OPTIMUM REPLACEMENT OF A SYSTEM SUBJECT TO SHOCKS AND A MATHEMATICAL LEMMA*

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ABSTRACT Let a system be subject to shocks which occur randomly over time. Let $N(t)$ denote the number of shocks occurring in $[0,t]$. It is assumed that the system needs replacement after it has functioned for a random length of time τ, a moment of a major failure, which is a stopping time with respect to the process $\{N(t), t \geq 0\}$. The costs we consider are due to shocks, maintenance and replacement. Again it may be economical to replace the system at time $\min(t,\tau)$ prior to its failure, for some fixed but optimally chosen t. Based on cost considerations we introduce an optimality criterion which we then use to find an optimal time t_0. In the process, we prove a general mathematical lemma which helps in arriving at the optimal time, for general classes of processes $\{N(t)\}$, stopping times τ and the cost structures involved. A discrete time version of the lemma is also given.

The present work was inspired by two recent pieces of work involving 'shock models' by Boland and Proschan [1982, 1983]. However for previous other similar work, the reader may also refer to Taylor [1975], A-Hameed [1977], A-Hameed and Shimi [1978], Zuckerman [1978, 1980], Feldman [1976, 1977a, 1977b], Boland [1982], Tilquin and Cleroux [1975] and Nakagawa [1981].

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A typical optimization problem that appears in most of these papers involves minimization of a ratio of two known functions (see (1.5)) with respect to their common argument. In section 2, we prove a basic mathematical lemma (lemma 2.1) which solves this problem subject to certain conditions on the two functions. Our lemma is fairly general and covers many cases solved in the literature. In particular it applies to the models considered by the above authors. For convenience of presentation we shall consider, in some detail, a generalized shocks-based model involving a stopping time τ, as described below.

Suppose a new system is subject to shocks which occur randomly over time. Let N(t) denote the number of shocks occurring in [0,t]. The random process $\{N(t),\ t \geq 0\}$ is assumed to be a separable point process with $N(0) = 0$, $N(t) < \infty$, for all $t \geq 0$ a.s. and right continuous sample paths with unit steps at $0 < \tau_1 \leq \tau_2 \leq \cdots$, with $\tau_0 = 0$. As a rule, it is assumed that the system needs replacement after it has functioned for a random length of time τ, a stopping time with respect to the process $\{N(t),\ t \geq 0\}$. In many situations, shocks may be interpreted as small, reparable breakdowns of the system and τ is the time when it undergoes a major irreparable breakdown needing a replacement. When the costs that are involved are due to shocks, maintenance and replacement etc., it may be economical to replace the system at time $\min(\tau, t)$ for some fixed but optimally chosen t. One of the objectives of the present work is to define an appropriate optimality criterion which is then used to arrive at an optimal time, t_0.

Generally the stopping time τ is beyond our control. For instance, when the arrivals of shocks are interpreted as occurrences of small reparable breakdowns of the system, the risk of having a major breakdown needing replacement may depend upon the number N(t) of the reparable damages. For example, when the system cannot stand more than k small breakdowns after which it then has to be replaced, $\tau = \tau_k$. Another example is a system consisting of k components running in parallel. The system is considered to have failed only when all the k components stop functioning. If we interpret jth shock as the failure of the jth component, $j = 1,2,\dots, k$, then $\tau = \tau_k$. Other examples are situations where each shock may cause a random damage to the system and τ is the time the cumulative damage caused by the shocks exceeds a fixed (or a random) threshold.

In the above examples τ is dependent upon the process $N(t)$. However, in some situations τ could be considered independent of the process $N(t)$. For instance, consider a system which is functioning with the help of electric power. Shocks may be interpreted as minor reparable breakdowns due to wear and tear. The system may get totally destroyed after it receives a rather high power surge. Here τ, the time of such an eventuality may be approximately independent of the process $N(t)$. Finally, in the event no such risk that we have represented by τ exists, we take $\tau = \infty$.

