Stochastic Integration Without Tears
(with Apology to P. A. Meyer)

by

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1. INTRODUCTION

It has become clear that the Itô integral for the Wiener process is insufficient for applications as well as for mathematical questions. A more general, semimartingale integral has been developed, but there has been resistance to it because of the large amount of technical knowledge required as a prerequisite. A recent result of K. Bichteler [1] and C. Dellacherie [4], building on an approach of Kussmaul [7] and Métivier and Pellaumail [10], however, has lead to the possibility of a new pedagogic approach for stochastic integration. This approach was suggested by P. A. Meyer [12] and sketched by C. Dellacherie [4]. E. Lenglart [8] has followed up on Dellacherie's work, and much of this article follows Lenglart's approach.

We attempt here to develop a theory of stochastic integration with an absolute minimum of technical prerequisites. Much of the deep theory is hidden in Theorem (10.1), the only theorem stated without proof; this theorem is not necessary for the first nine paragraphs. The approach is related to that of Bichteler [1], Kussmaul [7], and Métivier and Pellaumail [10], but it is not the same. Some of the theorems which are rather difficult in a traditional approach (such as Métivier [9], Dellacherie and Meyer [5], Meyer [11], and to some extent Schwartz [14]) are startlingly simple here (e.g., those of paragraphs four and five). A more complete and leisurely treatment, along with a theory of stochastic differential equations, is given in [13].

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2. PRELIMINARIES

Throughout this article we will assume given and fixed a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t \geq 0}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$; $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$; and $\bigcap_{u>t} \mathcal{F}_u = \mathcal{F}_t$. A <u>stopping time</u> T will be a nonnegative random variable T such that $\{T \leq t\} \in \mathcal{F}_t$, each $t \geq 0$. The stopping time σ-algebra, for T a stopping time, is $\mathcal{F}_T = \{\Lambda \in \mathcal{F}_\infty : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t, \text{ all } t \geq 0\}$. A process X will be said to be <u>cadlag</u> (respectively <u>caglad</u>) if its paths are right (resp. left) continuous and have left (resp. right) limits. (The acronym is from the French, as the English one would be the horrific rcwll and lcwrl). Two processes X and Y are <u>indistinguishable</u> if $P(\{\omega: \text{there exists } t \text{ and } X_t(\omega) \neq Y_t(\omega)\}) = 0$.

If T is a stopping time and X is a process, then

$$X_t^T = \begin{cases} X_t & \text{if } t < T(\omega) \\ X_T & \text{if } t \geq T(\omega). \end{cases}$$

If X is cadlag (or caglad) then

$$X_t^{T-} = \begin{cases} X_t & \text{if } t < T(\omega) \\ X_{T-} & \text{if } t \geqq T(\omega). \end{cases}$$

If S and T are two stopping times, then $]\!]S,T]\!] = \{(t,\omega): S(\omega) < t \leqq T(\omega)\}$. Then $1_{]\!]S,T]\!]}$ is the (adapted) process that is the indicator function of this "stochastic interval".

We put a σ-algebra on $\mathbb{R}_+ \times \Omega$ as follows: the predictable σ-algebra $\mathcal{P}$ is the smallest σ-algebra making all the left continuous, adapted processes measurable.

Finally we say that a process X is a local martingale if there exist stopping times T^k increasing to ∞ a.s. such that $X^{T^k} 1_{\{T^k>0\}}$ is a uniformly integrable martingale for each k.

The symbol L^0 will be used to denote finite-valued random variables; $\mathbb{D}$ will denote adapted processes with cadlag paths; and $\mathbb{L}$ will denote adapted processes with left continuous paths. A "b" in front of $\mathbb{L}$, or $\mathcal{P}$, will denote the bounded left continuous adapted processes and the bounded predictably measurable processes, respectively.

3. SEMIMARTINGALES

Let $\sum$ denote the space of simple predictable processes: that is, $H \in \sum$ if H has the representation

(3.1) $$H_t = H_o 1_{\{0\}} + \sum_{i=1}^{n} H_i 1_{]]T_i, T_{i+1}]]}$$

where $0 = T_0 < T_1 < \cdots < T_n < \infty$ are stopping times, and $H_i \in \mathcal{F}_{T_i}$. Given a process X, we define the linear mapping

$I_X \colon \sum \to L^o$ as follows:

(3.2) $$I_X(H) = H_o X_o + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i}),$$

where $H \in \sum$ has the representation (3.1).

(3.3) Definition: A process X is a total semimartingale if X is cadlag, adapted, and $I_X \colon \sum \to L^o$ is continuous, with $\sum$ having uniform convergence, and L^o having convergence in probability.

