POSITIVE DEPENDENCE IN MARKOV CHAINS

By

Leon J. Gleser and David S. Moore

Technical Report #84-22

Department of Statistics

Purdue University

July 1984

POSITIVE DEPENDENCE IN MARKOV CHAINS

Leon J. Gleser and David S. Moore
Purdue University, USA

SUMMARY

A general notion of positive dependence among successive observations in a finite-state stationary process is studied, with particular attention to the case of a stationary ergodic Markov chain. Some useful conditions equivalent to positive dependence are obtained for reversible chains, but shown not to be equivalent for nonreversible chains. Statistical implications of positive dependence are considered in detail elsewhere.

Keywords: REVERSIBLE MARKOV CHAINS; POSITIVE DEPENDENCE

1. INTRODUCTION

In this paper, we study conditions under which a stationary finite-state stochastic process, and in particular a stationary ergodic Markov chain, is positive dependent. The definition of positive dependence adopted here arises naturally in the study of the effects of dependent data on statistical procedures that assume iid observations. Gleser and Moore (1983) showed that positive dependence causes classical tests of fit (such as the Pearson chi-squared and Kolmogorov-Smirnov tests) to reject a true null hypothesis too often. Gleser and Moore (1984) extended this result to tests for categorical data. It is in this last setting that our notion of positive dependence for finite-state processes is of interest. Markov chains are a common model, and various aspects of the effect of Markov dependence among successive observations on tests for categorial data have been studied by Tavaré (1983) and Tavaré and Altham (1983).

We call two jointly distributed variables X,Y with common sample space $\mathcal{S}$ <u>positively dependent</u> if

$$\text{cov}\,\{h(X),\ h(Y)\} \geq 0 \tag{1.1}$$

for every function h: $\mathcal{S} \to (-\infty,\infty)$ such that $E|h(X)h(Y)| < \infty$. A stochastic process $\{X_t\}$ defined on the product space $\mathcal{S}^\infty$ is positively dependent if X_t, X_s are positively dependent for all t,s.

It is easy to see that when X,Y have identical marginal distributions, (1.1) is equivalent to

$$E\{h(X)h(Y)\} \geq 0 \tag{1.2}$$

for all h such that $E\{h^2(X)\} < \infty$. Thus (1.1) and (1.2) are equivalent for stationary processes.

For categorical variables, we may take $\mathcal{S} = \{1,\ldots,M\}$ for some integer M. The joint distribution of two variables X,Y on $\mathcal{S}$ is specified by the M x M matrix of joint probabilities.

$$R = ((r_{ij})), \qquad r_{ij} = P(X=i,\ Y=j).$$

Any function h: $\mathcal{S} \to (-\infty,\infty)$ can be represented as an M-vector h with ith component h(i). When X,Y have identical marginal distributions, (1.2) is equivalent to

$$h'R\,h \geq 0, \text{ all M-vectors } h. \tag{1.3}$$

When R is symmetric (i.e., when X,Y are exchangeable), (1.3) states that R is a positive semidefinite (p.s.d) matrix. For convenience, we will extend this terminology, and call a general M x M matrix R p.s.d. if (1.3) holds. The following theorem summarizes our discussion.

Theorem 1. Two jointly distributed categorical variables X,Y on the sample space $\mathcal{S} = \{1,\ldots,M\}$, with joint probability matrix R and identical marginal distributions are positive dependent if and only if R is p.s.d.

A number of concepts of positive dependence have been introduced; see chapter 5 of Barlow and Proschan (1975), Shaked (1979, 1982) and the references therein. Our definition is of interest for its statistical implications, and is appropriate for categorical variables because it does not depend on the arbitrary ordering of the values (states) 1,...,M of X and Y. There is no implication in either direction between our notion of positive dependence and such notions as association and orthant dependence (see Shaked (1982) for a survey) that assume a meaningful ordering of values. On the other hand, common definitions of dependence among successive categorical outcomes in terms of conditional probabilities do not imply a sign for dependence as we wish to do.