1. PRELIMINARIES

We begin by defining the various costs involved as follows:

C_i = Cost incurred due to occurrence of ith shock (a minor breakdown which is assumed to be reparable instantaneously), $i = 1,2,\dots$.

$f(i,t)$ = Cost per unit time of maintenance of the system at time $t \in [\tau_i, \tau_{i+1})$, $i = 1,2,\dots$.

R_1 = Cost of replacement of the system when it is replaced at time $t = \min(\tau,t)$ (i.e. when it is replaced before a major failure).

R_2 = Cost of replacement of the system at time τ with $\tau = \min(\tau,t)$ (i.e. when it is replaced at a major failure).

$C(t)$ = Total cost incurred during $[0,t]$.

Here it is assumed that for every fixed t, $P(\tau = t) = 0$. Furthermore the costs $\{C_i, i = 1,2,\ldots\}$ may be random and also for each $i \geq 0$, $\{f(i,t), t \geq 0\}$ may be assumed to be a stochastic process.

In this paper we deal with the problem of obtaining the optimal value of t which minimizes

$$\overline{C}(t) = \frac{E(C(t \wedge \tau))}{E(t \wedge \tau)} \quad , \tag{1}$$

where E denotes the expectation sign and $t \wedge \tau = \min(t,\tau)$. Under the stationary conditions of various costs etc., the measure $\overline{C}(t)$ could be interpreted as the long run average cost per replacement per unit time.

The model considered here is fairly general in the sense that the process N(t), stopping time τ, costs C_i, f(i,t) etc. are all left open with the hope that it will serve as an approximation to many practical situations. Several special cases are considered here as illustrations of the general theory.

Some special cases of the above model are considered more recently by Boland and Proschan [1982, 1983]. In their second paper they assume $C_j = 0$ for all j, $f(j,t) = \gamma + \delta j$, where γ and δ are fixed constants, $\tau = \infty$ and $\{N(t), t \geq 0\}$ is a Poisson process. Instead in their first paper they take $f \equiv 0$, $\tau = \infty$, $C_j = \alpha + \beta j$, where α and β are fixed constants, while the process $\{N(t), t \geq 0\}$ is as before assumed to be a Poisson process. These two cases follow immediately from our general model (see section 3.1). Moreover an alternative expression for $\overline{C}(t)$ is given here which becomes much simpler for the two cases of Boland and Proschan eliminating much of the algebra.

1.1 EXPRESSION FOR $\overline{C}(t)$.

For convenience let $C_0 = 0$. Also let I_A denote the indicator function of the event A,

$$N(t-) = \lim_{u \uparrow t} N(u) \tag{2}$$

and

$$N^*(t \wedge \tau) = \begin{cases} N(t) , & \tau > t \\ N(\tau -) & \tau \leq t . \end{cases} \tag{3}$$

Then it is easily seen that

$$C(t \wedge \tau) = R_1 I_{\{\tau > t\}} + R_2 \, I_{\{\tau \leq t\}} + \sum_{i=0}^{N^*(t \wedge \tau)} C_i + \int_0^t f(N(u), u) I_{\{\tau > u\}} du. \tag{4}$$

Here the first two terms in the above expression represent costs due to replacement at time min (τ, t) according as $t < \tau$ or $t \geq \tau$; the third term is the total cost due to shocks (or due to repairs of minor failures) and the last term is the total maintenace cost incurred during the time $[0, t \wedge \tau]$. Thus an expression for $\overline{C}(t)$ of (1) is given by

$$\overline{C}(t) = \frac{R_1 + \psi(t)}{\phi(t)} \tag{5}$$

where

$$\psi(t) = (R_2 - R_1) P(\tau \leq t) + E\left(\sum_{i=0}^{N^*(t \wedge \tau)} C_i\right) + \int_0^t E(f(N(u), u) I_{\{\tau > u\}}) du, \tag{6}$$

and

$$\phi(t) = E(t\wedge\tau) = \int_0^t P(\tau > u)du. \tag{7}$$

Our objective is to determine t_0 which minimizes $\overline{C}(t)$. For this purpose a simple mathematical lemma given in the next section provides sufficient conditions in terms of $\psi(t)$, $\phi(t)$ and R_1 for the existence of a finite t_0. A method for finding such a t_0 is also given in the lemma.

