ON NINIDENTIFIABILITY PROBLEMS AMONG SOME STOCHASTIC MODELS IN RELIABILITY THEORY*,**

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ON NONIDENTIFIABILITY PROBLEMS AMONG SOME STOCHASTIC MODELS IN RELIABILITY THEORY*

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ABSTRACT: Two classes of shock-models (i) threshold type and (ii) nonthreshold type are considered, where it is assumed that the shocks arrive over time according to a mixed Poisson process. These models arise in reliability theory for the study of life distributions and certain reliability properties of a system. The present paper deals with the problem of nonidentifiability among the members of the family of distributions that each model generates for certain observable quantities. The two families of distributions generated by the two models are also compared for their mutual nonidentifiability.

1. INTRODUCTION: When it came to stochastic modeling in live situations, Neyman was much concerned about the problems of nonidentifiability of probability distributions arising in their study. One of his early works, where he encountered the problem of nonidentifiability was his joint work with Dr. G. E. Bates in 1952 ([2],[3]) on the theory of accident proneness. This being the case, it was felt quite befitting to choose such a topic for the present paper dedicated to his memory. Here we shall be mainly concerned with problems of nonidentifiability in particular reference to certain existing stochastic models in the area of reliability theory.

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In order to study a natural random phenomenon arising in a live-situation, often an attempt is made to idealize the underlying stochastic mechanism, while still trying to keep it close enough to reality and then construct a stochastic model based on this mechanism. As a result one usually ends up with a family of distributions of the observable 'random quantity' X, generated through a set of unknown 'parameters'. Let this family of probability measures corresponding to X, as given by the stochastic model, be represented by

$$\Omega = \{P_\theta, \theta \varepsilon \Theta\} \quad ,$$

where θ is a labeling parameter taking values in an appropriate set Θ. Here X could be a random vector or the realization of a stochastic process over a fixed time interval, etc., as the case may be. Similarly the possible values of the parameter θ could be over a class Θ of vector-valued functions or over a subset of k-dimensional Euclidean space, etc. The following definitions are standard and have been adopted here from [17].

DEFINITION 1. Based on the random quantity X, the family Ω of distributions is said to be identifiable if the members of Ω are distinct for distinct θ's. Otherwise we say that it is nonidentifiable.

Again it is not uncommon to try to explain the same observable random quantity X, through more than one stochastic model, each based on its own set of underlying assumptions and on possibly different mechanisms. Each then generates its own family of probability distributions for the observable quantity X in question. Let

$$\Omega_i = \{P_{\theta_i}, \theta_i \varepsilon \Theta_i\}, \ i = 1,2,$$

be two such families corresponding to two different stochastic models.

DEFINITION 2. Based on the random quantity X, the family Ω_1 is said to be non-identifiable with respect to the family Ω_2, if for each $\theta_1 \varepsilon \Theta_1$ there exists $\theta_2 \varepsilon \Theta_2$ such that $P_{\theta_1} = P_{\theta_2}$, so that $\Omega_1 \subset \Omega_2$. If however $\Omega_1 = \Omega_2$, we say that Ω_1 and Ω_2 are completely mutually nonidentifiable. Again, Ω_1 and Ω_2 are said to be partially mutually nonidentifiable, if $\Omega_1 \cap \Omega_2$ is nonemty and is a proper subset of each Ω_i, i = 1,2. Finally, if $\Omega_1 \cap \Omega_2$ is emty, the families Ω_1 and Ω_2 are said to be mutually identifiable based on the random quantity X.

The problem of nonidentifiability of distributions of observable variables in stochastic modeling is of fundamental importance. As was pointed out through examples (see [17], [19]), this problem is typically much more acute than is usually thought of or looked into, or even reported. If on the other hand such a problem does exist, it needs to be investigated first, before the model is put to any practical use for the purposes of predictions. Otherwise, as indicated by Clifford [5] through numerical examples in his case, one may arrive at quite conflicting predictions by using them. Thus it is important to watch for the presence of a nonidentifiability problem and when present it appears that there are two courses open in eliminating it. The first one is by cutting down the size of the original family, if possible. This is possible, for instance based on the past experience, if one can conveniently know apriori the values of some of the parameters or some acceptable relations among them, one may be able to cut down the size of the family, with the hope that the reduced family poses much less of the problem. The second course is to look for another observable random quantity Y and then enhance the original family of probability distributions to consist now of the joint probability distributions of X and Y hoping that the new enhanced family has less or no problem of nonidentifiability.

The present paper attempts to study from the point of view of nonidentifiability, two fairly general classes of stochastic models for life distributions in the area of reliability theory. These models or special cases thereof, have been the subject of extensive study in recent literature in this area (see [1], [6], [10], [11]). It is appropriate to mention however at the outset that we shall not be concerned here at all about their validity for one live situation or the other. In the next section we introduce these two models with their relevant details. While sections 3 and 4 deal with the problem of nonidentifiability of distributions arising within each model separately, section 5 deals with the mutual identifiability aspects between the families of distributions generated by the two models.