For exchangeable random variables, Gleser and Moore (1983) discuss relations between (1.2) and other notions of positive dependence. In particular, (1.2) is equivalent in this case to positive definite dependence as defined by Shaked (1979). We shall show below that for Markov chains, positive dependence in the exchangeable case (i.e., reversible chains) has a number of properties that do not extend to nonreversible chains.

Section 2 presents a necessary condition and a sufficient condition for a joint probability matrix R to be p.s.d. Sections 3 through 5 concern stationary ergodic (i.e. aperiodic positive recurrent) Markov chains $\{X_t\}$. Section 3 considers reversible chains. A reversible chain is shown to be positive dependent if and only if all characteristic roots of the matrix of one-step transition probabilities are nonnegative. Equivalently, a reversible chain $\{X_t\}$ is a positive dependent process if and only if X_1, X_2 are positive dependent variables. Section 4 shows by example that these facts are false for nonreversible chains.

One interesting special case of a Markov chain is a chain in which $X_t = (Y_t, Z_t)$ has states (y,z), where y is the row index and z is the column index of a contingency table (Tavaré and Altham, 1983; Tavaré, 1983), and in which $\{Y_t\}$ and $\{Z_t\}$ are (conditionally) independent Markov chains. In Section 5, we briefly state some results concerning positive dependence of $\{X_t\}$ when at least one of the component processes $\{Y_t\}$, $\{Z_t\}$ is reversible.

2. A NECESSARY AND A SUFFICIENT CONDITION FOR POSITIVE DEPENDENCE

Let X,Y be two jointly distributed categorical variables on $\mathcal{S} = \{1,\ldots,M\}$ with common marginal distribution defined by

$$p_i = P\{X=i\} = P\{Y=i\},\ i = 1,\ldots,M.$$

Let $p = (p_1,\ldots,p_M)'$, $1_M = (1,1,\ldots,1)'$,

$$R = ((r_{ij})), \qquad r_{ij} = P(X=i,\ Y=j),$$

and note that

$$p' = 1_M'R = 1_M'R', \qquad p'1_M = 1. \tag{2.1}$$

The following are necessary conditions for X,Y to be positive dependent.

Theorem 2. If X,Y are positive dependent, then

$$r_{ii} - p_i^2 \geq 0, \qquad i = 1,\ldots,M, \tag{2.2}$$

or equivalently,

$$\sum_{j\neq i} [r_{ij} + r_{ji} - 2p_i p_j] \leq 0, \; i = 1,2,\ldots,M. \tag{2.3}$$

Proof. If X,Y are positive dependent, then R is p.s.d. in the sense of (1.3). For a given i, define

$$h_1 = e_i, \; h_2 = \sum_{j\neq i} e_j,$$

where e_j is the jth column of the M x M identify matrix I_M. Since R is p.s.d., it must be the case that

$$0 \leq (xh_1-h_2)'R(xh_1-h_2) = x^2 r_{ii} - x\{\sum_{j\neq i}(r_{ij} + r_{ji})\} + \sum_{j,k\neq i} r_{jk}$$

for all x, and hence for $x = \sum_{j\neq i}(r_{ij} + r_{ji})/2r_{ii}$. Thus,

$$0 \leq -\frac{\left\{\sum_{j\neq i}(r_{ij} + r_{ji})\right\}^2}{4r_{ii}} + \sum_{j,k\neq i} r_{jk},$$