A process X is a semimartingale if, for each $t \in [0,\infty[$, X^t is a total semimartingale.

4. PROPERTIES OF SEMIMARTINGALES

(4.1) Theorem. *The set of (total) semimartingales is a vector space.*

Proof. This is immediate from the definition. □

(4.2) Theorem. *If $Q \ll P$, then every P (total) semimartingale is a Q (total) semimartingale.*

Proof. Convergence in P-probability implies convergence in Q-probability and the result is immediate. □

(4.3) Theorem. X a semimartingale. Suppose X is adapted to $(\mathcal{G}_t)_{t\geq 0}$, a subfiltration of $(\mathcal{F}_t)_{t\geq 0}$. Then X is a $\mathcal{G}$-semimartingale.

Proof. Since $\sum(\mathcal{G}) \subseteq \sum(\mathcal{F})$, this is immediate from the definition. □

(4.4) Theorem. Let $(P_k)_{k\geq 1}$ be a sequence of probabilities such that X is a P_k-semimartingale for each k. Let $R = \sum_{k=1}^{\infty} \lambda_k P_k$, where $\lambda_k \geq 0$, each k, and $\sum_{k=1}^{\infty} \lambda_k = 1$. Then X is a semimartingale under R as well.

Proof. Suppose $H^n \in \sum$ such that H^n converges to H under R. Then H^n converges to H for all P_k with $\lambda_k > 0$. Therefore $I_X(H^n)$ converges to $I_X(H)$ in P_k-probability for all such k as well. This then implies $I_X(H^n)$ converges to $I_X(H)$ under R. □

(4.5) Theorem. Let X be a cadlag adapted process; let (T_n) be a sequence of positive random variables increasing to ∞ a.s. Let (X^n) be a sequence of semimartingales such that for each n $X^{T_n -} = (X^n)^{T_n -}$. Then X is a semimartingale.

Proof. We will show X^t is a total semimartingale for each $t > 0$. Define $R_n = T_n 1_{(T_n \leq t)} + \infty 1_{(T_n > t)}$. Then $P\{|I_X(H)| \geq C\} \leq P\{|I_{X^n}(H)| \geq C\} + P\{R_n < \infty\}$. But $\lim_{n\to\infty} P\{R_n < \infty\} = \lim_{n\to\infty} P\{T_n \leq t\} = 0$. Let H^k tend to 0 in $\sum$

(uniform convergence). For $\varepsilon > 0$, choose n such that $P(R_n < \infty) < \varepsilon/2$, and then k such that $P\{|I_{X^n}(H^k)| \geqq C\} < \varepsilon/2$. Thus for k large enough, $P\{|I_{X^t}(H^k)| \geqq C\} < \varepsilon$. □

(4.6) Corollary. X a process. If there exist stopping times (T_n) increasing to ∞ a.s. such that X^{T_n} is a semimartingale for each n, then X is a semimartingale.

5. EXAMPLES OF SEMIMARTINGALES

(5.1) Theorem. Let X be adapted, cadlag, with paths of finite variation on compacts. Then X is a semimartingale.

Proof. Note that $|I_{X^t}(H)| \leqq \|H\|_u \int_{0-}^t |dX_s|$, where $\int_{0-}^t |dX_s|$ denotes the total variation of the path on $[0,t]$. The result then follows. □

(5.2) Theorem. Let X be cadlag and a square integrable martingale. Then X is a semimartingale.

Proof. Without loss of generality assume $X_0 = 0$. Let $H \in \sum$. Then

$$
\begin{aligned}
E\{|I_X(H)|^2\} &= E\{(\sum_{i=0}^n H_i(X_{T_{i+1}} - X_{T_i}))^2\} \\
&= E\{\sum_{i=0}^n H_i^2(X_{T_{i+1}} - X_{T_i})^2\} \\
&\leqq \|H\|_u^2 E\{\sum_{i=0}^n (X_{T_{i+1}} - X_{T_i})^2\}
\end{aligned}
$$

$$= \|H\|_u^2 E\{\sum_{i=0}^{n} (X_{T_{i+1}}^2 - X_{T_i}^2)\}$$

$$\leq 2\|H\|_u^2 E(X_\infty^2),$$

and the result follows. □

X is a *locally square-integrable martingale* if there exist stopping times T_n increasing to   a.s. such that $X^{T_n} 1_{\{T_n > 0\}}$ is a cadlag, square-integrable martingale for each n.

(5.3) *Corollary*. *If* X *is a locally square-integrable martingale, then* X *is a semimartingale*.