2. THRESHOLD AND NONTHRESHOLD TYPE RELIABILITY MODELS: The two classes of models we shall be concerned with may conveniently be referred to as (i) threshold type models and (ii) the nonthreshold type models, both are meant for the study of life distributions and certain reliability properties of a system. It is commonly assumed that the system receives over time the so called 'shocks' or 'blows' in some random manner (see for instance Esary, Marshall and Proschan [6], Barlow and Proschan [1] and Proschan and Serfling [11]). Each shock results into possibly a random amount of 'damage' or 'wear' to the system. This damage keeps accumulating as the shocks arrive over time. If $N(t)$ denotes the number of shocks arriving during $(0,t)$, the total damage received by the system during this time is then given by

$$Z(t) = \sum_{n=0}^{N(t)} Y_n ,$$

where the random variables (r.v) Y_n, assumed to be independent and identically distributed (I.I.D), represent the random amounts of damages due to various shocks for $n = 0,1,2,\ldots$, with $Y_0 = 0$, a.s. In a threshold type model one postulates the existence of a threshold, either of the number of shocks or for the total damage, so that the system is assumed to fail as soon as this threshold is reached. Again this threshold is assumed to be random varying with a common distribution over the underlying conceptual population of similar components. In this paper, for simplicity, we shall restrict ourselves to the case where the threshold is given only in terms of the number of shocks or equivalently we may assume that the damages Y_n's are all equal to the same positive constant. Thus if K denotes the random threshold in terms of the number of shocks with

$$p_k = P(K=k),\ \overline{P}_k = \sum_{i=k+1}^{\infty} p_i,\ \overline{P}_0 = 1,\ k = 0,1,2,\ldots,$$

and T denotes the length of life of the system, we have for $t > 0$,

$$\overline{H}_1(t) \equiv P(T > t) = \sum_{k=0}^{\infty} \overline{P}_k Q_k(t)\ , \tag{1}$$

where $Q_k(t) = P(N(t) = k)$, $k = 0,1,2,\ldots$.

Again in the case of nonthreshold type models, instead of a threshold, one postulates the existence of an appropriate nonnegative risk function $f(\underset{\sim}{\xi}(t);t)$ such that

$$P(\text{system fails during } (t,t+\tau)\,|\,\underset{\sim}{\xi}(T) = \underset{\sim}{x}, T > t) = f(\underset{\sim}{x};t)\tau + o(\tau), \tag{2}$$

where $\underset{\sim}{\xi}(t) = (N(t), Z(t))$. A standard argument (see [12], [14], [16]) then leads to

$$\overline{H}_2(t) \equiv P(T > t) = E\{\exp[-\int_0^t f(\underset{\sim}{\xi}(\tau);\tau)d\tau]\}\ . \tag{3}$$

In the present paper we shall restrict ourselves to the case with

$$f(\underset{\sim}{\xi}(t);t) = \alpha(t)N(t), \tag{4}$$

where $\alpha(\cdot)$ is a nonnegative function satisfying certain regularity conditions, so that (3) now becomes

$$\overline{H}_2(t) \equiv P(T > t) = E\{\exp[-\int_0^t \alpha(\tau)N(\tau)d\tau]\}. \tag{5}$$

Again, in [1] and [6], the authors have studied certain reliability properties of the function (1) under the assumption that N(t) is a time homogeneous Poisson process with parameter $\lambda > 0$, so that (1) is given by

$$\overline{H}_1(t) = \sum_{k=0}^{\infty} \overline{P}_k \frac{(\lambda t)^k}{k!} \exp(-\lambda t). \tag{6}$$

For the nonthreshold case of (4) and (5), we assume that (i) the function $\alpha(\cdot)$ is continuous for $t > 0$ and (ii) satisfies the condition

$$\int_0^\infty [1-\exp\{-\int_\tau^\infty \alpha(u)du\}]d\tau = \infty. \tag{7}$$

Under these conditions, when N(t) is a time homogeneous Poisson process with parameter $\lambda > 0$, it can be easily shown (see Puri [14]) that (5) becomes

$$\overline{H}_2(t) = \exp[-\lambda \int_0^t \{1-\exp(-\int_\tau^t \alpha(u)du)\}d\tau]. \tag{8}$$

It may be noted here that while the continuity requirement for the function $\alpha(\cdot)$ can be somewhat relaxed, nevertheless in view of (8), the condition (7) is essential in order that T be an honest r.v. Also the reader may find the above nonthreshold model as exhibited by (8) in a paper by Mercer [10]. However for the present study, we consider the more realistic case for the above models, where the Poisson parameter λ itself is assumed to vary randomly over the population of components (or systems) with cumulative distribution function (c.d.f.) F (.) with F(0) = 0, so that the shock arrival process N(t) is a mixed Poisson process. The mixed Poisson processes form a fairly rich class of Markov processes (see Feigen [7], LeCam [8] and Puri [18]). In particular they include the linear birth processes after a time-scale change, among others. Also they are characterized as being the only point processes (save time-scale changes) with an order statistic property and with $\lim_{t\to\infty} E(N(t)) = \infty$ (see Puri [18]). Again the reader may find in [17], a brief mention of few of the results (without proofs) of the present paper for the special case, where λ was assumed to be nonrandom. For the present generalized case, (6) and (8) respectively now become

$$\bar{H}_1(t) = \sum_{k=0}^{\infty} \bar{P}_k \int_0^{\infty} \frac{(\lambda t)^k}{k!} \exp(-\lambda t)\, dF(\lambda), \tag{9}$$

and

$$\bar{H}_2(t) = F^*\left(\int_0^t \{1-\exp(-\int_\tau^t \alpha(u)du)\}d\tau\right), \tag{10}$$

where $F^*(.)$ is the Laplace-Stieltjes transform (L.S.T.) of F(.). An alternative representation of (10), which will be useful, is given by

$$\bar{H}_2(t) = \sum_{k=0}^{\infty} [\phi(t)]^k \int_0^{\infty} \frac{(\lambda t)^k}{k!} \exp(-\lambda t)\, dF(\lambda), \tag{11}$$

where

$$\phi(t) = \frac{1}{t} \int_0^t \exp[-\int_\tau^t \alpha(u)du]d\tau . \tag{12}$$

