BLOCK DESIGNS: GENERAL OPTIMALITY RESULTS WITH APPLICATIONS TO SITUATIONS WHERE BALANCED DESIGNS DO NOT EXIST

by

Dexter C. Whittinghill III[1]
Purdue University

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Department of Statistics
Purdue University

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Corrections to Purdue University Department of Statistics T.R. #84-21

Page	Line	Incorrect	Correct
40	9		put "sym" in lower left of matrix
69	8	$r - \frac{\lambda_0}{k}$	$r - \frac{\lambda_0}{v}$
74	-9	"=" at end of line	should be a "$\leq$"
74	-8	$B(r_{dvj};b,k)$	$B(r_{dv};b,k)$
82	top		put "sym" in lower left of matrix
86	-3	equi-	extra-
91	-2	$\frac{n_{dij}^2}{k}$ at end of line	should be $\frac{n_{d1j}^2}{k}$
93	1,6,8	(5.3.2)	(5.3.1)
97	7	... cary through the same ...	... carry through all of the same ...
103	1	$\mathcal{P}_4$	$\mathcal{P}_U$
107	3	$\mathcal{D}(v,b+m,pv+q)\ \mu_{d1} \leq$	$\mathcal{D}(v,b+m,pv+q),\ \mu_{d1} \leq$
116	middle		put "sym" in lower left of matrix
120	-6	$v \geq 3.$	$v \geq 3,\ e \geq 1,\ p \geq 1.$
139	-1	$v \geq 3.$	$v \geq 3,\ p \geq 1.$
140	-1	$v \geq 3.$	$v \geq 3,\ p \geq 1.$
151	-7	... if (9.1.3) ...	... if the left hand side of (9.1.3) ...
156	$12\frac{1}{2}$	d^o	d_1
156	5	$\mathcal{P}$-better	$\mathcal{P}_4$-better

Page	Line	Incorrect	Correct
104	7		correction to proof - separate sheet
150	-8		add to the end of the paragraph: Let $\underline{B}_d = b^{-1} \underline{\Delta}'_{+d}\, \underline{\Phi}_p\, \underline{\Phi}'_p\, \underline{\Delta}_{+d}$.

Corrected proof to Lemma 6.2.2 of Whittinghill (1984) pp. 104-105.
- Jan 1985

<u>Proof</u>. The nonzero eigenvalues of C_{d*} are $v\lambda_1 k^{-1}+np+1-2k^{-1}$ and $v\lambda_1 k^{-1}+np+1$ with multiplicities $(v/2)-1$ and $v/2$, respectively. We also have without loss of generality

$$C_{d*} = \frac{1}{k}\begin{bmatrix} a & -(b+1) & -b & -b & & -b & -b \\ -(b+1) & a & -b & -b & & -b & -b \\ & & & & \cdots & & \\ -b & -b & a & -(b+1) & & -b & -b \\ -b & -b & -(b+1) & a & & -b & -b \\ & \vdots & & & \ddots & & \\ -b & -b & -b & -b & & a & -(b+1) \\ -b & -b & -b & -b & & -(b+1) & a \end{bmatrix}$$

with $a=(v-1)\lambda_1 + k(np+1) - (np^2+2p+1)=kr-\lambda_0 + k(np+1) - (np^2+2p+1)$ and $b = \lambda_1 + np^2 + 2p$.

Lemma 6.2.1 says d* E-betters all d with $\min_{1 < i < v}(r_{di}) < r+np$. Therefore any d that is to E-better d* must have $r_{di} = r+np+1 = r_{d*i}$ $(1 < i < v)$. Lemma 6.2.1 also says d* E-betters any d with $r_{di} = r_{dj} = r+np+1$ and $\lambda_{dij} < \lambda_1+np^2+2p-(q-1)=\lambda_1+np^2+2p-1$, for $i \neq j$. Therefore the only d that d* might not E-better are those with all $r_{d_i} = r+np+1$ and all $\lambda_{dij} > \lambda_1+np^2+2p$ for $i \neq j$. This leaves us with only two possibilities.

d can have maximum trace, or tr C_d=tr C_{d*}. In this case each row (and column) of C_d has its diagonal element equal to that of C_{d*}, v-2 of the λ_{dij}'s equal to λ_1+np^2+2p and one λ_{dij} equal to λ_1+np^2+2p+1. Then C_d is equivalent to C_{d*} in the sense that it will have the same eigenvalues.

The other case is where d is not of full trace. Then at least two rows of KC_d have all off diagonal elements equal to $\lambda_1+np+2p$. Assuming without loss of generality these rows correspond to treatments 1 and 2 then $c_{d11} = c_{d22} = r+np+1-k^{-1}(\lambda_0+np^2+2p+2)$ and $c_{d12} = -k^{-1}(\lambda_1+np^2+2p)$. Using (ii) of Theorem 2.2.1,

$$\begin{aligned}\mu_{d1} &< r+np+1 - k^{-1}(\lambda_0+np^2+2p+2) + k^{-1}(\lambda_1+np^2+2p) \\ &= v\lambda_1 k^{-1}+np+1 - 2k^{-1} = \mu_{d*1}.\end{aligned}$$

This completes the proof.

CHAPTER 1

INTRODUCTION

1.1 The Model and Notation

In the one-way elimination of heterogeneity setting of analysis of variance we are applying v treatments or varieties to N experimental units which are organized into b blocks of size k_j, $j = 1,\ldots,b$. If we define r_i $(i = 1,\ldots,v)$ to be the number of times treatment i is applied to an experimental unit in the whole experiment, we see that $N = \sum_{j=1}^{b} k_j = \sum_{i=1}^{v} r_i$. Since for v, b, and k_1 through k_b fixed we can apply the treatments in different ways, we will label particular applications of the treatments with d, d*, or a subscripted d, calling each application a design. This also means that quantities like the r_i may be subscripted, and for the remainder of this chapter quantities that change from design to design will be subscripted for illustration.

We assume the additive model $E(Y_{dmij}) = \alpha_i + \beta_j$ $(m = 1,\ldots,N;$ $i = 1,\ldots,v;$ $j = 1,\ldots,b)$ with Y_{dmij} being the observation on the m-th experimental unit, which is in block j and was treated with treatment i. The N observations are assumed to be uncorrelated with common variance σ^2. Defining the vectors $\underline{\alpha}' = (\alpha_1,\ldots,\alpha_v)$, $\underline{\beta}' = (\beta_1,\ldots,\beta_b)$, $\underline{Y}_d' = (Y_{d1ij},\ldots,Y_{dNij}) = (Y_{d1},\ldots,Y_{dN})$, $\underline{k}' = (k_1,\ldots,k_b)$ and $\underline{r}_d' = (r_{d1},\ldots,r_{dv})$ we can write the model in matrix form:

$$E(\underline{Y}_d) = X_d\underline{\theta} \equiv [X_d^{(1)}, X_d^{(2)}]\binom{\underline{\alpha}}{\underline{\beta}}.$$

Here X_d is the N by (v+b) matrix whose m-th row consists of all 0's except for a 1 in the i-th and (v+j)-th columns, since $E(Y_{dm}) = E(Y_{dmij}) = \alpha_i + \beta_j$. $X_d^{(1)}$ and $X_d^{(2)}$ are N by v and N by b, respectively, and partition X_d into columns corresponding to treatments and blocks.

The normal equations for the full model are $X_d'X_d\hat{\underline{\theta}} = X_d'\underline{Y}_d$ where

$$X_d'X_d = \begin{bmatrix} X_d^{(1)'}X_d^{(1)} & \vdots & X_d^{(1)'}X_d^{(2)} \\ \cdots & \vdots & \cdots \\ X_d^{(2)'}X_d^{(1)} & \vdots & X_d^{(2)'}X_d^{(2)} \end{bmatrix} \equiv \begin{bmatrix} \underline{r}_d^{\delta} & \vdots & \eta_d \\ \cdots & \vdots & \cdots \\ \eta_d' & \vdots & \underline{k}^{\delta} \end{bmatrix}.$$

Here $\underline{x}^{\delta}$ represents the diagonal q by q square matrix with x_i in the ii-th position, x_i being from the vector $\underline{x}' = (x_1,\ldots,x_q)$. $\eta_d = (n_{dij})$ is the v by b <u>incidence matrix</u> whose elements n_{dij} equal the number of times treatment i appears in block j. Note that $\sum_{j=1}^{b} n_{dij} = r_{di}$ and $\sum_{i=1}^{v} n_{dij} = k_j$.

In the one-way elimination of heterogeneity setting we are not interested in the block effects, only the treatment effects. The reduced normal equations for $\underline{\alpha}$ are

$$[\underline{r}_d^{\delta} - \eta_d\underline{k}^{-\delta}\eta_d']\hat{\underline{\alpha}} = [X_d^{(1)'} - \eta_d\underline{k}^{-\delta}X_d^{(2)'}]\underline{Y}_d,$$

where $\underline{k}^{-\delta} = (\underline{k}^{\delta})^{-1}$. $C_d = r_d^{\delta} - \eta_d\underline{k}^{-\delta}\eta_d'$ is the coefficient or information matrix for the α_i in the design d and is also known as the <u>C-matrix</u> for d. C_d is a symmetric, v by v, positive semi-definite matrix whose row (and column) sums equal zero. Hence $\text{rank}(C_d) \leq v-1$ and C_d will always have at least one zero eigenvalue. In fact $\text{rank}(C_d) = v-1$ if and only if all treatment contrasts $\sum_{i=1}^{v} w_i \alpha_i$ (where $\sum_{i=1}^{v} w_i = 0$) are estimable. Designs with $\text{rank}(C_d) = v-1$ are known as connected designs.

The eigenvalues of C_d will be used in determining the optimality of a given design d. Let $\mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$ be the ordered eigenvalues of C_d.

It will be useful to name the collection of possible designs for a given v, b and $\underline{k}$. So we define $\mathcal{D}(v,b,\underline{k}) = \{d\colon\ d \text{ has parameters } v,b,\underline{k}\}$. $\mathcal{D}$ will be used instead of $\mathcal{D}(v,b,\underline{k})$ when v,b and $\underline{k}$ are clear from context, and k instead of $\underline{k}$ will mean $k_1 = \ldots = k_b = k$.

Another useful matrix of parameters is $\Lambda_d = (\lambda_{dij}) = \mathcal{N}_d \mathcal{N}_d'$. This v by v matrix gives $\lambda_{dij} = \sum_{s=1}^{b} n_{dis} n_{djs}$, and for $i \neq j$ λ_{dij} is the total number of times treatments i and j are paired in the design. If all the blocks are of the same size, then $C_d = \underline{r}_d^{\delta} - \frac{1}{k} \mathcal{N}_d \mathcal{N}_d' = \underline{r}_d^{\delta} - \frac{1}{k} \Lambda_d$.

At this point we present an example. Here and always a design will be represented as an array of elements of $\{1,2,\ldots,v\}$ with the b columns representing the blocks.

Example 1.1.1: Hedayat and Federer (1974). $v = 5$, $b = 6$, and d below is variance balanced (see Definition 1.3.1).

$$d: \begin{matrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 5 & 5 & 5 \\ 3 & 3 & & & & \\ 4 & 4 & & & & \end{matrix}$$

$\underline{r}'_d = (3,3,3,3,4)$, $\underline{k}' = (4,4,2,2,2,2)$,

$$\eta_d = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\Lambda_d = \eta_d \eta'_d = \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ & 3 & 2 & 2 & 1 \\ & & 3 & 2 & 1 \\ & & & 3 & 1 \\ \text{sym} & & & & 4 \end{bmatrix},$$

where "sym" will indicate the matrix is symmetric, and

$$C_d = \underline{r}_d^{\delta} - \eta_d \underline{k}^{-\delta} \eta'_d$$

$$= \frac{1}{2} \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ & 4 & -1 & -1 & -1 \\ & & 4 & -1 & -1 \\ & & & 4 & -1 \\ \text{sym} & & & & 4 \end{bmatrix}.$$

1.2 The Problem and Motivation

The problem considered in this thesis will be how we can best allocate the treatments to the experimental units so that C_d has as large a trace as possible and its nonzero eigenvalues are as equal as possible. More rigorously, we will search for a design $d^* \in \mathcal{D}(v,b,\underline{k})$ that will minimize the value of a function $\Phi(C_d)$ for all $d \in \mathcal{D}$. d^* will be called <u>Φ-optimal</u>, and the function Φ is an <u>optimality criterion</u>. Before listing the classes of optimality criteria in the next section, some heuristic motivation for the problem will be given.

Historically experimenters have used designs like complete block designs (CBD's, with $k_1 = \ldots = k_b = v$) and balanced incomplete block designs (BIBD's, with $k_1 = \ldots = k_b < v$). These were chosen because they make the r_{di} equal and the λ_{dij} $(i \neq j)$ equal.

<u>Example 1.2.1</u>:

a)

	1	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2	2
d_1:	3	3	3	3	3	3	3	3	3
	4	4	4	4	4	4	4	4	4

This is a CBD.

b)

	1	1	1	2	2	3	1	1	1	2	2	3
d_2:	1	1	1	2	2	3	2	3	4	3	4	4
	2	3	4	3	4	4	2	3	4	3	4	4

This is variance balanced by Definition 1.3.1. It is a BIBD by some definitions, but not in this thesis.

c)	1	1	1	1	1	1	1	1	1	2	2	2
d_3:	2	2	2	2	2	2	3	3	3	3	3	3
	3	3	3	4	4	4	4	4	4	4	4	4

This is a BIBD according to Definition 1.2.1.

Each of the designs in Example 1.2.1 has C_d of the form $(a+b)I_v - bJ_{v,v}$ where I_v is the v by v identity matrix, and $J_{v,v}$ is the v by v matrix with all elements equal to 1. C_d in this form is called <u>completely symmetric</u>, and has v-1 equal nonzero eigenvalues. Also note that since the trace of C_d, call it $tr(C_d)$, is $\sum_{i=1}^{v} c_{dii} = \sum_{i=1}^{v} (r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k}) = N - \frac{1}{k} \sum_{j=1}^{b} (\sum_{i=1}^{v} n_{dij}^2)$, it is maximized when for each j the n_{dij} are as equal as possible. (This is under the restrictions $n_{dij} \geq 0$, $\sum_{i=1}^{v} n_{dij} = k$ for each j.) This means the designs in a) and c) of the example also maximize $tr(C_d) = \sum_{i=0}^{v-1} \mu_{di} = \sum_{i=1}^{v-1} \mu_{di}$.

In this thesis the BIBD will be required to have the maximum possible trace in its class, so the CBD and BIBD will maximize the sum of their v-1 nonzero and equal eigenvalues. Kiefer (1958, 1975) generalized these maximum trace and balanced designs with the following definition.

<u>Definition 1.2.1</u>: For given v, b, and k a design d is said to be a <u>balanced block design</u> (BBD) if (1) all the r_{di} are equal, (2) all the λ_{dij} are equal $(i \neq j)$, (3) all the λ_{dii} are equal, and (4) for each (i,j) $|n_{dij} - k/v| < 1$. The CBD has k = v and we will define a BIBD to be a BBD with k < v.

Optimality criteria used in this thesis to measure the "goodness" of a design will allow the BBD defined above to be optimal, and reward larger $\sum_{i=1}^{v-1} \mu_{di}$ and μ_{di} $(i = 1,\ldots,v-1)$ that are nearly equal when a BBD does not exist. Two such criteria will be $\Phi_A(C_d) = \sum_{i=1}^{v-1} \mu_{di}^{-1}$ and $\Phi_D(C_d) = \prod_{i=1}^{v-1} \mu_{di}^{-1}$ (defined to be $+\infty$ if rank $(C_d) < v-1$).

Alternately, one can look at a solution to the normal equations for $\hat{\underset{\sim}{\alpha}}$. If $C_d\hat{\underset{\sim}{\alpha}} = (X_d^{(1)'} - \eta_d \underset{\sim}{k}^{-\delta} X_d^{(2)'})\underset{\sim}{Y}_d$, then one solution is $\hat{\underset{\sim}{\alpha}} = C_d^-(X_d^{(1)'} - \eta_d \underset{\sim}{k}^{-\delta} X_d^{(2)'})\underset{\sim}{Y}_d$ where C_d^- is the Moore-Penrose inverse of C_d. The covariance matrix of $\hat{\underset{\sim}{\alpha}} = \mathrm{Cov}(\hat{\underset{\sim}{\alpha}}) = \sigma^2 C_d^-$ in this case. We want to make this covariance matrix small in some sense, and one sense is through $\mathrm{tr}(\mathrm{Cov}(\hat{\underset{\sim}{\alpha}})) = \sum_{i=0}^{v-1}$ (variances of $\hat{\alpha}_i$) $= \sum_{i=0}^{v-1} \sigma^2 \lambda_{di}$ where the λ_{di} are the eigenvalues of C_d^-. Now $\mathrm{rank}(C_d) = \mathrm{rank}(C_d^-)$ and each nonzero eigenvalue of C_d^- is the reciprocal of a nonzero eigenvalue of C_d. Without loss of generality let $\lambda_{di} = \mu_{di}$ if $\mu_{di} = 0$ and $\lambda_{di} = \mu_{di}^{-1}$ if $\mu_{di} > 0$ $(i = 0,1,\ldots,v-1)$. Then minimizing $\mathrm{tr}(\mathrm{Cov}(\hat{\underset{\sim}{\alpha}}))$ corresponds to minimizing $\sigma^2 \sum_{i=1}^{v-1} \lambda_{di} = \sigma^2 \sum_{i=1}^{v-1} \mu_{di}^{-1} = \sigma^2 \Phi_A(C_d)$.

In regression settings the determinant $\det(\mathrm{Cov}(\hat{\alpha}))$ is minimized if $\mathrm{Cov}(\hat{\alpha})$ is nonsingular. So in this model minimization of $\prod_{i=1}^{v-1} \lambda_{di} = \prod_{i=1}^{v-1} \mu_{di}^{-1} = \Phi_D(C_d)$ for $\mu_{d1} > 0$ is also considered as a means of making $\mathrm{Cov}(\hat{\underset{\sim}{\alpha}})$ small.

Finally, we choose to work with C_d and $\mu_{d1} \leq \cdots \leq \mu_{d,v-1}$ because we are not interested in any particular v-1 linearly independent contrasts. If that were the case, we would use the appropriate reparameterization of the model.

1.3 The Optimality Criteria

In this section we will give the various optimality criteria $\Phi(C_d)$ to be minimized by some $d^* \in \mathcal{D}(v,b,\underline{k})$. Recall that C_d has the eigenvalues $\mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$, and note that the symbol $\mathcal{J}$ will be used with subscripts to label large classes of optimality criteria. Let $\mathcal{B}_{v,0}$ be the set of v by v symmetric non-negative definite matrices with zero row sums. Then any optimality criteria $\Phi(\cdot)$ will be a function from $\mathcal{B}_{v,0}$ into $(-\infty,\infty]$ satisfying restrictions that will vary from criterion to criterion. Unless otherwise stated, $\Phi(C_d) = +\infty$ if $\mu_{d1} = 0$.

(1) $\Phi_A(C_d) = \sum_{i=1}^{v-1} \mu_{di}^{-1}$. This is the A-optimality criterion from Kiefer (1959, 1975).

(2) $\Phi_D(C_d) = \prod_{i=1}^{v-1} \mu_{di}^{-1}$ or equivalently $\Phi_D^*(C_d) = -\sum_{i=1}^{v-1} \log \mu_{di}$. This is the D-optimality criterion of Kiefer (1958, 1959, 1975).

(3) $\Phi_E(C_d) = \max_{1 \leq i \leq v-1} (\mu_{di}^{-1}) = \mu_{d1}^{-1}$. This is the E-optimality criterion of Kiefer (1958, 1959, 1975). It is sometimes easier to think of this as maximizing μ_{d1}.

(4) $\mathcal{J}_p = \{\Phi_p(C_d) = \sum_{i=1}^{v-1} \mu_{di}^{-p}:\ 0 < p < \infty\}$. These criteria are essentially the Φ_p^* criteria from Kiefer (1975).

(5) $\mathcal{J}_1 = \{\Phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{di})$: f is real valued on $[0,c_0]$; f is continuous, strictly convex and strictly decreasing on $[0,c_0]$; f is continuously differentiable, and f' is strictly concave on $(0,c_0)\}$, where $c_0 = \max_{d\in\mathcal{D}}(\text{tr } C_d)$. Also we allow $f(0) = \lim_{x\to 0+} f(x) = +\infty$. These are the type 1 optimality criteria of Cheng (1978).

(6) $\mathcal{J}_2 = \{\Phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{di})$: f is real valued on $[0,c_0]$; f is continuous, strictly convex and continuously differentiable on $(0,c_0)$; f' is strictly concave on $(0,c_0)$; $\lim_{x\to 0+} f(x) = f(0) = +\infty\}$. Cheng (1981a).

(7) $\mathcal{J}_3 = \{\Phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{di})$: f is real valued, nonincreasing, convex and continuous on $[0,c_0]\}$. This class of criteria was introduced by Cheng (1979).

(8) $\mathcal{J}_4 = \{\Phi_F(C_d) = F(\mu_{d,v-1},\ldots,\mu_{d1})$: F is Schur-convex and nonincreasing in each argument on $[0,c_0]^{v-1}\}$. This was introduced to the author by Cheng via a personal communication, but is also in Constantine (1983).

(9) $\mathcal{J}_U = \{\Phi$: Φ is convex on $\mathcal{B}_{v,0}$; $\Phi(bC_d)$ is nonincreasing in the scalar $b \geq 0$; Φ is invariant under each permutation of rows and (the same) columns$\}$. This is the Universal Optimality class of Kiefer (1975).

All these criteria are minimized by a design d* if C_{d*} is completely symmetric and of maximum trace in $\mathcal{D}(v,b,\underline{k})$. A BBD of Kiefer will satisfy these conditions, as also will any variance balanced design d* with C_d^* of maximum trace (see Definition 1.3.1, Corollary 1.4.4, and Theorem 2.3.1). If such designs do not exist for a given $\mathcal{D}(v,b,\underline{k})$, then these criteria allow competing designs to be ranked. This is not the case with the following classical criteria, where a design satisfying the conditions does or does not exist, and no optimality function is given.

Definition 1.3.1: A design d is variance balanced if and only if $C_d = \mu(I_v - \frac{1}{v} J_v)$ with $\mu = tr(C_d)/(v-1)$. Rao (1958) and Puri and Nigam (1977).

Definition 1.3.2: A design d is efficiency balanced if and only if $C_d = \lambda(\underline{r}_d^\delta - \underline{r}_d\underline{r}_d'/N)$ with λ a constant of proportionality. Williams (1975) and Puri and Nigam (1977).

Definition 1.3.3: A design d is pairwise balanced if and only if $\Lambda = \underline{\theta}^\delta + c\underline{1}_v\underline{1}_v'$ where θ_i and c are real, $c > 0$, and $\underline{1}_v$ is the v by 1 vector of ones. See Puri and Nigam (1977).

It should be said that variance balance and efficiency balance are special cases of Calinski's (1977) X^{-1} balance, which is new but still does not allow for ranking competing designs that are not X^{-1}-balanced.

Finally we must mention the hierarchical relationships among the optimality criteria. Φ_A is a member of $\mathcal{J}_p$, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_U$. Φ_D is a member of $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_U$. Φ_E is a member of $\mathcal{J}_4$ and $\mathcal{J}_U$. $\mathcal{J}_p \subset \mathcal{J}_1 \subset \mathcal{J}_3 \subset \mathcal{J}_4 \subset \mathcal{J}_U$ and $\mathcal{J}_2 \subset \mathcal{J}_3$. And with the definition of generalized criteria (given below) we see that Φ_D

and Φ_E are limiting cases of $\mathcal{J}_p$, and Φ_E is also a limiting case of $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$.

Definition 1.3.4: A wide sense criterion Φ^o for a class $\mathcal{J}$ is the composition $\Phi^o = F \circ \Phi$ where $\Phi \in \mathcal{J}$ and F is non-decreasing.

Definition 1.3.5: A generalized criterion Φ^* for a class $\mathcal{J}$ is the pointwise limit of a sequence of wide sense criteria Φ_i^o for $\mathcal{J}$: $\Phi^*(C_d) = \lim_{i\to\infty} \Phi_i^o(C_d)$.

Cheng gave these two definitions for $\mathcal{J}_1$ in his paper of (1981b) so Φ_E would be a generalized $\mathcal{J}_1$ criterion. Without the nonincreasing F, Φ_E is not a limit of type 1 criteria, as he originally stated in his 1978 paper. Actually $\Phi_D(C_d) = \lim_{p\to 0+} (\frac{1}{v-1} \sum_{i=1}^{v-1} \mu_{di}^{-p})^{1/p}$ and $\Phi_E(C_d) = \lim_{p\to\infty}(\frac{1}{v-1} \sum_{i=1}^{v-1} \mu_{di}^{-p})^{1/p}$.

The symbol that will be used to represent a class of criteria and its generalized criteria will be $g\mathcal{J}$, where $\mathcal{J}$ was the original class of criteria. Note that $\Phi \in \mathcal{J}$ is a generalized criterion for $\mathcal{J}$ so $\mathcal{J} \subset g\mathcal{J}$. In the case of $\mathcal{J}_p$, $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$, Φ_E is not included in $\mathcal{J}$ but in $g\mathcal{J}$. $\mathcal{J}_4 = g\mathcal{J}_4$ as $\mathcal{J}_4$ is a closed convex cone of functions.

1.4 Results of Others

We will begin with four theorems that are important to this thesis and to this area of statistics. Then some other results that are called upon in later chapters will be presented.

Theorem 1.4.1: (Cheng (1978) Theorem 2.2) Let $\mathcal{C} = \{C_d: d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v > 2$. Suppose $C_{d*} \in \mathcal{C}$ has two distinct nonzero eigenvalues $\mu_2 > \mu_1$ with multiplicities 1 and v-2, respectively, and that

(i) $tr(C_{d*}) = \max_{d \in \mathcal{D}}(tr\ C_d)$,

(ii) $tr(C_{d*}^2) < (tr\ C_{d*})^2/(v-2)$, and

(iii) C_{d*} maximizes $tr\ C_d - [(v-1)/(v-2)]^{\frac{1}{2}}\ [tr(C_d^2) - (tr\ C_d)^2/(v-1)]^{\frac{1}{2}}$ for all $d \in \mathcal{D}$.

Then d* is optimal with respect to all generalized type 1 criteria ($g\mathcal{J}_1$-optimal) over all $d \in \mathcal{D}$.

Cheng applied this theorem to a subclass of the following class of designs.

Definition 1.4.1: A design $d \in \mathcal{D}(v,b,k)$ will be called a regular graph design (RGD) if (1) $r_{d1} = \ldots = r_{dv}$, (2) for each (i,j) we have $|n_{dij} - k/v| < 1$, and (3) for $i \neq j$, $i' \neq j'$ we have $|\lambda_{dij} - \lambda_{di'j'}| = 0$ or 1. An extreme RGD of type 1 has C_d with two distinct nonzero eigenvalues $\mu_2 > \mu_1$ with multiplicities 1 and v-2 respectively.

Let $\lambda_1(d)$ be the smaller value that the λ_{dij} take on, and $\lambda_2(d) = \lambda_1(d)+1$. Also let n(d) = the number of λ_{dij} in each row of Λ_d equal to $\lambda_1(d)$.

Theorem 1.4.2: (Cheng (1978) Theorem 3.1) Let $d^* \in \mathcal{D}(v,b,k)$ be an extreme RGD of type 1 with $\lambda_1(d^*) > 0$ or $\lambda_1(d^*) = 0$ and $n(d^*) < v/2$. Then d* is $g\mathcal{J}_1$-optimal over all $d \in \mathcal{D}(v,b,k)$.

Note: In Cheng's Theorem there is a uniqueness result not stated here.

In particular Theorem 1.4.2 can be applied to $d^* \in \mathcal{D}(v,b,k)$ when d^* is a most balanced group divisible partially balanced block design of type 1 (MB GD PBBD of type 1). That is d is a GD PBBD with 2 groups and $\lambda_2 = \lambda_1+1$ in the notation of GD designs.

Theorem 1.4.1 is important in the following way. If no design exists with maximum trace and all its eigenvalues equal, then Cheng's eigenvalue structure of $0 < a = \ldots = a < b$ is a logical guess for the next best form, and as it turns out, this structure with maximum trace and two conditions does give optimality over a large class of Φ. In Chapter 2 an optimality result with the eigenvalue structure $0 < a < b = \ldots = b$ will be proved as Theorem 2.1.1.

Theorem 1.4.2 has a proof for the situation where the r_{di} are all equal. This proof is modified in Chapter 3 to apply Theorem 1.4.1 to two situations where the r_{di} are not equal.

Constantine (1981) investigated some situations where BIBD's did not exist and proved the E-optimality of some designs. In this thesis we will adopt his terminology for those classes where no BIBD exists, some of the E-optimal designs will be proved optimal over a larger class of criteria, and his techniques for proving E-optimality will be extensively applied. Other authors, including Cheng (1980), Jacroux (1980a, 1980b, 1982), and Constantine (1982), have written papers on E-optimality techniques, but the straightforward techniques of Constantine (1981) are most useful here.

First we explain the terminology used for classes $\mathcal{D}(v,b,k)$ where no BIBD exists. The class $\mathcal{D}(4,5,3)$ does not contain a BIBD, but $\mathcal{D}(4,4,3)$ does. We can refer to $\mathcal{D}(4,5,3)$ as a class defined by a "BIBD plus one block". Another example of such a class is $\mathcal{D}(4,7,2)$ since $\mathcal{D}(4,6,2)$ contains a BIBD. In general the "BIBD plus m blocks" ($1 \leq m \leq v/k$) defines a parameter set $\mathcal{D}(v,b+m,k)$ where no BIBD exists, but one exists in $\mathcal{D}(v,b,k)$. Also the "BIBD minus m blocks" ($1 \leq m \leq v/k$) defines $\mathcal{D}(v,b-m,k)$ which does not contain a BIBD but where $\mathcal{D}(v,b,k)$ does.

In many cases a BIBD plus or minus m blocks will also describe a particular design in $\mathcal{D}(v,b+m,k)$ or $\mathcal{D}(v,b-m,k)$ that is constructed by adding or removing m blocks pairwise disjoint with respect to the treatments that they contain.

<u>Example 1.4.1</u>: Here is a design in $\mathcal{D}(4,8,2)$ that is a BIBD plus (the last) two disjoint blocks.

1	1	1	2	2	3	1	3
2	3	4	3	4	4	2	4

Secondly Constantine proved the E-optimality of a BIBD plus m binary ($n_{dij} = 0,1$) blocks for $1 \leq m < v/k$ in the classes $\mathcal{D}(v,b+m,k)$ where $\mathcal{D}(v,b,k)$ allowed a BIBD to be constructed. He proved the E-optimality of a BIBD minus m (binary) blocks for $v/k^2 \leq m \leq v/k$ in the classes $\mathcal{D}(v,b-m,k)$ where $\mathcal{D}(v,b,k)$ allowed a BIBD, and he proved the E-optimality of a GD PBIBD plus s binary blocks where $s < (v-m)/k$, the GD PBIBD has $\lambda_2 = \lambda_1+1$, m groups, and the blocks are compatible with the partition defined by the m groups. These theorems were Theorems 3.1, 3.3, and 3.4 in his paper. As will be shown in Chapter 3, some of these E-optimal designs are actually optimal for larger classes of criteria.

The third aspect of Constantine's paper that is relevant to this thesis involves his Theorem 1.4.3 below and the techniques used to prove it. As he points out, if $\{\sigma_i\}$ is a collection of n permutations on the symbols 1,2,...,v and C_d is a v by v information matrix, then we can define $\bar{C}_d = (1/n)\sum_{i=1}^{n} C_d^{\sigma_i} = (1/n)\sum_{i=1}^{n} P_i C_d P_i'$; P_i being the v by v matrix representation of σ_i. If $\mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$ and $\mu_{\bar{d}0} = 0 \leq \mu_{\bar{d}1} \leq \cdots \leq \mu_{\bar{d},v-1}$ are the eigenvalues of C_d and $\bar{C}_d$, respectively, then $\bar{C}_d$ is nonnegative definite with row sums zero, tr $\bar{C}_d$ = tr C_d, $\mu_{d1} \leq \mu_{\bar{d}1}$ and $\mu_{d,v-1} \geq \mu_{\bar{d},v-1}$. This technique can be applied to an information matrix C_d by selecting subsets of $\{1,2,\ldots,v\}$ and applying all the permutations of the symbols in each subset to that same subset. This smooths out the matrix C_d by averaging out the c_{dij}'s in blocks defined by those subsets. In particular Constantine used the fact that $\mu_{\bar{d}1} \geq \mu_{d1}$. Here are two of the three parts of his theorem.

Theorem 1.4.3: (Constantine (1981) Theorem 3.2) In any block design $d \in \mathcal{D}(v,b,k)$ the following inequalities hold:

(i) $\mu_{d1} \leq \frac{v}{v-1} \min_{1\leq i\leq v} \left(r_{di} - \frac{1}{k}\sum_{j=1}^{b} n_{dij}^2\right)$

(ii) $\mu_{d1} \leq \min_{2\leq n\leq v} \left[\frac{k-1}{kn}\sum_{i=1}^{n} r_{di} + \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n} \lambda_{dij}\right]$

This theorem will be extended to unequal k_j with the same method of proof in Theorem 2.2.1. For (i) above $(v/(v-1))(r_{di} - k^{-1} \sum_{j=1}^{b} n_{dij}^2)$ is an eigenvalue of $\bar{C}_d$ if you permute over the set of treatments excluding i. Then $\mu_{d1} \leq \mu_{\bar{d}1}$ which is less than or equal to this particular eigenvalue.

Kiefer (1975) proved the following theorem and applied it to BBD's.

Theorem 1.4.4: (Kiefer (1975) Proposition 1) Suppose there exists a $d^* \in \mathcal{D}(v,b,k)$ such that (1) C_{d^*} is completely symmetric, and (2) $\text{tr}\, C_{d^*} = \max_{d \in \mathcal{D}}(\text{tr}\, C_d)$. Then d* is Universally Optimal ($\mathcal{J}_U$-optimal) in $\mathcal{D}(v,b,k)$.

Corollary 1.4.4: (Kiefer (1974) p. 28) If $d^* \in \mathcal{D}(v,b,k)$ is a BBD then d* is $\mathcal{J}_U$-optimal in $\mathcal{D}(v,b,k)$.

Theorem 1.4.4 will be applied in the proof of Theorem 2.3.1, an extension of Corollary 1.4.4 to unequal k_j.

We conclude this section by presenting some less pivotal but nonetheless needed results due to others.

Theorem 1.4.5: (Cheng (1983)) Let $\mathcal{G} \subset \mathbb{R}^{v-1}$. If there exists a $\underline{\mu}^{*\prime} = (\mu_{v-1}^*, \ldots, \mu_1^*) \in \mathcal{G}$ such that

(i) $\mu_{v-1}^* = \ldots = \mu_2^* > \mu_1^*$,

(ii) $\underline{\mu}^*$ maximizes $\sum_1^{v-1} \mu_i$ over $\mathcal{G}$,

(iii) $\underline{\mu}^*$ minimizes $\sum_1^{v-1} \mu_i^{-q}$ over $\mathcal{G}$ for some $q > 0$, then $\underline{\mu}^*$ minimizes $\sum_{i=1}^{v-1} \mu_i^{-p}$ for all $0 < p \leq q$ and minimizes $\prod_{i=1}^{v-1} \mu_i^{-1}$, the generalized limiting case.

Lemma 1.4.1: (See Constantine (1981) Lemma 2.2.) Let C_d be an information matrix of the form

$$\begin{bmatrix} (a+b)I_p - bJ_{p,p} & -cJ_{p,v-p} \\ -cJ_{v-p,p} & (d+e)I_{v-p} - eJ_{v-p,v-p} \end{bmatrix}.$$

Then C_d has the eigenvalues $(a+b)$, $(d+e)$, vc and 0 with multiplicities $p-1$, $v-p-1$, 1 and 1 respectively.

Lemma 1.4.2: (a: Rao (1973) p. 68. b: See Remark 1.4.1)

Let the matrices $A = (a+b)I_q - bJ_{q,q}$, $B = -cJ_{q,q}$, $D = -dJ_{q,v-mq}$ and and $E = (e+f)I_{v-mq} - fJ_{v-mq,v-mq}$.

a) The v by v information matrix

$$\begin{bmatrix} A & B & \cdots & B & B \\ & A & \cdots & B & B \\ & & \ddots & \vdots & \vdots \\ & & & A & B \\ \text{sym} & & & & A \end{bmatrix}$$

with $v = mq$, $q \geq 3$ has the eigenvalues 0, vc, and $(a+b)$ with multiplicities 1, $m-1$ and $m(q-1)$ respectively.

b) The v by v information matrix

$$\begin{bmatrix} A & B & \cdots & B & B & D \\ & A & \cdots & B & B & D \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & A & B & D \\ & & & & A & D \\ & & & & & E \end{bmatrix}$$

with $v-mq \geq 1$, $q \geq 2$ has the eigenvalues 0, vd, $(a-(q-1)b+qc)$, $(a+b)$ and $(e+f)$ with multiplicities 1, 1, m-1, m(q-1) and v-mq-1 respectively.

Remark 1.4.1: Part b) might have been used by Constantine (1981). In any case the eigenvectors corresponding to the eigenvalues in the order given are $\underset{\sim}{1}_v$;

$$\begin{bmatrix} -\left(\frac{v-mq}{mq}\right)\underset{\sim}{1}_{mq} \\ \underset{\sim}{1}_{v-mq} \end{bmatrix};$$

$$\begin{bmatrix} \gamma_{11}\underset{\sim}{1}_q \\ \vdots \\ \gamma_{1m}\underset{\sim}{1}_q \\ \mathcal{O}_{v-mq,1} \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_{m-1,1}\underset{\sim}{1}_q \\ \vdots \\ \gamma_{m-1,m}\underset{\sim}{1}_q \\ \mathcal{O}_{v-mq,1} \end{bmatrix};$$

$$\begin{bmatrix} \underset{\sim}{\alpha}_1 \\ \mathcal{O}_{q,1} \\ \vdots \\ \mathcal{O}_{q,1} \\ \mathcal{O}_{v-mq,1} \end{bmatrix}, \ldots, \begin{bmatrix} \underset{\sim}{\alpha}_{q-1} \\ \mathcal{O}_{q,1} \\ \vdots \\ \mathcal{O}_{q,1} \\ \mathcal{O}_{v-mq,1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathcal{O}_{q,1} \\ \mathcal{O}_{q,1} \\ \vdots \\ \underset{\sim}{\alpha}_1 \\ \mathcal{O}_{v-mq,1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathcal{O}_{q,1} \\ \mathcal{O}_{q,1} \\ \vdots \\ \underset{\sim}{\alpha}_{q-1} \\ \mathcal{O}_{v-mq,1} \end{bmatrix};$$

and

$$\begin{bmatrix} \mathcal{O}_{q,1} \\ \vdots \\ \mathcal{O}_{q,1} \\ \underline{\beta}_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathcal{O}_{q,1} \\ \vdots \\ \mathcal{O}_{q,1} \\ \underline{\beta}_{v-mq-1} \end{bmatrix}$$

where the q by 1 vectors $\underline{\alpha}_1,\ldots,\underline{\alpha}_{q-1}$, $\underline{1}_q$ are pairwise orthogonal; the v-mq by 1 vectors $\underline{\beta}_1,\ldots,\underline{\beta}_{v-mq-1}$, $\underline{1}_{v-mq}$ are pairwise orthogonal; the m by 1 vectors $\underline{\gamma}_1,\ldots,\underline{\gamma}_{m-1}$, $\underline{1}_m$ are pairwise orthogonal with $\underline{\gamma}_i' = (\gamma_{i1},\ldots,\gamma_{i,m})$; and $\mathcal{O}_{x,y}$ is the x by y zero matrix.

Lemma 1.4.3: (Essentially Cheng (1978) Lemma 3.1)

a) For $\sum_{i=1}^{v} n_{dij} = k_j$ $(j = 1,\ldots,b)$, $tr(C_d)$ is maximized if the integers n_{dij} $(i = 1,\ldots,v)$ satisfy $|n_{dij}-k_j/v| < 1$, for each j.

b) For $tr(C_d)$ fixed, $tr(C_d^2)$ is minimized if the c_{dii} are as equal as possible and the c_{dij} $(i \neq j)$ are as equal as possible.

c) $P_d = tr(C_d^2)-(tr\ C_d)^2/(v-1) \geq 0$ with equality if and only if C_d is completely symmetric.