Again for the special case considered by Boland and Proschan [1983], where $\tau = \infty$, $C_i = 0$ for $i = 1,2,\ldots$, $f(i,t) = \gamma + i\delta$ for $i = 0,1,2,\ldots$, $t \geq 0$, with γ and δ being some constants, and the process $\{N(t),\ t \geq 0\}$ is Poisson with mean function $E(N(t)) = \Omega(t) = \int_0^t \lambda(u)du$, the expression (5) reduces to

$$\overline{C}(t) = \gamma + (R_1 + \delta\int_0^t \Omega(u)du)t^{-1}\ , \tag{8}$$

which is same as obtained by Boland and Proschan [1983] after their Theorem 1.2. Similarly for the special case considered by Boland and Proschan [1982], where $\tau = \infty$, $f \equiv 0$, $C_i = \alpha + i\beta$, for $i = 1,2,\ldots$, and $N(t)$ is a Poisson process with mean function $\Omega(t)$, the expression (5) reduces, after some simplification, to

$$\overline{C}(t) = [R_1 + (\alpha + \beta)\Omega(t) + \frac{\beta}{2}\Omega^2(t)]/t\ . \tag{9}$$

Finally Lemma 2.1 of the next section can be used for obtaining sufficient conditions on $\Omega(t)$, α, β, γ, δ and R_1 in order that an optimal t_0 exists in each of the above two cases. The same lemma can also be used for obtaining this t_0.

2. A USEFUL LEMMA

In this section we present a simple mathematical lemma, which can be helpful in minimizing (or maximizing) a ratio of two functions such as (10) with respect to their common argument. Our lemma can be useful for optimization problems of this nature arising in several areas of operation research. The lemma itself has two cases, the continuous time case (Lemma 2.1) and its discrete time analog (Lemma 2.2) given below.

LEMMA 2.1. Let

$$C(t) = \frac{R+\psi(t)}{\phi(t)}, \quad t \geq 0, \tag{10}$$

where $R > 0$, $\phi(0) = \psi(0) = 0$, $\phi(t)$ and $\psi(t)$ are continuously differentiable functions on $[0,\infty)$ with strictly positive derivatives $\phi'(t)$ and $\psi'(t)$ for all $t \geq 0$. Also let $D(t)$ be nondecreasing (nonincreasing) over $[0,\infty)$, where

$$D(t) = \psi'(t)/\phi'(t). \tag{11}$$

Then we have

(i) $D(t)\phi(t) - \psi(t)$ is nondecreasing (nonincreasing) over $[0,\infty)$.

(ii) In the case $D(t)$ is nondecreasing and

$$\lim_{t\to\infty} (D(t)\phi(t) - \psi(t)) > R, \tag{12}$$

there exists at least one (finite) point t_0 of global minimum of $C(t)$. Such points are the only solutions of the equation.

$$D(t)\phi(t) - \psi(t) = R. \tag{13}$$

The minimum value of $C(t)$ subject to (12) is given by

$$C(t_0) = D(t_0) \ , \tag{14}$$

with t_0 being any solution of (13). The global point of minima (and hence the solution of the equation (13)) is unique if $D(t)$ is strictly increasing.

(iii) In case $D(t)$ is nondecreasing and

$$\lim_{t\to\infty} (D(t)\phi(t) - \psi(t)) = R \ , \tag{15}$$

$C(t)$ is a nondecreasing function of t so that a minimum of $C(t)$ occurs at $t = \infty$. However $C(t)$ may assume the minimum value at all points $t \in [u_0, \infty)$ for some finite u_0.