(5.4) *Corollary*. *A local martingale* X *with continuous paths is a semimartingale*.

Proof. Without loss of generality we assume $X_0 = 0$. Let $R_p = \inf\{t: |X_t| \geq p\}$. It is easy to check that X^{R_p} is a bounded (and hence square-integrable) martingale for each p. Clearly R_p increases to ∞ a.s. with p. Thus X is a locally square-integrable martingale, and hence a semimartingale. □

(5.5) *Corollary*. *Let* X *be a cadlag process with a decomposition* $X = M + A$ *with* M *a locally square-integrable martingale and* A *an adapted process with paths of finite variation on compacts; then* X *is a semimartingale*.

Proof. Since the space of semimartingales is a vector space, this follows from (5.1) and (5.3). □

Note that Theorem (10.1) is a converse of (5.5); that is, all semimartingales have such a decomposition.

(5.6) Theorem. Let X be a cadlag process with stationary and independent increments. Then X is a semimartingale.

Proof. Let $\Delta X_s = X_s - X_{s-}$, the jump at time s. Let $J_t = \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}$, and set $Y_t = X_t - J_t$. Then J is cadlag, adapted, and has paths of finite variation on compacts as a consequence of X having cadlag paths. The Lévy theory (cf, eg [2]) tells us that Y has stationary, independent increments as well, and also a finite mean; the stationarity implies the function $t \mapsto E(Y_t)$ is affine. Thus

$$X_t = \{Y_t - E(Y_t)\} + \{E(Y_t) + J_t\},$$

where $Y_t - E(Y_t)$ is a locally bounded martingale and where $E(Y_t) + J_t$ has paths of finite variation on compacts. Thus X is a semimartingale by (5.5). □

6. STOCHASTIC INTEGRALS

Let $\mathbb{D}$ represent all adapted processes with cadlag paths. On $\mathbb{D}$ we put the topology of uniform convergence on compacts in probability, abbreviated ucp. That is, for a process $Y \in \mathbb{D}$, let $Y_t^* = \sup_{s \leq t} |Y_s|$. Then Y^n converges to Y in ucp if $(Y^n - Y)_t^*$ converges to 0 in probability for every t. Note that this topology is metrizable, and the metric

$$d(Y,Z) = \sum_{n=1}^{\infty} \frac{1}{2^n} E\{\min(1, (Y-Z)_n^*)\}$$

makes $\mathbb{D}$ into a complete metric space compatible with ucp.

We let $\mathbb{L}$ denote all adapted processes with left continuous paths; $b\mathbb{L}$ denotes all processes in $\mathbb{L}$ with bounded paths. We begin with a preliminary result.

(6.1) Theorem. The space $\sum$ is dense in $\mathbb{L}$ under the ucp topology.

Proof. Let $Y \in \mathbb{L}$. Let $T_n = \inf\{t: |Y_t| > n\}$. Then $Y^n = Y^{T^n} 1_{\{T_n > 0\}}$ are in $b\mathbb{L}$ and converge to Y in ucp. So $b\mathbb{L}$ is dense in $\mathbb{L}$. Henceforth, assume $Y \in b\mathbb{L}$ Define Z by $Z_t = \lim\limits_{\substack{u \to t \\ u > t}} Y_u$. Then $Z \in \mathbb{D}$ is the cadlag modification of Y. For $\varepsilon > 0$, define

$$T_0^\varepsilon = 0$$

$$T_{n+1}^\varepsilon = \inf\{t: t > T_n \text{ and } |Z_t - Z_{T_n}| > \varepsilon\}.$$

Since Z is cadlag, the T_n^ε are stopping times increasing to ∞ a.s. as n increases. Let $Z^\varepsilon = \sum_n Z_{T_n^\varepsilon} 1_{[[T_n^\varepsilon, T_{n+1}^\varepsilon[[}$, each $\varepsilon > 0$. Then Z^ε are bounded and converge uniformly to Z as ε tends to 0. Let

$$U^\varepsilon = Z_0 1_{\{0\}} + \sum_n Z_{T_n^\varepsilon} 1_{]]T_n^\varepsilon, T_{n+1}^\varepsilon]]},$$

and the preceding implies U^ε converges uniformly on compacts to $Z_- = Y$.