From (2.1),

$$\sum_{j,k\neq i} r_{jk} = 1-r_{ii} - \left\{\sum_{j\neq i}(r_{ij} + r_{ji})\right\},$$

and

$$r_{ii} + \sum_{j\neq i} r_{ij} = p_i = r_{ii} + \sum_{j\neq i} r_{ji} \quad , \qquad (2.4)$$

so that

$$0 \leq -\frac{(p_i - r_{ii})^2}{r_{ii}} + 1 + r_{ii} - 2p_i = \frac{1}{r_{ii}}(r_{ii} - p_i^2),$$

verifying (2.2). Equation (2.4), and the fact that $\sum_{i=1}^{m} p_i = 1$, imply that

$$\begin{aligned} 2(r_{ii} - p_i^2) &= p_i - \sum_{j\neq i} r_{ij} + p_i - \sum_{j\neq i} r_{ji} - 2p_i^2 \\ &= 2p_i(1 - p_i) - \sum_{j\neq i}(r_{ij} + r_{ji}) \qquad (2.5) \\ &= \sum_{j\neq i}(2p_ip_j - r_{ij} - r_{ji}). \end{aligned}$$

Hence (2.2) and (2.3) are equivalent. □

Recall that if X and Y are independent,

$$r_{ii} = p_i^2 \quad , \quad r_{ij} = r_{ji} = p_ip_j \quad , \quad i,j = 1 \ldots, M.$$

Theorem 1 shows that for X and Y to be positive dependent, the event {X=Y=i} must be more probable than would be the case if X and Y were independent (assuming that p_i defines the marginal probabilities in both cases), i = 1,...,M. Equivalently, for X and Y to be positive dependent, P{X ≠ Y, X or Y = i} must be less probable than would be the case if X and Y were independent, i = 1,...,M. These requirements coincide with our intuition concerning positive dependence.

A sufficient condition for positive dependence is obtained by strengthening (2.3) to require that

$$r_{ij} + r_{ji} \leq 2p_ip_j \quad , \quad \text{all } i \neq j. \qquad (2.6)$$

Note that (2.6) implies (2.3), and also (2.2).

Theorem 3. If (2.6) holds, then X and Y are positive dependent.

Proof. Since $h'Rh = h'R'h$, R is p.s.d. if and only if the symmetric matrix $R + R'$ is p.s.d.

Let

$$A = R + R' - 2p'p \quad , \quad A = ((a_{ij})).$$

Since A is symmetric, it has M real roots $\lambda_1 \geq \dots \geq \lambda_M$. By Geršgorin's Theorem (Marcus and Minc, 1964, p. 146), for each $k = 1,2,\dots,M$ there exists an $i = i(k)$ such that

$$|\lambda_k - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \ . \tag{2.7}$$

However, by (2.6),

$$a_{ij} = r_{ij} + r_{ji} - 2p_i p_j \leq 0,$$

so that from (2.5),

$$\sum_{j \neq i} |a_{ij}| = \sum_{j \neq i} (2p_i p_j - r_{ij} - r_{ji})$$

$$= 2(r_{ii} - p_i^2) = a_{ii} \ .$$

Hence, it follows from (2.7) that $\lambda_k \geq 0$, $k = 1,\dots,M$, and hence A is p.s.d. Since

$$A \text{ p.s.d.} \Rightarrow h'(R+R')h \geq 2(h'p')^2 \geq 0 \quad \text{all } h\text{: } M \times 1,$$

$R+R'$ is p.s.d. Hence R is p.s.d., and X,Y are positive dependent. □

Although (2.6) is sufficient for X,Y to be positive dependent, it is not necessary. For example, if

$$R = \frac{1}{12}\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

Then $p_1 = p_2 = p_3 = (1/3)$, $r_{12} + r_{21} = (1/4) > (2/9) = 2p_1p_2$, but R is p.s.d. (so that X,Y are positive dependent). However, (2.6) is necessary and sufficient for X,Y to be positive dependent when M = 2, since in this case (2.2), (2.3) and (2.6) are equivalent assertions.