Let Ω_1 denote the family of distributions of T as given by (9) for the threshold models, generated by varying $\underset{\sim}{\theta} = (\{\overline{P}_k\}, F(.))$. Similarly let Ω_2 be the corresponding family of distributions given by (10) - (11) for the nonthreshold models, generated by varying $\theta = (\alpha(.), F(.))$, where $\alpha(.)$ is subject to conditions (i) and (ii) given above. Likewise the symbols $\Omega_1(F)$, $\Omega_1(\{P_k\})$, $\Omega_2(F)$, $\Omega_2(\alpha(.))$, etc., stand for the corresponding subfamilies where in each case the parameter given in the parentheses is considered known while the other parameter is assumed to generate the subfamily in question.

REMARK 1. As is evident in (11), the expression (3) for nonthreshold models often is like (9) except that $\overline{P}_k$'s in general depend on t for these models. In our present case these take the form $[\phi(t)]^k$ in (11), which admits an alternative interpretation namely that the various shocks affect the system independently with $\phi(t)$ representing the probability that the system is spared from a single shock during $(0,t)$. This interpretation results essentially because of the linearity assumption of the risk function made in (4). Although this linearity assumption may appear questionable in some practical situations, yet we shall show later that the family Ω_2 based on this assumption is already much richer than the family Ω_1, in that it contains many more distributions besides containing the family Ω_1 (see section 5).

In the next two sections, we consider first the problem of nonidentifiability within each of the two families Ω_1 and Ω_2.

3. NONIDENTIFIABILITY WITHIN FAMILY Ω_1:

The following theorem deals with the nonidentifiability among the members of the threshold family of distributions.

THEOREM 1: Based on the r.v. T, the family of distributions Ω_1 is not identifiable. More specifically, for any arbitrary member of Ω_1 corresponding to $\underset{\sim}{\theta} = (\{\overline{P}_k\}, F(.))$ and for every $0 < a < 1$, there exists $\underset{\sim}{\theta}_a = (\{\overline{P}_k(a)\}, F_a(.))$ such that the expression (9) corresponding to $\underset{\sim}{\theta}$ and $\underset{\sim}{\theta}_a$ both coincide. On the other hand the corresponding enhanced family Ω_1^* of joint distributions of T and N_T, generated by varying $\underset{\sim}{\theta} = (\{\overline{P}_k\}, F(.))$ is indeed identifiable, where the r.v. N_T denotes the number of shocks arriving by the failure time T.

PROOF: Consider an arbitrary but fixed member of Ω_1 corresponding to $\underset{\sim}{\theta} = (\{\overline{P}_k\}, F(.))$. For $0 < a < 1$, let

$$\overline{P}_k(a) = \sum_{j=0}^{k} \binom{k}{j} (1-a)^{k-j} a^j \, \overline{P}_j, \; k = 0,1,2,\ldots \; .$$

Since $\overline{P}_0 = 1$, we have $\overline{P}_0(a) = 1$. Also using the fact that $\overline{P}_j$ is increasing with j, it can be shown that so is $\overline{P}_j(a)$. In fact it follows by noting that $\overline{P}_k(a) = E(\overline{P}_X)$, where X is a binomial r.v. with k number of trials and the probability of success for each trial equal to a and the fact that the binomial distribution has monotone likelihood ratio in the parameter k (see Lehmann [9]). Let for every $x \geq 0$, $F_a(x) = F(ax)$. Then using (9) it can be easily seen that the expression for $\overline{H}_1(t)$ for $\theta_a = (\{\overline{P}_k(a)\}, F_a(.))$ coincides with the one for $\underset{\sim}{\theta}$. This result for the special case when F is degenerate is known (see [4], [6]). For the second part, let $g(s) = E(s^K)$, $|s| \leq 1$, be the probability generating function (p.g.f.) of K, with $g(0) = 0$. Then it can be easily shown that

$$E(s^{N_T} \exp(-uT)) = \int_0^\infty g(\tfrac{\lambda s}{\lambda+u}) \, dF(\lambda), \tag{13}$$

where $Re(u) \geq 0$ and $|s| \leq 1$. Consequently the identity of the joint distributions of T and N_T for any two members of Ω_1^* corresponding to $\underset{\sim}{\theta}_1 = (\{\overline{P}_k\}, F_1(.))$ and $\underset{\sim}{\theta}_2 = (\{\overline{Q}_k\}, F_2(.))$ is equivalent to the identity

$$\int_0^\infty g_1\left(\frac{\lambda s}{\lambda+u}\right)dF_1(\lambda) \equiv \int_0^\infty g_2\left(\frac{\lambda s}{\lambda+u}\right)dF_2(\lambda), \tag{14}$$

for $\text{Re}(u) \geq 0$, $|s| \leq 1$, where g_1 and g_2 are the p.g.f's corresponding to $\{\bar{P}_k\}$ and $\{\bar{Q}_k\}$ respectively. Putting $u = 0$ in (14) immediately yields $g_1(s) \equiv g_2(s)$. Using this while equating the powers of s on both sides of (14), we must have for some $k \geq 1$,

$$\int_0^\infty \left(\frac{\lambda}{\lambda+u}\right)^k dF_1(\lambda) \equiv \int_0^\infty \left(\frac{\lambda}{\lambda+u}\right)^k dF_2(\lambda),\ \text{Re}(u) \geq 0, \tag{15}$$

which in turn is equivalent to

$$\int_0^\infty \lambda^k \exp(-\lambda t)dF_1(\lambda) \equiv \int_0^\infty \lambda^k \exp(-\lambda t)dF_2(\lambda),\ t \geq 0. \tag{16}$$