Finally, define

$$Y^{n,\varepsilon} = Y_0 1_{\{0\}} + \sum_{i=1}^{n} Y_{T_i^\varepsilon} 1_{]]T_i^\varepsilon \wedge n, T_{i+1}^\varepsilon \wedge n]]}$$

and this can be made arbitrarily close to $Y \in b\mathbb{L}$ by taking ε small enough and then n large enough. □

Recall that a process $H \in \sum$ has a representation

$$H = H_0 1_{\{0\}} + \sum_{i=1}^{n} H_i 1_{]]T_i, T_{i+1}]]} \tag{6.2}$$

for $H_i \in \mathcal{F}_{T_i}$ and $0 = T_0 < T_1 < \cdots < T_n < \infty$ stopping times. For a cadlag process X we define the linear mapping $J_X \colon \sum \to \mathbb{D}$ by:

$$J_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X^{T_{i+1}} - X^{T_i}), \tag{6.3}$$

for H with a representation of the form (6.2). We call $J_X(H)$ the <u>stochastic integral of</u> H <u>with respect to</u> X. We will use freely the notations

$$J_X(H) = \int H_s dX_s = H \cdot X. \tag{6.4}$$

(6.5) <u>Theorem</u>. <u>Let</u> X <u>be a semimartingale</u>. <u>Then the mapping</u> $J_X \colon \sum_{ucp} \to \mathbb{D}_{ucp}$ <u>is continuous</u>.

<u>Proof</u>. $\sum_{ucp}$ denotes the space $\sum$ endowed with the ucp topology. Since we are only dealing with convergence on compact sets, without loss of generality we take X to be a total semimartingale. It suffices to show that if H^k converges to 0 (ucp), then one can extract a subsequence k_n such that $J_X(H^{k_n})$ converges

to 0 ucp. First suppose H^k tends to 0 uniformly, and is uniformly bounded. Let

$$T^k = \inf\{t: |(H^k \cdot X)_t| \geqq \delta\}.$$

Then $P\{(H^k \cdot X)^*_t \geqq \delta\} = P\{(H^k \cdot X)^*_{t \wedge T^k} \geqq \delta\}$

$$\leqq P\{(H^k \cdot X)^*_{T^k} \geqq \delta\} = P\{|H^k 1_{[0,T^k]} \cdot X| \geqq \delta\}$$

$$= P\{|I_X(H^k 1_{[0,T^k]})| \geqq \delta\}$$

which tends to 0 by the definition of total semimartingale. Thus we have that for $\varepsilon > 0$, for $t > 0$, there exists a c such that for $H \in \sum$ with $\|H\|_u < c$, $P(J_X(H)^*_t > \delta) < \varepsilon/2$. We need only consider compact intervals of the form $[0,t]$, $t > 0$. For fixed t we can find k_n such that if

$$T_n = \inf\{s: |H^{k_n}_{s \wedge t}| > c\},$$

then $P(T_n < \infty) < \varepsilon/2$. Moreover, by the left continuity $|(H^{k_n})^{T_n}| \leqq c$. Thus

$$P\{(H^{k_n} \cdot X)^*_t \geqq \delta\} \leqq P\{(H^{k_n} \cdot X)^*_{T^n} > \delta\} + P\{T_n < \infty\}$$

$$\leqq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and the continuity is established. □

Since the integration operator J_X has been seen to be continuous on $\sum_{ucp}$, and $\sum_{ucp}$ dense in $\mathbb{L}$ (6.1), we can extend the integration from $\sum$ to $\mathbb{L}$ by continuity. Thus we have:

(6.6) Definition: Let X be a semimartingale. The continuous linear mapping $J_X: \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ obtained as the extension of $J_X: \sum \to \mathbb{D}$ is called the stochastic integral.

7. PROPERTIES OF THE STOCHASTIC INTEGRAL

Throughout this paragraph X will denote a semimartingale and H will denote an element of $\mathbb{L}$. The stochastic integral defined in paragraph 6 will also be denoted:

$$J_X(H) = H \cdot X = \int H_s dX_s .$$

(7.1) Theorem. Let T be a stopping time. Then $(H \cdot X)^T = H1_{[[0,T]]} \cdot X = H \cdot (X^T)$.

(7.2) Theorem. The jump process $\Delta(H \cdot X)_s$ is indistinguishable from $H_s(\Delta X_s)$.

Proofs: Both properties are clear when $H \in \sum$, and hence they follow when $H \in b\mathbb{L}$ by passing to the limit under ucp. □

If Q is another probability, we denote $H_Q \cdot X$ as the stochastic integral computed under the law Q.

(7.3) Theorem. Let $Q << P$. Then $H_Q \cdot X$ is Q-indistinguishable from $H_P \cdot X$.