(iv) For the remaining cases, such as if the limit in (12) is less than R or if $D(t)$ is nonincreasing over $[0,\infty)$, $C(t)$ is a nonincreasing function of t and the minimum of $C(t)$ occurs only at $t = \infty$.

PROOF. For $0 \le t_1 < t_2$, we have

$$\begin{aligned}
&[D(t_2)\phi(t_2) - \psi(t_2)] - [D(t_1)\phi(t_1) - \psi(t_1)] \\
&\quad = [D(t_2) - D(t_1)]\phi(t_1) + [D(t_2)\phi(t_2) - \psi(t_2)] - [D(t_2)\phi(t_1) - \psi(t_1)] \\
&\quad = [D(t_2) - D(t_1)]\phi(t_1) + \int_{t_1}^{t_2} [D(t_2) - D(s)]\phi'(s)ds \ ,
\end{aligned} \tag{16}$$

which is nonnegative or nonpositive according as $D(t)$ is nondecreasing or nonincreasing, thereby proving (i). Again $dC(t)/dt = 0$ yields

$$D(t)\phi(t) - \psi(t) = R. \tag{17}$$

Also if $D(t)$ is nondecreasing so will be $D(t)\phi(t) - \psi(t)$, as shown above. Thus if (12) holds, it follows that equation (13) must have at least one finite solution, say t_0. In the event $D(t)$ is strictly increasing so will be $D(t)\phi(t) - \psi(t)$ and the solution t_0 will be unique in that case. We now show that t_0 is a point of global minimum of $C(t)$. For $0 < t < t_0$, we have

$$C(t) - C(t_0) = \frac{[R(\phi(t_0) - \phi(t)) - \psi(t_0)\phi(t) + \psi(t)\phi(t_0)]}{\phi(t)\phi(t_0)} \tag{18}$$

Substituting $R = D(t_0)\phi(t_0) - \psi(t_0)$ in (18) we have

$$\begin{aligned} C(t) - C(t_0) &= \frac{D(t_0)(\phi(t_0) - \phi(t)) - (\psi(t_0) - \psi(t))}{\phi(t)} \\ &= [\phi(t)]^{-1} \int_t^{t_0} (D(t_0) - D(s))\phi'(s)ds\ , \end{aligned} \tag{19}$$

which is nonnegative since $D(t)$ is nondecreasing. Similarly for $t > t_0$, we have

$$C(t) - C(t_0) = [\phi(t)]^{-1} \int_{t_0}^{t} (D(s) - D(t_0))\phi'(s)\, ds\ , \tag{20}$$

which again, being nonnegative, establishes that t_0 is a point of global minimum of $C(t)$.

The proofs of (iii) and (iv) are similar and hence omitted. □

In the following lemma we give the discrete time version of Lemma 2.1.

<u>LEMMA</u> 2.2. <u>Let for</u> $i = 0,1,2,\ldots$

$$C(i) = \frac{R+\psi(i)}{\phi(i)}\ ; \quad D(i) = \frac{\psi(i+1) - \psi(i)}{\phi(i+1) - \phi(i)}\ , \tag{21}$$

where $\phi(0) = \psi(0) = 0$, $\phi(i+1) > \phi(i)$, $\psi(i+1) > \psi(i)$, and $R > 0$. Let $D(i)$ be a nondecreasing (nonincreasing) function of i. Then we have

(i) $D(i)\phi(i) - \psi(i)$ is a nondecreasing (nonincreasing) function of i.