3. POSITIVE DEPENDENCE FOR REVERSIBLE MARKOV CHAINS

Let now $\{X_t\}$ be a stationary ergodic M-state Markov chain with transition matrix

$$T = ((t_{ij})) \quad , \quad t_{ij} = P(X_2=j|X_1=i). \tag{3.1}$$

Because $\{X_t\}$ is ergodic, the stationary (marginal) probabilities $p_i = P(X_t=i)$ satisfy $p_i > 0$ for all i. Further,

$$p'T = p' \tag{3.2}$$

and

$$T1_M = 1_M \quad , \tag{3.3}$$

where $p = (p_1,\ldots,p_M)'$, $1_M = (1,1,\ldots,1)'$. Define R_{st} to be the joint probability matrix for X_s, X_t with (i,j)th entry $P(X_s=i,\ X_t=j)$. Then

$$\begin{aligned} R_{st} &= DT^k \qquad t = s+k,\ k = 1,2,\ldots, \\ &= (T')^k D \quad t = s-k,\ k = 1,2,\ldots, \end{aligned}$$

where $D = \text{diag}\,(p_1,p_2,\ldots,p_M)$. According to Theorem 1, $\{X_t\}$ is positive dependent if and only if R_{st} is p.s.d. for all $s \neq t$.

The necessary condition (2.2) for positive dependence of successive terms (say X_1,X_2) is now equivalent to $t_{ii} \leq p_i$ for all i. The sufficient condition (2.6) is equivalent to $p_it_{ij} + p_jt_{ji} \leq 2p_ip_j$ for all $i \neq j$. Since T determines the properties of the chain, we might hope that positive dependence of $\{X_t\}$ is implied by positive dependence of X_1,X_2, and has a simple characterization in terms of T. We shall realize these hopes for reversible chains.

The following facts will be helpful.

Lemma 1. Let $U = D^{\frac{1}{2}} T D^{-\frac{1}{2}}$. Then U has nonnegative elements and

(a) R_{st} is symmetric for all $s \neq t$ if and only if U is symmetric;

(b) R_{st} is p.s.d. if and only if $U^{|s-t|}$ is p.s.d.;

(c) U and T have identical characteristic roots.

Proof. Note that for $t > s$, $R_{st} = DT^{|s-t|} = D^{\frac{1}{2}} U^{|s-t|} D^{\frac{1}{2}}$, while $R_{st} = R'_{ts}$ gives similar expressions for $t < s$. Parts (a) and (b) follow from this, and (c) holds because U,T are similar matrices. □

Recall that a chain $\{X_t\}$ is reversible if $R_{ts} = R_{st}$ for all $s \neq t$, or equivalently if all R_{st} are symmetric. Here is the main result of this section.

Theorem 4. Let $\{X_t\}$ be a reversible stationary ergodic Markov chain. Then the following statements are equivalent:

(a) $\{X_t\}$ is positive dependent;

(b) $Q = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{t,s=1 \\ t\neq s}}^{n} D^{-\frac{1}{2}}(R_{st} - pp')D^{-\frac{1}{2}}$ is p.s.d.;

(c) DT is p.s.d. (i.e., X_1 and X_2 are positive dependent);

(d) All characteristic roots of T are nonnegative.

Proof. If $h: \mathcal{S} \to (-\infty,\infty)$ is represented as an M-vector $h = (h(1),\ldots,h(M))'$,

$$\text{cov}\,\{h(X_s),\, h(X_t)\} = h'(R_{st} - pp')h,$$

so that by (1.1), (a) implies (b) whenever the limit defining Q exists. Let

$$L = U - (D^{\frac{1}{2}} 1_M)(D^{\frac{1}{2}} 1_M)', \tag{3.4}$$

where U is as in Lemma 1. Then from the fact that the non-1 characteristic roots of T are less than 1 in absolute value, it can be shown that $(I_M - L)^{-1}$ exists and that

$$Q = (I_M - L)^{-1} L + L'(I_M - L')^{-1} \tag{3.5}$$

so that Q also exists. Note that Q is always symmetric. When $\{X_t\}$ is reversible, all R_{st} are symmetric. Hence U and L are symmetric by Lemma 1(a) and (3.4), so that

$$Q = 2(I_M - L)^{-1} L.$$

Consequently, the characteristic roots $\lambda_i(L)$ of L are related to the characteristic roots $\lambda_i(Q)$ of Q by

$$\lambda_i(Q) = 2\{1 - \lambda_i(L)\}^{-1}\lambda_i(L), \qquad i = 1,\ldots,M.$$