Lemma 1.4.4: (Essentially Takeuchi (1961) Lemma 1)

Let $F = aC_d + bI_v + (c/v)J_{v,v}$ where a,b and c are constants. If $0 = \mu_0 \leq \mu_1 \leq\cdots\leq \mu_{v-1}$ are the eigenvalues of C_d with corresponding eigenvectors $\underline{\lambda}_0 = \underline{1}_v$, $\underline{\lambda}_1,\ldots,\underline{\lambda}_{v-1}$ then the eigenvalues of F are b+c, $a\mu_1+b,\ldots,a\mu_{v-1} + b$ with the same corresponding eigenvectors.

1.5 Final Assumptions and Definitions with Outline of the Text

We begin with two restrictions on the collections $\mathcal{D}(v,b,\underline{k})$ of designs that will be adhered to in the remainder of this thesis. $\mathcal{D}(v,b,\underline{k})$ will contain only connected designs and connected designs where every block contains at least two treatments.

The first restriction is chosen because we want to be able to estimate no less than v-1 linearly independent contrasts. This is always possible except in the most trivial cases, and in fact most of the optimality criteria take on the value $+\infty$ in those cases.

The second restriction comes from the fact that a block with only one treatment applied to its experimental units provides no information in C_d. Since $C_d = C_d^{(1)} +\ldots+ C_d^{(b)}$ where $C_d^{(j)}$ is the C-matrix for the j-th block, if block b, say, has only treatment 1, then

$$C_d^{(b)} = \begin{bmatrix} k_b & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} - \begin{bmatrix} k_b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot k_b^{-1} \cdot [k_b, 0, \ldots, 0] = \mathcal{O}_{v,v}.$$

Therefore $C_d = C_d^{(1)} +\ldots+ C_d^{(b-1)}$.

Definition 1.5.1: A binary block is a block of size k_j where the n_{dij} $(i = 1,\ldots,v)$ are as equal as possible, thus taking on only two consecutive values.

In an incomplete block, $k_j < v$, the n_{dij} take on 0 or 1, and this corresponds to the classical definition. If $k_j = p_j v+q_j$, then q_j of the $n_{dij} = p_j+1$ and $v-q_j$ of the $n_{dij} = p_j$ $(p_j \geq 0)$.

Definition 1.5.2: The extra-replicated treatments of a binary block j are those for which $n_{dij} = p_j+1$.

Definition 1.5.3: Binary blocks are called disjoint if the extra-replicated treatments of each block, when considered as sets, are disjoint sets.

Definitions 1.5.1 through 1.5.3 are useful for $k_j \geq v$. Note that our BBD has binary blocks by definition.

If k = 2, then our second assumption above and Definition 1.5.1 force all blocks to be binary in $\mathcal{D}(v,b,2)$ for any v and b. Lemma 1.4.3 says all $d \in \mathcal{D}(v,b,2)$ will have the maximum trace.

A design $d \in \mathcal{D}$ with all blocks binary will be of maximum trace in $\mathcal{D}$ by Lemma 1.4.3.

A BBD in $\mathcal{D}(v,b,k)$ is sometimes represented as a BBD(v,b,k,r,λ) where r and λ are the common values of the r_{di} and λ_{dij} $(i \neq j)$ respectively. We will use this notation when it is useful.

Definition 1.5.4 d_1 is said to be $\mathcal{J}$-better than d_2 if and only if $\Phi(c_{d1}) \leq \Phi(c_{d2})$ for all $\Phi \in \mathcal{J}$. Strictly $\mathcal{J}$-better is used when the inequality is strict.

The notation int[x] will denote the greatest integer less than or equal to x.

Recall that the subscript d on various quantities may be dropped in later chapters if the context allows it.

Before concluding, a brief outline of the remainder of the thesis follows.

Chapter 2 contains original works inspired by Theorems 1.4.1, 1.4.3 and 1.4.4. Lemma 2.4.1 was inspired by Lemma 1.4.1 and necessity.

Chapter 3 looks at applications of Theorem 1.4.1 and Theorem 2.1.1 to BIBD's plus or minus 1 block.

Chapter 4 looks at applications of Theorem 2.1.1 to the BBD plus one block of size v-1, and Chapter 5 at the BBD minus one observation.

Chapters 6 and 7 discuss the BBD plus or minus m blocks, respectively, with applications of Theorem 2.1.1.

Chapter 8 gives selected A-, D- and E-efficiencies for various interesting designs discussed in Chapters 3 through 7.

Finally Chapter 9 discusses trend-free block designs of Bradley and Yeh (1980) in the light of applying Theorem 2.1.1 and extending one of their later results.

CHAPTER 2

GENERAL THEORETICAL RESULTS

2.1 Results for Schur-convex Optimality Criteria

Before stating and proving the original results in this section some definitions and theorems from the theory of majorization must be presented. They will be paraphrased from Marshall and Olkin (1979) with their location in that book given parenthetically. For this discussion on majorization, any vector $\underline{x}' = (x_1,\ldots,x_n) \in \mathbb{R}^n$ will have $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$ be the ordered x_i.

Definition 2.1.1: (Definition 1.A.1) For $\underline{x}$, $\underline{y} \in \mathbb{R}^n$, $\underline{y}$ majorizes $\underline{x}$, written $\underline{y} \succ \underline{x}$, if and only if we have

(i) $$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1,\ldots,n-1$$

and $$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

or equivalently we have

(ii) $$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \quad k = 1,\ldots,n-1$$

and $$\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Definition 2.1.2: (Definition 1.A.2) For $\underline{x}$, $\underline{y} \in \mathbb{R}^n$; $\underline{y}$ weakly majorizes $\underline{x}$, written $\underline{y} \overset{w}{\succ} \underline{x}$, if and only if we have

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \quad k = 1,\ldots,n.$$

Remark 2.1.1: This is the weak supermajorization defined by Marshall and Olkin (1979).

Definition 2.1.3: (Definition 3.A.1) A function $F\colon \mathcal{A} \subset \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex on $\mathcal{A}$ if and only if $\underline{x} \prec \underline{y}$ on $\mathcal{A}$ implies $F(\underline{x}) \leq F(\underline{y})$. F is strictly Schur-convex if in addition $F(\underline{x}) < F(\underline{y})$ whenever $\underline{x} \prec \underline{y}$ but $\underline{x}$ is not a permutation of $\underline{y}$.

Lemma 2.1.1: (Proposition 4.B.2) $\underline{x} \prec^w \underline{y}$ if and only if $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ holds for all continuous, convex nonincreasing $f\colon \mathbb{R} \to \mathbb{R}$.

Lemma 2.1.2: (Theorem 3.A.8 with definition p. 443) A function $F\colon \mathcal{A} \subset \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\underline{x} \prec^w \underline{y} \text{ on } \mathcal{A} \text{ implies } F(\underline{x}) \leq F(\underline{y})$$

if and only if F is Schur-convex and nonincreasing in each argument on $\mathcal{A}$.

Lemma 2.1.3: (Theorem 3.A.8a) F is a real-valued function on $\mathcal{A} \subset \mathbb{R}^n$. Then $\underline{x} \prec^w \underline{y}$ on $\mathcal{A}$ and $\underline{x}$ is not a permutation of $\underline{y}$ imply $F(\underline{x}) < F(\underline{y})$ if and only if F is strictly Schur-convex and strictly decreasing in each argument on $\mathcal{A}$.

Lemmas 2.1.1 and 2.1.2 provide some motivation for $\mathcal{J}_3$ and $\mathcal{J}_4$ optimality criteria, respectively. Lemma 2.1.3 is presented to illustrate what is needed for strict inequality in $F(\underset{\sim}{x}) \leq F(\underset{\sim}{y})$.

Since optimality criteria are thought of as functions of C_d we write $\Phi(C_d) = F(\underset{\sim}{\mu}_d)$ where $\underset{\sim}{\mu}' = (\mu_{d,v-1},\ldots,\mu_{d1})$ is the vector of eigenvalues of C_d and $0 < \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$.

Now we begin the original work.

<u>Theorem 2.1.1</u>: Let $\mathcal{C} = \{C_d: d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v \geq 3$. Suppose $C_{d^*} \in \mathcal{C}$ has

(i) two distinct eigenvalues $\mu_2^* > \mu_1^* > 0$ with multiplicities $v-2$ and 1 respectively,

(ii) $\operatorname{tr} C_{d^*} = \max_{d\in\mathcal{D}}(\operatorname{tr} C_d)$,

(iii) d^* E-optimal in $\mathcal{D}$.

Then d^* (or C_{d^*}) is optimal with respect to all $\mathcal{J}_4$ criteria over all $d \in \mathcal{D}$.

<u>Proof</u>: Let d be an arbitrary design in $\mathcal{D}$ with the eigenvalues of C_d being $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{v-1}$. Let $\operatorname{tr} C_d^* = (v-2)\mu_2^* + \mu_1^* = \operatorname{tr} C_d + \varepsilon = \sum_{i=1}^{v-1} \mu_i + \varepsilon$ with $\varepsilon \geq 0$. Now $(\mu_{v-1},\ldots,\mu_1) \overset{w}{\succ} (\mu_{v-1} + \varepsilon, \mu_{v-2},\ldots,\mu_1) \succ ((\varepsilon + \sum_{i=2}^{v-1} \mu_i)/(v-2),\ldots,(\varepsilon + \sum_{i=2}^{v-1} \mu_i)/(v-2),\mu_1) \succ (\mu_2^*,\ldots,\mu_2^*,\mu_1^*)$ with the last majorization justified as follows. $\mu_1^* \geq \mu_1$ by E-optimality implies $(v-2)\mu_2^* \leq \varepsilon + \sum_{i=2}^{v-1} \mu_i$. Using the first definition of majorization we have $k\mu_2^* \leq k(\varepsilon + \sum_{i=2}^{v-1} \mu_i)/(v-2)$ for $k = 1,2,\ldots,v-2$

and $(v-2)\mu_2^* + \mu_1^* = (v-2)(\varepsilon + \sum_{i=2}^{v-1} \mu_i)/(v-2) + \mu_1$.

Noting that $\underset{\sim}{y} \succ \underset{\sim}{x} \Rightarrow \underset{\sim}{y} \overset{w}{\succ} \underset{\sim}{x}$ we have $(\mu_{v-1},\ldots,\mu_1) \overset{w}{\succ} (\mu_2^*,\ldots,\mu_2^*,\mu_1^*)$ so for any $\mathcal{J}_4$ criterion $F(\mu_2^*,\ldots,\mu_2^*,\mu_1^*) \leq F(\mu_{v-1},\ldots,\mu_1)$, and that completes the proof.

The following two theorems are similar in statement and proof, but are for the eigenvalue structures $0 < a = \ldots = a < b$ (of Cheng) and $0 < a < b = \ldots = b < c$.

Theorem 2.1.2: Let $\mathcal{C} = \{C_d: \ d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v \geq 3$. Suppose $C_{d^*} \in \mathcal{C}$ has

(i) two distinct eigenvalues $\mu_2^* > \mu_1^* > 0$ with multiplicities 1 and v-2 respectively,

(ii) $\text{tr } C_{d^*} = \max_{d\in\mathcal{D}}(\text{tr } C_d)$,

(iii) $\mu_2^* = \mu_{d^*,v-1} \leq \mu_{d,v-1}$ for all $d \in \mathcal{D}$.

Then d* is optimal with respect to all $\mathcal{J}_4$ criteria over all $d \in \mathcal{D}$.

Proof: Let $d \in \mathcal{D}$ be arbitrary with C_d having eigenvalues $0 \leq \mu_1 \leq \cdots \leq \mu_{v-1}$ and $\text{tr } C_{d^*} = \mu_2^* + (v-2)\mu_1^* = \text{tr } C_d + \varepsilon = \sum_{i=1}^{v-1} \mu_i + \varepsilon$, $\varepsilon \geq 0$. $(\mu_{v-1},\ldots,\mu_1) \succ (\mu_{v-1}, (\sum_{i=1}^{v-2} \mu_i)/(v-2),\ldots,(\sum_{i=1}^{v-2} \mu_i)/(v-2)) \overset{w}{\succ} (\mu_{v-1}, (\varepsilon + \sum_{i=1}^{v-2} \mu_i)/(v-2),\ldots,(\varepsilon + \sum_{i=1}^{v-2} \mu_i)/(v-2)) \succ (\mu_2^*,\mu_1^*,\ldots,\mu_1^*)$ with the last majorization following from the fact that $\mu_2^* \leq \mu_{v-1}$ implies $(v-2)\mu_1^* \geq \varepsilon + \sum_{i=1}^{v-2} \mu_i$ so

$k\mu_1^* \geq k(\varepsilon + \sum_{i=1}^{v-2} \mu_i)/(v-2)$ for $k = 1,\ldots,v-2$ and $(v-2)\mu_1^* + \mu_2^* = (\varepsilon + \sum_1^{v-2} \mu_i) + \mu_{v-1}$.

Theorem 2.1.3: Let $\mathcal{C} = \{C_d\colon\ d \in \mathcal{D}\}$ be a class ~~of~~ matrices in $\mathcal{B}_{v,0}$ with $v \geq 4$. Suppose $C_{d*} \in \mathcal{C}$ has

(i) three distinct eigenvalues $\mu_3^* > \mu_2^* > \mu_1^* > 0$ with multiplicities 1, v-3 and 1 respectively,

(ii) $\text{tr } C_{d*} = \max_{d\in\mathcal{D}} (\text{tr } C_d)$,

(iii) d* E-optimal and $\mu_3^* = \mu_{d*,v-1} \leq \mu_{d,v-1}$ for all $d \in \mathcal{D}$.

Then D* is optimal with respect to all $\mathcal{J}_4$ criteria over all $d \in \mathcal{D}$.

Proof: Let $d \in \mathcal{D}$ be arbitrary with C_d having eigenvalues $0 \leq \mu_1 \leq \cdots \leq \mu_{v-1}$. Let $\text{tr } C_{d*} = \text{tr } C_d + \varepsilon_2$ $(\varepsilon_2 \geq 0)$, $\mu_{v-1} - \mu_3^* = \varepsilon_3 \geq 0$, and $\mu_1^* - \mu_1 = \varepsilon_1 \geq 0$.

$$(\mu_{v-1},\mu_{v-2},\ldots,\mu_2,\mu_1)$$

$$\succ (\mu_{v-1},(\sum_2^{v-2} \mu_i)/(v-3),\ldots,(\sum_2^{v-2} \mu_i)/(v-3),\mu_1)$$

$$= (\mu_{v-1},(\text{tr } C_d-\mu_1-\mu_3)/(v-3),\ldots,(\text{tr } C_d-\mu_1-\mu_3)/(v-3),\mu_1)$$

$$= (\mu_3^* + \varepsilon_3,\ (\text{tr } C_{d*} - \varepsilon_2 - (\mu_1^* - \varepsilon_1)-(\mu_3^* + \varepsilon_3))/(v-3),$$

$$\ldots,(\text{tr } C_{d*} - \varepsilon_2 - (\mu_1^* - \varepsilon_1)-(\mu_3^* + \varepsilon_3))/(v-3),\ \mu_1^* - \varepsilon_1)$$

$$= (\mu_3^* + \varepsilon_3, \mu_2^* + \frac{\varepsilon_1-\varepsilon_2-\varepsilon_3}{v-3}, \ldots, \mu_2^* + \frac{\varepsilon_1-\varepsilon_2-\varepsilon_3}{v-3}, \mu_1^* - \varepsilon_1)$$

$$\overset{w}{>} (\mu_3^*, \mu_2^*, \ldots, \mu_2^*, \mu_1^*)$$

where the last step follows from the definition of weak majorization as follows:

a) $\mu_1^* - \varepsilon_1 \leq \mu_1^*$

b) $\mu_1^* - \varepsilon_1 + k(\mu_2^* + \frac{\varepsilon_1-\varepsilon_2-\varepsilon_3}{v-3}) = \mu_1^* + k\mu_2^*$

$- \varepsilon_1(1 - \frac{k}{v-3}) - \frac{k}{v-3}(\varepsilon_2 + \varepsilon_3) \leq \mu_1^* + k\mu_2^*$.

for $k = 1,2,\ldots,v-3$.

c) $\mu_1^* - \varepsilon_1 + (v-3)(\mu_2^* + \frac{\varepsilon_1-\varepsilon_2-\varepsilon_3}{v-3}) + \mu_3^* + \varepsilon_3$

$= \mu_1^* + (v-3)\mu_2^* + \mu_3^* - \varepsilon_2 \leq \mu_1^* + (v-3)\mu_2^*$

$+ \mu_3^*$.

Hence $(\mu_{v-1}, \mu_{v-2}, \ldots, \mu_2, \mu_1) \overset{w}{>} (\mu_3^*, \mu_2^*, \ldots, \mu_2^*, \mu_1)$ which implies $\Phi(C_{d*}) \leq \Phi(C_d) = F(\underline{\mu}_d)$ for all Schur-convex F nonincreasing in each argument.

<u>Remark 2.1.2</u>: Theorems 2.1.2 and 2.1.3 are not as useful as Theorem 2.1.1 for the following reason. One can often find a connected design in any $\mathcal{D}$ that has $\mu_{d,v-1} < \mu^*_{d,v-1}$ if you let tr C_d be much less than tr C_{d*}. If $\mathcal{D}$ allows only one value for tr C_d, however, then the theorems could prove helpful.

If Theorems 1.4.1 or 2.1.1 fail to work, then the following Theorems 2.1.4 and 2.1.5, respectively, might be of use.

Theorem 2.1.4: Let $\mathcal{C} = \{C_d: d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ and $v \geq 3$. Suppose C_{d^0}, $C_{d^*} \in \mathcal{C}$ satisfy

1) $0 < \mu_1^0 = \ldots = \mu_{v-2}^0 < \mu_{v-1}^0$ for eigenvalues of C_{d^0},

2) $tr(C_{d^0}) = tr(C_{d^*}) = \max_{d \in \mathcal{D}}(tr\ C_d)$,

3) $\mu_1^0 \leq \mu_1^* \leq \mu_2^* \leq \ldots \leq \mu_{v-1}^*$ with the μ_i^*'s being the eigenvalues of C_{d^*}.

Then $\Phi(C_{d^*}) \leq \Phi(C_{d^0})$ for any $\Phi(C_d) = F(\mu_{d,v-1},\ldots,\mu_{d1})$ where F is Schur-convex on $[0,\infty)^{v-1}$. If in 3) we also have $\mu_i^0 < \mu_i^*$ for some $i \in \{1,2,\ldots,v-2\}$ then $\Phi(C_{d^*}) < \Phi(C_{d^0})$ for any $\Phi(C_d) = F(\mu_{d,v-1}, \ldots,\mu_{d1})$ where F is strictly Schur-convex on $[0,\infty)^{v-1}$.

Proof: $\sum_{i=1}^{k} \mu_i^* \geq k\mu_1^0 = \sum_{i=1}^{k} \mu_1^0$, $k = 1,\ldots,v-2$ and $\sum_{i=1}^{v-1} \mu_i^* = \sum_{i=1}^{v-1} \mu_i^0$ imply $(\mu_{v-1}^0,\ldots,\mu_1^0) \succ (\mu_{v-1}^*,\ldots,\mu_1^*)$ which implies $\Phi(C_{d^0}) \geq \Phi(C_{d^*})$ for the Schur-convex Φ. If in addition some $\mu_i^0 < \mu_i^*$ for some $i \in \{1,\ldots,v-2\}$ then $\underline{\mu}^*$ is not a permutation of $\underline{\mu}^0$ and $\Phi(C_{d^0}) > \Phi(C_{d^*})$ if Φ is strictly Schur-convex.

Corollary 2.1.4: Let $\mathcal{C} = \{C_d: d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v > 2$. Suppose $C_{d^*} \in \mathcal{C}$ has

1) $0 < \mu_1^* = \ldots = \mu_2^* < \mu_{v-1}^*$ for its eigenvalues.

2) $tr\ C_{d^*} = \max_{d \in \mathcal{D}}(tr\ C_d)$,

3) $\Phi(C_{d*}) \leq \Phi(C_d) = F(\mu_{d,v-1},\ldots,\mu_{d1})$ for any $d \in \mathcal{D}$ and F strictly Schur-convex.

Then any strictly E-better $d^0 \in \mathcal{D}$ has tr $C_{d^0} <$ tr C_{d*}. Hence d* is E-optimal among full trace designs.

Proof: Suppose C_{d^0} had eigenvalues $0 < \mu_1^0 \leq \ldots \leq \mu_{v-1}^0$ and $\mu_1^0 > \mu_1^*$. If tr C_{d^0} = tr C_{d*} then by Theorem 2.1.4 we would have $\Phi(C_{d^0}) < \Phi(C_{d*})$, contradicting Condition 3). Therefore an E-better d_0 is not of full trace.

Theorem 2.1.5: Let $\mathcal{C} = \{C_d: \ d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v \geq 3$. Suppose $C_{d^0}, C_{d*} \in \mathcal{C}$ satisfy

1) $0 < \mu_1^0 < \mu_2^0 = \ldots = \mu_{v-1}^0$ for the eigenvalues of C_{d^0},

2) tr $C_{d_1^0}$ = tr $C_{d*} = \max_{d \in \mathcal{D}}$ (tr C_d),

3) $0 < \mu_1^* \leq \ldots \leq \mu_{v-1}^* \leq \mu_{v-1}^0$ with the μ_i^*'s being the eigenvalues of C_{d*}.

Then $\Phi(C_{d*}) \leq \Phi(C_{d^0})$ for any $F(\mu_{d,v-1},\ldots,\mu_{d1})$ where F is Schur-convex on $[0,\infty)^{v-1}$. If in 3) we also have $\mu_i^* < \mu_i^0$ for some $i \in \{2,\ldots,v-1\}$ then $\Phi(C_{d*}) < \Phi(C_{d^0})$ for any $\Phi(C_d) = F(\mu_{d,v-1},\ldots,\mu_{d1})$ where F is strictly Schur-convex.

Proof: $\sum_{k}^{v-1} \mu_i^0 = (v-k)\mu_{v-1}^0 \geq \sum_{k}^{v-1} \mu_i^*$, $k = 2,\ldots,v-1$ and

$\sum_1^{v-1} \mu_i^0 = \sum_1^{v-1} \mu_i^*$ so we have $(\mu_{v-1}^0,\ldots,\mu_1^0) \succ (\mu_{v-1}^*,\ldots,\mu_1^*)$ which implies

$\Phi(C_{d^0}) \geq \Phi(C_{d^*})$. If in addition we have $\mu_i^* < \mu_i^0$ for some

$i \in \{2,\ldots,v-1\}$ then $\underline{\mu}^*$ is not a permutation of $\underline{\mu}^0$ and $\Phi(C_{d^0}) > \Phi(C_{d^*})$

if Φ is strictly Schur-convex.

Corollary 2.1.5: Let $\mathcal{C} = \{C_d\colon\ d \in \mathcal{D}\}$ be a class of matrices in

$\mathcal{B}_{v,0}$ with $v > 2$. Suppose $C_{d^*} \in \mathcal{C}$ has

1) $0 < \mu_1^* < \mu_2^* = \ldots = \mu_{v-1}^*$ for its eigenvalues,

2) $\text{tr } C_{d^*} = \max_{d\in\mathcal{D}} (\text{tr } C_d)$,

3) $\Phi(C_{d^*}) \leq \Phi(C_d) = F(\mu_{d,v-1},\ldots,\mu_{d1})$ for any $d \in \mathcal{D}$ and F strictly Schur-convex.

Then any $d^0 \in \mathcal{D}$ with $\mu_{v-1}^0 < \mu_{v-1}^*$ has $\text{tr } C_{d^0} < \text{tr } C_{d^*}$.

Proof: Suppose $\mu_{v-1}^0 < \mu_{v-1}^*$ for some d_0, C_{d^0} having eigenvalues

$0 < \mu_1^0 \leq\ldots\leq \mu_{v-1}^0$. If $\text{tr } C_{d^0} = \text{tr } C_{d^*}$ then by Theorem 2.1.5 we would

have $\Phi(C_{d^0}) < \Phi(C_{d^*})$, contradicting condition 3).

Theorem 2.1.1 is a $\mathcal{J}_4$ optimality result for d* with a given eigenvalue structure, C_{d^*} of maximum trace, and d* optimal for some criterion in the class $\mathcal{J}_4$. The eigenvalue structure is $0 < a < b = \ldots = b$ and the single criterion is $\Phi_E(C_d)$. A logical question is whether other global optimality results for $\mathcal{J}_4$ exist when given 1) an eigenvalue structure, 2) maximum trace and 3) optimality for one or two of the criteria in $\mathcal{J}_4$.

Here we present four numerical examples where a d* in a hypothetical $\mathcal{D}$ has one of four eigenvalue structures and is of maximum trace. Then for d* considered optimal for one or two of A-, D- and E-optimality examples are given which are strictly better than d* for the remaining criteria, indicating other global results of a similar kind do not exist.

Before we begin we must discuss v = 2 and v = 3. For v = 2 the lone eigenvalue is just tr C_d and the reader is referred to Corollary 2.3.1. For v = 3 there are just two eigenvalues and they are equal or different. The eigenvalue structure corresponding to them being different is just a special case of Theorems 1.4.1, 2.1.1 or 2.1.2 and the reader is referred to those.

In the tables below the nonzero eigenvalues for the hypothetical designs are given along with the criteria for which d* is optimal (in the column "d*"), the criteria for which d_i is optimal (in the column "d_i") and how tr C_d compares to tr C_{d*} (in the column "tr").

Example 2.1.1: $0 < \mu_{d^*_1} < \mu_{d^*_2} = \cdots = \mu_{d^*,v-1}$, $v \geq 4$. If d* is A-optimal we have Theorem 1.4.5. If d* is E-optimal we have Theorem 2.1.1. Otherwise see Table 2.1.1.

Table 2.1.1: Examples for $0 < a < b = \ldots = b$.

	μ_{d_1}	μ_{d_2}		$\mu_{d,v-1}$	d*	d_i	tr
d*:	4	5	(5) ... (5)	5	*	*	*
d_1:	4.1	4.5	(5) ... (5)	5.4	A,D	E	=
d_2:	4.1	4.6	(5) ... (5)	5.3	D	A,E	=

Example 2.1.2: $0 < \mu_{d_1^*} = \ldots = \mu^*_{d,v-2} < \mu_{d^*,v-1}$, $v \geq 4$. See Table 2.1.2. Note that if d* is not E-optimal but A- or D-optimal Corollary 2.1.4 says the E-better design cannot be of full trace. For d_1 $\varepsilon > 0$ and is taken as small as needed. For d_4 $\varepsilon,\sigma > 0$ and each is taken as small as needed.

Table 2.1.2: Examples for $0 < a = \ldots = a < b$

	μ_{d_1}		...		$\mu_{d,v-2}$	$\mu_{d,v-1}$	d*	d_i	tr
d*:	4	(4)	...	(4)	4	5	*	*	*
d_1:	$4+\varepsilon$		...		$4+\varepsilon$	$4+\varepsilon$	A,D	E	<
d_2:	3.7	(4)		(4)	4.65	4.65	A,E	D	=
d_3:	3.9	(4)		(4)	4.185	4.9	D,E	A	<
d_4:	$4+\varepsilon$		....		$4+\varepsilon$	$\dfrac{\left(\prod_1^{v-1} \mu_i^* - \sigma\right)}{(4+\varepsilon)^{v-2}}$	D	A,E	<
d_5:	3.8	(4)		(4)	4.6	4.6	E	A,D	=

Example 2.1.3: $0 < \mu_{d_1^*} = \dots = \mu_{d_i^*} < \mu_{d_{i+1}^*} = \dots = \mu_{d_{v-1}^*}$, $v \geq 5$,

$2 \leq i \leq v-3$. See Table 2.1.3. Note that d_1, d_5, d_6 and d_7 are special cases. d_7 is for $2 < i < v-3$ only. In d_2 and d_6 take $\varepsilon > 0$ small enough. In d_7 take $\varepsilon, \sigma > 0$ as needed with $(i-2)\varepsilon = (v-i-3)\sigma$.

Table 2.1.3: Examples for $0 < a = \dots = a < b = \dots = b$

d	μ_1	$(\mu_2)\dots(\mu_{i-1})$	μ_i	μ_{i+1}	$(\mu_{i+2})\dots(\mu_{v-2})$	μ_{v-1}	d^*	d_i	tr
d^*	4	(4) ... (4)	4	5	(5) ...(5)	5	*	*	*
d_1	4.1	(i=2)	4.1	4.1	(5) ...(5)	5.7	A,D	E	=
d_2	4+ε	...	4+ε	4+ε	(5) ...(5)	(6-(i+1)ε)	A,D	E	=
d_3	3.85	(4) ... (4)	4.19	4.96	(5) ...(5)	5	A,E	D	=
d_4	3.995	(4) ... (4)	4.105	4.6	(5) ...(5)	5.3	D,E	A	=
d_5	4.1	(i=2)	4.2	4.3	(5) ...(5)	5.4	D	A,E	=
d_6	4+ε	... 4.1	4.2	4.3	(i=v-3)	(5.4-(v-5)ε)	D	A,E	=
d_7	4+ε	... 4.1	4.2	4.3	(5-σ)...(5-σ)	5.4	D	A,E	=
d_8	3.9	(4) ... (4)	4.5	4.5	(5) ...(5)	5.1	E	A,D	=

Example 2.1.4: $0 < \mu_{d_1^*} < \mu_{d_2^*} = \dots = \mu_{d^*,v-2} < \mu_{d^*,v-1}$, $v \geq 4$.

The examples in Table 2.1.4 are for $v = 4$, but for $v \geq 5$ just add the eigenvalue 4 with multiplicity $v-4$.

Table 2.1.4: Examples for $0 < a < b = \ldots = b < c$

	μ_{d_1}	μ_{d_2}	μ_{d_3}	d*	d_i	tr
d*	3	4	5	*	*	*
d_1	3.2	3.4	5.4	A,D	E	=
d_2	2.9	4.5	4.6	A,E	D	=
d_3	2.96	4.5	4.5	D,E	A	<
d_4	3.2	3.6	5.2	D	A,E	=
d_5	2.98	4.51	4.51	E	A,D	=

Remark 2.1.3: In Examples and Tables 2.1.2, 2.1.3 and 2.1.4 no single d_i was found to be D- and E-better than an A-optimal d*. For our purposes, d_1 and d_2 must be taken together to show an A-optimal d* is D- and E-bettered. There may exist such a single design, or a result saying such a single design will not exist, but it is unknown to the author.

2.2 Bounding the Extreme Nonzero Eigenvalues

In this section we will extend Theorem 1.4.3 of Constantine to unequal k_j and to give two lower bounds on $\mu_{d,v-1}$.

We begin with two upper bounds on μ_{d_1}.

Theorem 2.2.1: In any block design $d \in \mathcal{D}(v,b,\underline{k})$ with replications $\underline{r}_d$ and eigenvalues of C_d being $0 = \mu_{d_0} \leq \mu_{d_1} \leq \cdots \leq \mu_{d,v-1}$ we have

$$\text{(i)}\quad \mu_{d1} \leq \left(\frac{v}{v-1}\right) \min_{1\leq i\leq v} \left[r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right]$$

and

$$\text{(ii)}\quad \mu_{d1} \leq \min_{2\leq n\leq v} \left\{\frac{1}{n} \sum_{i=1}^{n} \left(r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right) + \frac{2}{n(n-1)} \sum\sum_{1\leq i<j\leq n} \left(\sum_{\ell=1}^{b} \frac{n_{di\ell}n_{dj\ell}}{k_\ell}\right)\right\}.$$

Proof: (i) Let $d \in \mathcal{D}$ be arbitrary. By definition of C_d, $c_{dii} = r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}$ and $c_{dij}(i \neq j) = - \sum_{\ell=1}^{b} \frac{n_{di\ell}n_{dj\ell}}{k_\ell}$. We now take all permutations of the rows and (the same) columns except for the ith row. Averaging the resulting (v-1)! matrices we get $\bar{C}_d$. Without changing the eigenvalues of $\bar{C}_d$ we can interchange the first and i-th rows and columns. Then we have $\bar{C}_d$ equivalent to

$$\begin{bmatrix} r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j} & -\beta_i \underline{1}_{v-1} \\ -\beta_i \underline{1}'_{v-1} & (A_i+B_i)I_{v-1} - B_i J_{v-1,v-1} \end{bmatrix}$$

where $\beta_i = \frac{1}{v-1} \sum_{\substack{j=1 \\ j\neq i}}^{v} \left(\sum_{\ell=1}^{b} \frac{n_{di\ell}n_{dj\ell}}{k_\ell}\right)$. By Lemma 1.4.1 we know that one eigenvalue of $\bar{C}_d$ is $v\beta_i$. Since $\bar{C}_d$ has zero-row sums along with C_d,

$$v\beta_i = \frac{v}{v-1}\left(r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right).$$

Hence $\mu_{d1} \leq \mu_{\bar{d}1} \leq v\beta_i = \frac{v}{v-1}\left(r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right)$.

Since i was arbitrary, we get μ_{d1} less than $v\beta_i$ for each i and hence the result.

(ii) Now we average over varieties 1,...,n and n+1,...,v separately, for some $2 \leq n \leq v$. The averaged matrix $\bar{C}_d$ equals

$$\begin{bmatrix} (a_n+\alpha_n)I_n - \alpha_n J_{n,n} & -\beta_n J_{n,v-n} \\ -\beta_n J_{v-n,n} & (d_n+\Delta_n)I_{v-n} - \Delta_n J_{v-n,v-n} \end{bmatrix}$$

where $a_n = \frac{1}{n}\sum_{i=1}^{n}\left(r_{di} - \sum_{j=1}^{b}\frac{n_{dij}^2}{k_j}\right)$ and $\alpha_n =$

$$\frac{2}{n(n-1)}\sum_{1\leq i<j\leq n}\left(\sum_{\ell=1}^{b}\frac{n_{di\ell}n_{dj\ell}}{k_\ell}\right).$$

By Lemma 1.4.1 one eigenvalue of $\bar{C}_d$ is $a_n+\alpha_n$, so we have $\mu_{d1} \leq \mu_{\bar{d}1} \leq a_n+\alpha_n$. Therefore $\mu_{d1} \leq \min_{2\leq n\leq v}(a_n+\alpha_n)$ and we get the result. This ends the proof.

Remark 2.2.1: Part (i) of the theorem was proved by Kiefer (1958) and Chackrabarti (1963) by another method.

Theorem 2.2.2: In any block design $d \in \mathcal{D}(v,b,\underline{k})$ with replications $\underline{r}_d$ and C_d having eigenvalues $0 = \mu_{d0} \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1}$ we have

(i) $$\mu_{d,v-1} \geq \frac{v}{v-1}\max_{1\leq i\leq v}\left(r_{di} - \sum_{j=1}^{b}\frac{n_{dij}^2}{k_j}\right)$$

and (ii) $$\mu_{d,v-1} \geq \max_{2\leq n\leq v}\left\{\frac{1}{n}\sum_{\ell=1}^{n}\left(r_{d\ell} - \sum_{j=1}^{b}\frac{n_{d\ell j}^2}{k_j}\right) + \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n}\left(\sum_{\ell=1}^{b}\frac{n_{di\ell}n_{dj\ell}}{k_\ell}\right)\right\}.$$

Proof: The proof of (i) is analogous to the proof of (i) in Theorem 2.2.1. One eigenvalue of $\bar{C}_d$ is $\frac{v}{v-1}\left(r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right)$ except that here we use $\mu_{d,v-1} \geq \mu_{\bar{d},v-1} \geq \frac{v}{v-1}\left(r_{di} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j}\right)$ for each i. Hence $\mu_{d,v-1}$ must be bigger than the largest of the $(v/(v-1))c_{dii}$. Similarly for the proof of (ii) we use $\mu_{d,v-1} \geq \mu_{\bar{d},v-1} \geq \bar{a}_n + \alpha_n$ of the $\bar{C}_d$ given in the proof of (ii) of Theorem 2.2.1.

2.3 Uniform Optimality and Unequal Block Sizes

The following result extends that of Kiefer (1975), given earlier as Corollary 1.4.4, to unequal k_j. Hedayat (1974) had extended the result to unequal $k_j < v$, but here no such restriction is made. All three proofs are analogous.

Theorem 2.3.1: (Corollary to Theorem 1.4.4)

Suppose there exists a connected variance balanced block design $d^* \in \mathcal{D}(v,b,\underline{k})$ with parameters N, v, $\underline{k}$ and $\underline{r}_d^*$ satisfying for each block j

1) $n_{dij}^* = k_j/v$ when k_j/v is an integer $(i = 1,\ldots,v)$, or

2) the n_{dij}^*'s $(i = 1,\ldots,v)$ are as close together as possible when k_j/v is not an integer (that is $k_j - v\cdot\text{int}[k_j/v]$ of the n_{dij}^*'s equal $\text{int}[k_j/v] + 1$ and the rest of the n_{dij}^*'s equal $\text{int}[k_j/v]$.)

subject to $\sum_{i=1}^{v} n_{dij}^* = k_j$. Then d^* is $\mathcal{J}_u$-optimal in $\mathcal{D}(v,b,\underline{k})$.

Proof: By Definition 1.3.1 C_{d*} is completely symmetric, so all that remains is to prove C_{d*} is of maximum trace in $\mathcal{D}$. For an arbitrary design $d \in \mathcal{D}$,

$$\text{tr } C_d = \sum_{i=1}^{v} r_{di} - \sum_{i=1}^{v} \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j} = N - \sum_{j=1}^{b} \frac{1}{k_j} \left(\sum_{i=1}^{v} n_{dij}^2 \right).$$

Maximizing tr C_d subject to $\sum_{i=1}^{v} n_{dij} = k_j$ for each j is accomplished by making the n_{dij} ($i = 1,\ldots,v$) as equal as possible for each fixed j, since this minimizes $\sum_{i=1}^{v} (n_{dij}^2)$ for each fixed j. (See Lemma 1.4.3). But this has been accomplished by d*. Thus ends the proof.

Theorem 2.3.1 can be applied to the case of $v = 2$. This case is trivial but is given here for the sake of completeness.

Corollary 2.3.1: If $v = 2$ and $k_j \geq 2$ for $j = 1,\ldots,b$, then $d^* \in \mathcal{D}(2,b,\underline{k})$ is $\mathcal{J}_u$-optimal in $\mathcal{D}(2,b,\underline{k})$ if $n_{d*1j} = c_j$ for $k_j = 2c_j$ (k_j even) and $n_{d*1j} = c_j$ or c_j+1 for $k_j = 2c_j+1$ (k_j odd), where c_j is a positive integer.

Proof: For any $d \in \mathcal{D}(2,b,\underline{k})$,

$$C_d = \begin{bmatrix} c_{d11} & c_{d12} \\ c_{d12} & c_{d22} \end{bmatrix} = \begin{bmatrix} c_{d11} & -c_{d11} \\ -c_{d11} & c_{d11} \end{bmatrix}$$

since C_d is an information matrix. So C_{d*} is completely symmetric making d* variance balanced. d* is connected since $k_j \geq 2$ so treatments 1 and 2 are paired at least once in d*. Finally the n_{d*ij} are as equal as possible, so by Theorem 2.3.1 we are done.

2.4 An Eigenvalue Lemma

The following result gives the eigenvalues for an information matrix with three blocks down the diagonal. In a sense it extends Lemma 1.4.1 of Constantine for two blocks down the diagonal.