Proof. Note that by (4.2) X is known to be a Q-semimartingale. The theorem is clear if $H \in \sum$, and it follows for $H \in \mathbb{L}$ by passage to the limit in the ucp topology, since convergence in P-probability implies convergence in Q-probability. □

(7.4) Theorem. Let $(\mathcal{G}_t)_{t\geq 0}$ be another filtration such that $H \in \mathbb{L}(\mathcal{G}) \cap \mathbb{L}(\mathcal{F})$ and such that X is also a $\mathcal{G}$-semimartingale. Then $H_{\mathcal{G}}\cdot X = H_{\mathcal{F}}\cdot X$.

Proof. $\mathbb{L}(\mathcal{G})$ denotes left continuous processes adapted to the filtration $(\mathcal{G}_t)_{t\geq 0}$. Since $H \in \mathbb{L}(\mathcal{G}) \cap \mathbb{L}(\mathcal{F})$, we can find $H^n \in \sum(\mathcal{G}) \cap \sum(\mathcal{F})$ converging to H in ucp, and as the result is clear for $H^n \in \sum$, the full result follows by passing to the limit. □

(7.5) Theorem. Let P_k be a sequence of probabilities such that X is a P_k-semimartingale for each k. Let $R = \sum_{k=1}^{\infty} \lambda_k P_k$ where $\lambda_k \geq 0$, each k, and $\sum \lambda_k = 1$. Then $H_R\cdot X = H_{P_k}\cdot X$, P_k a.s., all k such that $\lambda_k > 0$.

Proof. If $\lambda_k > 0$, then $P_k << R$, and this result is a corollary of (7.3). Note that by (4.4) we know that X is an R-semimartingale. □

(7.6) Theorem. If the semimartingale has paths of finite variation on compacts, then H·X is indistinguishable from the Lebesgue-Stieltjes integral, computed path by path.

Proof. The result is evident for $H^n \in \sum$ converging to H in ucp. Then there exists a subsequence n_k such that $\lim_{n_k\to\infty} (H^{n_k} - H)^*_t = 0$ a.s., and the result follows by interchanging limits, justified by the uniform a.s. convergence. □

(7.7) Theorem. Let X, $\overline{X}$ be two semimartingales, and let $H, \overline{H} \in \mathbb{L}$. Let $A = \{\omega: H_{\cdot}(\omega) = \overline{H}_{\cdot}(\omega)$ and $X_{\cdot}(\omega) = \overline{X}_{\cdot}(\omega)\}$ and let $B = \{\omega: t \mapsto X_t(\omega)$ is of finite variation on compacts$\}$. Then $H \cdot X = \overline{H} \cdot \overline{X}$ on A, and $H \cdot X$ is equal to a path-by-path Lebesgue-Stieltjes integral on B.

Proof. Note that if $P(A) = 0$ the first assertion is trivially true. If $P(A) > 0$, define a probability Q by $Q(\Lambda) = P(\Lambda | A)$. Then under Q we have both H and $\overline{H}$, and X and $\overline{X}$, are indistinguishable. Thus $H_Q \cdot X = \overline{H}_Q \cdot \overline{X}$, and hence $H \cdot X = \overline{H} \cdot \overline{X}$ P-a.s. on A by (7.3), since $Q << P$.

As for the second assertion, if $B = \Omega$ the result is merely Theorem (7.6). Defining R by $R(\Lambda) = P(\Lambda | B)$ (assuming without loss that $P(B) > 0$), then $R << P$ and $B = \Omega$ a.s. dR, hence $H_R \cdot X$ equals the Stieltjes integral R-a.s.. The result again follows by (7.3). □

(7.8) Corollary. With the notations of Theorem (7.7), let $S < T$ be two stopping times. Let

$$C = \{\omega: H_t(\omega) = \overline{H}_t(\omega); X_t(\omega) = \overline{X}_t(\omega); S(\omega) < t \leqq T(\omega)\}$$

$$D = \{\omega: t \mapsto X_t(\omega) \text{ is of finite variation on } S(\omega) \leq t \leq T(\omega)\}.$$

Then $H\cdot X^T - H\cdot X^S = \bar{H}\cdot\bar{X}^T - \bar{H}\cdot\bar{X}^S$ *on* C, *and* $H\cdot X^T - H\cdot X^S$ *equals a path-by-path Lebesgue-Stieltjes integral on* D.

Proof. Let $Y_t = X_t - X_{t\wedge S}$. Then $H\cdot Y = H\cdot X - H\cdot X^S$, and Y does not charge the set $[[0,S]]$, as is evident, or alternatively is an easy consequence of (7.7). One can now apply (7.7) to Y to obtain the result. □

(7.9) *Theorem*. *The stochastic integral process* $Y = H\cdot X$ *is a semimartingale, and for* $G \in \mathbb{L}$ *we have*

$$G\cdot Y = G\cdot(H\cdot X) = (GH)\cdot X.$$

Proof. Suppose first $Y = H\cdot X$ is a semimartingale. Then $G\cdot Y = J_Y(G)$. If G, H are in $\sum$, then it is clear that $J_Y(G) = J_X(GH)$. The associativity then extends to $\mathbb{L}$ by continuity.