It follows that

$$Q \text{ p.s.d.} \Leftrightarrow \text{all } \lambda_i(Q) \geq 0 \Leftrightarrow \text{all } \lambda_i(L) \geq 0 \Leftrightarrow L \text{ p.s.d.}$$

From (3.4) we see that L p.s.d. implies U is p.s.d., and from Lemma 3(b) this in turn implies that $DT = R_{12}$ is p.s.d. Hence (b) implies (c).

To see that (c) implies (d), apply Lemma 1 to show that

$$DT \text{ p.s.d.} \Leftrightarrow U \text{ p.s.d.} \Leftrightarrow \text{all } \lambda_i(U) \geq 0 \Leftrightarrow \text{all } \lambda_i(T) \geq 0,$$

where $\lambda_i(U)$ and $\lambda_i(T)$ are the characteristic roots of U and T, respectively. Similarly

$$\lambda_i(T) \geq 0 \Leftrightarrow U \text{ p.s.d.} \Leftrightarrow U^{|s-t|} \text{ p.s.d., all } s \neq t$$

$$\Leftrightarrow R_{st} \text{ p.s.d., all } s \neq t$$

so that (d) implies (a). This completes the proof of Theorem 4. □

In Gleser and Moore (1984), it is shown that the positive semidefiniteness of the matrix Q defined in Theorem 4(b) implies that any classical chi-squared test of fit for a model $p_i = p_i(\theta)$, $i = 1,\ldots,M$, specifying the stationary (marginal) probabilities of $\{X_t\}$ will reject too often under the null hypothesis. For reversible chains, Theorem 4 shows that Q is p.s.d. if and only if $\{X_t\}$ is positive dependent.

It should be noted that the transition matrix T need not be symmetric, even for reversible chains. When T is not symmetric, it is not true that $\lambda_i(T) \geq 0$, all i, implies that T is p.s.d. in the sense of (1.3). Indeed, every M=2 state Markov Chain is reversible, but the transition matrix

$$T = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix},$$

which has characteristic roots .15 and 1.00, is not p.s.d. (For example, if $h = (8,-5)'$, then $h'T\,h < 0$.) Also T can be p.s.d., but not have nonnegative characteristic roots. An example is

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

which is p.s.d., but has two imaginary characteristic roots. Of course, the explanation for the lack of relationship between the signs of the characteristic roots of T and the positive semidefiniteness of T is that the positive semidefiniteness of T is related to the characteristic roots of $T + T'$, not T.

Theorem 4 provides two intuitively satisfying characterizations of positive dependence for Markov chains $\{X_t\}$ -- namely, the positive semidefiniteness of $R_{12} = D\,T$ and the nonnegativity of the characteristic roots of T. Unfortunately, such characterizations are generally valid only for reversible Markov chains, as will be seen in Section 4. A survey of Markov chain models used in scientific practice (Olkin, Gleser and Derman, 1980; see also Coleman, 1964) shows that the overwhelming majority of such models (except for those which have only 2 states) are not reversible. Thus, the applicability of Theorem 4 is likely to be rather limited.

4. POSITIVE DEPENDENCE FOR NONREVERSIBLE MARKOV CHAINS

Consider again the assertions (a), (b), (c), (d) of Theorem 4. In this section, we will show that for nonreversible Markov Chains these assertions are not generally equivalent. However, relationships among (a), (b) and (c) do exist, as shown in the following theorem.