Lemma 2.4.1: The eigenvalues of the v by v information matrix C =

$$\begin{bmatrix} (a_1+b_1)I_{m_1}-b_1J_{m_1,m_1} & -c_{12}J_{m_1,m_2} & -c_{13}J_{m_1,m_3} \\ & (a_2+b_2)I_{m_2}-b_2J_{m_2,m_2} & -c_{23}J_{m_2,m_3} \\ & & (a_3+b_3)I_{m_3}-b_3J_{m_3,m_3} \end{bmatrix}$$

(where $m_1 + m_2 + m_3 = v$ and $m_1, m_2, m_3 \geq 1$) are 0, (a_1+b_1), (a_2+b_2), (a_3+b_3), $(A+(B_1^2+B_2)^{\frac{1}{2}})/2$ and $(A-(B_1^2+B_2)^{\frac{1}{2}})/2$ with multiplicities 1, m_1-1, m_2-1, m_3-1, 1 and 1, respectively, where $A = (m_1+m_2)c_{12} + (m_1+m_3)c_{13} + (m_2+m_3)c_{23}$, $B_1 = (m_1-m_2)c_{12} - (m_1+m_3)c_{13} + (m_2+m_3)\ c_{23}$ and $B_2 = 4m_1m_2(c_{12}-c_{13})(c_{12}-c_{23})$.

Proof: The eigenvector for 0 is of course $\underline{1}_v$. Let $\underline{\alpha}_1,\ldots,\underline{\alpha}_{m_1-1}, \underline{1}_{m_1}$ be pairwise orthogonal, $\underline{\beta}_1,\ldots,\underline{\beta}_{m_2-1},\underline{1}_{m_2}$ be pairwise orthogonal and $\underline{\gamma}_1,\ldots,\underline{\gamma}_{m_3-1},\underline{1}_{m_3}$ be pairwise orthogonal. Then (a_i+b_i) i = 1,2,3 have the eigenvectors

$$\begin{pmatrix} \underline{\alpha}_j \\ \mathcal{O}_{m_2,1} \\ \mathcal{O}_{m_3,1} \end{pmatrix} \quad j = 1,\ldots,m_1-1; \quad \begin{pmatrix} \mathcal{O}_{m_1,1} \\ \underline{\beta}_\ell \\ \mathcal{O}_{m_3,1} \end{pmatrix} \quad \ell = 1,\ldots,m_2-1;$$

and $\begin{pmatrix} \mathcal{O}_{m_{1},1} \\ \mathcal{O}_{m_{2},1} \\ \underline{\gamma}_p \end{pmatrix}$ $p = 1,\ldots,m_3-1$ respectively. If $m_i = 1$, we do not get the eigenvalue (a_i+b_i) and of course also do not have the corresponding eigenvectors.

The last two eigenvalues remain the same if c_{12}, c_{13}, and c_{23} are distinct $c_{12} = c_{13}$, or $c_{12} = c_{13} = c_{23}$. The proofs in the three cases differ, but if c_{12}, c_{13} and c_{23} are distinct the proof is as follows.

By Lemma 1.4.4 if C has eigenvalues $0 \le \mu_1 \le \cdots \le \mu_{v-1}$ and corresponding eigenvectors $\underline{\xi}_0, \underline{\xi}_1,\ldots,\underline{\xi}_{v-1}$ then $F = aC + bI_v + (c/v)J_{v,v}$ has the eigenvalues $b+c$, $a\mu_1 + b,\ldots,a\mu_{v-1}+b$ with the same corresponding eigenvectors. This method was used to get the eigenvalues (a_i+b_i) and 0, and leaves

$$[C + c_{13}J_{v,v}-(a_1+c_{13})I_v] \begin{bmatrix} \underline{1}_{m_1} \\ y_j\underline{1}_{m_2} \\ -(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j)\underline{1}_{m_3} \end{bmatrix}$$

$$= \lambda_j \begin{bmatrix} \underline{1}_{m_1} \\ y_j\underline{1}_{m_2} \\ -(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j)\underline{1}_{m_3} \end{bmatrix} = \lambda_j\underline{\sigma}_j (i = 1,2.)$$

to be solved for the eigenvalues of $C + c_{13}J_{v,v}-(a_1+c_{13})I_v$ with their corresponding eigenvectors $\underline{\sigma}_j$. We get the following three equations, with the last a linear combination of the first two.

$$(m_1-1)(c_{13}-b_1) + m_2y_j(c_{13}-c_{12}) = \lambda_j \qquad (3.4.1)$$

$$m_1(c_{13}-c_{12}) + (a_2-a_1)y_j + (m_2-1)(c_{13}-b_2)y_j$$

$$- m_3(c_{13}-c_{23})(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j) = \lambda_j y_j$$

$$m_2y_j(c_{13}-c_{23})-(a_3-a_1)(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j)$$

$$- (m_3-1)(c_{13}-b_3)(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j) = -\lambda_j(\frac{m_1}{m_3} + \frac{m_2}{m_3} y_j).$$

Substituting the value of λ_j from (3.4.1) into the second equation we get

$$y_j^2m_2(c_{12}-c_{13})+y_j(a_2-a_1-(m_2-1)b_2+(m_1-1)b_1+m_2c_{23}-m_1c_{13})$$

$$+m_1(c_{23}-c_{12}) = 0.$$

Solving for y_j, plugging back into (3.4.1), and solving for $\lambda_j+(a_1+c_{13})$ which are the eigenvalues of C (recall they have the same eigenvectors) gives $(A \pm (B_1^2 + B_2)^{\frac{1}{2}})/2$ for the last two eigenvalues of C.

Remark 2.4.1: An analogous result for four blocks down the diagonal has three of the eigenvalues as the solutions of a cubic equation and did not prove useful.

CHAPTER 3

THE BIBD PLUS OR MINUS M BLOCKS

3.1 The Eigenvalue Structures of Cheng and this Thesis

In this section we present two lemmas that give the eigenvalues for the C-matrices of designs constructed by adding disjoint binary blocks to a BIBD or removing disjoint (binary) blocks from a BIBD.

Lemma 3.1.1: When adding m disjoint binary blocks of size k to a BIBD d in $\mathcal{D}(v,b,k)$ with $1 \leq m \leq v/k$, the eigenvalues of the C-matrix of the resulting design d* in $\mathcal{D}(v,b+m,k)$ are

$$0 \text{ with multiplicity } 1,$$

$$\frac{v\lambda}{k} \text{ with multiplicity } v-m(k-1)-1, \text{ and}$$

$$\frac{v\lambda}{k} + 1 \text{ with multiplicity } m(k-1).$$

In particular C_{d*} has the eigenvalue structure $0 < (v\lambda/k) < (v\lambda/k)+1 = \dots = (v\lambda/k)+1$ when m = 1 and k = v-1 ($v \geq 3$) or when m = 2 and k = v/2 ($v \geq 4$). C_{d*} has the eigenvalue structure $0 < (v\lambda/k) = \dots = (v\lambda/k) < (v\lambda/k)+1$ when m = 1 and k = 2 ($v \geq 3$).

Proof: $C_{d*} = C_d + C_0$ where

$$C_d = \left(\frac{r(k-1)}{k} + \frac{\lambda}{k}\right)I_v - \frac{\lambda}{k} J_{v,v}$$

and C_0 is the C-matrix for the blocks added. Without loss of generality we can relabel the treatments 1 through v so that C_0 is of the form

$$\begin{bmatrix} A_1 & \mathcal{O}_{k,k} & \cdots & \mathcal{O}_{k,k} & \mathcal{O}_{k,v-km} \\ & A_2 & \cdots & \mathcal{O}_{k,k} & \mathcal{O}_{k,v-km} \\ & & \ddots & \vdots & \vdots \\ & & & A_m & \mathcal{O}_{k,v-km} \\ \text{sym} & & & & \mathcal{O}_{v-km,v-km} \end{bmatrix} \qquad (3.1.1)$$

where $A_i = I_k - k^{-1}J_{k,k}$ for $i = 1,\ldots,m$. Then C_{d*} is of the form

$$\begin{bmatrix} B_1 & C & \cdots & C & D \\ & B_2 & \cdots & C & D \\ & & \ddots & \vdots & \vdots \\ & & & B_m & D \\ \text{sym} & & & & B_{m+1} \end{bmatrix} \qquad (3.1.2)$$

where

$$B_i = \left(\frac{(r+1)(k-1)}{k} + \frac{\lambda+1}{k}\right)I_k - \frac{\lambda+1}{k} J_{k,k} \qquad i = 1,\ldots,m,$$

$$B_{m+1} = \left(\frac{r(k-1)}{k} + \frac{\lambda}{k}\right)I_{v-km} - \frac{\lambda}{k} J_{v-km,v-km},$$

and $\qquad C = (-\lambda/k)J_{k,k}$ and $D = (-\lambda/k)J_{k,v-km}$.

Note here that if $v = mk$ then C_0 and C_{d*} are of the form, respectively,

$$\begin{bmatrix} A_1 & \mathcal{O}_{k,n} & \cdots & \mathcal{O}_{k,k} \\ & A_2 & \cdots & \mathcal{O}_{k,k} \\ & & \ddots & \vdots \\ \text{sym} & & & A_m \end{bmatrix} \qquad (3.1.3)$$

and

$$\begin{bmatrix} B_1 & C & \cdots & C \\ & B_2 & \cdots & C \\ & & \ddots & \vdots \\ \text{sym} & & & B_m \end{bmatrix} . \tag{3.1.4}$$

From Lemma 1.4.2 the eigenvalues of C_{d*} are $(v\lambda/k)+1$, $v\lambda/k$ and 0 with the multiplicities given in the hypothesis. $m(k-1) = v-2$ for $m = 2$, $v = 2k$ and $m = 1$, $k = v-1$, and $m(k-1) = 1$ for $m = 1$, $k = 2$.

Lemma 3.1.2: When deleting m disjoint (binary) blocks of size k from a BIBD d in $\mathcal{D}(v,b,k)$ with $1 \leq m \leq v/k$, the eigenvalues of the C-matrix of the resulting design d* in $\mathcal{D}(v,b-m,k)$ are

0 with multiplicity 1,

$\frac{v\lambda}{k} - 1$ with multiplicity $m(k-1)$, and

$\frac{v\lambda}{k}$ with multiplicity $v-m(k-1)-1$.

In particular C_{d*} has eigenvalue structure $0 < (v\lambda/k)-1 < (v\lambda/k) = \ldots = (v\lambda/k)$ when $m = 1$ and $k = 2$ ($v \geq 3$) and C_{d*} has eigenvalue structure $0 < (v\lambda/k)-1 = \ldots = (v\lambda/k)-1 < (v\lambda/k)$ when $m = 1$, $k = v-1$ ($v \geq 3$) and when $m = 2$, $v = 2k$ ($v \geq 4$).

Proof: $C_{d*} = C_d - C_0$ where C_d is as shown in the proof of Lemma 3.1.1. Without loss of generality we can relabel the treatments so that C_0 is as shown at (3.1.1).

Then C_{d*} is of the form of (3.1.2) but now with

$$B_i = \left(\frac{(r-1)(k-1)}{k} + \frac{\lambda-1}{k}\right)I_k - \frac{\lambda-1}{k} J_{k,k}, \quad i = 1,\ldots,m, \text{ and } B_{m+1},$$

C and D still of the form given in (3.1.2). As before if $v = mk$ then

C_0 and C_{d*} are as given in (3.1.3) and (3.1.4) respectively, but with B_1 through B_m as defined here.

From Lemmas 1.4.2 the eigenvalues of C_{d*} are $(v\lambda/k)+1$, $v\lambda/k$ and 0 with the multiplicities given in the hypothesis. The rest of the proof follows easily.

The BIBD plus 1 block for k = 2, the BIBD minus 1 block for k = v-1, and the BIBD minus 2 blocks for k = v/2 give Cheng's eigenvalue structure. The first two cases were proved E-optimal by Constantine (1981) and are shown to be $g\mathcal{J}_1$-optimal in Section 3.2. The third case is just that of a MB GD PBIBD with $\lambda_2 = \lambda_1+1$. This was proved $g\mathcal{J}_1$-optimal by Cheng (1978). See Theorem 1.4.2.

The BIBD plus 1 block for k = v-1, the BIBD plus 2 blocks for k = v/2, and the BIBD minus 1 block for k = 2 yield the eigenvalue structure $0 < a < b = \ldots = b$. The first was shown to be E-optimal by Constantine (1981) and will be shown in Section 3.3 to be $\mathcal{J}_4$-optimal. The second and third cases will also be discussed in that section, though the former is not necessarily optimal and the latter does not yet have a full solution.

3.2 The Eigenvalue Structure $0 < a = \ldots = a < b$ and $g\mathcal{J}_1$-optimality

We will begin with the BIBD plus 1 block, k = 2.

Theorem 3.2.1: Let d^0 be a BIBD in $\mathcal{D}(v,b,2)$, $v \geq 3$. Add to d^0 one binary block of size 2 and call the resulting design d*. Then d* is $g\mathcal{J}_1$-optimal in $\mathcal{D}(v,b+1,2)$.

Proof: We will apply Theorem 1.4.1. Since $k = 2$, d^o is the union of e copies ($e \geq 1$) of the BIBD in $\mathcal{D}(v,v(v-1)/2,k)$ that has the $v(v-1)/2$ unique pairs of the v treatments as blocks. Therefore $C_{d^o} = (e/2)(vI_v-J_{v,v})$. Without loss of generality assume we add the block consisting of treatments 1 and 2. Then

$$C_{d*} = \frac{1}{2}\left[\begin{array}{cc|cccc} e(v-1)+1 & -(e+1) & -e & -e & \cdots & -e \\ -(e+1) & e(v-1)+1 & -e & -e & \cdots & -e \\ \hline & & e(v-1) & -e & \cdots & -e \\ & & & e(v-1) & \ddots & -e \\ \text{sym} & & & & \ddots & \vdots \\ & & \text{sym} & & & e(v-1) \end{array}\right]$$

From Lemma 3.1.2 and the fact that $\lambda = e$ for the BIBD d^o, or directly from Lemma 1.4.1, we have for C_{d*}: $\mu_{d*o} = 0$, $\mu_{d*1} = \cdots = \mu_{d*,v-2} = ev/2$, and $\mu_{d*,v-1} = (ev/2)+1$.

Since $k = 2$ then from what was said in Section 1.5, d* satisfies condition (i) of Theorem 1.4.1 trivially.

Now $\operatorname{tr} C_{d*} = e(v-1)+1 + e(v-1)(v-2)/2 = (ev(v-1)+2)/2$ and

$$\begin{aligned} \operatorname{tr}(C_{d*}{}^2) &= \sum_{i=1}^{v} \sum_{j=1}^{b} c_{d*_{ij}}{}^2 \\ &= \frac{2}{4}\left[(e(v-1)+1)^2+(e+1)^2+(v-2)e^2\right] \\ &\quad + \frac{(v-2)}{4}\left[e^2(v-1)^2+(v-1)e^2\right] \\ &= \frac{1}{4}\,(e^2v^2(v-1)+4ve+4). \end{aligned}$$

For condition (ii) of Theorem 1.4.1 we need to check to see if

$(v-2)\ tr(C_{d*}^{2}) < (tr\ C_{d*})^2$, or if

$$\frac{(v-2)}{4}\,(e^2v^2(v-1)+4ve+4) < \frac{1}{4}\,(ev(v-1)+2)^2 = \frac{1}{4}\,(e^2v^2(v-1)^2+4ev(v-1)+4).$$

This is true if and only if

$$(v-2)(e^2v^2(v-1)+4ve+4) < e^2v^2(v-1)^2+4ev(v-1)+4,$$

if and only if

$$e^2(v^4-3v^3+2v^2) + e(4v^2-8v)+4v-8 < e^2(v^4-2v^3+v^2) + e(4v^2-4v)+4,$$

if and only if

$$0 < e^2(v^3-v^2) + e(4v) + 12-4v = e^2(v^3-v^2) + 4v(e-1) + 12,$$

which is true as $e \geq 1$ and $v \geq 3$.

Now since every design in $\mathcal{D}(v,b+1,2)$ has the same trace, condition (iii) of Theorem 1.4.1 is equivalent to proving C_{d*} minimizes $tr(C_d^2) - (tr\ C_d)^2/(v-1)$ or minimizing $tr(C_d^2) = \sum_{i=1}^{v} c_{dii}^2 + 2 \sum\sum_{1\leq i<j\leq v} c_{dij}^2$.

From Lemma 1.4.3 this is accomplished by making the c_{dii} as equal as possible and the c_{dij} as equal as possible. Clearly since $(-2c_{dij})$ must be an integer, making $(-2c_{d12}) = m+1$, $(-2c_{dij}) = m$ for $i = 1,2$ and $j = 3,\ldots,v$ and $(-2c_{dij}) = m$ for $3 \leq i < j \leq v$ does just that for the c_{dij}'s. In the case of $k = 2$, $2c_{dii}$ is an integer, so making $2c_{d11} = 2c_{d22} = m(v-1)+1$ and $2cd_{ii} = m(v-1)$ for $i = 3,4,\ldots,v$ makes the c_{dii} as equal as possible. Hence $tr(C_d^2)$ is minimized by d* and we satisfy condition (iii).

Therefore by Theorem 1.4.1 d* is $g\mathcal{J}_1$-optimal, and the proof is complete.

For an example of such a design use the design in Example 1.4.1 but without the last block.

Theorem 3.2.2: Let d^o be a BIBD in $\mathcal{D}(v,b,v-1)$, $v \geq 3$. Remove from d^o one (binary) block and call the resulting design d*. Then d* is $g\mathcal{J}_1$-optimal in $\mathcal{D}(v,b-1,v-1)$.

Proof: Again we will apply Theorem 1.4.1. Since $k = v-1$, d^o is the union of e copies ($e \geq 1$) of the BIBD in $\mathcal{D}(v,v,v-1)$ that has the v subsets of size v-1 from $\{1,\ldots,v\}$ as blocks. Therefore $b = ev$ and

$$C_{d^o} = \frac{1}{v-1}\begin{bmatrix} e(v-1)(v-2) & -e(v-2) & \ldots & -e(v-2) \\ & e(v-1)(v-2) & \ldots & -e(v-2) \\ & & \ddots & \vdots \\ \text{sym} & & & e(v-1)(v-2) \end{bmatrix}. \qquad (3.2.1)$$

That is $r_{d_1^o} = \ldots = r_{d_v^o} = e(v-1)$, $\lambda_{d_{ij}^o} = e(v-2)$ for $1 \leq i < j \leq v$ and $k-1 = v-2$. We then have $C_{d*} = C_{d^o} - C_0$ where C_0 is the C-matrix for the removed block. Without loss of generality

$$C_0 = \frac{1}{v-1}\begin{bmatrix} (v-2) & -1 & \ldots & -1 & 0 \\ & (v-2) & \ldots & -1 & 0 \\ & & \ddots & & \vdots \\ & & & (v-2) & 0 \\ \text{sym} & & & & 0 \end{bmatrix} \qquad (3.2.2)$$

and then

$$C_{d*} = \frac{1}{v-1}\left[\begin{array}{cccc|c} (e(v-1)-1)(v-2) & -(e(v-2)-1) & \cdots & -(e(v-2)-1) & -e(v-2) \\ & (e(v-1)-1)(v-2) & \cdots & -(e(v-2)-1) & -e(v-2) \\ & & \ddots & \vdots & \vdots \\ \text{sym} & & & (e(v-1)-1)(v-2) & -e(v-2) \\ \hline & \text{sym} & & & e(v-1)(v-2) \end{array}\right].$$

The eigenvalues of C_{d*} are 0, $(v\lambda/k)-1$ and $(v\lambda/k)$ with multiplicities 1, v-2 and 1 respectively, where $\lambda = e(v-2)$ and $k = v-1$. So d* satisfies the eigenvalue structure of Theorem 1.4.1.

The condition (i), C_{d*} being of full trace, is satisfied since d* is binary. That is for each j, n_{d*ij} is 0 or 1 for i = 1,...,v.

We must now check condition (ii), that $(v-2)\mathrm{tr}(C_{d*}^2) < (\mathrm{tr}\ C_{d*})^2$. Well

$$\mathrm{tr}\ C_{d*} = \frac{1}{v-1}\left((v-1)(e(v-1)-1)(v-2) + e(v-1)(v-2)\right)$$

$$= (v-2)(ev-1)$$

$$\mathrm{tr}(C_{d*}^2) = \left(\frac{1}{v-1}\right)^2[(v-1)\{(e(v-1)-1)^2(v-2)^2 + (v-2)(e(v-2)-1)^2$$

$$+ e^2(v-2)^2\} + (v-1)e^2(v-2)^2 + e^2(v-1)^2(v-2)^2]$$

$$= \frac{1}{v-1}[(v-2)^2(ev-e-1)^2 + (v-2)(ev-2e-1)^2$$

$$+ 2e^2(v-2)^2 + e^2(v-1)(v-2)^2].$$

Condition (ii) holds if and only if $\mathrm{tr}(C_{d*}^2) < (v-2)(ev-1)^2$ or

$(v-2)(ev-e-1)^2 + (ev-2e-1)^2 + 2e^2(v-2) + e^2(v-1)(v-2) < (v-1)(ev-1)^2$.

With algebra we get the equivalent statement $0 > -ev(ev-2)$. This is true since $v \geq 3$ and $e \geq 1$.

The proof that condition (iii) holds will be more difficult than in Theorem 3.2.1. We begin by defining $P_d = tr(C_d^2) - (tr\ C_d)^2/(v-1)$ for a design d. Then condition (iii) asks if C_{d*} maximizes

$$tr\ C_d - [(v-1)/(v-2)]^{\frac{1}{2}} P_d^{\frac{1}{2}}. \qquad (3.2.3)$$

Now $(v-1)P_d = (v-2)\sum_1^v c_{dii}^2 + 2(v-1) \sum_{1\leq i<j\leq v}\sum c_{dij}^2 - 2\sum_{1\leq i<j\leq v}\sum c_{dii}c_{djj}.$

For $\sum_{i=1}^v c_{dii}$ fixed, Lemma 1.4.3 says $(v-1)P_d$ is minimized, or (3.2.3) maximized, with the c_{dii} and the c_{dij} as equal as possible respectively.

Now $(v-1)P_{d*} = v-2$, which follows from simple substitution and some algebra. Hence $tr\ C_{d*} - [(v-1)/(v-2)]^{\frac{1}{2}} P_{d*}^{\frac{1}{2}} = (v-2)(ev-1)-1$. Clearly then for $d \in \mathcal{D}(v,b-1,v-1)$ with $tr\ C_d \leq (v-2)(ev-1)-1$ then (iii) holds for those d and d* because by Lemma 1.4.3 $P_d \geq 0$. Therefore all we need to do is consider designs d with $tr\ C_d = tr\ C_{d*} - \alpha/(v-1)$, $\alpha = 0,1,2,\ldots,v-2$.

For $\alpha = 0$, or d with the same trace as d*, we see d* maximizes (3.2.3) since the c_{d*ii} are as equal as possible and the c_{d*ij} are as equal as possible. If $tr\ C_d = tr\ C_{d*}$, d must also be binary, so $c_{dii} = r_{di}(k-1)/k = r_{di}(v-2)/(v-1)$. So the c_{dii} are as equal as possible if and only if the r_{di} are as equal as possible, which is true for d*. If $(-2)\sum_{1\leq i<j\leq v}\sum c_{dij} = tr\ C_{d*} = (v-2)(ev-1) =$

$$\frac{ev(v-1)(v-2)-(v-1)(v-2)}{v-1} = (v-1)(v-2)\left(\frac{e(v-2)-1}{v-1}\right)$$

$$+ 2(v-1) \cdot \frac{e(v-2)}{v-1}$$

and $(-(v-1)c_{dij})$ is always an integer, then the c_{d*ij} are as equal as possible.

The difficulty then is when $\alpha = 1,2,\ldots,v-2$. Here reducing tr C_d a little may minimize P_d enough to make condition (iii) fail. We will now fix $\alpha \in \{1,\ldots,v-2\}$ and let d be arbitrary in the class $\mathcal{D}(\alpha) \subset \mathcal{D}(v,b-1,v-1)$ where $\mathcal{D}(\alpha) = \{d: \text{ tr } C_d = \text{tr } C_{d*} - \alpha/(v-1)\}$. (3.2.3) will still be maximized (for the fixed trace) if the diagonal and off-diagonal elements of C_d are as equal as possible for all $d \in \mathcal{D}(\alpha)$. Actually we will work with a C_{d_α} which may not correspond to a design $d \in \mathcal{D}(\alpha)$, but satisfies this smoothness requirement. The hypothetical design, which we will designate d_α, will have the C-matrix that maximizes (3.2.3) over $\mathcal{D}(\alpha)$, and (3.2.3) evaluated for d_α will be less than (3.2.3) evaluated with d*.

d_α will have $(v-1)(v-2) + \alpha$ c_{dij}'s equal to $(e(v-2)-1)/(v-1)$ and $2(v-1) - \alpha$ c_{dij}'s equal to $e(v-2)/(v-1)$. Note that for any real design α would have to be even, but for the purposes of the proof it does not matter. The question of how equal the diagonal elements can be in $\mathcal{D}(\alpha)$ remains to be answered.

First note that tr C_d for $d \in \mathcal{D}(\alpha)$ is $\alpha(v-1)^{-1}$ less than tr C_{d*} because the design is no longer binary. tr $C_d = \sum_{i=1}^{v} r_{di} - \frac{1}{(v-1)} \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2$ and we cannot reduce $\sum_{i=1}^{v} r_{di}$, so the n_{dij}'s no longer satisfy $|n_{dij}-(v-1)/v| < 1$ for all (i,j).

Secondly we note that for $d \in \mathcal{D}(\alpha)$, $r_{di}(v-1)-\sum_{j=1}^{b} n_{dij}^2$

$$\leq r_{di}(v-1)-r_{di} = r_{di}(v-2) \leq (r_{di}+1)(v-1)-(r_{di}+1)-\alpha$$

$$= (r_{di}+1)(v-2)-\alpha = r_{di}(v-2)+(v-2)-\alpha.$$

Therefore given $r_{di} < r_{dj}$, then $(v-1)c_{dii} \le (v-1)c_{djj}$ even if we maximize c_{dii} and minimize c_{djj} since $1 \le \alpha \le v-2$.

We claim that the c_{dii} are most equal in $\mathcal{D}(\alpha)$ if v-1 of the $((v-1)c_{dii}) = (r-1)(v-1)-(r-1) = (r-1)(v-2)$ and the remaining $((v-1)c_{dii}) = r(v-1)-r-\alpha = r(v-2)-\alpha$, where $r = r_{d_1^o} = \ldots = r_{d_v^o} = e(v-1)$. Let an arbitrary $d \in \mathcal{D}(\alpha)$ have $r_{d1} \le r_{d2} \le \cdots \le r_{dv}$. $r_{d*1} = \ldots = r_{d*,v-1} = r-1$, $r_{d*v-1} = r$ is the most equal configuration of the r_{di}, and is the one used in the claim. Also let $\alpha_i \ge 0$ be such that $\sum_{i=1}^{v} \alpha_i = \alpha$. Then d has $((v-1)c_{dii}) = r_{di}(v-1)-\sum_{j=1}^{b} n_{ij}^2 = r_{di}(v-1)-r_{di} -\alpha_i = r_{di}(v-2)-\alpha_i$, $i = 1,\ldots,v$.

Now $(r_{dv}(v-2)-\alpha_v,\ldots,r_{d1}(v-2)-\alpha_1) \succ (r_{dv}(v-2)-\alpha, r_{d,v-1}(v-2), \ldots,r_{d1}(v-2)) \succ (r(v-2)-\alpha,(r-1)(v-2),\ldots,(r-1)(v-2)) = (r_{d\alpha,v}(v-2), \ldots,r_{d_\alpha,1}(v-2))$. Therefore d_α makes the c_{dii} as equal as possible and maximizes (3.2.3) for all $d \in \mathcal{D}(\alpha)$. So all that remains is to compare (3.2.3) for d* and d_α.

$$(v-1)P_{d_\alpha} = (v-2)\left[(v-1)\,\frac{(r-1)^2(v-2)^2}{(v-1)^2} + \frac{(r(v-2)-\alpha)^2}{(v-1)^2}\right]$$

$$- 2\left[\frac{(v-1)(v-2)}{2} \cdot \frac{(r-1)^2(v-2)^2}{(v-1)^2} + (v-1)\,\frac{(r-1)(v-2)(r(v-2)-\alpha)}{(v-1)^2}\right]$$

$$+ (v-1)\left[((v-1)(v-2)+\alpha)\,\frac{(e(v-2)-1)^2}{(v-1)^2} + (2(v-1)-\alpha)\,\frac{e^2(v-2)^2}{(v-1)^2}\right]$$

$$= (v-2) + \frac{\alpha^2(v-2)}{(v-1)^2} - \frac{2\alpha v}{(v-1)} + \frac{5\alpha}{(v-1)} \qquad (3.2.4)$$

with algebra and substitution of e(v-1) for r.

$$\text{tr } C_{d*} - [(v-1)/(v-2)]^{\frac{1}{2}} P_{d*}^{\frac{1}{2}} = (v-1)(ev-1)-1 \geq$$

$$\text{tr } C_{d_\alpha} - [(v-1)/(v-2)]^{\frac{1}{2}} P_{d_\alpha}^{\frac{1}{2}} = (v-1)(ev-1) - \frac{\alpha}{v-1} - [(v-1)/(v-2)]^{\frac{1}{2}} P_{d_\alpha}^{\frac{1}{2}}$$

if and only if $(v-1)P_{d_\alpha} \geq (v-2)(1-\alpha/(v-1))^2$. That is if and only if

$$(v-1)P_{d_\alpha} \geq (v-2) - \frac{2\alpha(v-2)}{(v-1)} + \frac{\alpha^2(v-2)}{(v-1)^2}$$

$$= (v-2) + \frac{\alpha^2(v-2)}{(v-1)^2} - \frac{2\alpha v}{v-1} + \frac{4\alpha}{v-1}$$

which is obvious from (3.2.4) and the fact that $\alpha > 0$.

Since α was arbitrary in $\{1,2,\ldots,v-2\}$ we have shown C_{d*} maximizes (3.2.3) for all $d \in \mathcal{D}(v,b-1,v-1)$. Therefore condition (iii) of Theorem 1.4.1 is satisfied and d* is $g\mathcal{J}_1$-optimal in $\mathcal{D}(v,b-1,v-1)$.

Example 3.2.1. In $\mathcal{D}(5,4,4)$ d* given below is $g\mathcal{J}_1$-optimal.

	1	1	1	1
	2	2	2	3
d*:	3	3	4	4
	4	5	5	5

3.3 The Eigenvalue Structure 0 < a < b = ... = b and $\mathcal{J}_4$-optimality

We will begin with the case of the BIBD plus one block, k = v-1.

Theorem 3.3.1: Let d^o be a BIBD in $\mathcal{D}(v,b,v-1)$, $v \geq 3$. Add to d^o one binary block of size v-1 and call the resulting design d*. Then d* is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,b+1,v-1)$.

Proof: Here we will apply Theorem 2.1.1. From Lemma 3.1.1 we know the eigenvalues of C_{d*} are $0 < v\lambda/k < (v\lambda/k)+1 = \ldots = (v\lambda/k)+1$ where $k = v-1$ and $\lambda_{d^o ij} = \lambda$ for $1 \leq i < j \leq v$. d* maximizes tr C_d for all $d \in \mathcal{D}(v,b+1,v-1)$ because the blocks in d^o and the block added to d^o are binary. Finally, as mentioned in Section 1.4, Constantine (1981) in his Theorem 3.1 proved the E-optimality of the BIBD plus one binary block. Hence we have satisfied conditions (i) - (iii) of Theorem 2.1.1 and the proof is complete.

Example 3.3.1: In $\mathcal{D}(5,4,6)$ d* given below is $\mathcal{J}_4$-optimal.

	1	1	1	1	2	1
	2	2	2	3	3	2
d*:	3	3	4	4	4	3
	4	5	5	5	5	4

Another case for the eigenvalue structure $0 < a < b = \ldots = b$ is the BIBD plus 2 disjoint binary blocks with $v = 2k$, $(v \geq 4)$. Constantine (1981) referred to an example of Cheng (1979). There the BIBD(8,14,4,7,3) plus two disjoint binary blocks is A-, D- and E-bettered by an RGD that was given by John and Mitchell (1977), which is the union of the two cyclic designs generated by the blocks (1,2,3,5) and (1,2,3,6). However the BIBD(4,6e,2,3e,e), $e \geq 1$, plus two disjoint binary blocks is $\mathcal{J}_3$-optimal in $\mathcal{D}(4,6e+2,2)$. This was proved by Cheng (1979).

The final case for the eigenvalue structure of this thesis coming from Section 3.1 is that of the BIBD minus 1 block, k = 2. A BIBD

$d^o \in \mathcal{D}(v,b,2)$ must be the union of e ($e \geq 1$) copies of the BIBD in $\mathcal{D}(v,v(v-1)/2,2)$, which contains all pairings of the treatments 1 through v, so $b = ev(v-1)/2$.

Cheng (1981a) and Constantine (1983) proved the $\mathcal{J}_2$- and $\mathcal{J}_4$-optimality, respectively, of d^o minus one block if $e = 1$. Both proofs were graph theoretic, and do not hold for $e > 1$.

Theorem 2.1.1 can be applied, and will be in some cases, but it is the proof of the E-optimality of d^o minus one block that is difficult.

We begin a discussion and a series of lemmas that will become the proof that $d^* \in \mathcal{D}(v,b-1,2)$, where d^* is d^o minus one block, is E-optimal for v = 3,4,5, and 6. Then Theorem 2.1.1 will be applied in the form of Theorem 3.3.2.

Constantine (1981) proved the E-optimality of the BIBD minus m disjoint (binary) blocks for $vk^{-2} \leq m \leq vk^{-1}$. If $k = 2$, $v/4 \leq 1$ for v = 3 and 4 only. Hence we will get $\mathcal{J}_4$-optimality for v = 3,4, $e \geq 1$.

$C_{d^o} = (e/2)(vI_v - J_{v,v})$, and so C_{d^*}, without loss of generality, can be written as

$$\frac{1}{2}\left[\begin{array}{cc|cccc} (r-1) & -(\lambda-1) & -\lambda & -\lambda & \cdots & -\lambda \\ -(\lambda-1) & (r-1) & -\lambda & -\lambda & \cdots & -\lambda \\ \hline & & r & -\lambda & \cdots & -\lambda \\ & & & r & \cdots & -\lambda \\ & & & & \ddots & \\ \text{sym} & & \text{sym} & & & r \end{array}\right]$$

where $r = e(v-1)$ and $\lambda = e$, the replication and pairing parameters of d^o, respectively.

As in Constantine (1981), let $\mathcal{D}(v,b-1,2) = S_1 \cup S_2 \cup S_3$ where $S_1 = \{d \in \mathcal{D}: \min(r_{di}) \leq r-2\}$, $S_2 = \{d \in \mathcal{D}: \min(r_{di}) \geq r-1$, and $\exists$ $r_1 = r_2 = r-1$, say, with $\lambda_{12} \leq \lambda-1\}$ and $S_3 = \{d \in \mathcal{D}: \min(r_{di}) \geq r-1$ and $\lambda_{ij} \geq \lambda$ for all i,j such that $r_i = r_j = r-1\}$.

Lemma 3.3.1: If $d^* \in \mathcal{D}(v,b-1,2)$, $v \geq 3$, then $\Phi_E(C_{d^*}) \leq \Phi_E(C_d)$ for all $d \in S_1 \cup S_2$.

Proof: The proof of Constantine's (1981) Theorem 3.3 holds for $1 \leq m \leq v/k$ on $S_1 \cup S_2$. It is for S_3 that he needs $m \geq vk^{-2}$.

Lemma 3.3.2: If we let m be the number of treatments with $r_{di} = r-1$ for $d \in S_3$, then $\Phi_E(C_{d^*}) \leq \Phi_E(C_d)$ for any d with $m \geq v/2$. Assume $v \geq 3$.

Proof: Here we will apply Proposition 2.1c of Jacroux (1982), which says that if for all $i \neq j$ with $r_{di} = r_{dj} = r-1$ we have $\lambda_{dij} \geq \lambda$, then

$$\mu_{d1} \leq \frac{v((r-1)(k-1)-(m-1)\lambda)}{(v-m)k} \quad (3.3.1)$$

For k = 2 the right hand side of (3.3.1) is less than or equal to $\mu_{d^*1} = (v\lambda/k)-1 = (ev/2)-1$ when

$$\frac{v[(e(v-1)-1)-(m-1)e]}{2(v-m)} = \frac{v}{2(v-m)}(e(v-m)-1)$$

$$\leq \frac{ev}{2} - 1.$$

This is true when $ve(v-m)-v \leq ev(v-m)-2(v-m)$, if and only if $v \leq 2m$. This ends the proof.

Letting $S_3 = S_{3.1} \cup S_{3.2}$ where $S_{3.1} = \{d \in S_3: \ m < v/2\}$ and $S_{3.2} = \{d \in S_3: \ m \geq v/2\}$, then d* is E-optimal in $S_1 \cup S_2 \cup S_{3.2}$. So only the $d \in S_{3.1}$ remain.

Lemma 3.3.3: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r-1$ and $r_{dj} = r+b$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \leq \lambda-a$, $a \geq (b+3)/2$ (a, b are non-negative integers).

Proof: Without loss of generality we let i = 1, j = 2 and we average C_d over the permutations of {1,2}, and {3,...,v} separately. Then $\bar{C}_d$ =

$$\frac{1}{2}\begin{bmatrix} C_{11} & C_{12} \\ C_{12}' & C_{22} \end{bmatrix} \tag{3.3.2}$$

where $C_{11} = (r+((b-1)/2)+\lambda-a)I_2 - (\lambda-a)J_{2,2}$. By Lemma 1.4.1 one eigenvalue of $\bar{C}_d$ is $(r+((b-1)/2)+\lambda-a)/2 = (ev/2)+((b-1)/4) - a/2$. But $\mu_{d1} \leq \mu_{\bar{d}1} \leq$

$$\frac{ev}{2} + \frac{b-1}{4} - \frac{a}{2} \leq \frac{ev}{2} - 1$$

if and only if $a \geq (b+3)/2$.

Lemma 3.3.4: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r+x$ and $r_{dj} = r+y$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \leq \lambda-a$, $a \geq ((x+y)/2)+2$ (x, y, a are nonnegative integers).

Proof: Without loss of generality let i = 1, j = 2 and average C_d over the permutations of {1,2} and {3,...,v} separately. Again

$\bar{C}_d$ is of the form of (3.3.2). From C_{11} and Lemma 1.4.1 $\mu_{d1} \leq \mu_{\bar{d}1} \leq (e\lambda/2) + ((x+y)/4) - a/2 \leq (e\lambda/2)-1$ if and only if $a \geq ((x+y)/2)+2$.

Lemma 3.3.5: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r_{dj} = r-1$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \geq \lambda+1$.

Proof: Without loss of generality let $i = 1$, $j = 2$ and $\lambda_{d12} = \lambda+a$, $a \geq 1$. Averaging over C_d as in the two previous lemma's we get $\bar{C}_d$ of the form (3.3.2) with $C_{11} = (r-1+\lambda+a)I_2-(\lambda+a)J_{2,2}$, $C_{12} = -(\lambda-(a+1)/(v-2))J_{2,v-2}$ and $C_{22} = (r+\lambda+(2+2a)/((v-2)(v-3)))I_{v-2} - (2+2a)/((v-2)(v-3))J_{v-2,v-2}$. One eigenvalue of $\bar{C}_d$, via Lemma 1.4.1, is from C_{12} and equals

$$\frac{v}{2}\left(\lambda - \frac{a+1}{v-2}\right) = \frac{ev}{2} - \frac{v(a+1)}{2(v-2)}.$$

This is no bigger than $(ev/2)-1$ (and bounds $\mu_{\bar{d}1} \geq \mu_{d1}$) if and only if $(v(1+a))/(2v-4) \geq 1$ if and only if $a \geq (v-4)/4$ or $a \geq 1$ (since a is an integer and $v \geq 5$).

Lemma 3.3.6: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r+x$ and $r_{dj} = r+y$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \geq \lambda+a$ for $a \geq ((x+y)/2)+2-4/v$. (x,y, a are nonnegative integers).

Proof: Follows as in Lemma 3.3.5 with C_{12} yielding $\mu_{d1} \leq \mu_{\bar{d}1} \leq$

$$\frac{v}{2}\left(e - \frac{2a-(x+y)}{2(v-2)}\right) \leq \frac{ev}{2} - 1$$

if and only if

$$a \geq \frac{x+y}{2} + 2 - \frac{4}{v}.$$

Lemma 3.3.7: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r-1$ and $r_{dj} = r+b$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \geq \lambda+a$ where $2a-b \geq 3-8/v$ (a,b are nonnegative integers).