In particular for a fixed $t_0 > 0$, we have

$$\int_0^\infty \lambda^k \exp(-\lambda t_0)dF_1(\lambda) = \int_0^\infty \lambda^k \exp(-\lambda t_0)dF_2(\lambda). \tag{17}$$

Using (16) and (17) we have $\forall\ u \geq 0$,

$$\int_0^\infty \exp(-\lambda u)d\hat{F}_1(\lambda) \equiv \int_0^\infty \exp(-\lambda u)d\hat{F}_2(\lambda), \tag{18}$$

where for $i = 1,2$,

$$d\hat{F}_1(\lambda) = [\lambda^k \exp(-\lambda t_0)dF_i(\lambda)][\int_0^\infty \lambda^k \exp(-\lambda t_0)dF_i(\lambda)]^{-1}. \tag{19}$$

Extending (18) through analytical continuation for complex u's, with $\text{Re}(u) \geq 0$, establishes the identity of the Laplace-Stieltjes transforms of $\hat{F}_1$ and $\hat{F}_2$, leading to $\hat{F}_1 = \hat{F}_2$ and hence finally to $F_1 = F_2$. □

The next theorem shows that while the subfamily $\Omega_1(F)$ is identifiable, the subfamily $\Omega_1(\{\overline{P}_k\})$ is also identifiable under mild conditions.

THEOREM 2. (a) The subfamily $\Omega_1(F)$ is identifiable.
(b) The subfamily $\Omega_1(\{\overline{P}_k\})$ is identifiable whenever the p.g.f $g(.)$ for the sequence $\{p_k\}$ is such that $L(x)$ defined by the c.d.f.

$$L(x) = g([1+\exp(-x)]^{-1}), \quad -\infty < x < \infty , \tag{20}$$

has a nonvanishing characteristic function (c.f.).

PROOF. (a) The proof follows from the fact that $\Omega_1(F) \subset \Omega_2(F)$ (see theorem 5, section 5) and that the subfamily $\Omega_2(F)$ is identifiable (see theorem 4, section 4).
(b) Note that since $g(0) = 0$, $L(x)$ is a bonafide c.d.f. For two members of $\Omega_1(\{\overline{P}_k\})$, corresponding to c.d.f's F_1 and F_2, using (13) the identify of the corresponding distributions of T becomes equivalent to the identity

$$\int_0^\infty g(\tfrac{\lambda}{\lambda+u})dF_1(\lambda) \equiv \int_0^\infty g(\tfrac{\lambda}{\lambda+u})dF_2(\lambda), \quad \mathrm{Re}(u) \geq 0. \tag{21}$$

Working with real $u \geq 0$ and setting $u = \exp(-y)$, it can be easily shown after an appropriate transformation that (21) is equivalent to

$$\int_{-\infty}^\infty L(y-x)d\tilde{F}_1(x) \equiv \int_{-\infty}^\infty L(y-x)d\tilde{F}_2(x), \tag{22}$$

for $-\infty < y < \infty$, where $\tilde{F}_i$ is the c.d.f. of $[-\ell n\Lambda_i]$, with Λ_i having the c.d.f.

F_i, i = 1,2. Since L(.) has a nonvanishing c.f, taking Fourier transform of the convolutions on the two sides of (22) establishes the identity of the c.f's of $\tilde{F}_1$ and $\tilde{F}_2$ and hence of the c.d.f's F_1 and F_2. □

4. NONIDENTIFIABILITY WITHIN FAMILY Ω_2. In order to establish analogous results for the family Ω_2, we need to derive the expression similar to (13) for the present case of nonthreshold models. We briefly outline this below, first for the case when λ is a constant, so that the shock-arrival process is Poisson. Following the methods of ([12], [16]), it can be easily shown that

$$E(s^{N_T} \exp(-uT)|\lambda) = s\int_0^\infty \exp(-ut)\alpha(t)G_s(s;t)dt, \tag{23}$$

where

$$G(s;t) = E(s^{N(t)} I(T > t)) = \sum_{k=0}^\infty s^k P_{k,1}(t), \tag{24}$$

$$P_{k,1}(t) = P(N(t) = k, T > t), \tag{25}$$

I(A) denotes the indicator function of the event A, and G_s, etc. denote the corresponding partial derivatives of G. By establishing the usual differential equations satisfied by the probabilities $P_{k,1}(t)$'s, one can then show that G satisfies the differential equation

$$G_t + s\alpha(t)G_s = -\lambda(1-s)G. \tag{26}$$

Furthermore (26) when solved subject to G(s;0) = 1, leads to the solution

$$G(s;t) = \exp[-\lambda t+\lambda s\Psi(t)], \tag{27}$$

where

$$\Psi(t) = \int_0^t \exp(-\int_\tau^t \alpha(v)dv)d\tau. \tag{28}$$

Using this and (23) yield for each λ,

$$E(s^{N_T}\exp(-uT)|\lambda)$$

$$= \lambda s\int_0^\infty \exp(-ut)\alpha(t)\Psi(t)\exp[-\lambda t+\lambda s\Psi(t)]dt. \tag{29}$$

Finally taking expectation over Λ, we have

$$E(s^{N_T}\exp(-ut)) = -s\int_0^\infty \exp(-ut)\alpha(t)\Psi(t)F^{*'}(t-s\Psi(t))dt, \tag{30}$$

for $|s| \leq 1$, $Re(u) \geq 0$, where $F^{*'}$ is the derivative of L.S.T of F(.). We now proceed in establishing the analog of theorem 1 for the nonthreshold family of distributions.