Theorem 5. If $\{X_t\}$ is a nonreversible M-state Markov chain ($M > 2$), then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). That is,

$$\{X_t\} \text{ positive dependent} \Rightarrow Q \text{ p.s.d.} \Rightarrow DT \text{ p.s.d.}$$

Proof. The proof that (a) $\Rightarrow$ (b) in Theorem 4 remains valid. Suppose next that Q is p.s.d. It follows from (3.5) that for any M-vector h,

$$h'Q\,h = [(I_M - L')^{-1}h]'\ \{L(I_M - L') + (I_M - L)L'\}(I_M - L')^{-1}h.$$

Consequently,

$$\begin{aligned} Q \text{ p.s.d.} &\Leftrightarrow L(I_M - L') + (I_M - L)L' \text{ is p.s.d.} \\ &\Leftrightarrow L + L' - 2LL' \text{ is p.s.d.} \end{aligned} \tag{4.1}$$

It is now easily shown that

$$L + L' - 2LL' \text{ p.s.d.} \Rightarrow U \text{ p.s.d.}$$

and this implies that $DT = R_{12}$ is p.s.d. by Lemma 1(b). □

The implication (a) $\Rightarrow$ (b) shows that positive dependence retains its statistical importance even in nonreversible cases.

We now give counterexamples for all implications in Theorem 4 other than those of Theorem 5. To show that for nonreversible Markov Chains, Q p.s.d. does not imply that $\{X_t\}$ is positive dependent, consider $\{X_t\}$ having transition matrix

$$T = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} , \qquad (4.2)$$

for which $p_1 = 1/5$, $p_2 = p_3 = 2/5$. Here,

$$L = \frac{1}{20} \begin{pmatrix} 6 & -4\sqrt{2} & 6\sqrt{2} \\ \sqrt{2} & 2 & -3 \\ -4\sqrt{2} & 2 & 2 \end{pmatrix}$$

and

$$L + L' - 2LL' = \frac{1}{8} \begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is p.s.d.}$$

Thus from (4.1), Q is p.s.d. However,

$$R_{13} = DT^2 = \frac{1}{20} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix}$$

is not p.s.d., since $(0,1,-1)\, R_{13}\, (0,1,-1)' < 0$. Hence $\{X_t\}$ is not positive dependent.

Next, we give an example of a Markov Chain $\{X_t\}$ for which $R_{12} = DT$ is p.s.d., but Q is not p.s.d. Let $\{X_t\}$ have transition matrix

$$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

for which $p_1 = p_2 = p_3 = 1/3$. Then $D = 3^{-1} I_3$, and

$$DT + T'D = \frac{2}{9}\, 1_3 1_3' \quad \text{is p.s.d.,}$$

implying that DT is p.s.d. On the other hand

$$L + L' - 2LL' = -\,(18)^{-1}(3I_3 + 11\, 1_3 1_3')$$

is clearly not p.s.d., so that by (4.1) Q is not p.s.d.

Finally, for nonreversible chains there are no general relationships between assertions (c) and (d) of Theorem 4. That is, DT p.s.d. neither implies or is implied by the nonnegativity of the characteristic roots of T. The transition matrix T in (4.2) is an example of a case where DT is p.s.d. but T does not have nonnegative characteristic roots. (Here, two characteristic roots of T are imaginary.) On the other hand,

$$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has characteristic roots 0,1/12, and 1, but DT is not p.s.d.

Note. We also considered the conjecture that the positive semidefiniteness of DT is related to nonnegativity of the real parts of the (possibly imaginary) characteristic roots of T. No such general relationship, in either direction, was found.

To summarize, for nonreversible Markov chains $\{X_t\}$, there seems to be no obvious way to use properties of the transition matrix T to infer that $\{X_t\}$ is positive dependent (or even that Q is p.s.d.). Since the transition matrix T determines the properties of the chain, a necessary and sufficient relationship between properties of T and the positive dependency of $\{X_t\}$ must exist. We merely have shown that this relationship is not the simple, intuitive relationship that Theorem 4 shows holds for reversible Markov chains.