(ii) In the case $D(i)$ is a nondecreasing and

$$\lim_{i\to\infty} (D(i)\phi(i) - \psi(i)) > R \ , \tag{22}$$

the quantity m_0 defined by

$$\max\{i: D(i)\phi(i) - \psi(i) \leq R\} = m_0 \tag{23}$$

is finite and a minimum of $C(i)$ occurs at m_0 or m_0+1. In the event $D(i)$ is strictly increasing, then

(a) minimum of $C(i)$ is attained only at $m_0 + 1$ if

$$D(m_0)\phi(m_0) - \psi(m_0) < R. \tag{24}$$

(b) minimum of $C(i)$ is attained at both the points m_0 and $m_0 + 1$, if

$$D(m_0)\phi(m_0) - \psi(m_0) = R. \tag{25}$$

(iii) Again if $D(i)$ is nondecreasing and

$$\lim_{i\to\infty} (D(i)\phi(i) - \psi(i)) = R, \tag{26}$$

then $C(i)$ is nonincreasing and a minimum of $C(i)$ occurs at $i = \infty$. However

<u>minimum of</u> $C(i)$ <u>may also occur at all</u> $i \geq i_0$ <u>for some positive integer</u> i_0.

(iv) <u>In the remaining cases</u> $C(i)$ <u>is a nonincreasing function of</u> i <u>and a minimum occurs only at</u> $i = \infty$.

<u>PROOF</u>. The proof follows along lines similar to those of Lemma 2.1 and the fact that

$$[D(i+1)\phi(i+1) - \psi(i+1)] - [D(i)\phi(i) - \psi(i)] \\ = [D(i+1) - D(i)]\phi(i+1) \ . \tag{27}$$

<u>REMARK</u>. For situations where instead the point of maximum of $\phi(t)/[R + \psi(t)]$ is required, the above lemmas can still be used since maximizing $\phi(t)/[R + \psi(t)]$ is equivalent to minimizing $[R + \psi(t)]/\phi(t)$. By absorbing the constant R in the function ψ, this becomes essentially equivalent to interchanging the roles of the functions ψ and ϕ in the lemmas.

3. SOME SPECIAL CASES

For the general model considered in section 1, it has been seen in (5) that $\bar{C}(t)$ coincides with (10) with $\psi(t)$ and $\phi(t)$ given by (6) and (7). Assuming $R_1 \leq R_2$ which is generally true, we observe that $\psi(t)$ and $\phi(t)$ are nonnegative functions with $\psi(0) = \phi(0) = 0$. If τ, C_i, f, R_1, R_2 and the process N(t) are such that the functions $\psi(t)$ and $\phi(t)$ also satisfy the other conditions of Lemma 2.1, then it can be seen by applying the lemma whether a finite optimum t_0 exists or $t_0 = \infty$. In case a finite t_0 exists, then it can be determined by solving the equation (13).

In the following subsections we discuss a few particular cases as illustrations of the above procedure.

3.1. CASE WITH $\tau = \infty$.

From (5) the expression for $\overline{C}(t)$ with $\tau = \infty$ reduces to

$$\overline{C}(t) = \frac{1}{t}\,[R_1 + E(\sum_{i=0}^{N(t)} C_i) + \int_0^t E(f(N(u),u))du]. \tag{28}$$

Furthermore if $C_i = 0$ for all i, we have

$$\overline{C}(t) = \frac{1}{t}\,[R_1 + \int_0^t E(f(N(u),u))du]. \tag{29}$$

In case $E(f(N(t),t))$ is a continuous and strictly increasing function of t and

$$\lim_{t\to\infty} [tE(f(N(t),t)) - \int_0^t E(f(N(u),u))du] > R_1 , \tag{30}$$

it follows from Lemma 2.1 that a unique finite t_0 exists which is the solution of the equation.

$$tE(f(N(t),t)) - \int_0^t E(f(N(u),u))du = R_1 . \tag{31}$$

Again in (28) if

$$\begin{cases} C_i = \alpha + i\beta, \; i = 1,2, \ldots \\ f(i,t) = \gamma(t) + i\delta(t), \; i = 0,1,2, \ldots , \end{cases} \tag{32}$$

where α and β may be nonnegative random variables; likewise if $\{\gamma(t)\}$ and $\{\delta(t)\}$ may be nonnegative random processes for $t \geq 0$, all together with the process $\{N(t)\}$ assumed to be mutually independent, then (28) reduces to

$$\overline{C}(t) = \frac{1}{t}[R_1 + (E(\alpha) + \frac{1}{2}E(\beta))E(N(t)) + \frac{1}{2}E(\beta)E(N^2(t)) + \int_0^t E(\gamma(u))du + \int_0^t E(\delta(u))E(N(u))du]. \quad (33)$$