Proof: Follows as in Lemma 3.3.5 with C_{12} yielding $\mu_{d1} \leq \mu_{\bar{d}1} \leq$

$$\frac{v}{2}\left(e - \frac{2a-b+1}{2(v-2)}\right) \leq \frac{ev}{2} - 1$$

if and only if $2a-b \geq 3-8/v$.

Lemma 3.3.8: Let $d^* \in \mathcal{D}(5,10e-1,2)$ be a BIBD from $\mathcal{D}(5,10e,2)$ with one block removed. Then d^* is E-optimal in $\mathcal{D}(5,10e-1,2)$.

Proof: Lemma 3.3.1 says we need only look at $d \in S_3$, and in fact we can eliminate consideration of all but a few of the $d \in S_{3.1}$ with Lemmas 3.3.2 through 3.3.7.

Lemma 3.3.2 with $v = 5$ implies $2 \leq m < 5/2$, or $m = 2$ of the r_{di} will equal $r-1$. The rest will be equal to r.

Lemmas 3.3.1 and 3.3.5 say that if $r_{di} = r_{d2} = r-1$ then $\lambda_{d12} = \lambda$.

Lemmas 3.3.3 and 3.3.7 with $b = 0$ eliminate $\lambda_{dij} \leq \lambda-3/2$ and $\lambda_{dij} \geq \lambda+.7$ for $i = 1,2$ and $j = 3,4,5$. That is λ_{dij} for those i and j will be $\lambda-1$ or λ.

Lemmas 3.3.4 and 3.3.6 with $x = y = 0$ eliminate $\lambda_{dij} \leq \lambda-2$ and $\lambda_{dij} \geq \lambda+1.2$ for $i \neq j \in \{3,4,5\}$. That is if $r_{di} = r_{dj} = r$ then $\lambda_{dij} = \lambda-1$, λ or $\lambda+1$.

With a little work it is easy to see only two possible cases remain, and we will call them d_1 and d_2. Without loss of generality we have

$$2C_{d_1} = \begin{bmatrix} (r-1) & -\lambda & -\lambda & -\lambda & -(\lambda-1) \\ & (r-1) & -\lambda & -\lambda & -(\lambda-1) \\ & & r & -(\lambda-1) & -(\lambda+1) \\ & & & r & -(\lambda+1) \\ \text{sym} & & & & r \end{bmatrix}$$

and

$$2C_{d_2} = \begin{bmatrix} (r-1) & -\lambda & -\lambda & -\lambda & -(\lambda-1) \\ & (r-1) & -\lambda & -(\lambda-1) & -\lambda \\ & & r & -\lambda & -\lambda \\ & & & r & -(\lambda+1) \\ & & & & r \end{bmatrix}.$$

Now if we think of C_{d_1} as a blocked matrix with C_{11} and C_{22} each being 2 by 2, then we can apply Lemma 2.4.1 to get the eigenvalues of C_{d_1}. They are 0, $(5e-1)/2$ twice, and $(A \pm (B_1^2+B_2)^{\frac{1}{2}})/4$ where $A = 4\lambda + 3(\lambda-1) + 3(\lambda+1) = 10\lambda$, $B_1 = 0-3(\lambda-1)+3(\lambda+1) = 6$ and $B_2 = 4 \cdot 2 \cdot 2 \cdot 1 \cdot (-1) = -16$. The last two eigenvalues are then $(5e \pm (5)^{\frac{1}{2}})/2$, and $(5e-(5)^{\frac{1}{2}})/2 < (5e/2)-1 = \mu_{d^*1}$.

For C_{d_2} we average over the parameter sets {1,2}, {3} and {4,5} to get the blocked matrix

$$\frac{1}{2}\bar{C}_{d_2} = \begin{bmatrix} (r-1) & -\lambda & -\lambda & -(\lambda-\frac{1}{2}) & -(\lambda-\frac{1}{2}) \\ & (r-1) & -\lambda & -(\lambda-\frac{1}{2}) & -(\lambda-\frac{1}{2}) \\ & & r & -\lambda & -\lambda \\ & & & r & -(\lambda+1) \\ & & & & r \end{bmatrix}.$$

Applying Lemma 2.4.1 we get the eigenvalues of $\bar{C}_{d_2}$ to be 0,

$(5e \pm 1)/2$, and $(5e-1 \pm 1)/2$ or $(5e-2)/2$ and $5e/2$. $(5e-2)/2 = (5e/2)-1 = \mu_{d*1}$.

Therefore d* has $\mu_{d*1} \geq \mu_{d1}$ for all $d \in S_{3.1}$ and hence for all $d \in \mathcal{D}(5,10e-1,2)$. This completes the proof.

For v = 6 we need one more lemma.

Lemma 3.3.9: For $d \in \mathcal{D}(v,b-1,2)$, $v \geq 5$, if $r_{d1} = r_{d2} = r-1$, $\lambda_{d12} = \lambda$, $r_{d3} = r$ and $\lambda_{d13} = \lambda-1+a$, $\lambda_{d23} = \lambda-1+b$ then $\mu_{d1} \leq (ev/2)-1$ for $a = b = 0$ and $a+b \geq 4$ (a,b are nonnegative integers).

Proof: We average C_d over the treatment sets {1,2}, {3} and {4,...,v}. $\bar{C}_d$ =

$$\frac{1}{2}\begin{bmatrix} (r-1) & -\lambda & -(\lambda-1+(a+b)/2) & -(\lambda-\frac{a+b}{2(v-3)})J_{1,v-3} \\ -\lambda & (r-1) & -(\lambda-1+(a+b)/2) & -(\lambda-\frac{a+b}{2(v-3)})J_{1,v-3} \\ & & r & -(\lambda-\frac{a+b-2}{v-3})J_{1,v-3} \\ \text{sym} & & & C_{33} \end{bmatrix}$$

The eigenvalues of $\bar{C}_d$ are $0,(ev-1)/2$, $(ev/2)+(a+b-1)/((v-3)(v-4))$ with multiplicity v-4 and then two from Lemma 2.4.1 of the form $(A \pm (B_1^2+B_2)^{\frac{1}{2}})/4$.

$$A = 3(\lambda-1+\frac{a+b}{2})+(2+v-3)(\lambda-\frac{a+b}{2(v-3)}) + (1+v-3)(\lambda-\frac{a+b-2}{v-3})$$

$$= 2v\lambda-3+\frac{2(v-2)}{v-3}-\frac{2(a+b)}{v-3}$$

$$B_1 = (\lambda-1 + \frac{a+b}{2})-(2+v-3)(\lambda - \frac{a+b}{2(v-3)}) + (1+v-3)(\lambda - \frac{a+b-2}{v-3})$$

$$= (v-1)/(v-3)$$

$$B_2 = 4\cdot 2\cdot 1(\lambda-1 + \frac{a+b}{2} - \lambda + \frac{a+b}{2(v-3)})(\lambda-1 + \frac{a+b}{2} - \lambda + \frac{a+b-2}{v-3})$$

$$= 8(\frac{(v-2)(a+b)}{2(v-3)} -1)(\frac{(v-1)(a+b)}{2(v-3)} - \frac{v-1}{v-3}).$$

Now $(A-(B_1^2+B_2)^{\frac{1}{2}})/4 \leq (v\lambda/2)-1$ if and only if $(A-2v\lambda+4)^2 \leq B_1^2 + B_2$. With algebra we see this is true if and only if

$$(v-3)[(a+b)^2 2v-8(a+b)(v-2)+8] \geq 0.$$

But $F(w) = 2w^2v-8w(v-2)+8 \geq 0$ for all v when $w = 0$ or $w \geq 4$.

Remark 3.3.1: $\mu_{d1} \leq (ev/2)-1$ in Lemma 3.3.9 for $a+b = 2$, $v = 5$ and $a+b = 3$, $v = 5,6,7,8,9$.

Lemma 3.3.10: Let $d^* \in \mathcal{D}(6,15e-1, 2)$ be a BIBD from $\mathcal{D}(6,15e,2)$ minus one block. Then d^* is E-optimal in $\mathcal{D}(6,15e-1,2)$.

Proof: Lemma 3.3.1 says we need only look at S_3. As in the proof of Lemma 3.3.8 we eliminate all but a few $d \in S_{3.1}$.

Lemma 3.3.2 with $v = 6$ says that exactly two $r_{di} = r-1$. Without loss of generality let $r_{d1} = r_{d2} = r-1$.

Lemmas 3.3.1 and 3.3.5 force $\lambda_{d12} = \lambda$.

Lemmas 3.3.3 and 3.3.7 with $b = 0$ force $\lambda_{dij} = \lambda-1$ or λ, $i = 1,2$ and $j = 3,4,5,6$.

Lemmas 3.3.4 and 3.3.6 with $x = y = 0$ force $\lambda_{dij} = \lambda-1, \lambda$ or $\lambda+1$ for $i \neq j \in \{3,4,5,6\}$.

There are only two cases to consider at this point for the first two rows of C_d. One case has the first two rows of $2C_d$ =

$$\begin{bmatrix} (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda & -\lambda \\ -\lambda & (r-1) & -(\lambda-1) & -\lambda & -\lambda & -\lambda \end{bmatrix}$$

This is eliminated by Lemma 3.3.9. Hence all that remains is the case where the first two rows of $2C_d$ look like

$$\begin{bmatrix} (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda & -\lambda \\ -\lambda & (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda \end{bmatrix}$$

Now there are only three choices for λ_{d34}: $\lambda-1$, λ, $\lambda+1$. $\lambda-1$ will not yield a C-matrix as it forces $\lambda_{d56} = \lambda-2$. But $\lambda_{d34} = \lambda$, $\lambda+1$ give possible designs. The only two cases will be called d_1 and d_2.

$$C_{d_1} = \frac{1}{2}\begin{bmatrix} (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda & -\lambda \\ & (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda \\ & & r & -\lambda & -(\lambda+1) & -\lambda \\ & & & r & -\lambda & -(\lambda+1) \\ & & & & r & -(\lambda-1) \\ \text{sym} & & & & & r \end{bmatrix}.$$

No averaging technique works here using Lemma 1.4.1 or Lemma 2.4.1. However it turns out that $\underline{x}' = (1,-1,1,-1,1,-1)$ is an eigenvector of C_{d_1} and yields the eigenvalue $(r+\lambda-2)/2 = (e(v-1)+e-2)/2 = (6e/2)-1 = \mu_{d*1}$.

$$C_{d_2} = \frac{1}{2}\begin{bmatrix} (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda & -\lambda \\ & (r-1) & -\lambda & -(\lambda-1) & -\lambda & -\lambda \\ & & r & -(\lambda+1) & -\lambda & -\lambda \\ & & & r & -\lambda & -\lambda \\ & & & & r & -\lambda \\ & & & & & r \end{bmatrix}.$$

If we try averaging over {1,2}, {3,4} and {5,6} we get a block matrix with $C_{12} = -(\lambda - \frac{1}{2})J_{2,2}$, $C_{13} = -\lambda J_{22} = C_{23}$. C_{11}, C_{22} and C_{33} yield the eigenvalues (6e-1)/2, (6e+1)/2 and 6e/2 respectively, and of course have 0 also. Lemma 2.4.1 yields $A = 12\lambda-2$, $B_1 = 0$, $B_2 = 4$ so the last two eigenvalues of $\bar{C}_{d_2}$ are $(12\lambda-2 \pm 2)/4$ or (6e/2)-1 and 6e/2. $(6e/2)-1 \leq \mu_{d^*1}$ so we are done.

Theorem 3.3.2: Let $d^o \in \mathcal{D}(v,ev(v-1)/2, 2)$ be a BIBD. Let d* be the design constructed from d^o by removing one block from d^o. Then d* is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,(ev(v-1)/2)-1,2)$ for v = 3,4,5 and 6.

Proof: d* maximizes the trace in $\mathcal{D}$ trivially since k = 2. The eigenvalues are of the form $0 < (ev/2)-1 < (ev/2) = \ldots = (ev/2)$ by Lemma 3.1.2. E-optimality for v = 3 and 4 comes from Constantine (1981) as mentioned above. E-optimality for v = 5 and 6 comes from Lemmas 3.3.8 and 3.3.10, respectively. Therefore by Theorem 2.1.1 we are finished.

Remark 3.3.2: Actually the author has proved the E-optimality of these d*'s for v = 7. In this case m of Lemma 3.3.2 can be 2 or 3, and so a few more lemmas were used, each being proved in a manner similar to Lemma 3.3.9. About a dozen cases remained to be checked with most not yielding to Lemmas 1.4.1 or 2.4.1. The eigenvectors of these cases were found for e = 1 on MINITAB, and they generalized to $e \geq 1$. Each case had one eigenvalue bounded above by $(7e/2)-1$. Because of its tedious nature, this method was not extended to $v \geq 8$.

For the cases $v \geq 7$ and $e \geq 2$ the A-, D- and E-efficiencies were calculated. They may be seen in Chapter 8.

CHAPTER 4

THE BBD PLUS A BLOCK OF SIZE V-1

4.1 Preliminaries and the Cases $k = v-1, v$

Throughout Chapter 4 let d^o and d^* be as follows. Let d^o be a BBD in $\mathcal{D}(v,b,k)$. Then $C_{d^o} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v,v}$ with r, λ_0 and λ_1 being the unique values of $r_{d^o i}$, $\lambda_{d^o ii}$ $(i = 1,\ldots,v)$ and $\lambda_{d^o ij}$ $(1 \leq i < j \leq v)$ respectively. Recall that this design has maximum trace in $\mathcal{D}(v,b,k)$ since all of its blocks are binary.

Let d^* be the design in $\mathcal{D}(v,b+1, \underline{k}^*)$ that has d^o for its first b blocks and the treatments 1 through v-1 in the (b+1)-st block, where $(\underline{k}^*)' = (k\underline{1}_b', v-1)$. Then $C_{d^*} = C_{d^o} + C_0$ where C_0 is the C-matrix for the added block. We have

$$C_0 = \begin{bmatrix} I_{v-1} - (v-1)^{-1} J_{v-1,v-1}, & \mathcal{O}_{v-1,1} \\ \\ \mathcal{O}_{1,v-1} & , \quad 0 \end{bmatrix}$$

and so $c_{d^*ii} = r-(\lambda_0/k) + (v-2)/(v-1)$ for $i = 1,\ldots,v-1$, $c_{d^*v,v} = r-(\lambda_0/k)$, $c_{d^*ij} = -(\lambda_1/k)-(v-1)^{-1}$ for $1 \leq i < j \leq v-1$ and $c_{d^*iv} = -(\lambda_1/k)$ for $i = 1,\ldots,v-1$. By Lemma 1.4.1 the eigenvalues of C_{d^*} are 0, $v\lambda_1/k$ and $r-(\lambda_0/k)+(v-2)/(v-1)+(\lambda_1/k)+(v-1)^{-1}$ with multiplicities 1, 1 and v-2 respectively. The last eigenvalue simplifies to $(v\lambda_1/k)+1$ since $r-(\lambda_0/k) = (v-1)\lambda_1/k$. Note that d* has binary blocks and is of maximum trace in $\mathcal{D}(v,b+1,\underline{k}^*)$.

Throughout this chapter, without loss of generality we will assume $r_{dv} = \min_{1\le i\le v} (r_{di})$ for any $d \in \mathcal{D}(v,b+1,\underline{k}^*)$.

Lemma 4.1.1: Let d, $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$, $v \geq 3$, where d is arbitrary and d* is as defined above. Then if

$$r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 - \frac{n_{dv,b+1}^2}{v-1} \leq r - \frac{\lambda_0}{k}$$

we have d* E-better than d.

Proof: By Theorem 2.2.1 we have

$$\mu_{d1} \leq \frac{v}{v-1}\left(r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 - \frac{n_{dv,b+1}^2}{v-1}\right).$$

Since $\mu_{d_1^*} = (v\lambda_1/k) = (v/(v-1))(r-\lambda_0/k)$, simple algebra gives us the rest.

We will now attempt to get the $\mathcal{J}_4$-optimality of d* by applying Theorem 2.1.1 and Lemma 4.1.1. The cases $k = v-1$ and $k = v$ will be discussed in this section. In Section 4.2 $k < v-1$ will be covered, and in Section 4.3 we shall see that d* is not $\mathcal{J}_4$-optimal for many cases of $k \geq v+1$.

The case of $k = v-1$ has been covered in Theorem 3.3.1. Here is the case $k = v$.

Theorem 4.1.1: Let $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$ be a BBD plus one binary block of size v-1 where $(\underline{k}^*)' = (v\underline{1}_b', v-1)$ and $v \geq 3$. Then d* is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,b+1,\underline{k}^*)$.

Proof: Since $k = v$, d^0 is a CBD and for any $d \in \mathcal{D}(v,b+1,\underline{k}^*)$, $\min_{1 \le i \le v} (r_{di}) \le r = b < b+1$. If this minimum r_{di} is r_{dv} then there exists at least one $j \in \{1,\ldots,b,b+1\}$ for which $n_{dvj} = 0$.

If $j = b+1$ then $r_{dv} - \frac{1}{v}\sum_{j=1}^{b} n_{dvj}^2 - \frac{0}{v-1} \le r_{dv} - \frac{r_{dv}}{v} = \frac{r_{dv}(v-1)}{v} \le \frac{r(v-1)}{v} = r - \frac{\lambda_0}{v}$ since $\lambda_0 = r$ for d^0.

If $j < b+1$ then without loss of generality let $j = 1$. We have $r_{dv} - \frac{1}{v}\sum_{j=2}^{b} n_{dvj}^2 - \frac{n_{dv,b+1}^2}{v-1} \le r_{dv} - \frac{1}{v}\sum_{j=1}^{b} n_{dvj}^2 - \frac{n_{dv,b+1}^2}{v} \le r_{dv} - \frac{r_{dv}}{v} \le r - \frac{\lambda_0}{k}$.

By Lemma 4.1.1 d* E-betters all d in $\mathcal{D}(v,b+1,\underline{k}^*)$. Since d* is of maximum trace, has binary blocks and the right eigenvalue structure, Theorem 2.1.1 gives us the $\mathcal{J}_4$-optimality.

4.2 The Cases $k < v-1$

In these cases we shall prove that d* is $\mathcal{J}_4$-better than some of the $d \in \mathcal{D}(v,b+1,\underline{k}^*)$.

Lemma 4.2.1: d* is $\mathcal{J}_4$-better than any $d \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $n_{dv,b+1} = 0$ $(v \ge 3)$.

Proof: We use Lemma 4.1.1. $r-(\lambda_0/k) = r-(r/k) = r(k-1)/k$, and by Lemma 1.4.3 $r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 - \frac{0}{v-1} = r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 \le r_{dv} - (r_{dv}/k) = \frac{r_{dv}(k-1)}{k} \le \frac{r(k-1)}{k}$.

Therefore d* is E-better than d, and by Section 4.1 and Theorem 2.1.1, d* is $\mathcal{J}_4$-better.

Lemma 4.2.2: d* is E-better than any $d \in \mathcal{D}(v,b+1,\underline{k}^*)$, $v \geq 3$, for which $n_{dv,b+1} = \ell \geq 1$ and any one of the following holds:

a) $r-r_{dv} \geq \frac{v-1}{4k(k-1)}$,

b) $k \geq \frac{\sqrt{v}+1}{2}$ and $r_{dv} < r$,

c) For r_{dv} fixed, $0 \leq r-r_{dv} < \frac{v-1}{4k(k-1)}$ with $1 \leq \ell \leq (v-1 - a(r_{dv})^{\frac{1}{2}})/2k$ or $\ell \geq (v-1 + a(r_{dv})^{\frac{1}{2}})/2k$ where $a(r_{dv}) = (v-1)^2 - 4k(k-1)(v-1)(r-r_{dv})$,

d) For fixed ℓ, $1 \leq r_{dv} \leq r - \frac{\ell}{k-1}\left(1- \frac{\ell k}{v-1}\right)$,

e) $\ell = 1$ and $1 \leq r_{dv} < r$.

Proof:

Suppose $n_{dv,b+1} = \ell \geq 1$. We have

$$r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 - \frac{\ell^2}{v} \leq r_{dv} - \frac{r_{dv}-\ell}{k} - \frac{\ell^2}{v-1}$$

$$= \frac{r_{dv}(k-1)}{k} + \frac{\ell}{k} - \frac{\ell^2}{v-1}$$

with $1 \leq \ell \leq \min(r_{dv}, v-2)$ and $1 \leq r_{dv} \leq r$. For E-optimality of d* we need

$$\frac{r_{dv}(k-1)}{k} + \frac{\ell}{k} - \frac{\ell^2}{v-1} \leq \frac{r(k-1)}{k} \qquad (4.2.1)$$

by Lemma 1.4.1, which is true if and only if $r_{dv}(k-1) + \ell - \ell^2k/(v-1) \leq r(k-1)$.

If we fix r_{dv}, (4.2.1) is true if and only if $A(\ell,r_{dv}) = \ell^2k - (v-1)\ell + (r-r_{dv})(k-1)(v-1) \geq 0$. Solving for ℓ, $r-r_{dv} \geq (v-1)/(4k(k-1))$ implies $A(\ell,r_{dv}) \geq 0$ for all ℓ. If $r-r_{dv} < (v-1)/(4k(k-1))$ then $A(\ell,r_{dv}) \geq 0$ if $0 \leq \ell \leq (v-1-a(r_{dv})^{\frac{1}{2}})/2k$ or $\ell \geq (v-1+a(r_{dv})^{\frac{1}{2}})/2k$ where $a(r_{dv}) = (v-1)^2-4k(k-1)(v-1)(r-r_{d1})$. This follows from the quadratic formula, and proves a) and c).

If we fix ℓ, (4.2.1) is true if and only if $1 \leq r_{dv} \leq r - \frac{\ell}{k-1}(1-\frac{\ell k}{v-1})$. In the special case of $\ell = 1$, we get $1 \leq r_{dv} \leq r - (k-1)^{-1}(1-k/(v-1))$ or since r_{dv} is an integer, $1 \leq r_{dv} \leq r-1$. This proves d) and e).

Going back to $A(\ell,r_{dv})$, if $(v-1)/(4k(k-1)) \leq 1$ then d* E-betters all designs except those for which $r_{dv} = r$, but $(v-1)/(4k(k-1)) \leq 1$ if and only if $k \geq (\sqrt{v} + 1)/2$. Thus b) holds and the proof is finished.

Looking at the left-hand-side of (4.2.1) we might ask when is

$$\frac{r_{dv}(k-1)}{k} + \frac{\ell}{k} - \frac{\ell^2}{v-1} \leq \frac{r_{dv}(k-1)}{k} + \frac{\ell-1}{k} - \frac{(\ell-1)^2}{v-1} \ ?$$

This is true if and only if $\ell \geq (v-1+k)/2k$. So if $\ell \geq (v-1+k)/2k$ we can get a larger upper bound for μ_{d1} using Theorem 2.2.1 by letting $n_{dv,b+1} = \ell-1$ and not ℓ. The maximum for the left-hand-side of (4.2.1) is

$$\frac{v}{v-1}\left(\frac{r_{dv}(k-1)}{k} + \frac{\ell_0}{k} - \frac{\ell_0^2}{v-1}\right)$$

where $\ell_0 = \frac{v-1+k}{2k} - 1$ if $\frac{v-1+k}{2k} = \mathrm{int}[\frac{v-1+k}{2k}]$ and $\ell_0 = \mathrm{int}[\frac{v-1+k}{2k}]$ if $\frac{v-1+k}{2k} > \mathrm{int}[\frac{v-1+k}{2k}]$.

We can reduce $n_{dv,b+1}$ to $\ell = 1$ if $(v-1+k)/(2k) \leq 2$ or $k \geq (v-1)/3$. From this we get the following lemma.

<u>Lemma 4.2.3</u>: For $r_{dv} = \min\limits_{1\leq i\leq v} (r_{di})$, $\sum\limits_{i=1}^{v} r_{di} = vr+v-1$,

a) $\frac{r_{dv}(k-1)}{k} + \frac{\ell}{k} - \frac{\ell^2}{v-1} \leq \frac{r_{dv}(k-1)}{k} + \frac{\ell-1}{k} - \frac{(\ell-1)^2}{v-1}$ for $\ell \geq \frac{v-1+k}{2k}$ and

b) $\frac{r_{dv}(k-1)}{k} + \frac{\ell}{k} - \frac{\ell^2}{v-1}$ is maximized by $\ell_0 = \frac{v-1+k}{2k} - 1$ if

$\ell_0+1 = \mathrm{int}[\frac{v-1+k}{2k}]$ and $\ell_0 = \mathrm{int}[\frac{v-1+h}{2k}]$ if $\frac{v-1+k}{2k} > \mathrm{int}[\frac{v-1+h}{2k}]$.

To summarize what has been said about the optimality of d* we have the following theorem.

<u>Theorem 4.2.1</u>: Suppose $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$, $v \geq 3$, is a BBD plus one binary block of size v-1. Then d* is $\mathcal{J}_4$-better than all $d \in \mathcal{D}(v,b+1,\underline{k}^*)$ except possibly d for which $n_{dv,b+1} = \ell$ and

a) $v \geq 7$, $k \geq \frac{v-1}{3} > \frac{\sqrt{v}+1}{2}$, $r_{dv} = r$, $0 < \ell < \frac{v-1}{k}$, (and $\ell_0 = 1$),

b) $v \geq 7$, $\frac{v-1}{3} > k \geq \frac{\sqrt{v}+1}{2}$, $r_{dv} = r$, $0 < \ell < \frac{v-1}{k}$ (and $\ell_0 > 1$),

c) $v \geq 7$, $\frac{v-1}{3} > \frac{\sqrt{v}+1}{2} > k$, $0 \leq r-r_{dv} < \frac{v-1}{4k(k-1)}$, $(v-1-a(r_{dv})^{\frac{1}{2}})/2k < \ell < (v-1+a(r_{dv})^{\frac{1}{2}})/2k$ (and $\ell_0 > 1$),

d) $3 \leq v \leq 6$, $\frac{v-1}{3} < \frac{\sqrt{v}+1}{2} < k$, $r_{dv} = r$, $0 < \ell < \frac{v-1}{k}$ (and $\ell_0 = 1$), where $a(r_{dv}) = (v-1)^2 - 4k(k-1)(v-1)(r-r_{d1})$ and ℓ_0 is defined in Lemma 4.2.3.

Proof: $\frac{v-1}{3} > \frac{\sqrt{v}+1}{2}$ for $v \geq 7$ and $\frac{v-1}{3} < \frac{\sqrt{v}+1}{2}$ for $v \leq 6$. Then a) through d) follow from Lemma 4.2.2, parts a), b) and c), Lemma 4.2.3, Section 4.1 and Theorem 2.1.1.

For the cases of a), c) and d) of Theorem 4.2.1 A-, D- and E-efficiencies will be calculated in Chapter 8.

4.3 The Cases $k \geq v+1$

We begin with two general lemmas for the case of $k \geq v+1$ and another for this situation analogous to Lemma 4.2.1.

Lemma 4.3.1: Let $B(cb+d;\ b,k) = cb+d-k^{-1}(d(c+1)^2+(b-d)c^2)$ where $k \geq 2$, $c \geq 0$, $b \geq 0$, $0 \leq d \leq b$ and b, c, d and k are integers. Then $B(c_1b+d_1;\ b,k) \leq B(c_2b+d_2;\ b,k)$ if either of the following holds

a) $c_1 = c_2$ and $d_1 = d_2$ or $d_1 < d_2$ with $(k-1)/2 \geq c_2 \geq 0$.

b) $c_1 < c_2$ and $(k-1)/2 \geq c_2 > c_1 \geq 0$.

Proof: If $c_1 = c_2$ then $B(c_2b+d_2;\ b,k) - B(c_1b+d_1;\ b,k) = \frac{d_2-d_1}{k}(k-(2c_2+1)) \geq 0$ if and only if $d_2 = d_1$ or $d_2 > d_1$ and $k \geq 2c_2+1$. If $c_1 < c_2$, $B(c_1b+d_1;\ b,k) \leq B(c_1b+b;\ b,k) = B((c_1+1)b+0;\ b,k)$. If $c_2 = c_1+1$ we finish with $\leq B((c_1+1)b+d_2;\ b,k)$. If $c_2 > c_1+1$ we finish with $\leq B((c_1+1)b+b;\ b,k) \leq \ldots \leq B(c_2b+0;\ b,k) \leq B(c_2b+d_2;\ b,k)$. In each case we need $(k-1)/2 \geq c_2 > c_1 \geq 0$ and the inequalities follow as in the case for $c_1 = c_2$.

Lemma 4.3.2: Suppose we are investigating $d \in \mathcal{D}(v,b;k)$ with $k \geq v \geq 3$ and $\min_{1\leq i\leq v}(r_{di}) \leq r = \text{int}[bk/v]$. Then $B(\min_{1\leq i\leq v}(r_{di}); b,k) \leq B(r; b,k)$.

Proof: Let $r = c_2b+d_2$ and $\min_{1\leq i\leq v}(r_{di}) = c_1b+d_1$. We apply Lemma 4.3.1. If $c_1 = c_2$ and $d_1 = d_2$ we are done. Otherwise we must show $(k-1)/2 \geq c_2$. But $vr = v(c_2b+d_2) = v\cdot\text{int}[bk/v] \leq bk$ so $c_2 \leq (bk-vd_2)/vb \leq (bk)/vb = k/v \leq (k-1)/2$ with the last inequality following from $k \geq v \geq 3$.

Lemma 4.3.3: $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$, a BBD plus one binary block of size v-1, is E-better than any $d \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $n_{dv,j} = 0$, $j = 1,2,\ldots,b,b+1$. $(v \geq 3)$.

Proof: If $j = b+1$, $r_{dv} - \frac{1}{k}\sum_{j=1}^{b} n_{dvj}^2 - \frac{0}{v-1} \leq B(r_{dv}; b,k) \leq B(r; b,k) = r-\lambda_0/k$ with $r = \text{int}[bk/v]$. If $j = 1$, say, $r_{dv} - \sum_{j=1}^{b}\frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} = r_{dv} - \sum_{j=2}^{b}\frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq r_{dv} - \sum_{j=2}^{b+1}\frac{n_{dvj}^2}{k} = B(r_{dvj}; b,k) \leq B(r; b,k) = r-\lambda_0/k$. Therefore by Lemma 4.1.1 we are done.

This last lemma is most useful when $j = b+1$. It says a design that is E-better than d* must have its minimum replicated treatments (at least treatment v) appearing in the block of size v-1 at least once.

We will now look at the cases $q = 0$, 1, v-1 and $2 \leq q \leq v-2$ separately, where $k = pv+q$ $(p \geq 1)$. In each case a design $\hat{d}$ will be described as a candidate to be $\mathcal{J}_4$-better than d*. Then a theorem will be given where d* is proven $\mathcal{J}_4$-better than a subclass of the $d \in \mathcal{D}(v,b+1,\underline{k}^*)$, and possibly $\hat{d}$ is proven $\mathcal{J}_4$-better than d*. Proving d* is $\mathcal{J}_4$-better than the subclass will consist of four steps. Lemma 4.3.3 will be applied to show an E-better (than d*) d needs $n_{dvj} \geq 1$, $1 \leq j \leq b+1$. Then Lemma 4.1.1 will be applied to show an E-better d needs $r_{dv} = r$ for $n_{dv,b+1} \geq 1$. Thirdly Lemma 4.1.1 will be applied to show that an E-better d needs $n_{dv,b+1} = 1$ or possibly 2 with some conditions. Finally Theorem 2.1.1 will be applied to d* (recall it is of maximum trace and has eigenvalues $0 < (v\lambda_1/k) < (v\lambda_1/k) + 1 = \ldots = (v\lambda_1/k)+1)$ and the designs it E-betters.

For $k = pv$ $(p \geq 2)$ d^0 has b blocks, each containing p applications of the v treatments. d* has these blocks as its first b blocks with the (b+1)-st block of size v-1 and containing treatments 1 through v-1. Let $\hat{d}$ have d*'s first b-1 blocks, the b-th block containing p applications of treatments 1 through v-2, p+1 applications of v-1 and p-1 applications of v, and its last block containing treatments 1 through v-2 and v. To create $\hat{d}$ we have traded an application of v from the b-th block with the application of v-1 in the (b+1)-st. Note that $\lambda_0 = \lambda_1 = bp^2$.

Theorem 4.3.1. For $k = pv$, $p \geq 2$, d* is $\mathcal{J}_4$-optimal over all $d \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $v \geq 3$ and $(\underline{k}^*)' = (pv\underline{1}_b', v-1)$ except possibly those for which $r_{dv} = r$, $n_{dv,b+1} = 1$, $n_{dvj} \geq 1$ $(1 \leq j \leq b)$ and $v \geq 3$ with $p \geq 3$ or $v \geq 4$. There exists a $\hat{d}$ with $r_{\hat{d}v} = r$ and $n_{\hat{d}v,b+1} = 1$

which E-betters d* for $p \geq v/(v-2)$ but is E-worse for $p < v/(v-2)$ and is always A- and D-worse.

Proof: We apply Lemma 4.3.3.

If $r_{dv} \leq r-1$ and $b \geq 2$ then $r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq$ $B(r_{dv}-1;\ b,k)+1-(v-1)^{-1} \leq B(r-2;\ b,k)+1-(v-1)^{-1} = r-2 - \frac{\lambda_0-2p^2+2(p-1)^2}{k} + 1 - \frac{1}{v-1} = r-\lambda_0/k+(\frac{4p-2}{k} - 1 - \frac{1}{v-1})$. The first inequality follows from the fact that

$$-\frac{\ell^2}{k} - \frac{n^2}{v-1} \leq -\frac{(\ell+1)^2}{k} - \frac{(n-1)^2}{v-1} \qquad (4.3.1)$$

for $n \geq 2$, $\ell \geq 0$. If $b = 1$ then $B(r-2;\ b,k)+1-(v-1)^{-1} = r-(\lambda_0/k) + (\frac{4p-4}{k} - 1 - \frac{1}{v-1})$ since $p \geq 2$. But $\frac{4p-4}{k} - 1 - \frac{1}{v-1} \leq \frac{4p-2}{k} - 1 - \frac{1}{v-1} \leq 0$ for $p \geq 2$, $v \geq 3$. We apply Lemma 4.1.1, so an E-better d needs $r_{dv} = r$.

If $r_{dv} = r$, $n_{dv,b+1} \geq 1$ and $b \geq 2$ then

$$c_{dvv} = r-\sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r-x;\ b,k) + x - \frac{x^2}{v-1}$$

$$= r-(\lambda_0/k) + \frac{x(2p-1)}{k} - \frac{x^2}{v-1} \qquad (4.3.2)$$

for $x = 2$ or 1 with (4.3.2) larger for $x = 1$. The first inequality follows from (4.3.1). $\frac{x(2p-1)}{k} - \frac{x^2}{v-1} \leq 0$ for $v = 3$, $p = 2$, $x = 1$ and 2, but for $x = 2$ only when $v = 3$, $p \geq 3$ or $v \geq 4$, $p \geq 2$. If $b = 1$ and $n_{dv,b+1} = 1$ we can use (4.3.2) above and the same result holds. If $b = 1$ and $n_{dv,b+1} \geq 2$ we must again use $p \geq 2$:

$$r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq r-(\lambda_0/k) + \frac{4p-4}{k} - \frac{4}{v-1}$$

which is less than $r-(\lambda_0/k)$ for $p \geq 2$, $v \geq 3$. We apply Lemma 4.1.1, so an E-better d needs $r_{dv} = r$, $n_{dv,b+1} = 1$ (unless $v = 3$, $p = 2$).

Now we apply Theorem 2.1.1 to d* and all d except those with $r_{dv} = r$, $n_{dv,b+1} = 1$, $n_{dvj} \geq 1$ $(1 \leq j \leq b)$ and $v = 3$, $p \geq 3$ or $v \geq 4$.

For the second part of the theorem

$$C_{\hat{d}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym} & & C_{33} \end{bmatrix} \qquad (4.3.3)$$

where $C_{11} = (r+1)I_{v-2}-(\frac{bp^2}{k} + \frac{1}{v-1})J_{v-2,v-2}$, $C_{12} = -k^{-1}(bp^2+p)\underline{1}_{v-2}$, $C_{13} = -(\frac{bp^2-p}{k} + \frac{1}{v-1})\underline{1}_{v-2}$, $C_{22} = r+1 - \frac{bp^2+2p+1}{k}$, $C_{23} = -k^{-1}(bp^2-1)$ and $C_{33} = r - \frac{bp^2-2p+1}{k} - \frac{1}{v-1}$. From Lemma 2.4.1 the eigenvalues of $C_{\hat{d}}$ are $(v\lambda_1/k)+1 = r+1$ with multiplicity $v-3$, 0, and

$$\frac{v\lambda_1}{k} + \frac{1}{2} - \frac{1}{k} \pm \frac{1}{2}\left(\frac{v^2p^2-4vp^2+4}{p^2v^2} + \frac{4(p+1)}{pv(v-1)}\right)^{\frac{1}{2}}.$$

$\hat{d}$ is E-worse than d* if and only if

$$\frac{1}{2} - \frac{1}{k} - \frac{1}{2}\left(\frac{v^2p^2-4vp^2+4}{p^2v^2} + \frac{4(p+1)}{pv(v-1)}\right)^{\frac{1}{2}} < 0$$

which is true if and only if $p < v/(v-2)$. Straightforward but lengthy calculations comparing $\Phi_A(C_{\hat{d}})$ with $\Phi_A(C_{d*})$ and $\Phi_D(C_{\hat{d}})$ with $\Phi_D(C_{d*})$ show d* is strictly A- and D-better than $\hat{d}$. The proof is now complete.

For $k = pv+1$ ($p \geq 1$) d^0 has $b = ev$ blocks ($e \geq 1$) and comprises e copies of the BBD in $\mathcal{D}(v,v,pv+1)$ with each treatment appearing one more time than all others in exactly one block. Assume the b-th block of d* (or d^0) has treatment v extra-replicated. Then let $\hat{d}$ be the design with blocks 1 through b-1 exactly those of design d*, the b-th block with $n_{d,v-1,b} = p+1$ and $n_{dib} = p$ for $i = 1,\ldots,v-2,v$, and the (b+1)-st block with treatments 1 through v-2 and v. We have traded an application of v in a block where treatment v-1 is not extra-replicated (here block b) with the application of v-1 in block b+1.

Theorem 4.3.2: For $k = pv+1$, $p \geq 1$, d* is $\mathcal{J}_4$-optimal over all $d \in \mathcal{D}(v,ev+1,\underline{k}^*)$, (with $v \geq 3$ and $(\underline{k}^*)' = ((pv+1)\underline{1}_b', v-1)$, except possibly those d for which $r_{dv} = r$, $n_{dvj} \geq 1$ ($1 \leq j \leq b$), $n_{dv,b+1} = 1$ with $e \geq 1$ or $n_{dv,b+1} = 2$ with $e \geq 2$, $p < \frac{v-3}{2}$. There exists a $\hat{d}$ with $r_{\hat{d}v} = r$, $n_{\hat{d}v,b+1} = 1$ that is $\mathcal{J}_4$-better and strictly A-, D- and E-better than d*.

Proof: We apply Lemma 4.3.3.

If $r_{dv} \leq r-1$, $e = 1$, $r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r_{dv}-1;\, b,k) + 1-(v-1)^{-1} \leq B(r-2;\, b,k) + 1 - (v-1)^{-1} = r-(\lambda_0/k) + ((4p/k) - 1 - (v-1)^{-1})$ by the same argument used preceding (4.3.1). If $e \geq 2$ $c_{dvv} \leq r-(\lambda_0/k)+(\frac{4p+2}{k} - 1 - \frac{1}{v-1})$ by a similar argument, with $(\frac{4p}{k} - 1 - \frac{1}{v-1}) \leq (\frac{4p+2}{k} - 1 - \frac{1}{v-1}) \leq 0$ for $p \geq 1$, $v \geq 3$. We apply Lemma 4.1.1, so an E-better d must have $r_{dv} = r$ for $n_{dv,b+1} \geq 1$.