THEOREM 3. Based on the r.v. T, the family of distributions Ω_2 is not identifiable. More specifically, for any arbitrary member of Ω_2 corresponding to $\underset{\sim}{\theta} = (\alpha(.),F(.))$ and for every $0 < a < 1$, there exists $\underset{\sim}{\theta}_a = (\alpha_a(.),F_a(.))$ such that the expression (10) corresponding to $\underset{\sim}{\theta}$ and $\underset{\sim}{\theta}_a$ both coincide. On the other hand the corresponding enhanced family Ω_2^* of joint distributions of T and N_T, generated by varying $\underset{\sim}{\theta} = (\bar{\alpha}(.),F(.))$ is identifiable provided each F(.) has a moment generating function (m.g.f.) and the function $\alpha(.)$ satisfies the additional condition (besides (i) and (ii) given in section 2)

$$\lim_{t\to 0} [\Psi(t)/t] = 1. \tag{31}$$

PROOF. Consider an arbitrary but fixed member of Ω_2 corresponding to $\underset{\sim}{\theta} = (\alpha(.), F(.))$ with $F(0) = 0$, and $\alpha(.)$ satisfying conditions (i) and (ii) of section 2. For $0 < a < 1$, let $F_a(x) = F(ax)$ and

$$\alpha_a(t) = a\alpha(t)\Psi(t)[(1-a)t + a\Psi(t)]^{-1}, \tag{32}$$

where $\Psi(t)$ is as defined in (28). Then it can be easily verified that $\alpha_a(.)$ too satisfies (i) and (ii). Furthermore the expression (10) for $\overline{H}_2(t)$ for $\underset{\sim}{\theta}_a = (\alpha_a(.), F_a(.))$ coincides with the one for $\underset{\sim}{\theta} = (\alpha(.),F(.))$. This establishes that the family Ω_2 is not identifiable. We now consider the enhanced family Ω_2^* subject to the conditions on $F(.)$ and $\alpha(.)$ given in the theorem above. For two such members of Ω_2^* corresponding to $\underset{\sim}{\theta}_1 = (\alpha_1(.),F_1(.))$ and $\underset{\sim}{\theta}_2 = (\alpha_2(.),F_2(.))$, the identity of the corresponding expressions for (30) for $\mathrm{Re}(u) \geq 0$ and $|s| \leq 1$, can be easily shown to be equivalent to the identity

$$\alpha_1(t)[\Psi_1(t)]^k E(\Lambda_1^k \exp(-\Lambda_1 t))$$
$$\equiv \alpha_2(t)[\Psi_2(t)]^k E(\Lambda_2^k \exp(-\Lambda_2 t)), \tag{33}$$

valid for $t > 0$ and $k \geq 1$, where for $i = 1,2$, the r.v. Λ_i has the c.d.f $F_i(.)$ and

$$\Psi_i(t) = \int_0^t \exp(-\int_\tau^t \alpha_i(v)dv)d\tau. \tag{34}$$

Since $\Psi_i(t) > 0$, for $t > 0$, $i = 1,2$, it follows from (33) that for every $t > 0$, either both $\alpha_i(t)$'s are positive or both are simultaneously zero. Thus without loss of generality we may assume that there exists a sequence $\{t_n\}$ converging to zero with $\alpha_i(t_n) > 0$, for $i = 1,2$, and $n \geq 1$. Replacing t in (33) by t_n and dividing both sides of (33) by t_n^k and letting $n \to \infty$, using (31) it is easily seen that

$$\lim_{n\to\infty} \frac{\alpha_1(t_n)}{\alpha_2(t_n)} = c \tag{35}$$

exists, is positive and that

$$E(\Lambda_2^k) = cE(\Lambda_1^k),\ k = 1,2, \dots \ . \tag{36}$$

On the other hand using the fact that $F_i(0) = 0$, $i = 1,2$, and Λ_i's have m.g.f's, from (36) it easily follows that $F_1 = F_2$. Finally using this in (33), for t's with $\alpha_i(t) > 0$, we have

$$[\alpha_1(t)/\alpha_2(t)] = [\Psi_2(t)/\Psi_1(t)]^k \ . \tag{37}$$

This being valid $\forall k \geq 1$, implies that $\Psi_1(t) = \Psi_2(t)$ so that $\alpha_1(t) = \alpha_2(t)$. □