5. POSITIVE DEPENDENCE FOR MARKOV CHAINS WITH INDEPENDENT COMPONENTS

Let $\{Y_t\}$, $\{Z_t\}$ be independent stationary ergodic Markov chains with state spaces $\mathcal{S}_y = \{1,2,\ldots,r\}$, $\mathcal{S}_z = \{1,2.\ldots,c\}$ and transition matrices T_y, T_z, respectively. Further, let $p_y = (p_{y1},\ldots,p_{yr})'$, $p_z = (p_{z1},\ldots,p_{zc})'$ be the vectors of stationary (marginal) probabilities for $\{Y_t\}$, $\{Z_t\}$, and define

$$D_y = \text{diag}\,(p_{y1},\, p_{y2},\ldots,p_{yr}),\; D_z = \text{diag}\,(p_{z1},\, p_{z2},\ldots,p_{zc}).$$

It is then well known (see Tavaré, 1983) that $\{X_t\} = \{(Y_t,Z_t)\}$ is an rc-state stationary ergodic Markov Chain with transition matrix

$$T = T_y \otimes T_z,$$

where "$\otimes$" is the Kronecker product. Further, the vector of stationary (marginal) probabilities for $\{X_t\}$ is

$$p = p_y \otimes p_z = (p_1,\ldots,p_{rc})',$$

and

$$D = \text{diag}\,(p_1,\ldots,p_{rc}) = D_y \otimes D_z.$$

Note that the joint probability matrix for X_s, X_t is, for $t \geq s$,

$$R_{st} = DT^{|t-s|} = D_y T_y^{|t-s|} \otimes D_z T_z^{|t-s|}. \tag{5.4}$$

Clearly $\{X_t\}$ is reversible (i.e., all R_{st} are symmetric) if and only if $\{Y_t\}$ and $\{Z_t\}$ are reversible; this is the situation treated by Tavaré (1983), and Tavaré and Altham (1983).

Theorem 6 If $\{X_t\}$ is reversible, then the following assertions are equivalent:

(a) $\{X_t\}$ is positive dependent;

(b) both $\{Y_t\}$ and $\{Z_t\}$ are positive dependent;

(c) $D_y T_y$ and $D_z T_z$ are p.s.d.;

(d) T_y and T_z have nonnegative characteristic roots.

Proof When $\{X_t\}$ is reversible, R_{st}, $D_y T_y^{|t-s|}$ and $D_z T_z^{|t-s|}$ are symmetric for all $s \neq t$. Consequently, the characteristic roots $\lambda_{11}(R_{st}),\ldots,\lambda_{rc}(R_{st})$ are the products

$$\lambda_{ij}(R_{st}) = \lambda_i(D_yT_y^{|t-s|})\lambda_j(D_zT_z^{|t-s|}), \qquad i = 1,\ldots,r,\ j = 1,\ldots,c, \tag{5.1}$$

of the characteristic roots of $D_yT_y^{|t-s|}$ and $D_zT_z^{|t-s|}$. These latter matrices are just the joint probability matrices for Y_s, Y_t and Z_s, Z_t, respectively. When (b) holds, these matrices are p.s.d. by Theorem 1, and (a) follows by (5.1). Conversely, if (a) holds, then $\lambda_{ij}(R_{st}) \geq 0$ all i,j, implying by (5.1) that $\lambda_i(D_yT_y^{|t-s|})$ and $\lambda_j(D_zT_z^{|t-s|})$ must have the same sign for all i,j. This sign must be positive (≥ 0) since $D_yT_y^{|t-s|}$ and $D_zT_z^{|t-s|}$ are not zero matrices and have nonnegative elements, and hence by Frobenius' Theorem (Marcus and Minc, 1964; p. 142) must each have one positive characteristic root. Consequently, $\lambda_i(D_yT_y^{|t-s|}) \geq 0$, $\lambda_j(D_zT_z^{|t-s|}) \geq 0$, all i, all j, all $t \neq s$. Therefore, by Theorem 1, $\{Y_t\}$ and $\{Z_t\}$ are positive dependent, proving that (b) holds. Thus, (a) and (b) are equivalent. The equivalence of (a) and (b) to (c) and (d) is now a consequence of Theorem 4. □