It remains to show that $Y = H\cdot X$ is a semimartingale. Let (H^n) be in $\sum$ converging in ucp to H. Then $H^n\cdot X$ converges to $H\cdot X$ in ucp (6.5). Thus there exists a subsequence (n_k) such that $H^{n_k}\cdot X$ converges a.s. to $H\cdot X$.

Let $G \in \sum$, and let $Z^{n_k} = H^{n_k}\cdot X$, $Z = H\cdot X$. The Z^{n_k} are semimartingales converging pointwise to the process Z. For $G \in \sum$, $J_Z(G)$ is defined for any process Z; so we have

$$\begin{aligned} J_Z(G) &= G\cdot Z = \lim_{n_k\to\infty} G\cdot Z^{n_k} \\ &= \lim_{n_k\to\infty} G\cdot(H^{n_k}\cdot X) \end{aligned}$$

$$= \lim_{n_k\to\infty} (GH^{n_k})\cdot X = \lim_{n_k\to\infty} J_X(GH^{n_k})$$

$$= J_X(GH),$$

since X is a semimartingale. Therefore $J_Z(G) = J_X(GH)$ for $G \in \sum$.

Let G_n converge to G in $\sum_u$. Then G_nH converges to GH in $\mathbb{L}_{ucp}$, and since X is a semimartingale $\lim_{n\to\infty} J_Z(G_n) = \lim_{n\to\infty} J_X(G_nH) = J_X(GH) = J_Z(G)$. This implies Z^t is a total semimartingale, and $Z = H\cdot X$ is a semimartingale. □

(7.10) Theorem. Let X be a locally square integrable martingale, and let $H \in \mathbb{L}$. Then the stochastic integral $H\cdot X$ is also a locally square integrable martingale.

Proof. We have seen (5.3) that a locally square integrable martingale is a semimartingale, so we can formulate $H\cdot X$. Without loss of generality we may assume $X_0 = 0$. Also, if $T^k \uparrow \infty$ a.s. and $(H\cdot X)^{T^k}$ is a locally square integrable martingale for each k, it is simple to check that $H\cdot X$ itself is one. Thus without loss we assume X is a square integrable martingale. By stopping H, we may further assume H is bounded, by ℓ. Let $H^n \in \sum$, and H^n converge to H in ucp. We can then modify each H^n such that $\tilde{H}^n$ is bounded by ℓ, $\tilde{H}^n \in \sum$, and $\tilde{H}^n$ converges uniformly to H in probability in $[0,t]$. Then

$$E\{(\tilde{H}^n\cdot X)_t^2\} = E\{\sum_{i=1}^{k_n} \tilde{H}_i^n (X_t^{T_{i+1}} - X_t^{T_i})^2\}$$

$$\leqq \ell^2 E\{\sum_{i=1}^{k_n} (X_{T_{i+1}}^2 - X_{T_i}^2)\} \leqq \ell^2 E(X_\infty^2),$$

and hence $(\tilde{H}^n\cdot X)_t$ are uniformly bounded in L^2 and thus uniformly integrable. Passing to the limit then shows both that $H\cdot X$ is a martingale and that it is square integrable. □

The preceding property can be improved: a stochastic integral with respect to a general local martingale is again a local martingale (e.g., [8, p. 116]). However the proof is more difficult unless one assumes Theorem (10.1), after which it becomes an easy corollary.

(7.11) Definition: Let σ denote a sequence (finite or infinite) of stopping times: σ: $0 = T_0 \leqq T_1 \leqq T_2 \leqq \cdots \leqq T_i \leqq \cdots$. The sequence σ is called a random partition.

A sequence of random partitions σ_n: $0 = T_0^n \leqq T_1^n \leqq \cdots \leqq T_i^n \leqq \cdots$ is said to tend to the identity if $\sup_k T_k^n < \infty$ a.s., each n, and further

(i) $\lim_n \sup_k T_k^n = \infty$ a.s.

(ii) $\|\sigma_n\| = \sup_k (T_{k+1}^n - T_k^n)$ converges to 0 a.s..