We observe that in this case $\overline{C}(t)$ depends on the process $N(t)$ only through its first and second moments. Again with α,β, $\gamma(t) \equiv \gamma$, $\delta(t) \equiv \delta$ all nonrandom constants the expression (33) becomes

$$\overline{C}(t) = \gamma + \frac{1}{t}[R_1 + (\alpha + \frac{\beta}{2})E(N(t)) + \frac{\beta}{2}E(N^2(t)) + \delta\int_0^t E(N(u))du]. \quad (34)$$

Along with (34) we consider below in some detail the case where $N(t)$ is a mixed Poisson process with mixing distribution given by the random variable λ with distribution function F, i.e. given λ, the distribution of $N(t)$ is Poisson with mean $\lambda A(t)$, where $A(t)$ is a continuously differentiable function of t with $A'(t) \geq 0$. In this case $N(t)$ is a pure birth process with state-dependent birth rates

$$\lambda_k(t) = A'(t)\frac{\int_0^\infty \lambda^{k+1}\exp(-\lambda A(t))dF(\lambda)}{\int_0^\infty \lambda^k \exp(-\lambda A(t))dF(\lambda)} \quad (35)$$

$k = 0,1,2, \ldots, t \geq 0$ (see Feigen [1981] and Puri [1982] for these and other details about mixed Poisson processes.), and we have

$$E(N(t)) = E(\lambda)A(t) \quad (36)$$

and

$$E(N^2(t)) = E(\lambda)A(t) + E(\lambda^2)A^2(t) \ . \tag{37}$$

Substituting these in (34) we have

$$\overline{C}(t) = \gamma + \frac{1}{t}\,[R_1 + (\alpha+\beta)E(\lambda)A(t) + \frac{\beta}{2}\,E(\lambda^2)A^2(t) + \delta E(\lambda)\int_0^t A(u)du]. \tag{38}$$

Consider the special case with $A(t) = t$ so that

$$\overline{C}(t) = \gamma + \frac{1}{t}\,[R_1 + (\alpha+\beta)E(\lambda)t + \frac{\beta}{2}\,E(\lambda^2)t^2 + \frac{\delta}{2}\,E(\lambda)t^2]. \tag{39}$$

By applying Lemma 2.1 or directly it is seen that $\overline{C}(t)$ is minimized at the point t_0 given by

$$t_0 = [2R_1/(\beta E(\lambda^2) + \delta E(\lambda))]^{\frac{1}{2}} \ . \tag{40}$$

In particular if $F(\cdot)$ is degenerate at point λ_0, so that $N(t)$ is a homogeneous Poisson process with parameter λ_0, we have

$$t_0 = [2R_1/(\beta\lambda_0^2 + \delta\lambda_0)]^{\frac{1}{2}} \ . \tag{41}$$

The solutions of example 1.4 in Boland and Proschan [1983] and example 2.5(a) in Boland and Proschan [1982] can be obtained as particular cases of this case.

Again still with $A(t) = t$ if λ has an exponential distribution with parameter μ, then $N(t)$ is a Polya process with $\lambda_k(t) = (k+1)/(\mu+t)$. Here while the intensity of shocks increases with the number of shocks, it decreases with the age of the system. This may be the case where the system weakens with the shocks, but the person who handles the system gets experienced and maturity with time. Here the optimum replacement time t_0 is given by

$$t_0 = \mu[2R_1/(2\beta + \mu\delta)]^{\frac{1}{2}} \tag{42}$$

On the other hand if $A(t) = (\exp(\rho t) - 1)$ with $\rho > 0$ and λ has an exponential distribution with unit mean, the process $N(t)$ is a birth process with birth rates $\lambda_k = (k+1)\rho$, $k = 0,1,2, \ldots$. In this case with $C_j = 0$, $j = 1,2,\ldots$ (i.e. $\alpha = \beta = 0$) we have from (38)

$$\overline{C}(t) = \gamma + \frac{1}{t}[R_1 + \delta\int_0^t (\exp(\rho u) - 1)du]. \tag{43}$$