REMARK 2. It is easily seen from (33) that if F(.) were known the subfamily $\Omega_2^*(F)$ is identifiable without any additional conditions on $\alpha(.)$ or $F(.)$, as in that case (33) reduces to

$$\alpha_1(t)[\Psi_1(t)]^k \equiv \alpha_2(t)[\Psi_2(t)]^k \ , \tag{38}$$

valid for $t > 0$ and $k \geq 1$, from which one can easily argue for the identity of α_i's. If on the other hand $\alpha(.)$ were known, the subfamily $\Omega_2^*(\alpha)$ is again identifiable without any additional condition. In that case from (33) it follows that

$$E(\Lambda_1^k \exp(-\Lambda_1 t)) \equiv E(\Lambda_2^k \exp(-\Lambda_2 t)) \ , \tag{39}$$

for all $k \geq 1$ and $t \geq 0$. Multiplying now both sides of (39) by $(t^k/k!)$, adding over $k \geq 1$ and subtracting the sum from one yield the identity

$$E(\exp(-\Lambda_1 t)) \equiv E(\exp(-\Lambda_2 t)),\ t \geq 0, \tag{40}$$

from which it easily follows that $F_1 = F_2$.

The following theorem is analog of theorem 2 for the subfamilies $\Omega_2(\alpha)$ and $\Omega_2(F)$.

THEOREM 4. *The subfamilies* $\Omega_2(\alpha)$ *and* $\Omega_2(F)$ *are both identifiable*.

PROOF. For a given $\alpha(.)$ satisfying the conditions (i) and (ii) of section 2, the identity of two members of $\Omega_2(\alpha)$ corresponding to c.d.f.'s F_1 and F_2, is equivalent to (in view of (10))

$$F_1^*(t-\Psi(t)) \equiv F_2^*(t-\Psi(t)),\ t \geq 0. \tag{41}$$

Since $t-\Psi(t)$ as a function of t, varies continuously from 0 to ∞ as t varies from 0 to ∞, (41) implies the identify of the transforms $F_1^*(u)$ and $F_2^*(u)$ for $u \geq 0$, and hence of F_1 and F_2. Again for a given F with $F(0) = 0$, the identity of two members of $\Omega_2(F)$ corresponding to $\alpha_1(.)$ and $\Omega_2(.)$ is equivalent to

$$F^*(t-\Psi_1(t)) \equiv F^*(t-\Psi_2(t)), \tag{42}$$

for $t \geq 0$, where Ψ_i's are as defined in (34). Since $F^*(u)$ for $u \geq 0$ is strictly decreasing, (42) immediately implies $\Psi_1(t) \equiv \Psi_2(t)$ so that $\alpha_1(t) \equiv \alpha_2(t)$. □

5. MUTUAL IDENTIFIABILITY OF THE TWO FAMILIES. In order to distinguish between the two models in live situations it is of considerable interest to compare the

two families of distributions Ω_1 and Ω_2 from the point of view of their mutual identifiability. Since the underlying shock-arrival process is same for the two models and they differ only in their assumptions concerning the causation of the failure (threshold versus nonthreshold) of the system, such a comparison appears much more natural between the corresponding subfamilies $\Omega_1(F)$ and $\Omega_2(F)$, where the common F is the same although left unspecified. The following theorem is concerned with such a comparison.

THEOREM 5. (a) For every F with $F(0) = 0$, $\Omega_1(F) \subset \Omega_2(F)$, so that based on the r.v. T, the family $\Omega_1(F)$ is nonidentifiable with respect to the family $\Omega_2(F)$.

(b) Let the d.f. F with $F(0) = 0$ and $\int_0^\infty \lambda dF(\lambda) < \infty$ satisfy the condition

$$\limsup_{t \to \infty} [I_1(t)/J_\beta(t)] > 1, \tag{43}$$

for all $\beta > 0$, where

$$I_1(t) = \int_0^\infty v \exp(-v)F(v/t)dv, \tag{44}$$

$$J_\beta(t) = \int_0^\infty \exp[-v\, \rho_\beta(t)]F(v/t)dv, \tag{45}$$

and

$$\rho_\beta(t) = 1 - [1-\exp(-\beta t)](\beta t)^{-1}. \tag{46}$$

Then $\Omega_2(F)$ is strictly larger than the family $\Omega_1(F)$. In particular subject to (43), for each member of $\Omega_2(F)$ with a constant $\alpha(.) \equiv \alpha > 0$, there exists no corresponding member of $\Omega_1(F)$ with a matching distribution for T.

PROOF. (a) The proof follows from the fact that for each member of $\Omega_1(F)$ with a $\{\bar{P}_k\}$ and the corresponding function

$$\bar{H}_1(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{t^k}{k!} \int_0^{\infty} \lambda^k \exp(-\lambda t)dF(\lambda), \tag{47}$$

there exists a unique member of $\Omega_2(F)$ with a corresponding function

$$\bar{H}_2(t) = F^*(\int_0^t \{1-\exp(-\int_\tau^t \alpha(u)du)\}d\tau) \tag{48}$$

coinciding with $\bar{H}_1(t)$ of (47). The corresponding function $\alpha(.)$ which makes it possible, is given by

$$\alpha(t) = [dF^{*-1}(H_1(t))/dt][t-F^{*-1}(H_1(t))]^{-1} , \tag{49}$$

where $F^{*-1}(.)$ is the inverse function of L.S.T. F^*. Using the properties of the function $\bar{H}_1(.)$, it can be easily verified that the function $\alpha(.)$ given by (49) does indeed satisfy the desired conditions (i) and (ii) of section 2. Conversely for a given member of $\Omega_2(F)$ with a corresponding function $\alpha(.)$, it is not always possible to find a meaningful solution for $\{\bar{P}_k\}$ by equating (47) and (48).