Let Y be a process and let σ be a random partition. Then

$$Y^\sigma = Y_0 1_{\{0\}} + \sum_i Y_{T_i} 1_{]\!]T_i, T_{i+1}]\!]}.$$

If X is a semimartingale and if $H \in \mathbb{L}$, then $H^{\sigma} \cdot X = \int H_s^{\sigma} dX_s = H_0 X_0 + \sum_i H_{T_i}(X^{T_{i+1}} - X^{T_i})$, as is easy to check.

The next result has appealing intuitive content.

(7.12) <u>Theorem</u>. <u>Let</u> X <u>be a semimartingale, and let</u> Y <u>be a process with cadlag (or caglad) paths. Let</u> (σ_n) <u>be a sequence of random partitions tending to the identity. Then</u> $Y^{\sigma_n} \cdot X = Y_0 X_0 + \sum Y_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})$ <u>tends to the stochastic integral</u> $(Y_-) \cdot X$ <u>in ucp as</u> n <u>tends to</u> ∞.

<u>Proof</u>. If Y is caglad, then $Y_- = Y$; if Y is cadlag, then $(Y_-)_s = \lim_{\substack{u \to s \\ u < s}} Y_u$, the left continuous version. (Here $(Y_-)_0 = Y_0$.) We give the proof for Y cadlag, as the caglad case is analogous. Note that $Y_- \in \mathbb{L}$, and we let Y^k be in $\sum$ such that Y^k converges to Y_- in ucp. Then:

$$(7.13)\quad (Y_- - Y^{\sigma_n}) \cdot X = (Y_- - Y^k) \cdot X + (Y^k - (Y^k)^{\sigma_n}) \cdot X + ((Y^k)^{\sigma_n} - Y^{\sigma_n}) \cdot X.$$

The first term on the right in (7.13) equals $J_X(Y_- - Y^k)$; since J_X is continuous on $\mathbb{L}_{ucp}$, we have that $(Y_- - Y^k) \cdot X$ tends to 0 in ucp. Analogously we have $((Y^k)^{\sigma_n} - Y^{\sigma_n}) \cdot X$ tends to 0 in ucp for fixed n as k tends to ∞.

Consider then the middle term on the right side of (7.13). For fixed k and ω, $Y_s^k(\omega) - (Y^k)_s^{\sigma_n}(\omega)$ converges to 0 uniformly on compacts; moreover since $Y^k \in \sum$, we write explicitly the stochastic integrals as finite sums. Since X is right continuous,

$(Y^k - (Y^k)^{\sigma_n})\cdot X$ tends to 0 as n tends to ∞ for fixed k and ω. Thus one need only choose k so large that the first and third terms on the right of (7.13) are small; then for fixed k choose n so large that the middle term is small. The result follows. □

8. THE QUADRATIC VARIATION PROCESS OF A SEMIMARTINGALE

Throughout this paragraph X *will denote a semimartingale*.

(8.1) *Definition*. The *quadratic variation process* of X, denoted $[X,X] = ([X,X]_t)_{t\geqq 0}$, is defined to be for $t \geqq 0$:

$$[X,X]_t = X_t^2 - 2(X_- \cdot X_t)$$

where $(X_-)_0 = 0$.

(8.2) *Theorem*. *The quadratic variation process of a semimartingale* X *is an adapted process with cadlag, non-decreasing paths. Moreover*:

(i) $[X,X]_0 = X_0^2$ *and* $\Delta[X,X] = (\Delta X)^2$

(ii) *if* σ_n: $0 = T_0^n \leqq T_1^n \leqq \cdots \leqq T_i^n \leqq \cdots$ *is a sequence of random partitions tending to the identity, then*

$$X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})^2 \to [X,X]$$

in ucp.

Proof. Since X and the stochastic integral are both cadlag, so also is $[X,X]$. Since $\Delta(X_- \cdot X) = X_-\Delta X$ by (7.2), we have

$$
\begin{aligned}
(\Delta X_s)^2 &= (X_s - X_{s-})^2 \\
&= X_s^2 - X_{s-}^2 + 2X_{s-}(X_{s-} - X_s) \\
&= \Delta(X^2)_s - 2X_{s-}(\Delta X_s),
\end{aligned}
$$

from which (i) follows.