Now suppose that exactly <u>one</u> of $\{Y_t\}$, $\{Z_t\}$ is reversible. (This includes the possibility mentioned by Tavare (1983) that $\{Y_t\}$ or $\{Z_t\}$ could be an iid sequence, since any iid sequence is reversible.) Assuming (without loss of generality) that $\{Y_t\}$ is reversible, it is easily shown that

$$\begin{aligned} R_{st} + R'_{st} &= D_yT_y^{|t-s|} \otimes (D_zT_z^{|t-s|} + (T'_z)^{|t-s|}D_z) \\ &\equiv D_yT_y^{|t-s|} \otimes B_{st}. \end{aligned}$$

Consequently, the characteristic roots of the symmetric matrix $R_{st} + R'_{st}$ are the pairwise products

$$\lambda_{ij}(R_{st} + R'_{st}) = \lambda_i(D_yT_y^{|t-s|})\lambda_j(B_{st})$$

of the roots of the symmetric matrices $D_yT_y^{|t-s|}$ and B_{st}. (Note that since $\{Y_t\}$ is reversible, $D_yT_y^{|t-s|}$ is symmetric.) Using an argument similar to that used to prove Theorem 6, plus the fact that $D_zT_z^{|t-s|}$ is p.s.d. if and only if B_{st} is p.s.d., the following result can be obtained.

Theorem 7. If either $\{Y_t\}$ or $\{Z_t\}$ is reversible, then $\{X_t\}$ is positive dependent if and only if both $\{Y_t\}$ and $\{Z_t\}$ are positive dependent.

If neither $\{Y_t\}$ nor $\{Z_t\}$ is reversible, it is difficult to find conditions which guarantee that $\{X_t\}$ is positive dependent. In this situation, it is not clear even that positive dependence of $\{Y_t\}$ and $\{Z_t\}$ necessarily implies that $\{X_t\}$ is positive dependent, or vice versa. Note that any counterexamples to these assertions (if such counterexamples exist) must involve $\{X_t\}$ with at least 3x3=9 states, since if r=2 or c=2, at least one of $\{Y_t\}$, $\{Z_t\}$ is reversible, and Theorem 7 applies.

If at least one of $\{Y_t\}$, $\{Z_t\}$ is reversible, the determination of whether or not $\{X_t\}$ is positive dependent is simplified, since Theorem 4 can be used to check the positive dependence of the reversible component (either $\{Y_t\}$ or $\{Z_t\}$) of $\{X_t\}$.

Acknowledtement. This work was supported by NSF Grant DMS-8121948.

REFERENCES

BARLOW, R. E. and PROSCHAN, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

COLEMAN, JAMES S. (1964). Introduction to Mathematical Sociology. Free Press of Glencoe, Macmillan, N.Y.

GLESER, LEON JAY and MOORE, D. S. (1983). The effect of dependence on chi-squared and empiric distribution tests of fit. Ann. Statist. **11**, 1100-1108.

GLESER, LEON JAY and MOORE, D. S. (1984). The effect of positive dependence on chi-squared tests for categorical data. Mimeograph Series #84-20. Department of Statistics, Purdue University.

MARCUS, MARVIN and MINC, HENRYK (1964). A Survey of Matrix Theory and Matrix Inequalities, Prindle, Weber & Schmidt, Boston.

OLKIN, I., GLESER, L. J. and DERMAN, C. (1980). Probability Models and Applications. Macmillan, N.Y.

SHAKED, M. (1977). Some concepts of positive dependence for bivariate interchangeable distributions. Ann. Inst. Statist. Math. **31**, 67-84.

SHAKED, M. (1982). A general theory of some positive dependence notions. J. Mult. Anal. **12**, 199-218.

TAVARÉ, S. (1983). Serial dependence in contingency tables. J. Royal Statist. Soc. B **45**, 100-106.

TAVARÉ, S. and ALTHAM, P. M. E. (1983). Dependence in goodness of fit and contingency tables. Biometrika **70**, 139-144.