Assume $r_{dv} = r$. If $n_{dv,b+1} = 1$, $e \geq 1$, $r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{1}{v-1} \leq$

$r - \frac{\lambda_0 - (p+1)^2 + p^2}{k} - \frac{1}{v-1} > r - \lambda_0/k$. If $n_{dv,b+1} = 2$, $e = 1$ (and $v \geq 4$),

$r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{4}{v-1} \leq r - \frac{\lambda_0 - (p+1)^2 + p^2 - p^2 + (p-1)^2}{k} - \frac{4}{v-1} = r - (\lambda_0/k) +$

$\frac{4p}{k} - \frac{4}{v-1} \leq r - (\lambda_0/k)$ for $p \geq 1$, $v \geq 4$. If $n_{dv,b+1} = 2$, $e \geq 1$ (and

$v \geq 4$) then $c_{dvv} \leq r - \frac{\lambda_0 - 2(p+1)^2 + 2p^2}{k} - \frac{4}{v-1} = r - (\lambda_0/k) + \frac{4p+4}{k} - \frac{4}{v-1} \leq$

$r - (\lambda_0/k)$ if and only if $p \geq (v-3)/2$. If $3 \leq n_{dv,b+1} \leq v-2$ (and $v \geq 5$) then the three cases $e = 1$, $e = 2$ and $e \geq 3$ all yield d* as E-better with similar arguments. In each case above we used the fact that $c_{dvv} = r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r-x;\, b,k) + x - \frac{x^2}{v-1}$ for $n_{dv,b+1} \geq x = 3,2$ or 1 with the last statement increasing as x decreases. The argument follows as at (4.3.2). We apply Lemma 4.1.1, so an E-better d must have $r_{dv} = r$, $n_{dv,b+1} = 1$, or $n_{dv,b+1} = 2$ with $p < (v-3)/2$ and $e \geq 2$.

Now we apply Theorem 2.1.1 to d* and all d that d* E-bettered to prove the first part of the theorem.

For the second part of the theorem $C_{\hat{d}}$ is of the form given at (4.3.3). The eigenvalues of $C_{\hat{d}}$ are 0, $(v\lambda_1/k)+1$ with multiplicity $v-3$, and using Lemma 2.4.1,

$$\frac{v\lambda_1}{k} + \frac{1}{2} \pm \frac{1}{2} \left[\left(\frac{2p(v-2)}{k} - 1\right)^2 + \frac{4p}{k}\left(\frac{2p(v-2)}{k} - \frac{v-2}{v-1}\right)\right]^{\frac{1}{2}}.$$

$\hat{d}$ is of full trace, as is d*. Furthermore for $v \geq 3$, $p \geq 1$,

$$0 \leq \frac{1}{2}\,[(\frac{2p(v-2)}{k} - 1)^2 + \frac{4p}{k}\,(\frac{2p(v-2)}{k} - \frac{v-2}{v-1})]^{\frac{1}{2}} < \frac{1}{2}.$$

Therefore the eigenvalues of $\hat{d}$ majorize those of d*, and are not a permutation of them, making $\hat{d}$ $\mathcal{J}_4$-better and strictly A-, D- and E-better than d*.

Remark 4.3.1: $\hat{d}$ is $\mathcal{J}_4$-better than all d for which d* is $\mathcal{J}_4$-better. But $\hat{d}$ has not been proven $\mathcal{J}_4$-optimal in $\mathcal{D}(v,ev+1,\underline{k}^*)$ and its eigenvalue structure is not that of Theorem 2.1.1.

For k = pv+v-1, p $\geq$ 1, d^o comprises e copies (e $\geq$ 1) of the unique BBD in $\mathcal{D}(v,v,pv+v-1)$. d* is d^o union the block of size v-1 with treatments 1 through v-1. Without loss of generality assume the b-th block of d* contains p+1 copies of treatments 1 through v-2 and v, with p copies of v-1. To create $\hat{d}$ trade an application of v from the b-th block with the application of v-1 from the (b+1)-st block. Blocks 1 through b-1 of $\hat{d}$ remain identical to blocks 1 through b-1 of d*. Here b = ev and $\hat{d}$ is of maximum trace in $\mathcal{D}(v,ev+1,\underline{k}^*)$.

Theorem 4.3.3: For k = pv+v-1, p $\geq$ 1, d* is $\mathcal{J}_4$-optimal over all d $\in \mathcal{D}(v,ev+1,\underline{k}^*)$, (with v $\geq$ 3 and $(\underline{k}^*)' = ((pv+v-1)\underline{1}_b', v-1))$ except possibly those for which $r_{dv} = r$, $n_{dv,b+1} = 1$, and $n_{dvj} \geq 1$, $1 \leq j \leq b$. There exists a $\hat{d}$ with $r_{\hat{d}v} = r$ and $n_{\hat{d}v,b+1} = 1$ that is $\mathcal{J}_4$-better and strictly A-, D- and E-better than d*.

Proof: We apply Lemma 4.3.3.

If $r_{dv} \leq r-1$, $e \geq 1$, $r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r_{dv}-1;\, b,k) + 1 - (v-1)^{-1} \leq B(r-2;\, b,k) + 1 - (v-1)^{-1} = r-(\lambda_0/k) + (\frac{4p+2}{k} - 1 - \frac{1}{v-1})$. $\frac{4p+2}{k} - 1 - \frac{1}{v-1} < 0$ for $k = pv+q$, $p \geq 1$, $v \geq 3$ and $2 \leq q \leq v-1$, so r_{dv} must equal r.

If $r_{dv} = r$, $n_{dv,b+1} \geq 1$ then $r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq$

$r-(\lambda_0/k) + \frac{x(2p+1)}{k} - \frac{x^2}{v-1}$ for $x = 1$ or 2, the larger of which occurs for $x = 1$. $\frac{x(2p+1)}{k} - \frac{x^2}{v-1} \leq 0$ for $x \geq 2$, $p \geq 1$, $v \geq 3$, but is in fact greater than zero for $x = 1$. For $v = 3$ x can not be greater than 1.

We apply Lemma 4.1.1 to the preceding paragraphs, so an E-better d must have $r_{dv} = r$ and $n_{dv,b+1} = 1$.

We now apply Theorem 2.1.1 to d* and the d for which d* is E-better to finish the first part of the proof.

$C_{\hat{d}}$ is of the form given at (4.3.3), and has eigenvalues 0, $(v\lambda_1/k)+1$ with multiplicity v-3, and from Lemma 2.4.1

$$\frac{v\lambda_1}{k} + \frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)}\right)^{\frac{1}{2}}.$$

Since $0 \leq \frac{1}{2}\left(1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)}\right)^{\frac{1}{2}} < \frac{1}{2}$, these eigenvalues majorize and are not a permutation of those of d*, making $\hat{d}$ $\mathcal{J}_4$-better and strictly A-, D- and E-better than d*. So ends the proof.

For $k = pv+q$, $2 \leq q \leq v-2$, we can still construct a $\hat{d}$ which is potentially $\mathcal{J}_4$-better than d*. However for these q $C_{\hat{d}}$ is of the form

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ & C_{22} & C_{23} & C_{24} \\ & & C_{33} & C_{34} \\ & & & C_{44} \end{bmatrix} \qquad (4.3.4)$$

where $C_{11} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k} + 1)I_{q-1} - (\frac{\lambda_1}{k} + \frac{1}{v-1})J_{q-1,q-1}$, $C_{12} = -(\frac{\lambda_1}{k} + \frac{1}{v-1})J_{q-1,(v-2)-(q-1)}$, $C_{13} = -(\frac{\lambda_1}{k} + \frac{p+1}{k})\underline{1}_{q-1}$, $C_{14} = -(\frac{\lambda_1}{k} - \frac{p+1}{k} + \frac{1}{v-1})\underline{1}_{q-1}$, C_{22} is of the form of C_{11} but is $v-q-1$ on a side, $C_{23} = -(\frac{\lambda_1}{k} + \frac{p}{k})\underline{1}_{v-q-1}$, $C_{24} = -(\frac{\lambda_1}{k} - \frac{p}{k} + \frac{1}{v-1})\underline{1}_{v-q-1}$, $C_{33} = r - \frac{\lambda_0}{k} + 1 - \frac{2p+1}{k}$, $C_{34} = -\frac{\lambda_1}{k}$ and $C_{44} = r - \frac{\lambda_0}{k} + \frac{2p+1}{k} - \frac{1}{v-1}$. To get $\hat{d}$, without loss of generality assume the b-th block of d* has treatments 1 through q-1 and v as the extra-replicated treatments, with the (b+1)st block as usual. Then trade the application of v in the b-th block with that of v-1 in the (b+1)st. The first q-1 rows of $C_{\hat{d}}$ correspond to the treatments in the b-th block that were extra-replicated with treatment v.

In this case we cannot apply Lemma 2.4.1 to $C_{\hat{d}}$, and as was mentioned in Section 2.4, the 4-by-4 blocked matrix equivalent to Lemma 2.4.1 is unwieldy.

We shall be content here to give a theorem that shows d* is $\mathcal{J}_4$-optimal over many of the $d \in \mathcal{D}(v,b+1,\underline{k}^*)$, and then give some examples indicating that $\hat{d}$ with $r_{\hat{d}v} = r$, $n_{\hat{d}v,b+1} = 1$ is at least A-, D- and E-better than d*.

Theorem 4.3.4: For $k = pv+q$, $2 \leq q \leq v-2$, $p \geq 1$, $v \geq 3$, d^* is $\mathcal{J}_4$-optimal over all $d \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $(\underline{k}^*)' = ((pv+q)\underline{1}_b', v-1)$ except possibly those for which $r_{dv} = r$, $n_{vdj} \geq 1$ $(1 \leq j \leq b)$ and $n_{vd,b+1} = 1$ or $n_{vd,b+1} = 2$ with $p < (v-1-2q)/2$.

Proof: We apply Lemma 4.3.3.

The second paragraph in the proof of Theorem 4.3.3 holds for $2 \leq q \leq v-1$, so d^* E-betters d for which $n_{dv,b+1} \geq 1$ and $r_{dv} < r$.

If $r_{dv} = r$ and $n_{dv,b+1} \geq 1$ then $r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq$ $B(r-x;\, b,k)+x - \frac{x^2}{v-1}$ for $n_{dv,b+1} \geq x = 3,2,1$ with the last term increasing as X decreases. For $x = 3$, $e = 1$ (two cases) and $x = 3$, $e \geq 2$ we get $c_{dvv} \leq r-(\lambda_0/k) + \frac{6p+1}{k} - \frac{9}{v-1}$ or $r-(\lambda_0/k) + \frac{6p+3}{k} - \frac{9}{v-1}$, and the larger of which is less than $r-(\lambda_0/k)$ for $p \geq 1$, $v \geq 3$ (recall $n_{dv,b+1} \geq 3$ only for $v \geq 5$). For $x = 2$ we have $c_{dvv} \leq$ $r-(\lambda_0/k) + \frac{4p+2}{k} - \frac{4}{v-1}$. $\frac{4p+2}{k} - \frac{4}{v-1} \leq 0$ for $p \geq (v-1-2q)/2$. For $x = 1$ the bound on c_{dvv} is never less than or equal to $r-(\lambda_0/k)$. We apply Lemma 4.1.1, so an E-better d must have $n_{dv,b+1} = 1$ or $n_{dv,b+1} = 2$ and $p < (v-1-2q)/2$.

Now apply Theorem 2.1.1 to d^* and the designs that it E-betters.

Remark 4.3.2: For $q = 2$, d^0 is e copies $(e \geq 1)$ of the unique BBD in $\mathcal{D}(v,v(v-1)/2,\, pv+2)$, so we can get the general form for λ_0, λ_1, C_{d^*} and $C_{\hat{d}}$. For $3 \leq q \leq v-2$ this can not be accomplished in general.

Example 4.3.1 $q = 2$ or $q = v-2$. $\hat{d}$, $d^* \in \mathcal{D}(4,6+1,\underline{k}^*)$ with $k = 6$, and $\hat{d}$ is A-, D- and E-better.

d^*:

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	
1	1	1	2	2	3	
2	3	4	3	4	4	

$\hat{d}$:

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	4	4	
1	1	1	2	2	3	
2	3	3	3	4	4	

Example 4.3.2 $q = 3$ or $q = v-2$. $\hat{d}$, $d^* \in \mathcal{D}(5,10+1,\underline{k})$ with $k = 8$, and $\hat{d}$ is A-, D- and E-better than d^*, which is not given here.

$\hat{d}$:

1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	5
5	5	5	5	5	5	5	5	5	5	
1	1	1	1	1	1	2	2	2	3	
2	2	2	3	3	4	3	3	4	4	
3	4	4	4	5	5	4	5	5	5	

In Chapter 8 the following efficiencies for cases from this section will be presented: for $k = pv$ the A- and D- efficiencies of d^* and the E-efficiencies of d^* and $\hat{d}$, and for $k = pv+q > v$ the A-, D- and E-efficiencies for d^* $(1 \leq q \leq v-1)$ and $\hat{d}$ $(q = 1$ and $q = v-1)$.

CHAPTER 5

THE BBD MINUS ONE OBSERVATION

5.1 Preliminaries

Throughout Chapter 5 define d^o and d^* as follows. Let d^o be a BBD in $\mathcal{D}(v,b,k)$. Then $C_{d^o} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v,v}$ with $r_{d^oi} = r$, $\lambda_{d^oii} = \lambda_0$ $(1 \leq i \leq v)$ and $\lambda_{d^oij} = \lambda_1$ $(1 \leq i < j \leq v)$. d^o has all its blocks being binary and is of maximum trace in $\mathcal{D}(v,b,k)$.

Let d^* be the design in $\mathcal{D}(v,b,\underline{k}^*)$, $(\underline{k}^*)' = (k-1, k\underline{1}'_{b-1})$, whose last b-1 blocks are exactly those of d^o but whose first block is that of d^o but with an application of treatment 1 removed. Without loss of generality we will assume the first block of d^o has treatments 1 through k if $k < v$ or has treatments 1 through q extra-replicated if $k = pv+q$, $p \geq 1$.

d^* is of maximum trace in $\mathcal{D}(v,b,\underline{k}^*)$ since $n_{d^o11} = \max_{1 \leq i \leq v} (n_{d^oi1})$ and the blocks of d^* are binary. Note that if $k \leq v$, treatment 1 could only be removed from a block in which it appeared, but if $k \geq v+1$ we could remove an observation of 1 where it was not equi-replicated. This case gives a design $\tilde{d}$ that is E-worse (and $\mathcal{J}_4$-worse) than d^*, and will be considered later.

For d* the r_{d*i} are as equal as possible with $r_{d*1} = r-1$, $r_{d*i} = r$, $2 \leq i \leq v$. For any $d \in \mathcal{D}(v,b,\underline{k}^*)$ $\min_{1\leq i\leq v} (r_{di}) \leq r-1$, so without loss of generality assume r_{d1} equals this minimum for each d, though it may not be unique.

Note that we must have $k \geq 3$ by what was said in Section 1.5. For $k = 2$ loss of one observation in a block makes the other observation useless.

The cases $k < v$ and $k \geq v$ will be discussed separately in Sections 5.2 and 5.3 respectively.

5.2 The Cases $3 \leq k \leq v-1$

If the first block of d^o contained treatments 1 through k, then

$$C_{d*} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym} & & C_{33} \end{bmatrix}$$

where $C_{11} = r - 1 - \frac{1}{k}\sum_{j=2}^{b} n_{d*1j}^2 = (r-1)(k-1)/k$, $C_{12} = -(\frac{\lambda_1 - 1}{k})\underline{1}'_{k-1}$, $C_{13} = -\frac{\lambda_1}{k}\underline{1}'_{v-k}$, $C_{22} = (\frac{(r-1)(k-1)}{k} + 1 + \frac{\lambda_1 - 1}{k})I_{k-1} - (\frac{\lambda_1 - 1}{k} + \frac{1}{k-1})J_{k-1,k-1}$, $C_{23} = -\frac{\lambda_1}{k}J_{k-1,v-k}$ and $C_{33} = (\frac{r(k-1)}{k} + \frac{\lambda_1}{k})I_{v-k} - \frac{\lambda_1}{k}J_{v-k,v-k}$. From Lemma 2.4.1 we can get the eigenvalues of C_{d*}. They are 0, $\frac{r(k-1)}{k} + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k}$ with multiplicity v-k-1, $\frac{(r-1)(k-1)}{k} + 1 - \frac{1}{k} + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k}$ with multiplicity k-2, and $(A \pm (B_1^2 + B_2)^{\frac{1}{2}})/2k$

where $A = 2v\lambda_1 - k$, $B_1 = k-2$, and $B_2 = 4(k-1)$. $B_1^2 + B_2 = k^2$ so the last two eigenvalues become $(v\lambda_1/k)-1$ and $(v\lambda_1/k)$. Thus C_{d*} has eigenvalues $0 < (v\lambda_1/k)-1 < (v\lambda_1/k) = \ldots = (v\lambda_1/k)$, in the form of Theorem 2.1.1.

Theorem 5.2.1: d* is $\mathcal{J}_4$-better than all $d \in \mathcal{D}(v,b,\underline{k}^*)$, $(\underline{k}^*)' = (k-1, k\underline{1}'_{b-1})$, except possibly a design with $r_{di} \geq r-1$ $(1 \leq i \leq v)$, all blocks binary, and if $r_{di} = r_{dj} = r-1$, $\lambda_{dij} \geq \lambda_1$. $(v > k \geq 3)$.

Proof: Suppose d has $r_{d1} \leq r-2$. By Theorem 2.2.1

$$\mu_{d1} \leq \frac{v}{v-1}\left(r_{d1} - \frac{n_{d11}^2}{k-1} - \sum_{j=2}^{b} \frac{n_{dij}^2}{k}\right). \tag{5.2.1}$$

There is some $n_{d1j} = 0$, since $k_j < v$ $(1 \leq j \leq b)$ and $r_{d1} \leq r-2$. If $n_{d1b} = 0$, say, then we continue with (5.2.1)

$$= \frac{v}{v-1}\left(r_{d1} - \frac{n_{d11}^2}{k-1} - \sum_{j=2}^{b-1} \frac{n_{d1j}^2}{k}\right) \leq \frac{v}{v-1}\left(r_{d1} - \sum_{j=1}^{b-1} \frac{n_{d1j}^2}{k}\right)$$

$$\leq \frac{v}{v-1}\left(r_{d1} - \frac{r_{d1}}{k}\right) \leq \frac{v}{v-1}\,\frac{(r-2)(k-1)}{k}$$

$$= \frac{v\lambda_1}{k} - \frac{2v(k-1)}{k(v-1)} < \frac{v\lambda_1}{k} - 1$$

for $k \geq 3$, $v \geq 3$. If $n_{d11} = 0$ we continue with (5.2.1)

$$= \frac{v}{v-1}\left(r_{d1} - \sum_{j=2}^{b} \frac{n_{d1j}^2}{k}\right) \leq \frac{v}{v-1}\left(r_{d1} - \frac{r_{d1}}{k}\right)$$

and the rest follows identically.

Suppose $n_{d11} \geq 2$, $r_{d1} = r-1$. Again

$$\mu_{d1} \leq \frac{v}{v-1}\left(r-1 - \frac{n_{11}^2}{k-1} - \sum_2^b \frac{n_{ij}^2}{k}\right) \leq \frac{v}{v-1}\left(r-1 - \frac{r-1-n_{d11}}{k} - \frac{n_{d11}^2}{k-1}\right)$$

$$= \frac{v}{v-1}\left(\frac{(r-1)(k-1)}{k} + \frac{n_{d11}}{k} - \frac{n_{d11}^2}{k-1}\right)$$

$$= \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(\frac{n_{d11}^2}{k-1} - \frac{n_{d11}}{k} + \frac{k-1}{k}\right). \qquad (5.2.2)$$

Now $(a^2/(k-1))-a/k$ is increasing in $a \geq 1$, $k \geq 3$. If $n_{d11} = 2$, then (5.2.2) is less than $(v\lambda_1/k)-1$, and so must be for $n_{d11} \geq 2$.

Suppose $n_{d1j} \geq 2$ for some $j \in \{2,\ldots,v\}$ with $r_{d1} = r-1$. Without loss of generality let $j = b$.

$$\mu_{d1} \leq \frac{v}{v-1}\left(r-1 - \frac{n_{d11}^2}{k-1} - \frac{n_{d1b}^2}{k} - \sum_{j=2}^{b-1} \frac{n_{d1j}^2}{k}\right)$$

$$\leq \frac{v}{v-1}\left(r-1 - \frac{r-1}{k} + \frac{n_{d11}}{k} - \frac{n_{d11}^2}{k-1} + \frac{n_{d1b}}{k} - \frac{n_{d1b}^2}{k}\right)$$

$$= \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(\frac{n_{d11}^2}{k-1} - \frac{n_{d11}}{k} + \frac{n_{d1b}^2}{k} - \frac{n_{d1b}}{k}\right). \qquad (5.2.3)$$

$a(a-1)/k$ is increasing for $a \geq 1$. The proof that (5.2.3) is less than μ_{d*1} is analogous to that for (5.2.2), except that $n_{d11} = 0, 1$ are considered separately.

Suppose $r_{d1} = r_{d2} = r-1$, say, and $\lambda_{d12} \leq \lambda_1 - 1$. We also know n_{d1j}, n_{d2j} are 0 or 1 for all j. Theorem 2.2.1 says

$$\mu_{d1} \leq \frac{1}{2}\left(r-1 - \frac{n_{d11}^2}{k-1} - \sum_{j=2}^{b} \frac{n_{d1j}^2}{k} + r-1 - \frac{n_{d21}^2}{k-1} - \sum_{j=2}^{b} \frac{n_{k2j}^2}{k}\right) + $$
$$+ \frac{n_{d11}n_{d12}}{k-1} + \sum_{j=2}^{b} \frac{n_{d1j}n_{d2j}}{k}. \qquad (5.3.4)$$

If at least one of n_{d11}, n_{d21} is zero, (5.3.4) is

$$\leq r-1 - \frac{1}{2}\left(\sum_{j=1}^{b} \frac{n_{d1j}^2}{k} + \sum_{j=1}^{b} \frac{n_{d2j}^2}{k}\right) + \sum_{j=2}^{b} \frac{n_{d1j}n_{d2j}}{k}$$

$$\leq r-1 - \frac{r-1}{k} + \frac{\lambda_1 - 1}{k} = \frac{r(k-1)}{k} + \frac{\lambda_1}{k} - \frac{k-1}{k} - \frac{1}{k} = \frac{v\lambda_1}{k} - 1.$$

If $n_{d11} = n_{d21} = 1$, then (5.3.4) is

$$\leq r-1 - \frac{r-2}{k} - \frac{1}{k-1} + \frac{\lambda_1 - 2}{k} + \frac{1}{k-1} = \frac{r(k-1)}{k} - 1 + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k} - 1.$$

Therefore we have shown d* is E-better than all $d \in \mathcal{D}(v,b,\underline{k}^*)$ except possibly those with the characteristics given in the hypothesis. Application of Theorem 2.1.1 to d* and the designs it E-betters completes the proof.

Remark 5.2.1: The proof that d* E-betters any design with $n_{d11} \geq 2$ also shows d* is E-better than a design with $n_{d11} = 1$ if $v \leq k(k-1)/(k-2)$. The right hand side equals 6 for k = 3 and 4, and is increasing after that. So d* E-betters designs with $r_{d1} = r-1$, $n_{d11} = 1$ for v = 5 and v = 6, v = 7 with k = 6, but not necessarily for v = 7 with k = 3, 4 or 5.

A-, D- and E-efficiencies for d* will be presented in Chapter 8.

5.3 The Cases $k \geq v$

An application of treatment 1 is removed from block 1, where it is extra-replicated. Since the forms of C_{d^0} and C_{d^*} are different for each $k = pv+q$ $(0 \leq q \leq v-1,\ p \geq 1)$ we must consider the cases separately.

For $k = v$, by Theorem 4.1.1, d^* is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,b,\underline{k}^*)$.

For $k = pv$, $p \geq 2$, C_{d^*} is of the form

$$\begin{bmatrix} r - \frac{\lambda_0}{k} - 1 + \frac{p^2}{k} - \frac{(p-1)^2}{k-1} & -(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1})\underline{1}'_{v-1} \\ \text{sym} & (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_{v-1} - (\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p^2}{k-1})J_{v-1,v-1} \end{bmatrix}$$

From Lemma 1.4.1 the eigenvalues of C_{d^*} are 0, $r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k}$ with multiplicity $v-2$, and

$$v(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1}) = \frac{v\lambda_1}{k} - v(\frac{p^2}{k} - \frac{p(p-1)}{k-1})$$

$$= \frac{v\lambda_1}{k} - \frac{v}{v-1}\,(1-(\frac{p^2}{k} - \frac{(p-1)^2}{k-1})).$$

These are of the form of Theorem 2.1.1.

Theorem 5.3.1: d^* is $\mathcal{J}_4$-optimal over all $d \in \mathcal{D}(v,b,\underline{k}^*)$ where $(\underline{k}^*)' = (pv-1,\ pv\underline{1}'_{b-1})$, $p \geq 1$ and $v \geq 3$.

Proof: We assume $p \geq 2$, as the $p = 1$ case was proved in Theorem 4.1.1. Recall that d^* has the most balanced r_{di}: $r-1, r, \ldots, r$ with $r = bp^2$. Let $d \in \mathcal{D}(v,b,\underline{k}^*)$ be arbitrary and recall $r_{d1} = \min_{1\leq i\leq v} (r_{di})$.

Theorem 2.2.1 says that

$$\mu_{d1} \leq \frac{v}{v-1}\,(r_{d1} - \frac{n^2_{d11}}{k-1} - \sum_{j=2}^{b} \frac{n^2_{d1j}}{k}) = \frac{v}{v-1}\,(r_{d1} - n_{d11} - \sum_{j=2}^{b} \frac{n^2_{dij}}{k} +$$

$$+ n_{d11} - \frac{n^2_{d11}}{k-1})$$

$$\leq \frac{v}{v-1}\,(B(r_{d1}-n_{d11};\, b-1,k) + n_{d11} - \frac{n_{d11}^2}{k-1})$$

$$\leq \frac{v}{v-1}\,(B(r-1-n_{d11};\, b-1,k) + n_{d11} - \frac{n_{d11}^2}{k-1})$$

$$= \frac{v}{v-1}\,(r-1-\sum_{j=2}^{b} \frac{\tilde{n}_{1j}^2}{k} - \frac{n_{d11}^2}{k-1}) \qquad (5.3.1)$$

where the $\tilde{n}_{1j} \geq 0$ $(2 \leq j \leq b)$, are as equal as possible, and

$\sum_{j=2}^{b} \tilde{n}_{1j} + n_{d11} = r-1$. If the $\tilde{n}_{1j}$ $(2 \leq j \leq b)$ and n_{d11} are as equal as possible, then they take on the values $p-1,p,\ldots,p$. In this case

$$\sum_{j=2}^{b} \frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1} \qquad (5.3.2)$$

is minimized if the $p-1$ is assigned to n_{d11}. Now we must show (5.3.1) is maximized when the $\tilde{n}_{1j}$ and n_{d11} are as equal as possible with $n_{d11} = p-1$.

There are two other cases for the values of the $\tilde{n}_{1j}$ and n_{d11}. The first is for $n_{d11} < p-1$ and $\tilde{n}_{1j} \geq p$ $(2 \leq j \leq b)$ with at least one $\tilde{n}_{1j} > p$. The second is that $n_{d11} > p-1$ and $\tilde{n}_{1j} \leq p$ $(2 \leq j \leq b)$ with at least one $\tilde{n}_{1j} < p$.

If the first case is true, let $\tilde{n}_{1j} = p+\ell$, $\ell \geq 1$, and $n_{d11} = p-1-m$, $m \geq 1$. Then

$$\frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1} = \frac{(p+\ell)^2}{k} + \frac{(p-1-m)^2}{k} > \frac{(p+\ell-1)^2}{k} + \frac{(p-m)^2}{k-1} = \frac{(\tilde{n}_{1j}-1)^2}{k} + \frac{(n_{d11}+1)^2}{k-1}$$

for $v \geq 3$, $p \geq 2$. This means we can minimize (5.3.2) by pushing n_{d11} up to p-1 while making the remaining n_{d1j} equal to p.

If the second case is true, let $\tilde{n}_{1j} = p-\ell$, $\ell \geq 1$ with $n_{d11} = p-1+m$, $m \geq 1$. Then

$$\frac{(p-\ell)^2}{k} + \frac{(p-1+m)^2}{k-1} > \frac{(p-\ell+1)^2}{k} + \frac{(p+m-2)^2}{k-1}$$

for $v \geq 3$, $p \geq 2$. Again we maximize (5.3.2) by making the n_{d1j} $(1 \leq j \leq b)$ equal to p-1, p,...,p, respectively.

Therefore (5.3.2) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{(b-1)p^2}{k}-\frac{(p-1)^2}{k-1}\right) = \mu_{d*1},$$

the most balanced case, that of d*. d* is E-best in $\mathcal{D}(v,b,\underline{k}^*)$ so by Theorem 2.1.1, we are finished with the proof.

For $k = pv+1$, $p \geq 1$, C_{d*} is of the form

$$\begin{bmatrix} r-\frac{\lambda_0}{k}-1+\frac{(p+1)^2}{k}-\frac{p^2}{k-1} & -\left[\frac{\lambda_1}{k}-\frac{p(p+1)}{k}+\frac{p^2}{k-1}\right]\underline{1}'_{v-1} \\ \text{sym} & \left(r-\frac{\lambda_0}{k}+\frac{\lambda_1}{k}\right)I_{v-1}-\left(\frac{\lambda_0}{k}-\frac{p^2}{k}+\frac{p^2}{k-1}\right)J_{v-1,v-1} \end{bmatrix}$$

when we assume treatment 1 was extrareplicated in the first block. Recall that $r = e(p+1) + e(v-1)p$, where $b = ev$, $e \geq 1$.

The eigenvalues of C_{d*} are derived via Lemma 1.4.1, and are 0, $(v\lambda_1/k)$ with multiplicity v-2, and

$$\frac{v\lambda_1}{k} - v\left(\frac{p(p+1)}{k}-\frac{p^2}{k-1}\right) = \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(1-\left(\frac{(p+1)^2}{k}-\frac{p^2}{k-1}\right)\right).$$

They are of the form of Theorem 2.1.1.

Theorem 5.3.2: d^* is $\mathcal{J}_4$-optimal over all $d \in \mathcal{D}(v,b,\underline{k}^*)$ where $(\underline{k}^*)' = (pv,(pv+1)\underline{1}'_{b-1})$, $p \geq 1$ and $v \geq 3$.

Proof: d^* has the most balanced configuration of r_{di}. Let $d \in \mathcal{D}(v,b,\underline{k}^*)$ be arbitrary.

$$\mu_{d1} \leq \frac{v}{v-1}\left(r_{d1} - \frac{n_{d11}^2}{k-1} - \sum_{j=2}^{b} \frac{n_{d1j}^2}{k}\right)$$

$$\leq \frac{v}{v-1}\left(r-1-\sum_{j=2}^{b} \frac{\tilde{n}_{1j}}{k} - \frac{n_{d11}^2}{k-1}\right) \tag{5.3.3}$$

where the argument and definition of the $\tilde{n}_{1j}$ follow that of (5.3.1). The only difference is that the most equal the $\tilde{n}_{1j}$ $(2 \leq j \leq b)$ and n_{d11} can be is for e-1 of them to equal p+1 and e(v-1)+1 of them to equal p. (In d^o, for each treatment we had e blocks with $n_{dij} = p+1$ and e(v-1) blocks with $n_{dij} = p$.). In this most balanced case, (5.3.2) is minimized if a p is assigned to n_{d11}. Now we show (5.3.3) is maximized by this case.

One other possibility is that $n_{d11} < p$, e-1 of the $\tilde{n}_{1j} \geq p+1$, e(v-1) of the $\tilde{n}_{1j} \geq p$, and at least one inequality on the $\tilde{n}_{1j}$ is strict. Another possibility is that $n_{d11} > p$, e-1 of the $\tilde{n}_{1j} \leq p+1$, e(v-1) of the $\tilde{n}_{1j} \leq p$, and at least one inequality on the $\tilde{n}_{1j}$ is strict.

For the first case,

$$\frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1} = \frac{(p+\ell)^2}{k} + \frac{(p-m)^2}{k-1} > \frac{(p+\ell-1)^2}{k} + \frac{(p-m+1)^2}{k-1}$$

for $\ell \geq 1$, $m \geq 1$, $v \geq 3$, $p \geq 1$. Therefore we maximize (5.3.3) by pushing n_{d11} up to p, keeping the other n_{d1j} as equal as possible.

For the second case,

$$\frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1} = \frac{(p+1-\ell)^2}{k} + \frac{(p+m)^2}{k-1} > \frac{(p+2-\ell)^2}{k} + \frac{(p+m-1)^2}{k-1}$$

for $\ell \geq 1$, $m \geq 1$, $v \geq 3$, $p \geq 1$. We maximize (5.3.3) by pushing n_{d11} down to p, keeping the other n_{d1j} as equal as possible.

Finally then, (5.3.3) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{(e-1)(p+1)^2+e(v-1)p^2}{k}-\frac{p^2}{k-1}\right) = \mu_{d^*1},$$

and by Theorem 2.1.1 the proof is complete.

If we think of running an experiment with a BBD, k = pv+1 and $p \geq 1$, then we have just shown that if an observation must be thrown out, and it occurred in a block where that treatment was extra-replicated, the resulting design is $\mathcal{J}_4$-optimal. If the observation was from a block where the treatment was not extra-replicated, how much worse is the resulting design? If we call this design $\tilde{d}$ and assume treatment 1 was lost in block 1, then $C_{\tilde{d}}$ is of the form

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym} & & C_{33} \end{bmatrix} \qquad (5.3.4)$$

where

$$C_{11} = r-\frac{\lambda_0}{k} - 1 + \frac{p^2}{k} - \frac{(p-1)^2}{k-1},\quad C_{12} = -\left(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{(p+1)(p-1)}{k-1}\right),$$

$$C_{13} = -\left(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1}\right)\underline{1}_{v-2}',\quad C_{22} = r-\frac{\lambda_0}{k} + \frac{(p+1)^2}{k} - \frac{(p+1)^2}{k-1},$$

$C_{23} = -(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1})\underline{1}'_{v-2}$, $C_{33} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_{v-2} - (\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p^2}{k-1})J_{v-2,v-2}$, and without loss of generality treatment 2 was extra-replicated in block 1. Using Lemma 2.4.1 we get as eigenvalues of $C_{\tilde{d}}$: 0, $(v\lambda_1/k)$ with multiplicity v-2, and

$$\mu_{\tilde{d}1} = \frac{v\lambda_1}{k} - \frac{vp^2(v-1)}{k(k-1)} - \frac{2}{k-1} < \frac{v\lambda_1}{k} - \frac{vp^2(v-1)}{k(k-1)} = \mu_{d_1^*}.$$

The eigenvalues of $C_{\tilde{d}}$ differ from those of C_{d*} only in that $\mu_{\tilde{d}1} < \mu_{d_1^*}$ by exactly the amount that $tr(C_{\tilde{d}}) < tr(C_{d*})$.

For $k = pv+q$, $2 \leq q \leq v-1$, C_{d*} is of the form of (5.3.4) except that

$$C_{11} = r - \frac{\lambda_0}{k} - 1 + \frac{(p+1)^2}{k} - \frac{p^2}{k-1},\quad C_{12} = -(\frac{\lambda_1}{k} - \frac{(p+1)^2}{k} + \frac{p(p+1)}{k-1})\underline{1}'_{q-1},$$

$$C_{13} = -(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p^2}{k-1})\underline{1}'_{v-q},$$

$$C_{22} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_{q-1} - (\frac{\lambda_1}{k} - \frac{(p+1)^2}{k} + \frac{(p+1)^2}{k-1})J_{q-1,q-1},$$

$$C_{23} = -(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1})J_{q-1,v-q},$$

$$C_{33} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_{v-q} - (\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p^2}{k-1})J_{v-q,v-q}$$

and treatments 1 through q were assumed to be extra-replicated in block 1. Using Lemma 2.4.1 we get the eigenvalues 0, $(v\lambda_1/k)$ of multiplicity q-2 + v-q-1 = v-3, and two from $(A \pm (B_1^2+B_2)^{\frac{1}{2}})/2$. The last two eigenvalues are

$$\frac{v\lambda_1}{k} \text{ and } \frac{v\lambda_1}{k} - 1 + \frac{vp(p+1)}{k(k-1)},$$

so the eigenvalues of C_{d*} are still of the form $0 < a < b' = \ldots = b$.

Theorem 5.3.3: $d^* \in \mathcal{D}(v,b,\underline{k}^*)$ where $(\underline{k}^*)' = (pv+q-1, (pv+q)\underline{1}'_{b-1})$, $2 \le q \le v-1$, is $\mathcal{G}_4$-optimal over all d except possibly those with $r_{d1} = \min_{1 \le i \le v} (r_{di}) = r-1$ and $n_{d11} = p$ or $p+1$. $(v \ge 3)$.

Proof: d^* has the most balanced configuration of r_{di}. As in the previous two theorems we will show an E-better design must have $r_{d1} = r-1$, but we can no longer carry through the same arguments. This is because C_{d*} is not a 2-by-2 blocked matrix.

Let $d \in \mathcal{D}(v,b,\underline{k}^*)$ be arbitrary. Theorem 2.2.1 says

$$\mu_{d1} \le \frac{v}{v-1}\left(r_{d1} - \sum_{j=2}^{b} \frac{n_{d1j}^2}{k} - \frac{n_{d11}^2}{k-1}\right)$$

$$\le \frac{v}{v-1}\left(B(r_{d1}-n_{d11};\, b-1,k) + n_{d11} - \frac{n_{d11}^2}{k-1}\right)$$

$$\le \frac{v}{v-1}\left(B(r-1-n_{d11};\, b-1,k) + n_{d11} - \frac{n_{d11}^2}{k-1}\right)$$

$$= \frac{v}{v-1}\left(r-1-\sum_{j=2}^{n} \frac{\tilde{n}_{1j}^2}{k} - \frac{n_{d11}^2}{k-1}\right) \qquad (5.3.5)$$

where the arguments and definition of the $\tilde{n}_{1j}$ follow as at (5.3.1). If for d^0 we let $n_{d^01j} = p+1$ for b_0 blocks, and $n_{d^01j} = p$ for $b-b_0$ blocks, then for d^* $n_{d^*1j} = p+1$ for b_0-1 blocks and p for $b-b_0$ blocks. (5.3.2) is again minimized if a p is assigned to the n_{d11} and the $\tilde{n}_{1j}$ $(2 \le j \le b)$ equal p or p+1.

In the Theorems 5.3.1 and 5.3.2 we showed that any other configuration made the equivalent of 5.3.5 less than that for the most balanced case, which was bounded by $(v\lambda_1/k)-1$. Of course less balanced configurations of the n_{d1j} still give bounds lower than that for the balanced case, but the balanced case bound is too large.

If $n_{d11} = p$ (5.3.5) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{(b_0-1)(p+1)^2+(b-b_0)p^2}{k}-\frac{p^2}{k-1}\right)$$

but this is greater than $(v\lambda_1/k)-1$ for all v, p and q.

If $n_{d11} \leq p-1$ (5.3.5) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{b_0(p+1)^2+(b-b_0-1)p^2}{k}-\frac{(p-1)^2}{k-1}\right) \leq \mu_{d_1^*}.$$

If $n_{d11} \geq p+2$ (5.3.5) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{(b_0-3)(p+1)^2+(b-b_0+2)p^2}{k}-\frac{(p+2)^2}{k-1}\right) \leq \mu_{d_1^*}$$

If $n_{d11} = p+1$ (5.3.5) is

$$\leq \frac{v}{v-1}\left(r-1-\frac{(b_0-2)(p+1)^2+(b-b_0+1)p^2}{k}-\frac{(p+1)^2}{k-1}\right) \leq \mu_{d_1^*}$$

if and only if $q^2-q(v+1)+2v(p+1) \geq 0$. This holds for q = 2 and q = v-1 (any v and p) but not for all v, p and q.

Therefore d* is E-better than all d except possibly some for which $r_{d1} = r-1$, and $n_{d11} = p$ or $p+1$. Theorem 2.1.1 is now applied to the E-bettered (by d*) designs.

Remark 5.3.1: For the cases of $q = 2$ and $q = v-1$ an E-better (than d*) d must have for any minimum replicated treatment $r_{di} = r-1$ and $n_{di1} = p$. These are two cases where d* can be specified in general but no E-better $\hat{d}$ has been found, nor has it been proven not to exist.

