+1)}.
$$

Condition (3.8) and the remainder of (3.7) now follow. To show (3.9), observe that

$$
\begin{aligned}
&|S(K+k+1) - S'(K'+k+1)| \\
&= |(S(K+k) + u_{J(L+k+1)} - v_{J(L+k)}) \\
(3.13) \qquad &\quad - (S'(K'+k) + u_{J(L+k+1)} - v_{J(L+k)})| \\
&< \eta.
\end{aligned}
$$

Thus, the induction step is shown on $C \cap \{T(L+k+1) < \infty\}$. It holds vacuously on the remainder of C.

④ Proof of (3.7)-(3.9) for $n = 1$, on $C \cap C'(1)$:

$$S'(1) = \inf\{t \geq 0;\ (\tau_t, B_{\phi(t)}) \in G\}, \text{ so that}$$

$$\phi(S'(1)) = \inf\{r \geq \phi(0);\ (\tau_t, B_r) \in G \text{ for some } t \text{ with } \phi(t) = r\}.$$

Since $|\tau_t - \tau_r| < \delta$ whenever $\phi(t) = r$, we have that

(3.14) $\quad \phi(S'(1)) \geq T(1)$ everywhere on $\{T(1) < \infty\}$.

Moreover, on $C'(1) \cup C''(1)$ we have that $|B_t - B_{T(1)}| < \gamma$ for $t \in [T(1), T(1) + 2\eta]$, and hence that

(3.15) $\quad T(2) > T(1) + 2\eta$ and $(\tau_t, B_{\phi(t)})_{t \in [0, T(1)+\eta]}$ hits no box of G other than $[u_{J(1)}, v_{J(1)}] \times I_{J(1)}$.

On $C'(1) \cap C$, there is by definition an $r \in (T(1) + 2\delta, T(1) + \eta - \delta)$, with $B_r \in I_{J(1)}$. Then $r = \phi(t)$ for some $t \in (T(1) + \delta, T(1) + \eta)$. Since

$$u_{J(1)} \leq \tau_{T(1)} + \delta < \tau_{T(1)} + \eta \leq v_{J(1)},$$

we see that

$$(\tau_t, B_{\phi(t)}) \in [u_{J(1)}, v_{J(1)}] \times I_{J(1)}.$$

Recalling (3.15), we obtain (3.7) and (3.8). (3.9) follows by (3.2), (3.14) and the fact that $\phi(S'(1)) \leq r < T(1) + \eta$.

⑤ Proof of (3.6) and of (3.7)-(3.9) for $n = 1$, on $C \cap C"(1)$:

Fix a point in $C \cap C"(1)$. If $K' = 1$, then $S'(1) \leq v_{J(1)} - \tau_0 < T(1) + \eta$. Thus (3.6) follows by (3.14) and (3.15). Further (breaking things up into the two cases; that $\tau_0 + A'(S'(1)-)$ belongs to $(\tau_0 + A(T(1)-) - \delta, \tau_0 + A(T(1)-)]$ or $(\tau_0 + A(T(1)-), v_{J(1)}])$ we have by (3.1) that

$$(3.16) \qquad |A(T(1)+\eta) - A'(T(1)+\eta)| < \eta.$$

Likewise, if $K' = 0$, (3.16) still holds and also $T(2) \wedge S'(1) > T(1) + \eta$. Thus (3.16) holds on all of $C \cap C"(1)$, and irrespective of K',

$$S(K'+1) = \inf\{t > T(1)+\eta; (\tau_0 + A'(T(1)+\eta)+t-T(1)-\eta, B_{\phi(t)}) \in G\}.$$

Thus as in ③,

$$\phi(S'(K'+1)) = \inf\{r > \phi(T(1)+\eta); (\tau_0 + A'(T(1)+\eta)+t-T(1)-\eta, B_r) \in G \text{ for some } t \text{ with } \phi(t) = r\}.$$

We conclude as in ③ that $T(2) \leq \phi(S'(K'+1))$, now using the inequality

$$|(A'(T(1)+\eta)+t) - (A(T(1)+\eta)+r)| < \eta + \delta < 2\eta,$$

instead of (3.11).

On $C \cap C"(1) \cap C'(2)$ we may now proceed as in ③.

On $C \cap C"(1) \cap C"(2)$ we may do likewise, the only modifications being that instead of (3.12) we use that

$$\begin{aligned}
&|\tau_0 + A'(\hat{\phi}(T(2)-)-) - (u_{J(2)} - 2\eta)| \\
&= |\tau_0 + A'(T(1)+\eta) + \hat{\phi}(T(2)) - T(1) - \eta \\
&\qquad - (\tau_0 + A(T(1)+\eta) + T(2) - T(1) - \eta)| \\
&< \eta + \delta < 2\eta - 2\delta,
\end{aligned}$$

and that instead of (3.14), we have that

$$\begin{aligned}
&|S(K+1) - S'(K'+1)| \\
&= |T(1) + \eta + u_{J(2)} - A(T(1) + \eta) \\
&\qquad - (T(1) + \eta + u_{J(2)} - A'(T(1) + \eta))| \\
&< \eta.
\end{aligned}$$

Thus by induction, we have shown (3.6)-(3.9) on C, for $n = 1..m$.

⑥ Proof of (3.10):

Fix a point of C. Let $t_0 = 0$, and for $n \geq 1$ set $t_n = v_{J(n)} - \tau_0$ if $T(n) < \infty$ (and $t_n = \infty$ otherwise). If $n \geq 0$ is such that $[t_n, t_{n+1})$ is a finite interval, then conditions (3.1), (3.2), (3.4), (3.6), and (3.8) show that there are $t, t' \in [t_n, t_{n+1}]$ such that $\hat{A}$ and $\hat{A}'$ are linear (with unit slope) on $[t_n, t)$ and $[t_n, t')$ respectively, and then are constant on the remainder of $[t_n, t_{n+1})$. (Note that for $n = 0$ in particular, we may have that $t, t' = t_n$ or t_{n+1}). Similarly, if $t_n < \infty = t_{n+1}$, then both $\hat{A}$ and $\hat{A}'$ are linear on $[t_n, t_{n+1})$, with unit slope. Thus we will have that $|\hat{A}(r) - \hat{A}'(r)| < \varepsilon$ for every r, provided only that this holds for finite r of the form t_n. This in turn follows from (3.1), (3.6) and (3.9) (for example, if $n \geq 1$ then

$$|\hat{A}(v_{J(L+n)}-\tau_0) - \hat{A}'(v_{J(L+n)}-\tau_0)| = |S(K+n) - S'(K'+n)|$$
$$< \eta < \varepsilon$$

showing (3.10), and hence the Lemma. □

§4. Bibliography

[1] R. M. Blumenthal and R. K. Getoor. Markov Processes and Potential Theory. Academic Press, New York 1968

[2] E. Çinlar and J. Jacod. Representations of semimartingale Markov processes in terms of Wiener processes and Poisson random measures. Seminar on Stochastic Processes 1981, pp. 159-242. Birkhäuser, Boston 1981

Thomas S. Salisbury
Department of Mathematics
Purdue University
West Lafayette, Indiana 47907