(b) The proof is by contradiction. Let F satisfy the condition (43) for all $\beta > 0$. Consider a member of $\Omega_2(F)$ with a constant $\alpha(.) \equiv \alpha > 0$. Suppose that a corresponding member of $\Omega_1(F)$ with a sequence $\bar{P}_k$ exists such that

$$P_2(T > t|\alpha,F) \equiv P_1(T > t|\{\bar{P}_k\},F), \tag{50}$$

valid for all $t > 0$. Rewriting (9) and (10) differently, (50) can be equivalently written down as

$$\sum_{i=0}^{\infty} p_{i+1} I_i(t) \equiv \rho_\alpha(t) \cdot J_\alpha(t), \tag{51}$$

where for $i \geq 0$,

$$I_i(t) = \int_0^\infty \frac{v^i}{i!} \exp(-v) F(v/t) dv. \tag{52}$$

Using (51) and the fact that $E(\Lambda) < \infty$, we shall first show that in the present case p_1 must be zero. Differentiating both sides of (51) with respect to t (under the summation and the integral signs which is permissible here), one is led, after some simplification, to the identity

$$(1-\exp(-\alpha t))E[\Lambda t \exp\{-\Lambda t \rho_\alpha(t)\}]$$
$$\equiv \sum_{k=1}^{\infty} k\, p_k\, E[\frac{(\Lambda t)^k}{k!} \exp(-\Lambda t)], \tag{53}$$

where the r.v. Λ has the d.f. F, and $\rho_\alpha(t)$ is as defined in (46). Dividing both sides of (53) by t, and letting $t \to 0$, one obtains $p_1 = 0$, while using the fact that $E(\Lambda) < \infty$. Thus we may rewrite (51) as

$$\rho_\alpha(t) = \sum_{i=1}^{\infty} p_{i+1} I_i(t)[J_\alpha(t)]^{-1} . \tag{54}$$

Note that since

$$I_i(t) = E\{F(\chi^2_{2i+2}/2t)\},$$

where χ^2_ν is a chi-square r.v. with ν degrees of freedom, it follows that the integral $I_i(t)$ is strictly increasing with i. Again in view of (43) there exists a sequence $\{t_n\}$ with $t_n \to \infty$, such that

$$\lim_{n\to\infty} \{I_1(t_n)[J_{\bar{\alpha}}(t_n)]^{-1}\} > 1. \tag{55}$$

For the same sequence, using (55) and the increasing property of the integral $I_i(t_n)$, it follows from (54) that

$$\rho_\alpha(t_n) \geq I_1(t_n)[J_\alpha(t_n)]^{-1} . \tag{56}$$

Finally letting $n \to \infty$, since $\lim_{n\to\infty} \rho_\alpha(t_n) = 1$, in view of (55) we arrive at a contradiction. □

REMARK 3: The condition (43) essentially reflects the behaviour of F near the origin in terms of the rate with which F(t) tends to zero as $t \to 0$. Indeed it is satisfied by most of the standard distributions, such as Gamma distribution. In particular, when $F(x) = 1 - \exp(-x)$, $x \geq 0$, the limit in (43) is two, whereas for a degenerate F(.) at an arbitrary point $\lambda > 0$, the limit is ∞.

In order to find a way out of the mutual nonidentifiability problem between the families $\Omega_1(F)$ and $\Omega_2(F)$ as exhibited by the preceeding theorem, we now consider the corresponding enhanced families $\Omega_1^*(F)$ and $\Omega_2^*(F)$ of distributions of r.v.'s (T, N_T). The following theorem shows that the mutual nonidentifiability problem is substantially reduced by the additional information through the r.v. N_T.

THEOREM 6. The families of distributions $\Omega_1^*(F)$ and $\Omega_2^*(F)$ are partially mutually nonidentifiable. More specifically, no matter what the common F is, a member of $\Omega_1^*(F)$ corresponding to a sequence $\{\bar{P}_k\}$ yields the same distribution for (T,N_T)

as does a member of the family $\Omega_2^*(F)$ corresponding to a function $\alpha(.)$ if and only if for some $0 < p_1 < 1$,

$$P_k = p_1(1-p_1)^{k-1}, \; k = 1,2, \ldots , \tag{57}$$

and

$$\alpha(t) = \left(\frac{p_1}{1-p_1}\right)/t, \; t > 0. \tag{58}$$

PROOF. For a member of $\Omega_1^*(F)$, it follows from (13) that for $k \geq 1$, $Re(u) \geq 0$,

$$E(I(N(t)=k)\exp(-uT)) = p_k\int_0^\infty \frac{t^{k-1}}{(k-1)!} \exp(-ut)E(\Lambda^k \exp(-\Lambda t))dt. \tag{59}$$

Similarly for a member of $\Omega_2^*(F)$, it follows from (30) that for $k \geq 1$, $Re(u) \geq 0$,

$$E(I(N(t) = k)\exp(-uT)) = \frac{1}{(k-1)!}\int_0^\infty \exp(-ut)[\Psi(t)]^k \alpha(t)E(\Lambda^k \exp(-\Lambda t))dt, \tag{60}$$

where $\Psi(t)$ is as given by (28). The identity of (59) and (60) for all $k \geq 1$, and $Re(u) \geq 0$, immediately leads to the relations

$$P_k \equiv \alpha(t)[\psi(t)]^k \, t^{-(k-1)}, \; k \geq 1, \; t > 0, \tag{61}$$

from which (57)and (58) easily follow. □

6. A FEW CONCLUDING REMARKS (a) It is of interest to note that the question of distinguishability between the threshold type and the nonthreshold type models comes up often in several modeling situations in biology, medicine and public

health as well. In the experience of the author the nonthreshold type models appear to be much more reasonable in certain live situations than the threshold type models. (see Puri [12], [13], [15]).