If we remove treatment 1 from a block in d^0 where 1 was not extra-replicated, and call the design $\tilde{d}$, then $C_{\tilde{d}}$ is of the form (5.3.4). In this case

$$C_{11} = r- \frac{\lambda_0}{k} -1+ \frac{p^2}{k} - \frac{(p-1)^2}{k-1},\quad C_{12} = -(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p^2-1}{k-1})\underline{1}_q',$$

$$C_{13} = -(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1})\underline{1}_{v-q-1}',$$

$$C_{22} = (r- \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_q-(\frac{\lambda_1}{k} - \frac{(p+1)^2}{k} + \frac{(p+1)^2}{k-1})J_{q,q},$$

$$C_{23} = -(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1})J_{q,v-q-1}$$

$$C_{33} = (r- \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_{v-q-1}-(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p^2}{k-1})J_{v-q-1,v-q-1}$$

where treatments 2 through q+1 were extra-replicated in block 1. Lemma 2.4.1 yields the eigenvalues 0, $\mu_{\tilde{d}i} = \mu_{d^*i}$ $(2 \leq i \leq v-1)$, and

$$\mu_{\hat{d}1} = \frac{v\lambda_1}{k} - 1 + \frac{vp(p+1)}{k(k-1)} - \frac{2}{k-1} < \mu_{d^*1}.$$

In Chapter 8 A-, D- and E- efficiencies will be calcuated for $\tilde{d}$ when $k = pv+1$, and d* and $\tilde{d}$ for $k = pv+q$, $2 \leq q \leq v-1$.

CHAPTER 6

THE BBD PLUS M BLOCKS

6.1 Preliminaries

Throughout Chapter 6 define d^o and d^* as follows. Let $d^o \in \mathcal{D}(v,b,k)$ be a BBD with $k = pv+q$ $(1 \le q \le v-1)$. Add to d^o m binary blocks which are disjoint with respect to their extra-replicated treatments, with $1 \le m \le v/q$. If we call the new design d^*, then $d^* \in \mathcal{D}(v,b+m,k)$ and $C_{d^*} = C_{d^o} + C_0$ where C_0 is the C-matrix for the blocks added.

Lemma 6.1.1: C_0 is of the form

$$\begin{bmatrix} A & B & \cdots & B & B & D \\ & A & \cdots & B & B & D \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & A & B & D \\ & & & & A & D \\ & & & & & E \end{bmatrix}$$

where $A = (mp+1)I_q - (\frac{(m-1)p^2+(p+1)^2}{k})J_{q,q}$, $B = -(\frac{(m-2)p^2+2p(p+1)}{k})J_{q,q}$,

$D = -(\frac{(m-1)p^2+p(p+1)}{k})J_{q,v-mq}$ and $E = mpI_{v-mq} - \frac{mp^2}{k} J_{v-mq,v-mq}$.

Proof: The mq treatments extra-replicated in the added blocks appear mp+1 times in those blocks; the remaining treatments appear only mp times. The extra-replicated treatments are paired $(p+1)^2+(m-1)p^2$ times within added blocks, $2p(p+1)+(m-2)p^2$ times between added blocks, and $(m-1)p^2+p(p+1)$ times with the remaining treatments. The non-extra-replicated treatments are paired mp^2 times in the added blocks.

Lemma 6.1.2: C_{d*} has the eigenvalues 0,

$$\frac{v\lambda_1}{k} + mp \text{ with multiplicity } v-mq-1 \text{ (unless } mq \geq v-1),$$

$$\frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k} \text{ with multiplicity 1 (unless } mq = v),$$

$$\frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k} \text{ with multiplicity } m-1,$$

$$\frac{v\lambda_1}{k} + mp + 1 \text{ with multiplicity } m(q-1),$$

where $\lambda_1 = \lambda_{d^o ij}$ $(i \neq j)$ and for each case of m and q all realized eigenvalues are distinct. The eigenvalue structure is $0 < a < b =\ldots= b$ for $m = 2$ with $q = v/2$, $m = 1$ with $q = v-1$ or $m = v-1$ with $q = 1$. The eigenvalue structure is $0 < a =\ldots= a < b$ if $m = 1$ with $q = 1$ and is $0 < a =\ldots= a$ if $m = v$ and $q = 1$.

Proof: $C_{d^o} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v,v}$ with $\lambda_0 = \lambda_{d^o ii}$ and $r = r_{d^o i}$ $(1 \leq i \leq v)$. To $C_{d*} = C_{d^o} + C_0$ we apply Lemma 6.1.1 and Lemma 1.4.2 to get the eigenvalues.

It is easy to show that

$$\frac{v\lambda_1}{k} + mp \le \frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k} < \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}$$

with equality between the first pair of eigenvalues only when $mq = v$, but in that case they are not realized.

To get the parameter values for which the various eigenvalue structures occur consider $mq = v$, $mq = v-1$ and $1 \le mq \le v-2$ as separate cases. If $mq = v$, at most the two largest nonzero eigenvalues occur with $m = 2$ and $m = v$ yielding results. If $mq = v-1$ at most the three largest eigenvalues occur with $m = 1$ and $q = 1$ giving results. If $mq \le v-2$ then $m = q = 1$ gives the eigenvalue structure of Cheng. This ends the proof.

Note that for $q = 0$ all values of $b \ge 2$ allow a BBD in $\mathcal{D}(v,b,pv)$ which is $\mathcal{J}_U$-optimal. Therefore we only consider $1 \le q \le v-1$.

In Sections 6.2, 6.3 and 6.4 we will consider the cases $mq = v$, $mq = v-1$ and $1 \le mq \le v-2$, respectively. Throughout λ_0, λ_1 and r are defined as above.

6.2 The Case $mq = v$

For $mq = v$ the eigenvalues of C_{d*} are 0, $(v\lambda_1/k) + mp + 1 - (q/k)$ and $(v\lambda_1/k) + mp + 1$ with multiplicities 1, $m-1$ and $m(q-1)$ respectively.

All eigenvalues are equal if $m = v$ and $q = 1$, in which case d^o comprises e copies $(e \ge 1)$ of the unique BBD in $\mathcal{D}(v,v,pv+1)$, and d* will then be e+1 copies of the same. Therefore d* is a BBD and we can apply Corollary 1.4.4, making d* $\mathcal{J}_U$-optimal in $\mathcal{D}(v,ev+v,pv+1)$.

Note that if m = 1 and q = v we get $\mathcal{J}_4$-optimality of d*, but this is really just the case of q = 0.

The eigenvalue structure 0 < a < b =...= b is achieved in one case, that of m = 2 and q = v/2. The nonzero eigenvalues of d* become

$$\frac{v\lambda_1}{k} + mp + 1 - \frac{v}{2k} \quad \text{and} \quad \frac{v\lambda_1}{k} + mp + 1$$

with multiplicities 1 and v-2 respectively. Two examples will show, however, that each m and q pair must be dealt with separately.

Lemma 6.2.1: If $k = pv+q$, $p \geq 1$, $1 \leq q \leq v-1$, d^0 is a BBD in $\mathcal{D}(v,b,pv+q)$ and d* is d^0 plus m disjoint binary blocks with mq = v, then d* is E-better than any $d \in \mathcal{D}(v,b+m,pv+q)$ with $\min_{1\leq i\leq v} (r_{di}) \leq r+mp$, and any d for which $\lambda_{dij} \leq \lambda_1 + mp^2 + 2p-(q-1)$ whenever $r_i = r_j = r + mp + 1$. $v \geq 3$.

Proof: Using Theorem 1.4.3, and assuming without loss of generality that $r_{d1} = \min_{1\leq i\leq v} (r_{di})$ for an arbitrary d,

$$\mu_{d1} \leq \frac{v}{v-1}\left(r_{d1} - \sum_{j=1}^{b+m} \frac{n_{dij}^2}{k}\right) \leq \frac{v}{v-1} B(r_{d1};\, b+m,k)$$

$$\leq \frac{v}{v-1} B(r+mp;\, b+m,k) = \frac{v}{v-1}\left(r+mp - \frac{\lambda_0+mp^2}{k}\right)$$

$$= \frac{v\lambda_1}{k} + \frac{vmp^2}{k} + \frac{v^2p}{k(v-1)} \leq \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}.$$

If for d, $r_{d1} = r_{d2} = r + mp + 1$ and $\lambda_{d12} \leq \lambda_1 + mp^2 + 2p - (q-1)$,

$$\mu_{d1} \leq r+mp+1 - \frac{1}{2}\left(\sum_{j=1}^{b+m} \frac{n_{d1j}^2}{k} + \sum_{j=1}^{b+m} \frac{n_{d2j}^2}{k}\right) + \frac{\lambda_{d12}}{k}$$

$$\leq r+mp+1 - \frac{\lambda_0+mp^2+2p+1}{k} + \frac{\lambda_1+mp^2+2p-(q-1)}{k}$$

$$= \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k} \leq \frac{v\lambda_1}{k} + mp + 1.$$

If $m = 1$, $\frac{v\lambda_1}{k} + mp + 1$ is the smallest nonzero eigenvalue of d*. For $m \geq 2$ $\frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}$ is the smallest nonzero eigenvalue of d*. In either case $\mu_{d1} \leq \mu_{d*1}$.

Lemma 6.2.2: If $k = pv+2$, $p \geq 1$, $m = v/2$, d^0 and d* are as in Lemma 6.2.1, then d* is E-optimal in $\mathcal{D}(v,b+m,pv+2)$. $v \geq 3$.

Proof: The nonzero eigenvalues of C_{d*} are $(v\lambda_1/k)+mp+1-(2/k)$ and $(v\lambda_1/k)+mp+1$ with multiplicities $(v/2)-1$ and $(v/2)$, respectively. We also have

$$C_{d*} = \frac{1}{k}\begin{bmatrix} a & -(b+1) & -b & -b & \cdots & -b & -b \\ & a & -b & -b & \cdots & -b & -b \\ & & a & -(b+1) & \cdots & -b & -b \\ & & & a & \cdots & -b & -b \\ & & & & \ddots & \vdots & \vdots \\ & & & & & a & -(b+1) \\ & & & & & & a \end{bmatrix} \qquad (6.2.1)$$

with $a = (v-1)\lambda_1+k(mp+1)-(mp^2+2p+1)$ and $b = \lambda_1+mp^2+2p$.

Lemma 6.2.1 says d* E-bests all d with $\min_{1 \leq i \leq v} (r_{di}) \leq r+mp$. This means all r_{di} must equal $r + mp + 1 = r_{d*i}$ $(1 \leq i \leq v)$ for an E-better (than d*) d. Also Lemma 6.2.1 says d* E-betters any d with $r_{di} = r_{dj} = r+mp+1$ and $\lambda_{dij} \leq \lambda_1+mp^2+2p-1$. Therefore all that is left are d with all replications equal to r+mp+1 and $\lambda_{dij} \geq \lambda_1+mp^2+2p$. However at most one of these can be equal to λ_1+mp^2+2p+1 in each row. Therefore an E-better d must have C_d of the form (6.2.1). So d* must be E-optimal in $\mathcal{D}(v,b+m,pv+2)$.

Remark 6.2.1: The d* in Lemma 6.2.2 is unique up to a permutation of the treatment labels.

Theorem 6.2.1: Let v = 4, k = 4p+2 $(p \geq 1)$, d^o be a BBD in $\mathcal{D}(4,b,4p+2)$ and d* be d^o plus m = 2 disjoint binary blocks. Then d* is $\mathcal{J}_4$-optimal in $\mathcal{D}(4,b+2,4p+2)$.

Proof: d* has maximum trace, eigenvalues of the form $0 < a < b=b$ by the proof of Lemma 6.2.2 with (v/2) = 2, and d* is E-optimal by Lemma 6.2.2. Therefore Theorem 2.1.1 gives us $\mathcal{J}_4$-optimality.

Example 6.2.1: $d^* \in \mathcal{D}(4,8,6)$ given below is $\mathcal{J}_4$-optimal.

	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2
	3	3	3	3	3	3	3	3
d*:	4	4	4	4	4	4	4	4
	1	1	1	2	2	3	1	3
	2	3	4	3	4	4	2	4

Example 6.2.1 is the $k > v$ extension of the $\mathcal{J}_4$-optimal BIBD(4,6e,2,3e,e) (here e = 1) from Cheng (1979) that was mentioned in Section 3.3. In that same section the example of the BIBD(8,14,4,7,3) plus two disjoint binary blocks being A-, D- and E-bettered by an RGD can also be extended to $k > v$.

<u>Example 6.2.2</u>: Let $d^o \in \mathcal{D}(8,14,8p+4)$ be the BBD constructed by adding p copies of each treatment to each block of the BIBD(8,14,4,7,3). Then $d^* \in \mathcal{D}(8,16,8p+4)$ is d^o plus two disjoint binary blocks. The eigenvalues of C_{d^*} are 0, $(8\lambda_1/k) + (16p(p+1)/k) = (8A+24)/k$, and $(8\lambda_1/k) + 2p + 1 = (8A+28)/k$ where $\lambda_1 = 14p^2+14p+3$ and $A = 16p(p+1)$.

Now let $\hat{d}$ be the RGD in $\mathcal{D}(8,16,4)$ mentioned above with p copies of each treatment added to each block. $\hat{d} \in \mathcal{D}(8,16,8p+4)$ and the eigenvalues of $C_{\hat{d}}$ are 0, $(8A+26)/k$, $(8A+28-\sqrt{2})/k$, $(8A+28)/k$, and $(8A+28+\sqrt{2})/k$ with multiplicities 1, 2, 2, 1 and 2 respectively. $\hat{d}$ is easily seen to be E-better than d^* and with algebra is seen to be A- and D-better.

The eigenvalue structure $0 < a = \ldots = a < b$ is not realized for $mq = v$.

A-, D- and E-efficiencies for the cases of $m = 2$ and $q = v/2$ will be presented in Chapter 8, along with the A- and D-efficiencies for $q = 2$.

<u>6.3 The Case $mq = v-1$</u>

We begin with a lemma.

Lemma 6.3.1: For $k = pv+q$, $1 \le q \le v-1$, $p \ge 1$, $v \ge 3$, $mq \le v-1$ and d^* equal to d^o plus m disjoint binary blocks (d^o a BBD in $\mathcal{D}(v,b,pv+q)$), then for any $d \in \mathcal{D}(v,b+m,pv+q)$ $\mu_{d1} \le \frac{v\lambda_1}{k} + \frac{v}{v-1}(mp- \frac{mp^2}{k})$ $= \frac{v\lambda_1}{k} + \frac{vp}{k}(mp + \frac{mq}{v-1})$.

Proof: Since $mq \le v-1$, $\min_{1 \le i \le v} (r_{di}) \le r+mp$ for any $d \in \mathcal{D}(v,b+m,pv+q)$. Using Theorem 1.4.3, and assuming 1 is the minimum replicated treatment,

$$\mu_{d1} \le \frac{v}{v-1}\left(r_{d1} - \sum_{j=1}^{b+m} \frac{n_{d1j}^2}{k}\right) \le \frac{v}{v-1} B(r_{d1};\, b+m,k)$$

$$\le \frac{v}{v-1} B(r+mp;\, b+m,k) = \frac{v}{v-1}\left(r+mp- \frac{\lambda_0+mp^2}{k}\right).$$

Algebra completes the proof.

When $mq = v-1$ the eigenvalues of C_{d^*} are 0, $\frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k}$, $\frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}$, and $\frac{v\lambda_1}{k} + mp + 1$ with multiplicities 1, 1, m-1 and m(q-1), respectively.

Theorem 6.3.1: For $k = pv+q$, $1 \le q \le v-1$, $p \ge 1$, $v \ge 3$, $mq = v-1$, d^* is E-optimal in $\mathcal{D}(v,b+m,pv+q)$.

Proof: By Lemma 6.3.1 for any $d \in \mathcal{D}(v,b+m,pv+q)$

$$\mu_{d1} \le \frac{v\lambda_1}{k} + \frac{vp}{k}\left(mp + \frac{mq}{v-1}\right) = \frac{v\lambda_1}{k} + \frac{vp}{k}(mp+1),$$

the smallest nonzero eigenvalue of d^*. This completes the proof.

Now from Lemma 6.1.2 we have m = 1 with q = v-1 and m = v-1 with q = 1 as the only two cases giving d^* with one of the three eigenvalue structrues from Theorems 1.4.1, 1.4.4 and 2.1.1. The structure of the eigenvalues of C_{d*} is $0 < a < b = \ldots = b$ in both cases, with

$$a = \frac{v\lambda_1}{k} + \frac{vp}{k}(mp+1).$$

Theorem 6.3.2: For $k = pv+q$, $p \geq 1$, $v \geq 3$, $m = 1$ with $q = v-1$ or $m = v-1$ with $q = 1$, d* is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,b+m,pv+q)$.

Proof: Theorems 6.3.1 and 2.1.1 with the fact that C_{d*} is of maximum trace gives the result.

A- and D-efficiencies for these E-optimal designs when $mq = v-1$, $m \neq 1$, $q \neq 1$ will be presented in Chapter 8.

6.4 The Cases $1 \leq mq \leq v-2$

From Lemma 6.1.2 only one of the cases $1 \leq mq \leq v-2$ yields two distinct eigenvalues, and that is $m = q = 1$. The eigenvalue structure for d* with $mq = 1$ is that of Theorem 1.4.1 with C_{d*} having eigenvalues 0, $\frac{v\lambda}{k} + p$ with multiplicity $v-2$, and $\frac{v\lambda_1}{k} + \frac{vp(p+1)}{k}$.

For this case d^o is e copies ($e \geq 1$) of the unique BBD in $\mathcal{D}(v,v,pv+1)$, and d* is d^o plus a binary block with, say, treatment 1 extra-replicated. There is a $\hat{d}$ in $\mathcal{D}(v,ev+1,pv+1)$ with d* that is E-better than d* when $p \geq v-1$, but can only be constructed when $p \geq v-2$. We construct $\hat{d}$ by taking e-1 copies of that unique BBD in $\mathcal{D}(v,v,pv+1)$, plus v binary blocks with treatment 1 extra-replicated, and one block with treatments 2 through v appearing p+1 times and treatment 1 appearing p-(v-2) times.

$C_{\hat{d}} = C_{d^o} + C_0$ where

$$C_0 = \begin{bmatrix} p+1 - \frac{(p+1)^2+v^2-3v+2}{k} & -(\frac{p(p+1)-(v-2)}{k})1'_{v-1} \\ \text{sym} & (p+\frac{1}{k})I_{v-1} - \frac{p^2+1}{k}J_{v-1,v-1} \end{bmatrix}$$

and so $\hat{d}$ has the eigenvalues 0, $\frac{v\lambda_1}{k} + p + \frac{1}{k}$ with multiplicity v-2, and $\frac{v\lambda_1}{k} + \frac{vp(p+1)}{k} - \frac{v(v-2)}{k}$.

We need $p \geq v-1$ for

$$\frac{v\lambda_1}{k} + \frac{vp(p+1)}{k} - \frac{v(v-2)}{k} \geq \frac{v\lambda_1}{k} + p + \frac{1}{k},$$

with strict inequality if $p > v-1$ and equality for $p = v-1$. Note that for $p \geq v-1$ $\hat{d}$ is strictly E-better than d* but that $\text{tr } C_{\hat{d}} < \text{tr } c_{d^*}$ by $k^{-1}(v-1)(v-2)$. We summarize what is known about d* and $\hat{d}$ in this case of mq = 1 in the following theorem.

<u>Theorem 6.4.1</u>: d* is E-better than any $d \in \mathcal{D}(v,ev+1,pv+1)$ for which 1) $\min_{1 \leq i \leq v} (r_{di}) \leq r+p-1$ or 2) $\lambda_{dij} \leq \lambda_1+p^2$ when $r_{di} = r_{dj} = r+p$. For $p \geq v-1$ there exists a $\hat{d}$ (with $r_{\hat{d}1} = r+p+1$, $r_{\hat{d}i} = r+p$ for $2 \leq i \leq v$ and $\lambda_{\hat{d}ij} = \lambda_1+p^2+1$ for $2 \leq i < j \leq v$) which is strictly E-better than d*. Therefore d* is not $g\mathcal{J}_1$-optimal for $p \geq v-1$.

<u>Proof</u>: That d* is E-better than the d satisfying 1) or 2) follows from Lemma 6.4.1 at the end of the chapter. The results for $\hat{d}$ follow from the previous discussion.

Remark 6.4.1: From a few examples for this case of mq = 1 $\hat{d}$ appears to be A- and D-worse than d*. One such example follows.

Example 6.4.1: d*, $\hat{d} \in \mathcal{D}(3,4,7)$. $\hat{d}$ is E-better but A- and D-worse than d*.

	1	1	1	1		1	1	1	1
	1	1	1	1		1	1	1	2
	2	2	2	2		2	2	2	2
d*:	2	2	2	2	$\hat{d}$:	2	2	2	2
	3	3	3	3		3	3	3	3
	3	3	3	3		3	3	3	3
	1	2	3	1		1	1	1	3

Sometimes, as for the following example, when mq = 1 with p < v-1 we can prove d* is $g\mathcal{J}_1$-optimal using Theorem 1.4.1.

Example 6.4.2: $d^* \in \mathcal{D}(3,4,4)$ is $g\mathcal{J}_1$-optimal:

	1	1	1	1
	2	2	2	2
d*:	3	3	3	3
	1	2	3	1

In Chapter 8 the A-, D- and E-efficiencies of d* and the E-efficiency of $\hat{d}$ ($p \geq v-1$) are presented for this case of mq = 1.

For $2 \leq mq \leq v-2$ C_{d^*} has three (m = 1 or q = 1) or four distinct eigenvalues. It turns out that some of the ideas used to construct $\hat{d}$ for mq = 1, or other techniques, can be used to construct an E-better (than d*) design. Nothing has been generalized, but here are two such examples.

Example 6.4.3: $\hat{d}$ is E-better but A- and D-worse than d* in $\mathcal{D}(4,6,9)$. m = 2, q = 1, mq = 2 = v-2 and only $\hat{d}$ is given here.

	1	1	1	1	1	1
	1	1	1	1	1	2
	2	2	2	2	2	2
	2	2	2	2	2	3
$\hat{d}$:	3	3	3	3	3	3
	3	3	3	3	3	3
	4	4	4	4	4	4
	4	4	4	4	4	4
	1	1	1	2	2	4

Example 6.4.4: $\hat{d}$ is E-better but A- and D-worse than d* $\in \mathcal{D}(5,11,7)$. Given here are only the last two plots of each (binary) block to display the extra-replicated treatments. m = 1, q = 2 and mq = 2 = v-3.

d*:	1	1	1	1	2	2	2	3	3	4	1
	2	3	4	5	3	4	5	4	5	5	2
$\hat{d}$:	1	1	1	1	1	3	3	3	3	4	4
	2	2	2	2	2	4	4	5	5	5	5

Remark 6.4.2: The $\hat{d}$ for $\mathcal{D}(4,6,9)$ is not of full trace. The d* for $\mathcal{D}(4,6,5)$ is the p = 1 case corresponding to Example 6.4.3, and

can be proven E-optimal. The $\hat{d}$ for $\mathcal{D}(5,11,12)$ is of full trace. The d* for $\mathcal{D}(5,11,7)$ is the $p = 1$ case corresponding to Example 6.4.4, and can be proven E-optimal.

We conclude this chapter with the following lemma which was needed for Theorem 6.4.1 and $mq = 1$ but is also true for $2 \le mq \le v-2$.

Lemma 6.4.1: If $k = pv+q$, $1 \le q \le v-1$, $p \ge 1$, $v \ge 3$, $1 \le mq \le v-2$, then d* is E-better than any $d \in \mathcal{D}(v,b+m,pv+q)$ with $\min_{1\le i\le v} (r_{di}) \le r+mp-1$ or with $\lambda_{dij} \le \lambda_1+mp^2$ when $r_{di} = r_{dlj} = r+mp$.

Proof: Using Theorem 1.4.3 (i),

$$\mu_{d1} \le \frac{v}{v-1} B(\min_{1\le i\le v} (r_{di}); b+m,k)$$

$$\le \frac{v}{v-1} B(r+mp-1; b+m,k)$$

$$= \frac{v}{v-1} (r+mp-1- \frac{\lambda_0+(m+1)p^2-(p+1)^2}{k})$$

$$= \frac{v\lambda_1}{k} + \frac{v}{v-1} (mp-1- \frac{mp^2-2p-1}{k}) < \frac{v\lambda_1}{k} + mp.$$

Using Theorem 1.4.3 (ii),

$$\mu_{d1} \le r+mp- \frac{\lambda_0+mp^2}{k} + \frac{\lambda_1+mp^2}{k}$$

$$= \frac{v\lambda_1}{k} + mp.$$

This completes the proof.

CHAPTER 7

THE BBD MINUS M BLOCKS

7.1 Preliminaries

Throughout Chapter 7 define d^o and d^* as follows. Let $d^o \in \mathcal{D}(v,b,k)$ be a BBD with $k = pv+q$ $(1 \leq q \leq v-1)$. Remove from d^o m blocks disjoint with respect to their extra-replicated treatments, with $1 \leq m \leq v/q$. We will call this new design d^*. $d^* \in \mathcal{D}(v,b-m,k)$ and $C_{d^*} = C_{d^o} - C_0$ where C_0 is given in Lemma 6.1.1.

Lemma 7.1.1: C_{d^*} has the eigenvalues 0,

$$\frac{v\lambda_1}{k} - mp - 1 \text{ with multiplicity } m(q-1),$$

$$\frac{v\lambda_1{}^*}{k} - mp - 1 + \frac{q}{k} \text{ with multiplicity } m-1,$$

$$\frac{v\lambda_1}{k} - \frac{vp(mp+1)}{k} \text{ with multiplicity 1 (unless } mq = v),$$

$$\frac{v\lambda_1}{k} - mp \text{ with multiplicity } v-mq-1 \text{ (unless } mq \geq v-1),$$

where $\lambda_1 = \lambda_{d^o ij}$ $(i \neq j)$ and for each case of m and q all realized eigenvalues are distinct. The eigenvalue structure is $0 < a < b = \ldots = b$ if $m = q = 1$, $0 < a = \ldots = a < b$ if $m = 2$ with $q = v/2$, $m = 1$ with $q = v-1$ or $m = v-1$ with $q = 1$, and the nonzero eigenvalues are all equal if $m = v$ and $q = 1$.

Proof: $C_{d^o} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v,v}$ with $\lambda_0 = \lambda_{d^o ii}$ and $r = r_{d^o i}$ $(1 \leq i \leq v)$, and $C_{d*} = C_{d^o} - C_0$. We apply Lemma 6.1.1 and Lemma 1.4.2 to get the eigenvalues.

$$\frac{v\lambda_1}{k} - mp \geq \frac{v\lambda_1}{k} - \frac{vp(mp+1)}{k} > \frac{v\lambda_1}{k} - mp - 1 + \frac{q}{k}$$

with equality between the largest two only when $mq = v$, in which case those eigenvalues are not realized.

Looking at $mq = v$, $mq = v-1$ and $1 \leq mq \leq v-2$ separately we get the cases mentioned in the hypothesis for the eigenvalue structures of Theorems 2.1.1, 1.4.1 and 1.4.4.

Remark 7.1.1: We must of course have $m \leq b-2$ in order for d* to have at least two blocks. It may also not be possible to remove two disjoint binary blocks. The BBD in $\mathcal{D}(6,10,6p+3)$ does not have two disjoint binary blocks.

Note that for any $b \geq 2$ there is a BBD in $\mathcal{D}(v,b,pv)$ which is $\mathcal{J}_U$-optimal, so we only consider $1 \leq q \leq v-1$.

Before going on we present the following lemma.

Lemma 7.1.2: d* is E-better than any $d \in \mathcal{D}(v,b-m,k)$ for which 1) $\min_{1\leq i\leq v} (r_{di}) \leq r-mp-2$, 2) $\lambda_{dij} \leq \lambda_1 - mp^2 - 2p$ when $r_{di} = r_{dj} = r-mp-1$ and $q = 1$, or 3) $\lambda_{dij} \leq \lambda_1 - mp^2 - 2p - 1$ when $r_{di} = r_{dj} = r-mp-1$ and $q \geq 2$. $k = pv+q$ with $p \geq 1$, $1 \leq q \leq v-1$ $v \geq 3$ and $1 < mq \leq v-1$.

Proof: From Theorem 1.4.3 part (i) and assuming without loss of generality that $r_{d1} = \min_{1\leq i\leq v} (r_{di})$,

$$\mu_{d1} \leq \frac{v}{v-1}\left(r_{d1} - \sum_{j=1}^{b-m} \frac{n_{d1j}^2}{k}\right) \leq \frac{v}{v-1} B(r_{d1};\ b-m,k)$$

$$\leq \frac{v}{v-1} B(r-mp-2;\ b-m,k). \qquad (7.1.1)$$

For $q = 1$ and $b = v$ (7.1.1) equals

$$\frac{v}{v-1}\left(r-mp-2- \frac{\lambda_0-(m-1)p^2-(p+1)^2-p^2+(p-1)^2}{k}\right)$$

$$= \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(mp+2- \frac{mp^2+4p}{k}\right) \leq \frac{v\lambda_1}{k} -mp-1+ \frac{1}{k},$$

the smallest nonzero eigenvalue of d^* (since $mq > 1$). For $q \geq 2$ or $q = 1$ with $b \geq 2v$ (7.1.1) equals

$$\frac{v}{v-1}\left(r-mp-2- \frac{\lambda_0-(m-1)p^2-2(p+1)^2+p^2}{k}\right)$$

$$= \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(mp+2- \frac{mp^2+4p+2}{k}\right) \leq \frac{v\lambda_1}{k} - mp-1 \leq$$

the smallest nonzero eigenvalue of d^*.

If $r_{d1} = r_{d2} = r-mp-1$ and $q = 1$, by Theorem 1.4.3 part (ii) we have

$$\mu_{d1} \leq r-mp-1- \frac{1}{2} \sum_{j=1}^{b-m} \left(\frac{n_{d1j}^2+n_{d2j}^2}{k}\right) + \frac{\lambda_{d12}}{k}$$

$$\leq r-mp-1- \frac{\lambda_0-(m-1)p^2-(p+1)^2}{k} + \frac{\lambda_1-mp^2-2p}{k}$$

$$= \frac{v\lambda_1}{k} -mp-1+ \frac{1}{k} = \frac{v\lambda_1}{k} -mp-1+ \frac{q}{k},$$

the smallest nonzero eigenvalue of d^*. If $r_{d1} = r_{d2} = r-mp-1$ and $q \geq 2$,

$$\mu_{d1} \leq r-mp-1- \frac{\lambda_0-mp^2-2p-1}{k} + \frac{\lambda_1-mp^2-2p-1}{k}$$

$$= \frac{v\lambda_1}{k} -mp-1+ \frac{1}{k} - \frac{1}{k} = \frac{v\lambda_1}{k} -mp-1,$$

the smallest nonzero eigenvalue of d*. This completes the proof.

The cases of mq = v, mq = v-1 and $1 \leq mq \leq v-2$ will be covered in Sections 7.2, 7.3 and 7.4 respectively.

7.2 The Case mq = v

When mq = v C_{d*} is of the form

$$\begin{bmatrix} A & B & \cdots & B \\ & A & \cdots & B \\ & & \ddots & \vdots \\ & & & A \end{bmatrix}$$

where $A = (r- \frac{\lambda_0}{k} + \frac{\lambda_1}{k} -mp-1)I_q-(\frac{\lambda_1-mp^2-2p-1}{k})J_{q,q}$ and $B = -(\frac{\lambda_1-mp^2-2p}{k})J_{q,q}$. d* is just a GD PBBD with m groups of size q where in the notation of GD designs, $\lambda_2 = \lambda_1+1$. Then by Theorem 4.1 of Cheng (1978), C_{d*} is the unique E-optimal matrix in $\mathcal{C} = \{C_d: d \in \mathcal{D}(v,b-m,pv+q)\}$.

For m = v and q = 1 (and $b \geq 2v$) d* is a BBD in $\mathcal{D}(v,ev-v,pv+1)$ for $e \geq 2$. Corollary 1.4.4 says d* is $\mathcal{J}_U$-optimal. This is the case of $0 < a =\ldots= a$.

For m = 2 and q = v/2 d* has the eigenvalue structure $0 < a =\ldots= a < b$. Then d* is a MB GD PBBD and by Corollary 3.1.1 of Cheng (1978) d* is $g\mathcal{J}_1$-optimal.

No other cases of $mq = v$ considered here give one of the three eigenvalue structures of Lemma 7.1.1. $m = 1$ and $q = v$ gives $0 < a = \ldots = a$ but $k = pv+v$ allows a BBD for any $b \geq 2$, so we have required q to be less than v.

In Chapter 8 the A- and D-efficiencies of these E-optimal designs will be presented.

7.3 The Case $mq = v-1$

When $mq = v-1$ only two cases give d* with an eigenvalue structure mentioned in Lemma 7.1.1. They are $m = v-1$ with $q = 1$ and $m = 1$ with $q = v-1$. Each d* has the eigenvalue structure $0 < a = \ldots = a < b$.

For $m = v-1$ with $q = 1$ we require $d^0 \in \mathcal{D}(v,ev,pv+1)$ with $e \geq 2$. d* is then in $\mathcal{D}(v,ev-(v-1),pv+1) = \mathcal{D}(v,(e-1)v+1,pv+1)$ and equivalently could be constructed by adding one balanced block to a BBD in $\mathcal{D}(v,(e-1)v,pv+1)$. So this is just the case of $mq = 1$ of Section 6.4.

For $m = 1$ with $q = v-1$ C_{d*} has eigenvalues 0, $(v\lambda_1/k)-p-1$ and $(v\lambda_1/k)-(vp(p+1)/k)$ with multiplicities 1, v-2 and 1 respectively.

Theorem 7.3.1: d* is E-better than any $d \in \mathcal{D}(v,ev-1,pv+v-1)$ for which 1) $\min_{1 \leq i \leq v} (r_{di}) \leq r-p-2$ or 2) $\lambda_{dij} \leq \lambda_1-p^2-2p-1$ when $r_{di} = r_{dj} = r-p-1$. For $p \geq v-2$ there exists a $\hat{d}$ (with $r_{\hat{d}1} = r-p$, $r_{\hat{d}i} = r-p-1$ for $2 \leq i \leq v$ and $\lambda_{\hat{d}ij} = \lambda_1-p^2-2p$ for $2 \leq i < j \leq v$) which is strictly E-better than d*. Therefore d* is not $g\mathcal{J}_1$-optimal for $p \geq v-2$.

Proof: That d* E-betters the designs satisfying 1) and 2) follows from Lemma 7.1.2.

If d^o is e copies ($e \geq 1$) of the unique BBD in $\mathcal{D}(v,v,pv+v-1)$ let $\hat{d}$ be e-1 copies of that BBD in v blocks plus v-2 blocks with p applications of treatment 1 and p+1 applications of treatments 2 through v, and finally a block with p+v-1 applications of treatment 1 and p applications of treatments 2 through v.

$$C_{\hat{d}} = \begin{bmatrix} c_{11} & -c_{12}\underline{1}'_{v-1} \\ & \\ -c_{12}\underline{1}_{v-1} & C_{22} \end{bmatrix}$$

where $c_{11} = \frac{(v-1)e\Delta}{k} - \frac{v-1}{k}(p^2+p+v-2)$, $c_{12} = \frac{e\Delta}{k} - \frac{p^2+p+v-2}{k}$, and

$C_{22} = (\frac{ev\Delta}{k} - \frac{vp^2+(2v-1)p+v-2}{k})I_{v-1} - (\frac{e\Delta}{k} - \frac{p^2+2p}{k})J_{v-1,v-1}$. Here $\Delta = vp^2+2(v-1)p+v-1$ and is the λ_{ij} $(i \neq j)$ for the unique BBD in $\mathcal{D}(v,v,pv+1)$. Therefore $e\Delta = \lambda_1$ of d^o.

Therefore by Lemma 1.4.1 the eigenvalues of $C_{\hat{d}}$ are 0,

$$\frac{ve\Delta}{k} - \frac{vp^2+(2v-1)p+v-2}{k} = \frac{v\lambda_1}{k} - (p+1) + \frac{1}{k}$$

with multiplicity v-2 from C_{22} and

$$vc_{12} = v(\frac{e\Delta}{k} - \frac{p^2+p+v-2}{k})$$

$$= \frac{v\lambda_1}{k} - (p+1) + \frac{(v-1)(p+1)-v(v-2)}{k}.$$

To construct this design, we only need $p \geq 1$, but the single nonzero eigenvalue is at least as big as the eigenvalue with multiplicity v-2 if $p \geq v-2$ with equality for $p = v-2$ and strict inequality for $p > v-2$.

Since $\frac{v\lambda_1}{k} - (p+1) + \frac{1}{k} > \frac{v\lambda_1}{k} - (p+1) = \mu_{d^*1}$, $\hat{d}$ is strictly E-better than d* and d* cannot be $g\mathcal{J}_1$-optimal.

Remark 7.3.1: For $p < v-2$ Theorem 1.4.1 can not be applied in general. In trying to prove that the d* in the next example was $g\mathcal{J}_1$-optimal, the design $\tilde{d}$ of that example had a strictly larger value of $\text{tr } C_d - (\frac{v-1}{v-2})^{\frac{1}{2}}[\text{tr}(C_d^2)-(\text{tr } C_d)^2/(v-1)]^{\frac{1}{2}}$ than did d*. d* is E-optimal in $\mathcal{D}(4,3,7)$.

Example 7.3.1: $m = 1$, $q = v-1$, $p < v-2$. d*, $\tilde{d} \in \mathcal{D}(4,3,7)$. See Remark 7.3.1.

	1	1	1		1	1	1
	2	2	2		2	2	2
	3	3	3		3	3	3
d*:	4	4	4	$\tilde{d}$:	4	4	4
	1	1	1		1	1	3
	2	2	3		2	2	3
	3	4	4		3	4	4

It is also true that for $mq = v-1$ and $m > 1$ that d* is not necessarily E-optimal, and hence not necessarily $g\mathcal{J}_1$- or $\mathcal{J}_4$-optimal.

Example 7.3.2: $m = 2$, $q = 2$. $d^* \in \mathcal{D}(5,8,7)$ is E-worse than $\hat{d}$ given below.

	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2
	3	3	3	3	3	3	3	3
$\hat{d}$:	4	4	4	4	4	4	4	4
	5	5	5	5	5	5	5	5
	1	1	2	2	2	3	3	4
	1	1	3	4	5	4	5	5

The A-, D- and E-efficiencies for d* and the E-efficiency for $\hat{d}$ ($p \geq v-2$) with $m = 1$ (and $q = v-1$) will be presented in Chapter 8.

7.4 The Cases $1 \leq mq \leq v-2$

For all these m and q only $m = q = 1$ gives a d* with one of the eigenvalue structures in Lemma 7.1.1. The eigenvalues are 0, $\frac{v\lambda_1}{k} - \frac{vp(p+1)}{k}$ and $\frac{v\lambda_1}{k} - p$ with multiplicities 1, 1 and v-2 respectively.

Theorem 7.4.1: $d^* \in \mathcal{D}(v,ev-1,pv+1)$, the BBD in $\mathcal{D}(v,ev,pv+1)$ with one block removed, is $\mathcal{J}_4$-optimal in $\mathcal{D}(v,ev-1,pv+1)$. $v \geq 3$.

Proof: d* is of maximum trace and has eigenvalue structure $0 < a < b = \ldots = b$. For any $d \in \mathcal{D}(v,ev-1,pv+1)$,

$$\mu_{d1} \leq \frac{v}{v-1}\left(r-p-1-\frac{\lambda_0-(p+1)^2}{k}\right) = \frac{v\lambda_1}{k} - \frac{v}{v-1}\left(p+1-\frac{(p+1)^2}{k}\right)$$

$$= \frac{v\lambda_1}{k} - \frac{vp(p+1)}{k} = \mu_{d^*1}.$$

Therefore by Theorem 2.1.1 we are done.

The following examples illustrate that d* for $2 \leq mq' \leq v-2$ is not necessarily E- and hence $g\mathcal{J}_1$- or $\mathcal{J}_4$-optimal.