By applying Lemma 2.1, it is seen that there exists a unique finite minimum t_0 of $\overline{C}(t)$ which is given by the equation

$$(\rho t_0 - 1)\exp(\rho t_0) = [R_1\rho/\delta] - 1 . \tag{44}$$

Finally let λ be degenerate at one so that the process $N(t)$ is a non-homogeneous Poisson process with mean $A(t)$. For this case from (38) we have

$$\overline{C}(t) = \gamma + \frac{1}{t}[R_1 + (\alpha+\beta)A(t) + \frac{\beta}{2}A^2(t) + \delta\int_0^t A(u)du]. \tag{45}$$

The results of Boland and Proschan [1982, 1983] for infinite horizon time can now be obtained from (45) by applying Lemma 2.1. For instance taking $A(t) = At^2$ in (45) it is easily seen using the lemma that $\overline{C}(t)$ has a unique finite minima

t_0 given by the solution of the equation

$$9\beta A^2t^4 + 4\delta At^3 + 6(\alpha+\beta)At^2 = 6R_1 \ . \tag{46}$$

3.2. CASE WITH $\tau = \tau_k$, WHERE k IS A POSITIVE INTEGER.

In this case the system fails completely at the arrival of the kth shock, so that we may define

$$f(j,t) = C_j = 0 \ , \text{ for } j \geq k \ . \tag{47}$$

Consequently the expression (5) for $\overline{C}(t)$ takes the form

$$\overline{C}(t) = [R_1 + (R_2-R_1)P(\tau_k \leq t) + E(\sum_{i=0}^{N(t)} C_i) + \int_0^t E(f(N(u),u))du] \ (\int_0^t P(\tau_k > u)du)^{-1} \ . \tag{48}$$

An important special case is when k = 1, so that $\tau = \tau_1$ denotes the length of life of the system. Let G(t) denote the distribution function of τ_1 with probability density g(t) and r(t) = g(t)/[1-G(t)], the failure rate of G. Then in view of (48) the $\overline{C}(t)$ of (5) corresponds to the case with

$$\psi(t) = (R_2-R_1)G(t) + \int_0^t f(0,u)(1-G(u))du; \ \phi(t) = \int_0^t (1-G(u))du \ . \tag{49}$$

Also

$$D(t) = (R_2-R_1)r(t) + f(0,t). \tag{50}$$

In the event D(t) is a decreasing function of t, then by Lemma 2.1 the minimum of $\overline{C}(t)$ occurs at ∞, so that the optimal policy is never to replace the system prior to its natural ultimate failure. If on the other hand D(t) is an increasing function of t with

$$\lim_{t\to\infty} (D(t)\phi(t) - \psi(t)) > R_1 \quad , \tag{51}$$

which can be shown to be equivalent to

$$E(\tau_1)\lim_{t\to\infty} [(R_2-R_1)r(t) + f(0,t)] - \int_0^\infty f(0,u)(1-G(u))du > R_2 \; , \tag{52}$$

then $\overline{C}(t)$ has a global minimum at a finite point which can be determined with the help of Lemma 2.1.

Finally if $f(0,t) \equiv f_0$, a constant, it follows from above that in case r(t) is decreasing, the optimal policy is never to replace the system until τ_1 is reached. However if r(t) is increasing then the optimum t_0 will be finite if and only if

$$E(\tau_1)r(\infty) > R_2(R_2-R_1)^{-1} \; , \tag{53}$$

and the optimum t_0 is the solution of the equation

$$r(t_0)\int_0^{t_0}(1-G(u))du - G(t_0) = R_1(R_2-R_1)^{-1} \; . \tag{54}$$

<u>REMARK</u> From the above discussion it follows that there may be a system with increasing (nonconstant) failure rate for which the optimum policy is never to replace the system before its failure time τ_1. On the other hand there may also

be situations where although the system has a decreasing failure rate yet the optimal t_0 may be finite.