(b) The generalized models dealt with here involving mixed Poisson processes for the arrivals of shocks may be of some independent interest in the context of reliability theory, keeping in mind that the parameter λ may in fact vary over the population of components (or systems). In fact in their work on accident proneness Bates and Neyman ([2],[3]) also allowed λ to be random for similar reasons. These processes could of course be further generalized to mixed nonhomogeneous Poisson processes through a time scale change, allowing thereby the changes in the risk over time. Also since these processes are known to have an order statistic property, it makes them easily mathematically tractable. (see Feigen [7], Puri [8]).

(c) The considerations here of the r.v.'s such as N_T(besides T) makes it amply clear about how they could be used in reducing the problems of nonidentifiability. In many situations however, it is possible that the r.v. N_T considered here may not be even observable. In such cases one needs to look for some other observable variables which could be used instead.

(d) In theorem 2(b) the identifiability of $\Omega_1(\{\overline{P}_k\})$ was proved under the assumption that the c.d.f L(x) has a nonvanishing c.f. One could instead prove the same result under the assumption that the threshold K has all the moments and that the r.v. Λ^{-1} has m.g.f. This can be carried out by using the moment relation

$$E(T^n) = E\{\frac{(K+n-1)!}{(K-1)!}\} E(\Lambda^{-n}),\ n \geq 1\ , \tag{62}$$

which is established using (13). It is conjectured however that the family $\Omega_1(\{\overline{P}_k\})$ is identifiable without any conditions whatsoever. Similarly one could prove the identifiability of the family Ω_2^* in theorem 3, under somewhat different conditions than (31), to cover those cases where (31) does not hold. One such case in mind is where $\alpha(t) = (c/t)$, $t \geq 0$, $c > 0$, for which

$$[\Psi(t)/t] \equiv (c+1)^{-1} . \quad (63)$$

REFERENCES

[1] Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing, Holt, Rinehart and Winston, Inc. N.Y.

[2] Bates, G. E. and Neyman, J. (1952a). Contributions to the thoery of accident proneness. I. An optimistic model of the correlation between light and severe accidents, Univ. Calif. Pub. Statist., 1, 215-254.

[3] Bates, G. E. and Neyman, J. (1952b). Contributions to the thoery of accident proneness II. True or false contagion, Univ. Calif. Pub. Stat., 1, 255-276.

[4] Clifford, P. (1972). Nonthreshold models of the survival of bacteria after irradiation, Proc. 6th Berkeley Symp. on Math. Statist. and Prob., Univ. Calif. Press, 265-286.

[5] Clifford, P. (1977). Nonidentifiability in stochastic models of illness and death, Proc. Natl. Acad. Sci. USA, 74, 1338-1340.

[6] Esary, J. D., Marshall, A. W. and Proschan, F. (1973). Shock models and wear processes, Ann. Prob., 1, 627-649.

[7] Feigin, P. D. (1979). On characterization of point processes with the order statistic property, J. Appl. Prob. 16, 297-304.

[8] LeCam, L. (1982). Personal communication about an unpublished report of Professor LeCam written in fifties.

[9] Lehmann, E. (1959). Testing Statistical Hypothesis, J. Wiley, N.Y.

[10] Mercer, A. (1961). Some simple wear-dependent renewal processes, J. Roy Statist. Soc., Ser. B, 23, 368-376.

[11] Proschan, F. and Serfling, R. J. (1974). Reliability and Biometry, Statistical Analysis of Life Length, SIAM, Philadelphia.

[12] Puri, P. S. (1967). A class of stochastic models of response after infection in the absence of defense mechanism, Proc. 5th Berkeley Symp. on Math. Statist. and Prob. Univ. Calif. Press, 4, 511-535.

[13] Puri, P. S. (1969). Some new results in the mathematical theory of phage-reproduction, J. Appl. Prob., 6, 493-504.

[14] Puri, P. S. (1972). A method of studying the integral functionals of stochastic processes with applications III. Proc. 6th Berkeley Symp. on Math. Statist. and Prob. Univ. Calif. Press, 3, 481-500.

[15] Puri, P. S. and Senturia, J. (1972). On the mathematical theory of quantal response assays, Proc. 6th Berkeley Symp. on Math. Statist. and Prob., Univ. Calif. Press, 4. 231-247.

[16] Puri, P. S. (1973). On the distribution of the state of a process at the moment of a quantal response, J. Math. Biosciences, 18, 301-309.

[17] Puri, P. S. (1979). On certain problems involving nonidentifiability of distributions arising in stochastic modeling, Optimizing Methods in Statistics (Ed: J. S. Rustagi) Academic Press, Inc.

[18] Puri, P. S. (1982). On the characterization of point processes with the order statistic property without the moment condition, J. Appl. Prob., 19, 39-51.

[19] Puri, P. S. and Yackel, J. W. (1983). On a problem of nonidentifiability arising in simple stochastic models for stereological counts, Technical Report #83-25, Dept. of Statistics, Purdue Univ.