Example 7.4.1: $\hat{d}$ given below is strictly E-better than d* in $\mathcal{D}(4,5,6)$. Here m = 1 and q = v-2 = 2.

	1	1	1	1	1
	2	2	2	2	2
$\hat{d}$:	3	3	3	3	3
	4	4	4	4	4
	1	1	1	2	3
	2	2	4	3	4

Example 7.4.2: $\hat{d}$ is strictly E-better than d* in $\mathcal{D}(6,13,8)$, where both designs have only the extra-replicated treatments of their (binary) blocks displayed. Here m = 2 and q = (v-2)/2 = 2.

d*:	1	1	1	1	1	2	2	2	2	3	3	4	4
	2	3	4	5	6	3	4	5	6	5	6	5	6

$\hat{d}$:	1	1	1	1	1	2	2	3	3	3	4	4	5
	2	2	2	3	4	5	6	4	5	6	5	6	6

Example 7.4.3: $\hat{d}$ is strictly E-better than d* in $\mathcal{D}(7,6,10)$, where both designs have only the extra-replicated treatments of their (binary) blocks displayed. Here m = 1 and q = v-4.

	1	2	3	4	5	6
d*:	5	6	7	1	2	3
	6	7	1	2	3	4

	1	1	1	2	3	4
$\hat{d}$:	2	2	3	5	5	6
	3	4	4	6	7	7

CHAPTER 8

EFFICIENCIES FOR PREVIOUS CHAPTERS

8.1 Preliminaries

In this section we present two methods used to calculate most of the A-, D- and E-efficiencies in this chapter, as well as four lemmas used to prove the efficiencies are strictly increasing as a parameter related to the number of blocks increases.

In general we take a possibly nonexistent or nonconstructible design d_B whose eigenvalues $\bar{\mu}_i$ for $1 \leq i \leq v-1$ will be used to construct the lower bound for $\Phi(C_d)$. Then the true efficiency of a design d is greater than or equal to

$$e_\Phi(d) = \Phi(C_{d_B})/\Phi(C_d)$$

for any of Φ_A, Φ_D and Φ_E. This $e_\Phi(d)$ is really a lower bound since the Φ-best $d \in \mathcal{D}(v,b,\underline{k})$ may not be as good as d_B. Sometimes d_B will be a $\mathcal{J}_4$-optimal d* so $\bar{\mu}_i = \mu_{d^*i}$ for $1 \leq i \leq v-1$. Sometimes d_B will have $\bar{\mu}_1 = \mu_{d^*1}$ for an E-optimal d* with $\bar{\mu}_2 = \ldots = \bar{\mu}_{v-1} = (v-2)^{-1}(\text{tr } C_{d^*} - \mu_{d^*1})$.

In the numerical examples given the value of $e_\Phi(d)$ was calculated to seven or more places but is truncated to four decimal places for presentation. If $e_\Phi(d)$ had nonzero terms in the fifth decimal place and beyond we shall write $e_\Phi(d) > .9876$ (for example).

The first method for finding the $\bar{\mu}_i$ of d_B involves Theorem 2.2.1. The bound (i) for μ_{d1} is made as large as possible by taking $\min_{1\le i\le v}(r_{di}) = \min_{1\le i\le v}(r_{d*i})$ and minimizing $\sum_{j=1}^{v-1}(n_{d\ell j}^2/k_j)$ if $r_{d\ell} = \min_{1\le i\le v}(r_{di})$. If this maximum bound does not force the $tr(C_d)$ to be less than $tr(C_{d*}) = \max_{d\in\mathcal{D}}(tr\ C_d)$, then we let $\bar{\mu}_1$ equal this bound and $\bar{\mu}_2 = \ldots = \bar{\mu}_{v-1} = (v-2)^{-1}(tr\ C_{d*}-\bar{\mu}_1)$. Then Theorem 2.1.1 says this d_B is $\mathcal{J}_4$-optimal and hence A-, D- and E-optimal. We call this Method 1.

Method 2 is just letting $\bar{\mu}_1 = \ldots \bar{\mu}_{v-1} = (v-1)^{-1}(tr\ C_{d*})$. This is used if pushing bound (i) of Theorem 2.2.1 up too far decreases the trace of C_d. This can happen if there are blocks of different sizes. See for example the case of Theorem 4.2.1 part c) and its corresponding Theorem 8.3.2.

Now we present four lemmas used to prove that most of the efficiencies are strictly increasing in e, λ_1, or r while the remaining parameters v, p and q remain fixed. e was used as the number of copies of the unique BIBD in $\mathcal{D}(v,v(v-1)/2,2)$, so as e increases we are increasing the number of blocks of d* in $\mathcal{D}(v,ev(v-1)/2-1,2)$ in Section 8.2. As λ_1 and r increase for fixed v, p and q we are increasing the number of pairs in d^o used to construct d*. But $\lambda_1 = r(k-1)/(v-1) = bk(k-1)/(v(v-1))$ for d^o so both increase as b increases, for b the number of blocks in d^o.

<u>Lemma 8.1.1</u>: For A(x) a differentiable function of $x > 0$, b_i a constant with respect to x (for i = 0,1,2,3), $1 \le M, N \le v-2$, and $v \ge 3$ an integer we have

$$\frac{d}{dx}\left(\frac{\frac{1}{A(x)-b_2}+\frac{v-2}{A(x)-b_1}}{\frac{M}{A(x)-b_3}+\frac{N}{A(x)-b_0}}\right) > 0$$

if all of the following hold:

(i) $b_3 \geq b_2 \geq b_1 \geq b_0 \geq 0$ with at least one of the first three inequalities being strict,

(ii) $A'(x) > 0$, $A(x)-b_3 > 0$, and

(iii) $M(b_3-b_2) + (v-2)M(b_3-b_1) \geq N(b_2-b_0) + (v-2)N(b_1-b_0)$.

Proof: Suppose $M(b_3-b_2) \leq N(b_2-b_0)$. Then taking the derivative we get

$$A'(x)\left(\frac{M}{A(x)-b_3}+\frac{N}{A(x)-b_0}\right)^{-2}\left[\frac{M(b_3-b_2)}{(A(x)-b_3)^2(A(x)-b_2)^2}\right.$$

$$+\frac{M(v-2)(b_3-b_1)}{(A(x)-b_3)^2(A(x)-b_1)^2}-\frac{N(b_2-b_0)}{(A(x)-b_2)^2(A(x)-b_0)^2}$$

$$\left.-\frac{N(v-2)(b_1-b_0)}{(A(x)-b_1)^2(A(x)-b_0)^2}\right]$$

$$= A'(x)\left(\frac{M}{A(x)-b_3}+\frac{N}{A(x)-b_0}\right)^{-2} \times$$

$$\{M(b_3-b_2)[(A(x)-b_3)^{-2}(A(x)-b_2)^{-2}-(A(x)-b_2)^{-2}(A(x)-b_0]^{-2}]$$

$$+ (N(b_2-b_0)-M(b_3-b_2))[(A(x)-b_3)^{-2}(A(x)-b_1)^{-2}-(A(x)-$$

$$b_2)^{-2}(A(x)-b_0)^{-2}]$$

$$+N(v-2)(b_1-b_0)[(A(x)-b_3)^{-2}(A(x)-b_1)^{-2}-(A(x)-b_1)^{-2}(A(x)-b_0)^{-2}]$$

$$+\frac{(M(b_3-b_2+(v-2)(b_3-b_1))-N(b_2-b_0+(v-2)(b_1-b_0)))}{(A(x)-b_3)^{-2}(A(x)-b_1)^{-2}}\}.$$

This is strictly greater than zero by conditions (i) and (ii). The proof for $M(b_3-b_2) > N(b_2-b_0)$ is analogous.

Remark 8.1.1: Lemma 8.1.1 can actually be proved with a weak majorization argument that would allow for a more general result with a different condition (i).

Lemma 8.1.2: For A(x) a differentiable function of $x > 0$, b_i a constant with respect to x (for $i = 0,1,2,3$), $1 \leq M, N \leq v-2$ and $v \geq 3$ an integer we have

$$\frac{d}{dx}\left[\left(\frac{A(x)-b_3}{A(x)-b_2}\right)^M \left(\frac{A(x)-b_0}{A(x)-b_1}\right)^N\right] > 0$$

if all of the following hold:

(i) $b_3 \geq b_2 \geq b_1 \geq b_0 \geq 0$ with at least one of the first three inequalities being strict,

(ii) $A'(x) > 0$, $A(x)-b_3 > 0$, and

(iii) $M(b_3-b_2) \geq n(b_1-b_0)$.

Proof: Taking the derivative we get

$$\left(\frac{A(x)-b_3}{A(x)-b_2}\right)^{M-1} \left(\frac{A(x)-b_0}{A(x)-b_1}\right)^{N-1} \frac{A'(x)}{(A(x)-b_2)^2(A(x)-b_1)^2} \times$$

$$[M(b_3-b_2)(A(x)-b_0)(A(x)-b_1)-N(b_1-b_0)(A(x)-b_2)(A(x)-b_3)]$$

which is strictly greater than zero given conditions (i), (ii) and (iii).

Remark 8.1.2: A stronger result with the same proof is to replace (i) and (iii) with

$$M(b_3-b_2)(A(x)-b_0)(A(x)-b_1)-N(b_1-b_0)(A(x)-b_2)(A(x)-b_3) > 0$$

and (ii) with $A'(x) > 0$, $A(x)-b_i > 0$ for $i = 0,1,2,3$.

Lemma 8.1.3: For $A(x)$ a differentiable function of $x > 0$, a_i a constant with respect to x (for $i = 0,1,2,3$), and $v \geq 3$ an integer we have

$$\frac{d}{dx}\left[\frac{A(x)+a_3}{A(x)+a_1}\left(\frac{A(x)+a_0}{A(x)+a_2}\right)^{v-2}\right] > 0$$

if all of the following hold:

(i) $a_3 \geq a_2 \geq a_1 \geq a_0 \geq 0$ with at least one of the first and third inequalities being strict,

(ii) $A'(x) > 0$, $A(x)+a_0 > 0$, and

(iii) $(v-2)(a_2-a_0) \geq (a_3-a_1)$.

Remark 8.1.3: The proof of Lemma 8.1.3 is analogous to the proof of Lemma 8.1.2. Also (i) and (iii) could be replaced by

$$(v-2)(a_2-a_0)(A(x)+a_3)(A(x)+a_1)-(a_3-a_1)(A(x)+a_2)(A(x)+a_0) > 0$$

and (ii) by $A'(x) > 0$, $A(x)+a_i > 0$ for $i = 0,1,2,3$ to give a stronger result.

Lemma 8.1.4: For $A(x)$ a differentiable function of $x > 0$, $A'(x) > 0$, $b_3 > b_2 \geq 0$ we have

$$\frac{d}{dx}\left(\frac{A(x)-b_3}{A(x)-b_2}\right) > 0.$$

Proof: The proof is straighforward.

In the following sections we present selected efficiencies from Chapters 3 through 7.

8.2 Efficiencies for Chapter 3

Recall that $d^o \in \mathcal{D}(v,ev(v-1)/2,2)$ is e copies $(e \geq 1)$ of the BIBD in $\mathcal{D}(v,v(v-1)/2,2)$, and d* was d^o minus one (binary) block. d* has been proven $\mathcal{J}_4$-optimal for e = 1 and all v by others and $e \geq 2$ for v = 3,4,5 and 6 by the author.

Theorem 8.2.1: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v, \frac{ev(v-1)}{2} - 1,2)$ calculated with Method 1 are

$$e_A(d^*) = \frac{\frac{1}{ev-v(v-1)^{-1}} + \frac{v-2}{ev-(v-1)^{-1}}}{\frac{1}{ev-2} + \frac{v-2}{ev}},$$

$$e_D(d^*) = \left(\frac{ev-2}{ev-v(v-1)^{-1}}\right)\left(\frac{ev}{ev-(v-1)^{-1}}\right)^{v-2},$$

$$e_E(d^*) = \frac{ev-2}{ev-v(v-1)^{-1}},$$

with each strictly increasing in $e \geq 1$ and converging to 1 as e goes to infinity. $v \geq 3$.

Proof: From Section 3.1 $\mu_{d*1} = (v\lambda_1/2)-1$ and $\mu_{d*2} = \ldots = \mu_{d*,v-1} = (v\lambda_1/2)$. Method 1 gives $\mu_{d1} \leq \frac{v}{v-1}(r-1-\frac{r-1}{2}) = \frac{v\lambda_1}{2} - \frac{v}{2(v-1)} = \bar{\mu}_1$ and $\bar{\mu}_2 = \frac{v\lambda_1}{2} - \frac{1}{2(v-1)}$. We also have $\lambda_1 = e$ and $r = e(v-1)$.

Lemmas 8.1.1, 8.1.2 and 8.1.4 are used to get the efficiencies increasing in e with $b_0 = 0$, $b_1 = (v-1)^{-1}$, $b_2 = \frac{v}{v-1}$, $b_3 = 2$ and $A(e) = ev$. The convergence to 1 is trivial.

Remark 8.2.1: Because d* is intuitively the best design for any reasonable criteria the A-, D- and E-efficiencies were evaluated for $7 \leq v \leq 50$ and $1 \leq e \leq 5$. These were proved to be increasing in e for fixed v, and appear to be increasing in v for fixed e. In Table 8.2.1, p. 130, are presented $e_A(d^*)$, $e_D(d^*)$ and $e_E(d^*)$ for v = 7, 8, 9, 10, 15, 20, 30, 40, 50 and e = 1, 2, 3, 4, 5. They are truncated to four decimal places unless more places prove illustrative. Fewer than four places indicates equality with $e_E(d^*)$.

8.3 Efficiencies for Chapter 4

For $k < v-1$ we have the four cases given in Theorem 4.2.1, some of which will require different bounds than others. Recall that for d* a BBD plus a binary block of size (v-1), the eigenvalues of C_{d*} are $\mu_{d*1} = \frac{v\lambda_1}{k} = \frac{v}{v-1}(\frac{r(k-1)}{k})$ and $\mu_{d*2} = \ldots = \mu_{d*,v-1} = \frac{v\lambda_1}{k} + 1 = \frac{v}{v-1}(\frac{r(k-1)}{k} + \frac{v-1}{v})$.

Theorem 8.3.1: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$, $(\underline{k}^*)' = (k\underline{1}_b', v-1)$, $v \geq 7$ with $v-1 > k \geq \frac{v-1}{3}$ or $4 \leq v \leq 6$ with

Table 8.2.1: BIBD in $\mathcal{D}(v,ev(v-1)/2,2)$ minus one block

v	*	e = 1	e = 2	e = 3	e = 4	e = 5
7	A	.9878	.9974	.9989	.9994	.9996
	D	.9668	.9927	.9969	.9983	.9989
	E	.8571	.9350	.9579	.9689	.9753
8	A	.9921	.9983	.9992	.9996	.9997
	D	.9749	.9944	.9976	.9986	.9991
	E	.875	.9423	.9625	.9722	.9779
9	A	.9946	.9988	.9995	.9997	.9998
	D	.9803	.9955	.9980	.9989	.9993
	E	.8888	.9481	.9661	.9749	.9800
10	A	.9962	.9991	.9996	.9998	.9998
	D	.9841	.9963	.9984	.9991	.9994
	E	.9	.9529	.9692	.9771	.9818
15	A	.9989	.9997	.9998	.99994	.99996
	D	.9930	.9983	.9992	.9996	.9997
	E	.9333	.9679	.9788	.9842	.9874
20	A	.9995	.99990	.99995	.99997	.99998
	D	.9961	.9990	.9995	.9997	.9998
	E	.95	.9756	.9839	.9880	.9904
30	A	.9998	.99997	.99998	.999992	.999995
	D	.9983	.9995	.9998	.9998	.9999
	E	.9666	.9836	.9891	.9918	.9935
40	A	.99995	.99998	.999994	.999997	.999998
	D	.9990	.9997	.9998	.99994	.99996
	E	.975	.9876	.9918	.9938	.9951
50	A	.99997	.999993	.999997	.999998	.999999
	D	.9993	.9998	.99993	.99996	.99997
	E	.98	.9901	.9934	.9950	.9960

$v-1 > k \geq 2$, are calculated by Method 1 and equal

$$e_A(d^*) = \frac{(r+\frac{v-1-k}{(k-1)(v-1)})^{-1}+(v-2)(r+\frac{k(v-1)}{v(k-1)}-\frac{v-1-k}{(k-1)(v-1)(v-2)})^{-1}}{\frac{1}{r}+(v-2)(r+\frac{k(v-1)}{v(k-1)})^{-1}},$$

$$e_D(d^*) = \left(\frac{r}{r+\frac{v-1-k}{(k-1)(v-1)}}\right)\left(\frac{r+\frac{k(v-1)}{v(k-1)}}{r+\frac{k(v-1)}{v(k-1)}-\frac{v-1-k}{(k-1)(v-1)(v-2)}}\right)^{v-2},$$

$$e_E(d^*) = \frac{r}{r+\frac{v-1-k}{(k-1)(v-1)}},$$

with each strictly increasing in r and converging to 1 as r goes to infinity.

Proof: Method 1 gives $\mu_{d1} \leq \bar{\mu}_1 = \frac{v}{v-1}(\frac{r(k-1)}{k}+\frac{v-1-k}{k(v-1)})$ and $\bar{\mu}_2 = \frac{v}{v-1}(\frac{r(k-1)}{k}+\frac{v-1}{v}-\frac{v-1-k}{k(v-1)(v-2)})$ since the cases of this theorem are cases a) and d) of Theorem 4.2.1. Therefore the largest one can make $\bar{\mu}_1$ is by putting the treatment with $\min_{1\leq i\leq v}(r_{di}) = r$ in the block of size v-1 exactly once.

Lemmas 8.1.1, 8.1.2 and 8.1.4 show the efficiencies are increasing in r with $b_0 = 0$, $b_1 = \frac{v-1-k}{(k-1)(v-1)(v-2)}$, $b_2 = \frac{k(v-1)}{v(k-1)} - \frac{v-1-k}{(k-1)(v-1)}$, $b_3 = \frac{k(v-1)}{v(k-1)}$ and $A(r) = r+\frac{k(v-1)}{v(k-1)}$.

Theorem 8.3.2: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $\underline{k}^*$ as in Theorem 8.3.1, $v \geq 7$ with $2 \leq k < \frac{\sqrt{v}+1}{2}$ are calculated by Method 2 and equal

$$e_A(d^*) = \frac{(v-1)(r + \frac{k(v-2)}{v(k-1)})^{-1}}{\frac{1}{r} + (v-2)(r + \frac{k(v-1)}{v(k-1)})},$$

$$e_D(d^*) = \left(\frac{r}{r+\frac{k(v-2)}{v(k-1)}}\right)\left(\frac{r+\frac{k(v-1)}{v(k-1)}}{r+\frac{k(v-2)}{v(k-1)}}\right)^{v-2},$$

$$e_E(d^*) = \left(\frac{r}{r+\frac{k(v-2)}{v(k-1)}}\right),$$

with each strictly increasing in r and converging to 1 as r goes to infinity.

Proof: This is the case of Theorem 4.2.1 c) and so Method 2 is used giving $\bar{\mu}_1 = \ldots = \bar{\mu}_{v-1} = \frac{v}{v-1}\left(\frac{r(k-1)}{k} + \frac{v-2}{v}\right)$.

Lemmas 8.1.1, 8.1.2 and 8.1.4 show the efficiencies are increasing in r with $b_0 = 0$, $b_1 = b_2 = \frac{k}{v(k-1)}$, $b_3 = \frac{k(v-1)}{v(k-1)}$ and $A(r) = r + \frac{k(v-1)}{v(k-1)}$.

Remark 8.3.1: For the case of $v \geq 7$ and $\frac{v-1}{3} > k \geq \frac{\sqrt{v}+1}{2}$ we know a design E-better than d* needs $\min_{1 \leq i \leq v} (r_{di}) = r$ but we can put more than one allocation of a minimum treatment in the block of size v-1 to get a bigger $\bar{\mu}_1$. Therefore,

$$\bar{\mu}_1 = \frac{v}{v-1}\left(\frac{r(k-1)}{k} + \frac{\ell_0}{k} - \frac{\ell_0^2}{v-1}\right) \text{ and}$$

$$\bar{\mu}_i = \frac{v}{v-1}\left(\frac{r(k-1)}{k} + \frac{v-1}{v} + \frac{\ell_0^2}{v(v-1)(v-2)} - \frac{\ell_0(v-k)}{kv(v-2)}\right)$$

for $2 \leq i \leq v-1$ (where ℓ_0 is the value of $n_{di,b+1}$ which maximizes $\bar{\mu}_1$ when $r_{di} = r$) should be used to calculate at least the E-efficiency. Since $\ell_0 > 1$ does decrease the overall trace of C_d, Methods 2 is suggested for the A- and D-efficiencies.

The following examples illustrate how good the efficiencies are even for small values of v and r.

Example 8.3.1: Take $d^* \in \mathcal{D}(4,7,\underline{k}^*)$ with $k = 2$. Here $r = 3$ and it is a case from Theorem 8.3.1. $e_A(d^*) > .9791$, $e_D(d^*) = (.9)(27/26)^2 > .9705$ and $e_E(d^*) = .9$. To give some idea of how fast these efficiencies are increasing, for $d^* \in \mathcal{D}(4,13,\underline{k}^*)$ with $k=2$ $(r = 6)$ $e_A(d^*) > .9937$, $e_D(d^*) > .9909$ and $e_E(d^*) = (18/19) > .9473$.

Example 8.3.2: Take $d^* \in \mathcal{D}(10,11,\underline{k}^*)$ with $k = 2$. Here $r = 9$ and this is actually the smallest value of v for which Theorem 8.3.2 can be used since for $7 \leq v \leq 9$ we have $\frac{\sqrt{v}+1}{2} \leq 2$. Here $e_A(d^*) > .9967$, $e_D(d^*) > .9860$ and $e_E(d^*) > .8490$.

For $k = pv$ $(p \geq 2)$ it was shown in Theorem 4.3.1 that d^* was sometimes E-worse than but always A- and D-better than a $\hat{d}$. Here we will calculate E-efficiencies for d^* and $\hat{d}$ and A- and D-, efficiencies for d^* only.

Theorem 8.3.3: For $d^*, \hat{d} \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $k = pv$ $(p \geq 2, v \geq 3)$ the E-efficiencies calculated by Method 1 and the A- and D-efficiencies calculated by Method 2 are

$$e_E(d^*) = \frac{\lambda_1}{\lambda_1 + \frac{pv}{v-1}\left(\frac{2p-1}{pv} - \frac{1}{v-1}\right)},$$

$$e_E(\hat{d}) = \frac{\lambda_1 + \frac{p}{2} - \frac{1}{v} - \frac{p}{2}\left(\frac{v^2p^2-4vp^2+4}{v^2p^2} + \frac{4(p+1)}{vp(v-1)}\right)^{\frac{1}{2}}}{\lambda_1 + \frac{pv}{v-1}\left(\frac{2p-1}{pv} - \frac{1}{v-1}\right)},$$

$$e_A(d^*) = \frac{(v-1)(v\lambda_1+k(\frac{v-2}{v-1}))^{-1}}{\frac{1}{v\lambda_1} + (v-2)(v\lambda_1+k)^{-1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1}{v\lambda_1+k(\frac{v-2}{v-1})}\right)\left(\frac{v\lambda_1+k}{v\lambda_1+k(\frac{v-2}{v-1})}\right)^{v-2},$$

with each strictly increasing in λ_1 and converging to 1 and λ_1 increases to infinity.

Proof: Method 1 gives $\bar{\mu}_1 = \frac{v\lambda_1}{k} + \frac{v}{v-1}(\frac{2p-1}{k} - \frac{1}{v-1})$ with $k = pv$, and we use this $\bar{\mu}_1$ for $e_E(d^*)$ and $e_E(\hat{d})$. Lemma 8.1.4 is applied to $e_E(d^*)$, and to $e_E(\hat{d})$ for $p \geq v/(v-2)$ (which makes $e_E(\hat{d}) \geq e_E(d^*)$).

Method 2 gives $\bar{\mu}_i = \frac{v\lambda_1}{k} + \frac{v-2}{v-1}$ $(1 \leq i \leq v-1)$ and we apply Lemmas 8.1.1 and 8.1.2 to $e_A(d^*)$ and $e_D(d^*)$, respectively, with $b_0 = 0$, $b_1 = b_2 = k(v-1)^{-1}$, $b_3 = k$ and $A(\lambda_1) = v\lambda_1$.

Remark 8.3.2: For $k = pv$, putting an application of a minimum replicated treatment i with $r_{di} = r$ in the block of size v-1 pushes up $\bar{\mu}_1$ with Method 1 but decreases the overall trace of C_d. Therefore Method 2 is used for $e_A(d^*)$ and $e_D(d^*)$.

For $k = pv+q$ $(p \geq 1,\ 1 \leq q \leq v-1)$ we have the following theorems.

Theorem 8.3.4: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b+1,\underline{k}^*)$ with $k = pv+q$ $(p \geq 1,\ 1 \leq q \leq v-1)$ calculated with Method 1 are

$e_A(d^*) =$

$$\frac{\left(\lambda_1 + \frac{k}{v-1}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)\right)^{-1} + (v-2)\left(\lambda_1 + \frac{k}{v} - \frac{k}{(v-1)(v-2)}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)\right)^{-1}}{\frac{1}{\lambda_1} + (v-2)\left(\lambda_1 + \frac{k}{b}\right)^{-1}},$$

$$e_D(d^*) = \left(\frac{\lambda_1}{\lambda_1 + \frac{k}{v-1}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)}\right)\left(\frac{\lambda_1 + \frac{k}{v}}{\lambda_1 + \frac{k}{v} - \frac{k}{(v-1)(v-2)}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)}\right)^{v-2},$$

$$e_E(d^*) = \frac{\lambda_1}{\lambda_1 + \frac{k}{v-1}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)},$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 3$.

Proof: Analogous to that of Theorem 8.3.1 with $\bar{\mu}_1 = \frac{v\lambda_1}{k} + \frac{v}{v-1}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $\bar{\mu}_2 = \frac{v\lambda_1}{k} + 1 - \frac{v}{(v-1)(v-2)}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_0 = 0$, $b_1 = \frac{k}{(v-1)(v-2)}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_2 = \frac{k}{v} - \frac{k}{v-1}\left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_3 = \frac{k}{v}$ and $A(\lambda_1) = \lambda_1 + \frac{k}{v}$.

Theorem 8.3.5: The A-, D- and E-efficiencies for $\hat{d} \in \mathcal{D}(v,b+1,\underline{k}^*)$ calculated with Method 1 for $k = pv+1$ and $k = pv+v-1$, respectively, are

$e_A(\hat{d}) =$

$$\frac{(\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1}))^{-1}+(v-2)(\lambda_1+ \frac{k}{v} - \frac{k}{(v-1)(v-2)} (\frac{2p+1}{k} - \frac{1}{v-1}))^{-1}}{(\lambda_1+ \frac{k}{2v} (1-S(1)))^{-1}+(\lambda_1+ \frac{k}{2v} (1+S(1)))^{-1}+(v-3)(\lambda_1 + \frac{k}{v})^{-1}},$$

$$e_D(\hat{d}) = \frac{((\lambda_1+ \frac{k}{2v})^2-(\frac{k}{2v} S(1))^2)(\lambda_1+ \frac{k}{v})^{v-3}}{(\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1}))(\lambda_1+ \frac{k}{v} - \frac{k}{(v-1)(v-2)} (\frac{2p+1}{k} - \frac{1}{v-1}))^{v-2}},$$

$$e_E(\hat{d}) = \frac{\lambda_1+ \frac{k}{2v} (1-S(1))}{\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1})},$$ where

$S(1) = ((\frac{2p(v-2)}{k} - 1)^2 + \frac{4p}{k} (\frac{2p(v-2)}{k} - \frac{v-2}{v-1}))^{\frac{1}{2}}$, and

$e_A(\hat{d}) =$

$$\frac{(\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1}))^{-1}+(v-2)(\lambda_1+ \frac{k}{v} - \frac{k}{(v-1)(v-2)} (\frac{2p+1}{k} - \frac{1}{v-1}))^{-1}}{(\lambda_1+ \frac{k}{2v} (1-S(2)))^{-1}+(\lambda_1+ \frac{k}{2v} (1+S(2)))^{-1}+(v-3)(\lambda_1+ \frac{k}{v})^{-1}},$$

$$e_D(\hat{d}) = \frac{((\lambda_1+ \frac{k}{2v})^2-(\frac{k}{2v} S(2))^2)(\lambda_1+ \frac{k}{v})^{v-3}}{(\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1}))(\lambda_1+ \frac{k}{v} - \frac{k}{(v-1)(v-2)} (\frac{2p+1}{k} - \frac{1}{v-1}))^{v-2}},$$

$$e_E(\hat{d}) = \frac{\lambda_1+ \frac{k}{2v} (1-S(2))}{\lambda_1+ \frac{k}{v-1} (\frac{2p+1}{k} - \frac{1}{v-1})},$$ where

$S(2) = (1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)})^{\frac{1}{2}}$ with the E-efficiencies known to be strictly increasing in λ_1 and all converging to 1 as λ_1 goes to infinity, $p \geq 1$ and $v \geq 3$.

Proof: $\bar{\mu}_1$ and $\bar{\mu}_2$ were given in Theorem 8.3.4, the eigenvalues for $\hat{d}$ are found in the proofs of Theorems 4.3.2 and 4.3.3, and Lemma 8.1.4 is applied to the E-efficiencies.

Remark 8.3.3: Recall that for k = pv+q and q = 2 or q = v-1 $\hat{d}$ was shown to be strictly $\mathcal{J}_4$-better than d* in Theorems 4.3.2 and 4.3.3 respectively. Therefore even if we do not know how $e_A(\hat{d})$ and $e_D(\hat{d})$ behave, we know they are bounded between the efficiencies for d* and 1. For $2 \leq q \leq v-2$ $C_{\hat{d}}$ at (4.3.4) would not yield the eigenvalues for $\hat{d}$ in general. It is for these reasons that Theorem 8.3.4 is necessary.

The following examples with q = 1, 2 and v-1 respectively, show d* may be very good even for small v, p and q.

Example 8.3.3: d*, $\hat{d} \in \mathcal{D}(4,4+1,\underline{k}^*)$ with k = v+1 = 5. $e_A(d^*) >$ .9937 and $e_A(\hat{d}) >$ .9989. $e_D(d^*) >$ .9908 and $e_D(\hat{d}) >$ .9984. $e_E(d^*)$ = 54/58 > .9310 and $e_E(\hat{d}) >$ .9908.

Example 8.3.4: d*, $\hat{d} \in \mathcal{D}(4,6+1,\underline{k}^*)$ with k = v+2 = 6. See Example 4.3.1. $e_A(d^*) >$ .9985 and $e_A(\hat{d}) >$ .9996. $e_D(d^*) >$ .9978 and $e_D(\hat{d}) >$ .9994. $e_E(d^*)$ = 39/40 = .975 and $e_E(\hat{d}^*) >$ .9957.

Example 8.3.5: d*, $\hat{d} \in \mathcal{D}(4,4+1,\underline{k}^*)$ with k = v+v-1 = 7. $e_A(d^*) >$ .9985 and $e_A(\hat{d}) >$ .9999. $e_D(d^*) >$ .9978 and $e_D(\hat{d}) >$ .9998. $e_E(d^*)$ = 54/55 = $.98\overline{18}$ and $e_D(\hat{d}) >$ .9996.

8.4 Efficiencies for Chapter 5

For $3 \le k \le v-1$ recall that d*, a BIBD with one observation removed, had C_{d*} with eigenvalues 0, $\frac{v\lambda_1}{k} - 1 = \frac{v}{v-1}\left(\frac{r(k-1)}{k} - \frac{v-1}{v}\right)$, and $\frac{v\lambda_1}{k} = \frac{v}{v-1}\left(\frac{r(k-1)}{k}\right)$ with multiplicity v-2.

Theorem 8.4.1: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b,\underline{k}^*)$ with $(\underline{k}^*)' = (k1'_{b-1},k-1)$, $3 \le k \le v-1$, as calculated with Method 1 are

$$e_A(d^*) = \frac{\frac{1}{r-1} + (v-2)\left(r - \frac{(v-k)}{v(v-2)(k-1)}\right)^{-1}}{\left(r - \frac{k(v-1)}{v(k-1)}\right)^{-1} + \frac{v-2}{r}},$$

$$e_D(d^*) = \left(\frac{r - \frac{k(v-1)}{v(k-1)}}{r-1}\right)\left(\frac{r}{r - \frac{v-k}{v(v-2)(k-1)}}\right)^{v-2},$$

$$e_E(d^*) = \frac{r - \frac{k(v-1)}{v(k-1)}}{r-1},$$

with each strictly increasing in r and converging to 1 as r goes to infinity. $v \ge 3$.

Proof: Method 1 gives $\mu_{d1} \le \frac{v}{v-1}\left(r-1-\frac{r-1}{k}\right) = \frac{v}{v-1}\left(\frac{r(k-1)}{k} - \frac{k-1}{k}\right) = \bar{\mu}_1$ and $\bar{\mu}_2 = \frac{v}{v-1}\left(\frac{r(k-1)}{k}\right) - \frac{v-k}{k(v-1)(v-2)}$ since $\bar{\mu}_1$ is made largest by not putting the minimum replicated treatment in the smaller (b-th) block.

Lemmas 8.1.1, 8.1.2 and 8.1.4 are applied with $b_0 = 0$, $b_1 = \frac{v-k}{v(v-2)(k-1)}$, $b_2 = 1$ and $b_3 = \frac{k(v-1)}{v(k-1)}$ with $A(r) = r$.

Example 8.4.1: $d^* \in \mathcal{D}(5,10,\underline{k}^*)$ with $k = v-2 = 3$: $e_A(d^*) > .9961$, $e_D(d^*) > .9927$ and $e_E(d^*) = 24/25 = .96$.

Example 8.4.2: $d^* \in \mathcal{D}(5,5,\underline{k}^*)$ with $k = v-1 = 4$: $e_A(d^*) > .9968$, $e_D(d^*) > .9942$ and $e_E(d^*) = 44/45 = .97\overline{7}$.

Example 8.4.3: $d^* \in \mathcal{D}(4,4,\underline{k}^*)$ with $k = v-1 = 3$: $e_A(d^*) > .9840$, $e_D(d^*) > .9778$ and $e_E(d^*) = 15/16 = .9375$. $d^* \in \mathcal{D}(4,8,\underline{k}^*)$ with $k = 3$: $e_A(d^*) > .9969$, $e_D(d^*) > .9956$ and $e_E(d^*) = 39/40 = .975$.

For $k = pv$, $p \geq 1$ d^* was proved $\mathcal{J}_4$-optimal in Theorem 5.3.1. For $k = pv+q$, $p \geq 1$, $1 \leq q \leq v-1$ d^* was proved $\mathcal{J}_4$-optimal only for the case $q = 1$. However for all $1 \leq q \leq v-1$ the design $\tilde{d}$ (where the observation was removed from a block where it was not equireplicated) was always shown to be $\mathcal{J}_4$-worse than d^*. Now we shall give the A-, D- and E-efficiencies for d^* $(2 \leq q \leq v-1)$ and $\tilde{d}$ $(1 \leq q \leq v-1)$.

Theorem 8.4.2: The A-, D- and E-efficiencies for $\tilde{d} \in \mathcal{D}(v,ev,\underline{k}^*)$ with $e \geq 1$ and $k = pv+1$ are

$$e_A(\tilde{d}) = \frac{(v\lambda_1 - p(v-1))^{-1} + \frac{v-2}{v\lambda_1}}{(v\lambda_1 - p(v-1) - \frac{2k}{k-1})^{-1} + \frac{v-2}{v\lambda_1}},$$

$$e_D(\tilde{d}) = e_E(\tilde{d}) = \frac{v\lambda_1 - p(v-1) - 2k/(k-1)}{v\lambda_1 - p(v-1)},$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 3$.

Proof: Since Theorem 5.3.1 showed d* to be $\mathcal{J}_4$-optimal, we use its eigenvalues for $\tilde{\mu}_i$ $(1 \leq i \leq v-1)$. $\mu_{d*1} = \frac{v\lambda_1}{k} - \frac{p(v-1)}{k}$ and $\mu_{d*2} = \cdots = \mu_{d*,v-1} = \frac{v\lambda}{k}$, whereas $\mu_{\tilde{d}1} = \frac{v\lambda_1}{k} - \frac{p(v-1)}{k} - \frac{2}{k-1}$ and $\mu_{\tilde{d}i} = \mu_{d*i}$ for $2 \leq i \leq v-1$.

Lemmas 8.1.1 and 8.1.4 are applied to prove the increasing property.

Example 8.4.4: $\tilde{d} \in \mathcal{D}(5,5,\underline{k}^*)$ with $k = v+1 = 6$: $e_A(\tilde{d}) > .9775$ and $e_E(\tilde{d}) = 143/155 > .9225$.

Theorem 8.4.3: The A-, D-, and E-efficiencies for $d^* \in \mathcal{D}(v,b,\underline{k}^*)$ with $(\underline{k}^*)' = (k\underline{1}'_{b-1},k-1)$, $k = pv+q$, $2 \leq q \leq v-1$, as calculated by Method 1 are

$$e_A(d^*) = \frac{\left(v\lambda_1-k+\frac{vp(p+1)}{k-1}+\frac{(v-q)(q-1)}{(v-1)(k-1)}\right)^{-1}+(v-2)\left(v\lambda_1-\frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}\right)^{-1}}{\left(v\lambda_1-k+\frac{vp(p+1)}{k-1}\right)^{-1}+\frac{v-2}{v\lambda_1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1-k+\frac{vp(p+1)}{k-1}}{v\lambda_1-k+\frac{vp(p+1)}{k-1}+\frac{(v-q)(q-1)}{(v-2)(k-1)}}\right)\left(\frac{v\lambda_1}{v\lambda_1-\frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}}\right)^{v-2},$$

$$e_E(d^*) = \frac{v\lambda_1-k+\frac{vp(p+1)}{k-1}}{v\lambda_1-k+\frac{vp(p+1)}{k-1}+\frac{(v-q)(q-1)}{(v-1)(k-1)}},$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 3$.

Proof: By Method 1 $\mu_{d1} \leq \bar{\mu}_1 = \frac{v\lambda_1}{k} - 1 + \frac{vp(p+1)}{k(k-1)} + \frac{(v-q)(q-1)}{(v-1)k(k-1)} =$ $\mu_{d*1} + \frac{(v-q)(q-1)}{(v-1)k(k-1)}$. $\bar{\mu}_2 = \mu_{d*2} - \frac{(v-q)(q-1)}{(v-1)(v-2)k(k-1)}$. We apply Lemmas 8.1.1, 8.1.2 and 8.1.4 with $b_0 = 0$, $b_1 = \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}$, $b_2 = k - \frac{vp(p+1)}{k-1} - \frac{(v-q)(q-1)}{(v-1)(k-1)}$, $b_3 = k - \frac{vp(p+1)}{k-1}$ and $A(\lambda_1) = v\lambda_1$.

Theorem 8.4.4: The A-, D- and E-efficiencies of $\tilde{d} \in \mathcal{D}(v,b,\underline{k}^*)$ of Theorem 8.4.3 as calculated by Method 1 are

$$e_A(\tilde{d}) = \frac{\left(v\lambda_1 - k + \frac{vp(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)}\right)^{-1} + (v-2)\left(v\lambda_1 - \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}\right)^{-1}}{\left(v\lambda_1 - k + \frac{vp(p+1)}{k-1} - \frac{2k}{k-1}\right)^{-1} + \frac{v-2}{v\lambda_1}},$$

$$e_D(\tilde{d}) = \left(\frac{v\lambda_1 - k + \frac{vp(p+1)}{k-1} - \frac{2k}{k-1}}{v\lambda_1 - k + \frac{vp(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)}}\right)\left(\frac{v\lambda_1}{v\lambda_1 - \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}}\right)^{v-2},$$

$$e_E(\tilde{d}) = \frac{v\lambda_1 - k + \frac{vp(p+1)}{k-1} - \frac{2k}{k-1}}{v\lambda_1 - k + \frac{vp(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)}},$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 3$.