3.3. CASE WHERE THE STOPPING TIME τ IS INDEPENDENT OF THE PROCESS N(t).

We consider the special case where C_i's and f(i,t) satisfy (32) and α, β, τ and the processes $\gamma(t)$, $\delta(t)$ and N(t) are all mutually independent. The expression (5) for this case is given by

$$\begin{aligned}\overline{C}(t) = [&R_1 + (R_2-R_1)P(\tau \leq t) + E(\alpha)E(N^*(t\wedge\tau)) \\ &+ \frac{1}{2} E(\beta)\{E[N^*(t\wedge\tau)(N^*(t\wedge\tau)+1)]\} + \int_0^t E(\gamma(u))P(\tau > u)du \\ &+ \int_0^t E(\delta(u))E(N(u))P(\tau > u)du] \left(\int_0^t P(\tau > u)du\right)^{-1} . \end{aligned} \tag{55}$$

Also for m = 1,2,

$$E([N^*(t\wedge\tau)]^m) = E[N^m(t)]P(\tau > t) + \int_0^t E[N^m(u)]dF_\tau(u) , \tag{56}$$

where $F_\tau(\cdot)$ is the distribution function of τ. After substituting expressions (56) in (55) help of Lemma 2.1 can be taken in minimizing $\overline{C}(t)$ whenever the conditions of the lemma are satisfied. For example let $\alpha = \beta = \gamma(t) = 0$ and $\delta(t) \equiv \delta$, where δ is a positive constant. Also let N(t) be a Poisson process with mean λt and τ be exponentially distributed with parameter μ. Then from (55) $\overline{C}(t)$ of (5) corresponds to the case with

$$\begin{cases} \psi(t) = (R_2-R_1)(1 - \exp(-\mu t)) + \lambda\delta\int_0^t u \exp(-\mu u)du \\ \phi(t) = \frac{1}{\mu}(1 - \exp(-\mu t)). \end{cases} \tag{57}$$

Again here $D(t) = \lambda\delta t + \mu(R_2 - R_1)$ is a strictly increasing function of t. Moreover

$$\lim_{t\to\infty} [D(t)\phi(t) - \psi(t)] = \infty \quad . \tag{58}$$

Consequently by Lemma 2.1 there exists a unique finite point t_0 which minimizes $\overline{C}(t)$ and which satisfies the equation

$$\lambda\delta\mu t_0 - \lambda\delta(1 - \exp(-\mu t_0)) = R_1\mu^2 \ . \tag{59}$$

4. CONCLUDING REMARKS

It is often the case that the "expected" cost $\overline{C}(t)$ is expressed in the form $[R + \psi(t)]/\phi(t)$. In the cases where the conditions of our Lemma 2.1 are met, the lemma can be applied to find the optimal t_0, as exhibited through many cases considered above. However if the conditions of our lemma are not met, for example if $\psi'(t)/\phi'(t)$ is not monotone, our lemma fails to say anything about the optimum t_0. Some other technique needs to be applied for such situations.

Again in the event the assumptions of Lemma 2.1 for the existence of a unique finite optimum t_0 are satisfied, then it is given by the solution of the equation (13). However often it may be difficult to solve this equation directly. Any of the standard techniques (such as the one due to Newton-Raphson) well known in numerical analysis area (see Scarborough [1966], Chapter 10; Gerald [1978], Chapter 1), depending upon their suitability to the situation at hand, may be used in such cases. As an illustration the values of optimum t_0 for the equation (59) were obtained for different values of R_1 by Newton-Raphson method for the case with $\lambda = \mu = 1$, $\delta = 1/2$ and are given below.

R	1	2	3	4
t_0	2.950	4.990	6.999	9.000

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