Proof: The $\bar{\mu}_i$ are those from Theorem 8.4.3. The $\mu_{\tilde{d}i} = \mu_{d*i}$ for $2 \leq i \leq v-1$ and $\mu_{\tilde{d}1} = \mu_{d*1} - \frac{2}{k-1}$. Lemmas 8.1.1, 8.1.2 and 8.1.4 are applied to $e_A(\tilde{d})$ $e_D(\tilde{d})$ and $e_E(\tilde{d})$ with b_0, b_1, b_2 and $A(\lambda_1)$ as in the proof of Theorem 8.4.3. Here $b_3 = k - \frac{vp(p+1)}{k-1} + \frac{2k}{k-1}$.

Example 8.4.5: d^*, $\tilde{d} \in \mathcal{D}(4,6,\underline{k}^*)$ with $k = v+2 = 6$. $e_A(d^*) > .9998$ and $e_A(\tilde{d}) > .9814$. $e_D(d^*) > .9997$ and $e_D(\tilde{d}) > .9493$. $e_E(d^*) = 357/358 > .9972$ and $e_E(\tilde{d}) = 339/358 > .9469$.

8.5 Efficiencies for Chapter 6

In Chapter 6 $k = pv+q$ with $p \geq 1$ and $1 \leq q \leq v-1$. For the case $mq = v$ considered there, $m = 2$ with $q = v/2$ had d^* with the eigenvalue structure $0 < a < b = \ldots = b$ of Theorem 2.1.1. Examples 6.2.1 and 6.2.2 showed that each case must be looked at separately. Since d^* is the logical first guess at an optimal design, we present the following theorem.

Theorem 8.5.1: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b+2, pv+(v/2))$ as calculated by Method 1 are

$$e_A(d^*) = \frac{(v-1)(v\lambda_1+k(2p+1)-\frac{v}{2(v-1)})^{-1}}{(v\lambda_1+k(2p+1)-\frac{v}{2})^{-1}+(v-2)(v\lambda_1+k(2p+1)^{-1}},$$

$$e_D(d^*) = e_E(d^*)\left(\frac{v\lambda_1+k(2p+1)}{v\lambda_1+k(2p+1)-\frac{v}{2(v-1)}}\right)^{v-2},$$

$$e_E(d^*) = \frac{v\lambda_1+k(2p+1)-\frac{v}{2}}{v\lambda_1+k(2p+1)-\frac{v}{2(v-1)}},$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 4$.

Proof: In general for $mq = v$, Method 1 gives $\mu_{d1} \leq \bar{\mu}_1 = \frac{v\lambda_1}{k} + mp + 1 - \frac{v-q}{k(v-1)}$ and $\bar{\mu}_2 = \bar{\mu}_1$. Substitute $m = 2$, $q = v/2$.

$\mu_{d*1} = \frac{v\lambda_1}{k} + 2p+1 - \frac{v}{2k}$ and $\mu_{d*2} = \ldots = \mu_{d*,v-1} = \frac{v\lambda_1}{k} + 2p+1$. Now we apply Lemmas 8.1.1, 8.1.2 and 8.1.4 with $b_0 = 0$, $b_1 = b_2 = \frac{v}{2(v-1)}$, $b_3 = v/2$ and $A(\lambda_1) = v\lambda_1+k(2p+1)$ and calculus to get the strictly increasing and convergence properties.

Example 8.5.1: $d^*, \hat{d} \in \mathcal{D}(8,14+2,12)$ with $k = v+(v/2) = 12$. Recall that $\hat{d}$ was the RGD shown to be A-, D- and E-better than d^*. $e_A(d^*) > .9999753$ and $e_A(\hat{d}) > .9999756$, $e_D(d^*) > .9999140$ and $e_D(\hat{d}) > .9999147$, $e_E(d^*) = 245/248 > .9879$ and $e_E(\hat{d}) = 987/992 > .9949$.

It was also shown that for $mq = v$ and $q = 2$ d^* was E-optimal, and this was used to prove $\mathcal{J}_4$-optimality for d^* when $v = 4$, $m = 2$ and $q = 2$. For $q = 2$ in general we present the A- and D-efficiencies.

Theorem 8.5.2: The A- and D-efficiencies of $d^* \in \mathcal{D}(v,b+(v/2),pv+2)$ are

$$e_A(d^*) = \frac{(v\lambda_1+k(mp+1)-2)^{-1}+(v-2)(v\lambda_1+k(mp+1)-\frac{2(m-2)}{v-2})^{-1}}{(m-1)(v\lambda_1+k(mp+1)-2)^{-1}+(v-m)(v\lambda_1+k(mp+1))^{-1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1+k(mp+1)-2}{v\lambda_1+k(mp+1)-\frac{2(m-2)}{v-2}}\right)^{m-2}\left(\frac{v\lambda_1+k(mp+1)}{v\lambda_1+k(mp+1)-\frac{2(m-2)}{v-2}}\right)^{v-m}.$$

with both strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 4$.

Proof: Since by Lemma 6.2.2 d^* is E-optimal, we use $\bar{\mu}_1 = \mu_{d*1} = \frac{v\lambda_1}{k} + (mp+1) - \frac{2}{k}$ and $\bar{\mu}_2 = \frac{v\lambda_1}{k} + mp+1 - \frac{2(m-2)}{(v-2)k}$. Now calculus and Lemmas 8.1.1 and 8.1.2 are applied to complete the proof.

For $mq = v-1$ we saw that d^* is always E-optimal in Theorem 6.3.1 and for $m = 1$ or $m = v-1$ d^* is $\mathcal{J}_4$-optimal in Theorem 6.3.2. For $2 \leq m \leq v-2$ we have the following theorem.

Theorem 8.5.3: The A- and D-efficiencies for $d^* \in \mathcal{D}(v,b+m,pv+q)$ where $2 \leq m \leq v-2$ and $mq = v-1$ are

$$e_A(d^*) = \frac{(v\lambda_1+vp(mp+1))^{-1}+(v-2)(v\lambda_1+k(mp+1)-\frac{q(m-1)}{v-2})^{-1}}{(v\lambda_1+vp(mp+1))^{-1}+(m-1)(v\lambda_1+k(mp+1)-q)^{-1}+\frac{v-m-1}{v\lambda_1+k(mp+1)}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1+k(mp+1)-q}{v\lambda_1+k(mp+1)-\frac{q(m-1)}{v-2}}\right)^{m-1}\left(\frac{v\lambda_1+k(mp+1)}{v\lambda_1+k(mp+1)-\frac{q(m-1)}{v-2}}\right)^{v-m-1},$$

with both strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity. $v \geq 3$.

Proof: Since by Theorem 6.3.1 d^* is E-optimal, $\bar{\mu}_1 = \mu_{d^*1} = \frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k}$ and $\bar{\mu}_2 = \frac{v\lambda_1}{k} + mp+1-\frac{q(m-1)}{k(v-2)}$. $\mu_{d^*i} = \frac{v\lambda_1}{k} +mp+1-\frac{q}{k}$ for $2 \leq i \leq m$ and $\mu_{d^*i} = \frac{v\lambda_1}{k} +mp+1$ for $m+1 \leq i \leq v-1$. The limit properties are trivial, Lemma 8.1.2 is applied to $e_D(d^*)$, but $e_A(d^*)$ is shown to be strictly increasing in λ_1 directly. That completes the proof.

For $1 \leq mq \leq v-2$ only $m = q = 1$ gave C_{d^*} an eigenvalue structure of interest, that of Theorem 1.4.1. However if $p \geq v-1$ we saw d^* was E-worse than the design $\hat{d}$ of Theorem 6.4.1.

Theorem 8.5.4: $\mathcal{D}(v,ev+1,pv+1)$ with $v \geq 3$, $e \geq 1$, $p \geq 1$. The A- and D-efficiencies for d* and the E-efficiency for d* and $\hat{d}$ (for $p \geq v-1$) as calculated by Method 1 are

$$e_A(d^*) = \frac{(v\lambda_1+vp^2+\frac{vp}{v-1})^{-1}+(v-2)(v\lambda_1+kp+\frac{vp}{v-1})^{-1}}{(v-2)(v\lambda_1+vp^2+p)^{-1}+(v\lambda_1+kp+(v-1)p)^{-1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1+vp^2+vp}{v\lambda_1+vp^2+\frac{vp}{v-1}}\right)\left(\frac{v\lambda_1+kp}{v\lambda_1+kp+\frac{vp}{v-1}}\right)^{v-2},$$

$$e_E(d^*) = \frac{v\lambda_1+vp^2+p}{v\lambda_1+vp^2+\frac{vp}{v-1}},$$

$$e_E(\hat{d}) = \frac{v\lambda_1+vp^2+p+1}{v\lambda_1+vp^2+\frac{vp}{v-1}}, \quad \text{for } p \geq v-1,$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity.

Proof: By Method 1 $\mu_{d1} \leq \bar{\mu}_1 = \frac{v\lambda_1}{k} + \frac{vp}{k}(p+\frac{1}{v-1})$ and $\bar{\mu}_2 = \frac{v\lambda_1}{k}+p+\frac{vp}{k(v-1)}$. $\mu_{d^*i} = \frac{v\lambda_1}{k}+p$ for $1 \leq i \leq v-2$ and $\mu_{d^*,v-1} = \frac{v\lambda_1}{k} + \frac{vp(p+1)}{k}$. Recall $k = pv+1$. Lemma 8.1.1 is applied to $e_A(d^*)$ with $b_0 = 0$, $b_1 = (v-1)p-\frac{vp}{v-1}$, $b_2 = (v-1)p+kp-\frac{vp}{v-1} - vp^2$, $b_3 = (v-1)p$ and $A(\lambda_1) = v\lambda_1+kp+(v-1)p$. Lemma 8.1.4 is applied to $e_E(d^*)$ and $e_E(\hat{d})$. Lemma 8.1.3 is applied to $e_D(d^*)$ with $a_0 = kp = vp^2+p$, $a_1 = vp^2+\frac{vp}{v-1}$, $a_2 = kp + \frac{vp}{v-1} = vp^2+p+\frac{vp}{v-1}$, $a_3 = vp^2+vp$ and $A(\lambda_1) = v\lambda_1$.

Example 8.5.2: See Example 6.4.1. $e_A(d^*) = e_D(d^*) = 1364/1365 > .9992$ since $v = 3$ and $e_E(d^*) = 62/63 > .9841$ while $e_E(\hat{d}) = 1$ since $p+1 = vp/(v-1)$.

8.6 Efficiencies for Chapter 7

In Chapter 7 $k = pv+q$ with $p \geq 1$ and $1 \leq q \leq v-1$. It was noted that for the first case considered, $mq = v$, that all such designs were E-optimal. The following theorem gives with the A- and D-efficiencies of such designs not already proved A- and D-optimal.

Theorem 8.6.1: The A- and D-efficiencies for $d^* \in \mathcal{D}(v,b-m,pv+q)$ with $v \geq 3$, $b-m \geq 2$, $p \geq 1$, $mq = v$ and $3 \leq m \leq v-1$ are

$$e_A(d^*) = \frac{(v\lambda_1-k(mp+1))^{-1}+(v-2)(v\lambda_1-k(mp+1)+\frac{q(m-1)}{v-2})^{-1}}{(v-m)(v\lambda_1-k(mp+1))^{-1}+(m-1)(v\lambda_1-k(mp+1)+q)^{-1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1-k(mp+1)}{v\lambda_1-k(mp+1)+\frac{q(m-1)}{v-2}}\right)^{v-m-1}\left(\frac{v\lambda_1-k(mp+1)+q}{v\lambda_1-k(mp+1)+\frac{q(m-1)}{v-1}}\right)^{m-1},$$

with both strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity.

Proof: Since d^* is E-optimal we use $\bar{\mu}_1 = \frac{v\lambda_1}{k} - (mp+1)$. $\bar{\mu}_2 = \frac{v\lambda_1}{k} = (mp+1)+\frac{q(m-1)}{k(v-2)}$. $\mu_{d^*i} = \frac{v\lambda_1}{k} - (mp+1)$ for $1 \leq i \leq v-m$ and $\mu_{d^*i} = \frac{v\lambda_1}{k} - (mp+1) + \frac{q}{k}$ for $v-m+1 \leq i \leq v-1$. Lemmas 8.1.1 and 8.1.2 are applied to $e_A(d^*)$ and $e_D(d^*)$, respectively, with $b_0 = 0$, $b_1 = q(1-\frac{m-1}{v-2})$, $b_2 = q$ or b_1 (respectively), $b_3 = q$ and $A(\lambda_1) = v\lambda_1-k(mp+1)$.

The proof is complete after calculus is applied to get the limit properties.

For $mq = v-1$ only $m = 1$ and $m = v-1$ gave eigenvalue structures from Lemma 7.1.1, both structures being $0 < a = \ldots = a < b$. For $m = v-1$ $(q = 1)$ see the case of $m = q = 1$ in Section 6.4 and its corresponding Theorem 8.5.4.

When $m = 1$ and $q = v-1$ d^* has nonzero eigenvalues $\frac{v\lambda_1}{k} - (p+1)$ with multiplicity $v-2$ and $\frac{v\lambda_1}{k} - \frac{vp(p+1)}{k}$. There is also a $\hat{d}$ that is E-better than d^* for $p \geq v-2$, and is given in Theorem 7.3.1.

Theorem 8.6.2: $\mathcal{D}(v,ev-1,pv+v-1)$ with $v \geq 3$, $e \geq 1$, $p \geq 1$. The A- and D-efficiencies of d^* and the E-efficiency of d^* and $\hat{d}$ (for $p \geq v-2$) as calculated by Method 1 are

$$e_A(d^*) = \frac{(v\lambda_1 - k(p+1) + \frac{p+1}{v-1})^{-1} + (v-2)(v\lambda_1 - k(p+1) + \frac{v(p+1)}{v-1})^{-1}}{(v-2)(v\lambda_1 - k(p+1))^{-1} + (v\lambda_1 - k(p+1) + (v-1)(p+1))^{-1}},$$

$$e_D(d^*) = \left(\frac{v\lambda_1 - k(p+1) + (v-1)(p+1)}{v\lambda_1 - k(p+1) + \frac{(p+1)}{v-1}}\right)\left(\frac{v\lambda_1 - k(p+1)}{v\lambda_1 - k(p+1) + \frac{v(p+1)}{v-1}}\right)^{v-2},$$

$$e_E(d^*) = \frac{v\lambda_1 - k(p+1)}{v\lambda_1 - k(p+1) + \frac{p+1}{v-1}},$$

$$e_E(\hat{d}) = \frac{v\lambda_1 - k(p+1) + 1}{v\lambda_1 - k(p+1) + \frac{p+1}{v-1}}, \quad \text{for } p \geq v-2,$$

with each strictly increasing in λ_1 and converging to 1 as λ_1 goes to infinity.

Proof: Method 1 gives $\bar{\mu}_1 = \frac{v\lambda_1}{k} - \frac{vp(mp+1)}{k} - \frac{v}{v-1}\left(\frac{(mp+1)q-(p+1)}{k}\right)$ in general for $1 \le q \le v-1$. Substituting m = 1, q = v-1, $\bar{\mu}_1 = \frac{v\lambda_1}{k} - \frac{vp(p+1)}{k} - \frac{v(v-2)(p+1)}{k(v-1)} = \frac{v\lambda_1}{k} - (p+1) + \frac{p+1}{k(v-1)}$. $\bar{\mu}_2 = \frac{v\lambda_1}{k} - (p+1) + \frac{v(p+1)}{k(v-1)}$. The eigenvalues of d* and $\hat{d}$ come from Section 7.3. Lemma 8.1.4 is applied to the E-efficiencies while Lemma 8.1.1 is applied to $e_A(d^*)$ with $b_0 = 0$, $b_1 = (v-1)(p+1) - \frac{v(p+1)}{v-1}$, $b_2 = (v-1)(p+1) - \frac{p+1}{v-1}$, $b_3 = (v-1)(p+1)$ and $A(\lambda_1) = v\lambda_1 - k(p+1)$. Lemma 8.1.3 is applied to $e_D(d^*)$ with $a_0 = 0$, $a_1 = \frac{p+1}{v-1}$, $a_2 = \frac{v(p+1)}{v-1}$, $a_3 = (v-1)(p+1)$ and $A(\lambda_1) = v\lambda_1 - k(p+1)$.

Example 8.6.1: For d*, $\hat{d} \in \mathcal{D}(3,3\text{-}1,5)$ presented below, $e_A(d^*) = e_D(d^*) = 84/85 > .9882$ and $e_E(d^*) = 14/15 = .933\overline{3}$ while $e_E(\hat{d}) = 1$ since p+1 = v-1.

	1	1		1	1
	2	2		2	2
d*:	3	3	$\hat{d}$:	3	3
	1	1		1	2
	2	3		1	3

For $1 \le mq \le v-2$ the only case giving an eigenvalue structure of Theorems 1.4.1, 1.4.4 or 2.1.1 was m = q = 1 and that was proven $\mathcal{J}_4$-optimal. Therefore no efficiencies are presented for $2 \le mq \le v-2$.

CHAPTER 9

TREND-FREE BLOCK DESIGNS

9.1 Preliminaries

Bradley and Yeh (1980) introduced the theory of trend-free block designs. In this setting an experimenter is interested in applying v treatments to the plots of b blocks of size $k \leq v$, but there is thought to be a common polynomial trend over the k plots of each block.

The additive model for a block design with polynomial trends taken into consideration is

$$y_{jt} = \mu + \sum_{i=1}^{v} \delta_{djt}^{i}\tau_i + \beta_j + \sum_{\alpha=1}^{p} \theta_\alpha\phi_\alpha(t) + \varepsilon_{jt} \qquad (9.1.1)$$

where μ is the overall mean, τ_i is the treatment effect $(1 \leq i \leq v)$, β_j is the block effect $(1 \leq j \leq b)$, y_{jt} is the observation for plot t in block j $(1 \leq t \leq k)$, θ_α is the regression coefficient for $\phi_\alpha(t)$, $\phi_\alpha(t)$ is the value of the orthogonal polynomial of degree α at t $(1 \leq \alpha \leq p)$ and the ε_{jt} are assumed to be i.i.d. zero-mean random errors.

If we label the bk plots with ordered pairs (j,t) then $\delta_{djt}^{i} = 1$ and not 0 only if treatment i is applied to plot (j,t).

$\sum_{\alpha=1}^{p} \theta_\alpha \phi_\alpha(t)$ is the overall trend effect for plot t.

For the orthogonal polynomials we have $\sum_{t=1}^{k} \phi_\alpha(t) = 0$ and $\sum_{t=1}^{k} \phi_\alpha^2(t) = 1$ for $1 \leq \alpha \leq p$ and $\sum_{t=1}^{k} \phi_\alpha(t)\phi_{\hat{\alpha}}(t) = 0$ for $1 \leq \alpha < \hat{\alpha} \leq p$.

Bradley and Yeh (1980) derived the appropriate C-matrix and gave a condition for a design to be trend-free (TF) of degree p. The details of the derivations can be found there.

The C-matrix will be designated as C_d as before, but it has a new form:

$$C_d = \underline{r}_d^\delta - k^{-1} \eta_d \eta_d' - b^{-1} \underline{\Delta}_{+d}' \underline{\Phi}_p \underline{\Phi}_p' \underline{\Delta}_{+d}.$$

$\underline{\Delta}_{+d}$ is the k by v matrix with $r_{dit} = \sum_{j=1}^{b} \delta_{djt}^{i}$ in row t and column i, where r_{dit} is the number of times treatment i is applied to plot t in the design. Of course the column sums of $\underline{\Delta}_{+d}$ are $r_{di} = \sum_{t=1}^{k} r_{dit}$ and the row sums are $b = \sum_{i=1}^{v} r_{dit}$.

$\underline{\Phi}_p$ is the k by p matrix with $\phi_\alpha(t)$ in row t and column α. If we define $\underline{\phi}_\alpha' = (\phi_\alpha(1),\ldots,\phi_\alpha(k))$ for $1 \leq \alpha \leq p$, then $\underline{\Phi}_p = [\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_p]$. The reader should not confuse $\underline{\Phi}_p$ with the symbols Φ or Φ_p for optimality criteria.

<u>Lemma 9.1.1</u>: (Bradley and Yeh (1980)) Under the model at (9.1.1) a design d is a trend-free block design for degree p (TFBD(p)) if and only if

$$\sum_{j=1}^{b} \sum_{t=1}^{k} \delta^{i}_{djt}\phi_{\alpha}(t) = 0,\ \alpha = 1,\ldots,p;\ i = 1,\ldots,v; \qquad (9.1.2)$$

or equivalently

$$\sum_{\alpha=1}^{p} \sum_{i=1}^{v} \{\sum_{j=1}^{b} \sum_{t=1}^{k} \delta^{i}_{djt}\phi_{\alpha}(t)\}^{2} = 0 \qquad (9.1.3)$$

or equivalently

$$\underline{\Delta}'_{+d}\ \underline{\Phi}_{p} = \mathcal{O}_{v,p}. \qquad (9.1.4)$$

Remark 9.1.1: If d is a TFBD(p) then $\underline{\Delta}'_{+d}\ \underline{\phi}_{\alpha} = \mathcal{O}_{v,1}$ for $1 \leq \alpha \leq p$.

Yeh and Bradley (1983) presented results for the existence and construction of trend-free block designs (TFBD's). For situations where v, b, k, p and $\underline{r}$ do not allow a TFBD(p) to exist, Yeh and Bradley (1984) introduced two measures of closeness to the trend-free property. They are paraphrased here as definitions.

Definition 9.1.1: (their Definition 1) A block design under model (9.1.1) is said to be a nearly trend-free block design of Type A for incidence matrix η and degree p (NTFBD(A,p; η)) if (9.1.3) is minimized among all designs with the same η.

Definition 9.1.2: (their Definition 2) A block design under model (9.1.1) is said to be a nearly trend-free block design of Type B for incidence matrix η and degree p (NTFBD(B,p; η)) if among all designs with the same η the following holds:

(i) $\sum_{\alpha=1}^{p-1} \sum_{i=1}^{v} \{\sum_{j=1}^{b} \sum_{t=1}^{k} \delta^{i}_{djt}\phi_{\alpha}(t)\}^{2}$ is minimum, and

(ii) $\sum_{i=1}^{v}\{\sum_{j=1}^{b}\sum_{t=1}^{k}\delta_{djt}^{i}\phi_p(t)\}^2$ is minimum among all d satisfying (i).

Remark 9.1.2: If d is a NTFBD(A, p; η) then it is a NTFBD(B,p; η), but the converse does not necessarily hold. (9.1.3) is just the trace of $\underline{\Delta}'_{+d}\ \underline{\Phi}_p\ \underline{\Phi}'_p\ \underline{\Delta}_{+d}$. If (9.1.3) equals zero then d is a TFBD(p). If the minimum of (i) in Definition 9.1.2 is 0 then d is a TFBD(p-1) and a NTFBD(B,p; η).

Yeh and Bradley (1984) also proved the following theorem which is improved upon in Section 9.2.

Theorem 9.1.1: (their Theorem 1) If d^o is a BIBD or CBD for model (9.1.1) under a polynomial trend of degree p having incidence matrix η_0, and d^o is a TFBD(p-1) and a NTFBD(B,p; η_0) then d^o is A- and D-optimal among all d with $\eta_d = \eta_0$ and that are a TFBD(p-1).

Note that in Definitions 9.1.1 and 9.1.2 we restricted ourselves to d with the same η. In Theorem 9.1.1 the class of designs over which d^o is optimal is a subclass of the class in the definitions.

9.2 Extended Results

We introduce some notation first. Since we are keeping v, b and k fixed for all discussions we shall use $\mathcal{D}$ instead of $\mathcal{D}(v,b,k)$ for the class of all connected designs with each block containing at least two treatments. Now let $\mathcal{D}(\eta_0) = \{d: \eta_d = \eta_0\}$ and $\mathcal{D}(\Lambda_0,\underline{r}_0) = \{d: \Lambda_d = \Lambda_0$ and $\underline{r}_d = \underline{r}_0\}$ where $\Lambda_d = \eta_d\,\eta'_d$ and was defined in Chapter 1. Also let TF(p) stand for "trend-free for degree p".

We shall now extend the scope of Definitions 9.1.1 and 9.1.2 and Theorem 9.1.1.

Definition 9.2.1: A block design d_0 with $\Lambda_{d^0} = \Lambda_0$ and $\underline{r}_{d^0} = \underline{r}_0$ under model (9.1.1) is said to be a nearly trend-free block design of Type A for the pair $(\Lambda_0, \underline{r}_0)$ and degree p (NTFBD(A, p; $\Lambda_0,\underline{r}_0$)) if (9.1.3) is minimized over $\mathcal{D}(\Lambda_0,\underline{r}_0)$.

Definition 9.2.2: A block design d_0 with $\Lambda_{d^0} = \Lambda_0$ and $\underline{r}_{d^0} = \underline{r}_0$ under model (9.1.1) is said to be a nearly trend-free block design of Type B for the pair $(\Lambda_0,\underline{r}_0)$ and degree p (NTFBD(B, p; $\Lambda_0,\underline{r}_0$)) if the following holds:

(i) $\sum_{\alpha=1}^{p-1} \sum_{i=1}^{v} \{ \sum_{j=1}^{b} \sum_{t=1}^{k} \delta^{i}_{djt}\phi_\alpha(t)\}^2$ is a minimum over $\mathcal{D}(\Lambda_0,\underline{r}_0)$ and

(ii) $\sum_{i=1}^{v} \{ \sum_{j=1}^{b} \sum_{t=1}^{k} \delta^{i}_{djt}\phi_p(t)\}^2$ is minimum among all $d \in \mathcal{D}(\Lambda_0,\underline{r}_0)$ satisfying (i).

Remark 9.2.1: As in Section 9.1, if d is a NTFBD(A, p; $\Lambda_0,\underline{r}_0$) then it is a NTFBD(B, p; $\Lambda_0,\underline{r}_0$). The d^0 and d_3 in Example 9.2.1 illustrate that the converse is not necessarily true; d^0 is a NTFBD(B, 2, Λ_0, $\underline{r}_0$) but d_3 is "more free" than d^0 in the sense of Definition 9.2.1. If the minimum of (i) of definition 9.2.2 is 0, then d^0 is a TFBD(p-1) as well as a NTFBD(B, p; $\Lambda_0,\underline{r}_0$). The d^0 in each of Examples 9.2.1 and 9.2.2 is such a d^0. Finally (9.1.3) is still the trace of $\underline{\Delta}'_{+d}\, \underline{\Phi}_p\, \underline{\Phi}'_p\, \underline{\Delta}_{+d}$ and Lemma 9.1.1 still holds.

The scope of Definitions 9.1.1 and 9.1.2 were widened because the optimality and trend-free properties of a design d depend on η_d only through Λ_d and $\underline{r}_d$. Therefore taking a specific Λ_0 and $\underline{r}_0$ allows us to include any η_d with $\Lambda_d = \Lambda_0$ and $\underline{r}_d = \underline{r}_0$. This is particularly useful when we consider Λ_0 and $\underline{r}_0$ for a BIBD.

Theorem 9.2.1: If d^o is a BIBD (or CBD) in $\mathcal{D}(\Lambda_0,\underline{r}_0)$ for model (9.1.1) under a polynomial trend of degree p, and d^o is a TFBD(p-1) and a NTFBD(B, p; $\Lambda_0,\underline{r}_0$) then d^0 is $\mathcal{J}_4$-optimal over all TF(p-1) designs d in $\mathcal{D}(\Lambda_0,\underline{r}_0)$.

Proof: This is essentially the proof of Theorem 9.1.1. For all TF(p-1) designs $d \in \mathcal{D}(\Lambda_0,\underline{r}_0)$ we know $\underline{\Delta}_{+d}' \underline{\Phi}_{p-1} = 0$ so $\underline{\Delta}_{+d}' \underline{\Phi}_p = \underline{\Delta}_{+d}' \underline{\phi}_p$. We also know $\underline{r}_d^{\delta}-k^{-1}\Lambda_d = \underline{r}_0^{\delta}-k^{-1}\Lambda_0$ is completely symmetric, the matrix $b^{-1}\ \underline{\Delta}_{+d}' \underline{\phi}_p \underline{\phi}_p' \underline{\Delta}_{+d}$ has exactly one nonzero eigenvalue γ_d, and the eigenvalues of $C_d = \underline{r}_0^{\delta}-k^{-1}\Lambda_0-b^{-1}\underline{\Delta}_{+d}' \underline{\phi}_p \underline{\phi}_p' \underline{\Delta}_{+d}$ are $0 < \frac{v\lambda_1}{k} - \gamma_d, \frac{v\lambda_1}{k},\ldots, \frac{v\lambda_1}{k}$ as pointed out by Yeh and Bradley (1984). Since d^o is one of these designs and is also a NTFBD(B, p; $\Lambda_0,\underline{r}_0$), then $\gamma_{d^o} \leq \gamma_d = tr(b^{-1}\underline{\Delta}_{+d}' \underline{\phi}_p \phi_p' \Delta_{+d}) = \sum_{i=1}^{v} \{ \sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i}\phi_p(t)\}^2$. This makes the eigenvalue structure of C_{d^o} that of Theorem 2.1.1, d^o E-optimal over the TF(p-1) designs in $\mathcal{D}(\Lambda_0,\underline{r}_0)$, and C_{d^o} of maximum trace for the TF(p-1) designs in $\mathcal{D}(\Lambda_0,\underline{r}_0)$. So applying Theorem 2.1.1 we conclude the proof.

This optimality of d^o of Theorem 9.2.1 is still over a very small subclass of $\mathcal{D}$. Three questions naturally arise. Is such a d^o $\mathcal{J}_4$-optimal over all designs in $\mathcal{D}$? Is it optimal over all designs that are trend-free for at least degree p-1? Is it optimal over all designs in $\mathcal{D}(\Lambda_0, \underline{r}_0)$ but which do not even have to be TF(p-1)?

As the following two examples illustrate, in the BIBD case the answers to the last two questions, and therefore the first, can be no. The orthogonal polynomials used for the examples are those in Appendix 11 of Anderson and McLean (1974).

Example 9.2.1: $\mathcal{D} = \mathcal{D}(6,10,3)$ and $p = 2$. Below are d^o of Theorem 9.2.1, an E-better non-BIBD d_1 which is TF(1), a D-better non-BIBD d_2 which is TF(1) and a $\mathcal{J}_4$-better BIBD d_3 with $\Lambda_{d_3} = \Lambda_0$ and $\underline{r}_{d_3} = \underline{r}_0$, but which is not even TF(1).

	1	1	6	5	4	2	2	5	3	6
d^o:	2	5	3	4	1	3	4	6	4	3
	3	2	1	1	6	4	6	2	5	5
	1	3	6	5	4	2	2	5	1	6
d_1:	2	5	3	4	1	3	4	6	4	3
	3	2	1	1	6	4	6	2	5	5
	1	2	3	4	5	6	6	2	5	4
d_2:	2	3	4	5	6	1	1	3	3	6
	4	5	6	1	2	3	2	4	6	5

	1	1	3	4	4	2	2	5	3	6
d_3:	2	2	1	1	6	3	4	6	5	5
	3	5	6	5	1	4	6	2	4	3

<u>Example 9.2.2</u>: $\mathcal{D} = \mathcal{D}(6,15,4)$ and $p = 3$. Below we have d^o of Theorem 9.2.1, a $\mathcal{J}$-better non-BIBD d_1 which is TF(2), and an A- and D-better non-BIBD d_2 which is TF(3).

	1	1	3	2	2	5	4	6	6	5	5	3	6	4	4
d^o:	2	2	1	1	1	1	3	3	3	4	2	6	3	2	5
	3	5	2	4	6	6	1	4	5	6	4	4	5	5	6
	4	3	6	5	4	2	5	1	1	1	3	2	2	6	3

	1	1	2	3	3	6	4	4	5	2	4	5	6	6	5
d^o:	2	2	1	1	1	5	3	3	2	4	6	1	3	2	3
	4	4	6	5	6	3	5	2	6	5	1	4	5	4	6
	5	6	5	4	4	1	2	6	3	1	2	3	1	3	2

	1	6	2	5	4	6	5	4	3	1	4	3	2	5	6
d_2:	4	5	5	6	3	4	1	5	2	2	6	4	1	6	3
	5	4	6	4	6	3	2	1	5	4	3	2	6	1	5
	6	1	4	2	5	5	4	3	4	6	1	6	5	3	2

There are three easy to follow principles that worked for finding the counterexamples to the $\mathcal{J}_4$-optimality of d^o over all or part of $\mathcal{D}$ beyond that of Theorem 9.2.1.

The first was used to get d_1 in each of Examples 9.2.1 and 9.2.2. After finding d^o we knew the "smallest" B_d for equireplicated designs that are TF(p-1), and this was fixed. Using B_d as a guide, without changing the fact that the treatments are equireplicated, adjust the way the treatments are paired to make Λ_d less smooth so that the sum $k^{-1}\Lambda_d + B_d$ is more smooth than it was for d_0, therefore making C_d more smooth. For Example 9.2.2 we present $k^{-1}\Lambda_{d^o}$, $B_{d^o} = B_{d_1}$ and $k^{-1}\Lambda_{d_1}$, respectively.

$$k^{-1}\Lambda_{d^o} = \frac{1}{12}\begin{bmatrix} 30 & 18 & 18 & 18 & 18 & 18 \\ & 30 & 18 & 18 & 18 & 18 \\ & & 30 & 18 & 18 & 18 \\ & & & 30 & 18 & 18 \\ & & & & 30 & 18 \\ \text{sym} & & & & & 30 \end{bmatrix}$$

$$B_{d^o} = \frac{1}{12}\begin{bmatrix} 4 & 4 & 4 & -4 & -4 & -4 \\ & 4 & 4 & -4 & -4 & -4 \\ & & 4 & -4 & -4 & -4 \\ & & & 4 & 4 & 4 \\ & & & & 4 & 4 \\ \text{sym} & & & & & 4 \end{bmatrix}$$

$$k^{-1}\Lambda_{d'} = \frac{1}{12}\begin{bmatrix} 30 & 15 & 15 & 21 & 21 & 18 \\ & 30 & 15 & 21 & 18 & 21 \\ & & 30 & 18 & 21 & 21 \\ & & & 30 & 15 & 15 \\ & & & & 30 & 15 \\ \text{sym} & & & & & 30 \end{bmatrix}$$

The second principle was used to get d_2 in each of Examples 9.2.1 and 9.2.2. Here one finds different replications for some treatments that do not affect the TF(p-1) property but make a design nearer to TF(p). $\underline{r}'_{d_2}$ equals (4, 5, 5, 5, 5, 6) and (8, 8, 8, 12, 12, 12) instead of $5\underline{1}'_6$ and $10\underline{1}'_6$ in Examples 9.2.1 and 9.2.2, respectively.

The third principle was used to get d_3 in Example 9.2.1. Keeping the η of the BIBD generated d^o, just change the r_{dit}, $1 \leq t \leq k$, for each i so that $(r_{di1},\ldots,r_{dik})\underline{\phi}_\alpha$ is made small for each i and all $1 \leq \alpha \leq p$.

CHAPTER 10

SUMMARY

10.1 Optimality of Designs

In Table 10.1.1 are listed the classes $\mathcal{D}(v,b,\underline{k})$ for which a design d* was proved optimal in Chapters 3 through 7. The number of treatments (v), the number of blocks (b), and block size or sizes ($\underline{k}$) for which optimality was proved appear in the first three columns, respectively. Then the optimality criteria for which d* was proved optimal are listed. "$g\mathcal{J}_1$" stands for the generalized type 1 criteria of Cheng (1978) (see Theorem 1.4.1), "$\mathcal{J}_4$" stands for the criteria of Theorem 2.1.1, and "E" stands for the E-optimality criterion. Finally the theorem or lemma where the result was proved is given.

Note that b_0 is the number of blocks allowing a BBD of Kiefer (Definition 1.2.1) to be constructed.

10.2 Efficiencies

In Table 10.2.1 are presented the classes $\mathcal{D}(v,b,\underline{k})$ where efficiencies were obtained for some designs of interest. The table is set up just as Table 10.1.1 with one minor exception: The design for which the efficiency was obtained appears with the number of the respective theorem.

Table 10.1.1: Classes with optimalities

Values of v	Number of Blocks	Block Sizes	Optimality Criteria	Theorem or Lemma
≥ 3	$\frac{ev(v-1)}{2}+1$	(2,...,2)	$g\mathcal{J}_1$	T3.2.1
≥ 3	ev-1	(v-1,...,v-1)	$g\mathcal{J}_1$	T3.2.2
≥ 3	ev+1	(v-1,...,v-1)	$\mathcal{J}_4$	T3.3.1
3,4,5,6	$\frac{ev(v-1)}{2}-1$	(2,...,2)	$\mathcal{J}_4$	T3.3.2
≥ 3	b_0+1	(v,...,v,v-1)	$\mathcal{J}_4$	T4.1.1
≥ 3	b_0	(pv-1,pv,...,pv)	$\mathcal{J}_4$	T5.3.1
≥ 3	ev	(pv,pv+1,...,pv+1)	$\mathcal{J}_4$	T5.3.2
≥ 3	$b_0+v/2$	(pv+2,...,pv+2)	E	L6.2.2
4	6e+2	(4p+2,...,4p+2)	$\mathcal{J}_4$	T6.2.1
≥ 3	b_0+m	$(pv+\frac{v-1}{m},...,pv+\frac{v-1}{m})$ and $1 \leq m \leq v-1$	E	T6.3.1
≥ 3	ev+1	(pv+v-1,...,pv+v-1)	$\mathcal{J}_4$	T6.3.2
≥ 3	ev+v-1	(pv+1,...,pv+1)	$\mathcal{J}_4$	T6.3.2
≥ 3	ev-1	(pv+1,...,pv+1)	$\mathcal{J}_4$	T7.4.1

Table 10.2.1: Efficiencies obtained

Values of v	Number of Blocks	Block Sizes	Optimality Criteria	Theorem and Design
≥ 3	$\frac{ev(v-1)}{2}-1$	(2,...,2)	A,D,E	8.2.1,d*
≥ 7	b_0+1	(k,...,k,v-1) and $(v-1)/3 \leq k < v-1$	A,D,E	8.3.1,d*
4,5,6	b_0+1	(k,...,k,v-1) and $2 \leq k < v-1$	A,D,E	8.3.1,d*
≥ 7	b_0+1	(k,...,k,v-1) and $2 \leq k < (\sqrt{v}+1)/2$	A,D,E	8.3.2,d*
≥ 3	b_0+1	(pv,...,pv,v-1) and $p \geq 2$	A,D,E E	8.3.3,d* 8.3.3,$\hat{d}$
≥ 3	b_0+1	(pv+q,...,pv+q,v-1) and $1 \leq q \leq v-1$	A,D,E	8.3.4,d*
≥ 3	ev+1	(pv+1,...,pv+1,v-1)	A,D,E	8.3.5,$\hat{d}$
≥ 3	ev+1	(pv+v-1,...,pv+v-1,v-1)	A,D,E	8.3.5,$\hat{d}$
≥ 3	b_0	(k,...,k,k-1) and $3 \leq k \leq v-1$	A,D,E	8.4.1,d*
≥ 3	ev	(pv+1,...,pv+1,pv)	A,D,E	8.4.2,$\tilde{d}$
≥ 3	b_0	(pv+q,...,pv+q,pv+q-1) and $2 \leq q \leq v-1$	A,D,E A,D,E	8.4.3,d* 8.4.4,$\tilde{d}$
≥ 4	b_0+2	(pv+v/2,...,pv+v/2)	A,D,E	8.5.1,d*
≥ 4	$b_0+v/2$	(pv+2,...,pv+2)	A,D	8.5.2,d*
≥ 3	b_0+m	$(pv+\frac{v-1}{m},...,pv+\frac{v-1}{m})$ and $2 \leq m \leq v-2$	A,D	8.5.3,d*
≥ 3	ev+1	(pv+1,...,pv+1)	A,D,E E($p \geq v-1$)	8.5.4,d* 8.5.4,$\hat{d}$
≥ 3	$b_0-m \geq 2$	(pv+v/m,...,pv+v/m) and $3 \leq m \leq v-1$	A,D	8.6.1,d*
≥ 3	ev-1	(pv+v-1,...,pv+v-1)	A,D,E E($p \geq v-2$)	8.6.2,d* 8.6.2,$\hat{d}$

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