For part (ii), without loss assume $X_0 = 0$. Let $R_n = \sup_i T_i^n$. Then $R_n < \infty$ a.s., and by telescoping series $(X^2)^{R_n} = \sum_i \{(X^2)^{T_{i+1}^n} - (X^2)^{T_i^n}\}$ converges in ucp to X^2. Moreover series $\sum_i X_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})$ converges in ucp to $(X_- \cdot X)$ by (7.12). Since $X_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n}) = X^{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})$, using $b^2 - a^2 - 2a(b-a) = (b-a)^2$ and combining the two preceding statements yields the result. Note that (ii) reveals that $[X,X]$ has nondecreasing paths: if $s < t$ the approximating sums in (ii) contain at least as many (nonnegative) terms for t as for s. The cadlag nature of the paths allows the elimination of the null sets dependence on s and t. □

(8.3) Definition. Let X and Y be semimartingales. The *bracket process of X and Y*, $[X,Y]$, is defined by:

$$[X,Y] = \frac{1}{2}\{[X+Y,X+Y] - [X,X] - [Y,Y]\}.$$

(8.4) Corollary. *The bracket $[X,Y]$ of two semimartingales has cadlag paths of finite variation on compacts.*

Proof. By Theorem (8.2), $[X,Y]$ is the difference of two increasing, cadlag processes. □

(8.5) Theorem (Integration by Parts). Let X, Y be semimartingales. Then

$$XY = X_- \cdot Y + Y_- \cdot X + [X,Y].$$

Proof. By the definitions we have

$$[X,Y] = \frac{1}{2}\{(X+Y)^2 - 2(X_- + Y_-)\cdot(X+Y) - X^2 - Y^2 + 2X_- \cdot X + 2Y_- \cdot Y\},$$

and the result follows from the bilinearity of the stochastic integral $H \cdot X$ in (H,X). □

The next theorem is analogous to Theorem (8.2) and has essentially the same proof, so we simply state the theorem here.

(8.6) Theorem. Let X and Y be semimartingales.

(i) if $\sigma_n: 0 = T_0^n \leqq T_1^n \leqq \cdots \leqq T_i^n \leqq \cdots$ is a sequence of random partitions tending to the identity, then

$$[X,Y] = X_0 Y_0 + \lim_n \sum_i (X^{T_{i+1}^n} - X^{T_i^n})(Y^{T_{i+1}^n} - Y^{T_i^n})$$

in ucp;

(ii) $[X,Y]_0 = X_0 Y_0$ and $\Delta[X,Y] = \Delta X \Delta Y$.

The next theorem is a key result in our later extension of the stochastic integral.

(8.7) <u>Theorem</u>. <u>Let</u> X, Y <u>be two semimartingales; let</u> H, K <u>be in</u> $\mathbb{L}$. <u>Then</u>

$$[H\cdot X, K\cdot Y]_t = \int_0^t H_s K_s d[X,Y]_s$$

<u>and, in particular</u>,

$$[H\cdot X, H\cdot X]_t = \int_0^t H_s^2 d[X,X]_s .$$

<u>Proof</u>. The integrals on the right can be equivalently interpreted as semimartingale integrals or path by path Lebesgue-Stieltjes integrals, by (7.6).

Note that to establish the theorem it will suffice to show

$$(8.8) \qquad [H\cdot X, Y]_t = \int_0^t H_s d[X,Y]_s ;$$

one can then use the symmetry of the form $[\cdot,\cdot]$ and iterate (8.8) to obtain the theorem.

If $H = 1_{]\!]0,T]\!]}$ with T a stopping time, then (8.8) reduces to showing $[X^T, Y] = [X,Y]^T$, which is a clear consequence of Theorem (8.6).

If $H = U1_{]\!]S,T]\!]}$, where U is an $\mathscr{F}_S$-measurable r.v. and $S \leqq T$ a.s. are stopping times, then

$$H\cdot X = U(X^T - X^S)$$

and

$$\begin{aligned}[H\cdot X, Y] &= U([X^T,Y] - [X^S,Y]) \\ &= U([X,Y]^T - [X,Y]^S) \\ &= \int H_s d[H,Y]_s .\end{aligned}$$

We have the result then for $H \in \sum$ by linearity.

For $H \in \mathbb{L}$, we let $H^n \in \sum$ and converge in ucp to H. Let $Z^n = H^n \cdot X$, $Z = H \cdot X$. Then Z^n, Z are semimartingales and

$$(8.9) \qquad [Z^n, Y] = \int H_s^n d[X,Y]_s.$$

Integration by parts yields:

$$\begin{aligned} [Z^n, Y] &= YZ^n - (Y_-) \cdot Z^n - (Z_-^n) \cdot Y \\ &= YZ^n - ((Y_-)H^n) \cdot X - (Z_-^n) \cdot Y. \end{aligned}$$

By the definition of the stochastic integral we know $\lim Z^n = Z$ in ucp; since $\lim H^n = H$ in ucp as well, we conclude:

$$\begin{aligned} (8.10) \qquad \lim [Z^n, Y] &= YZ - ((Y_-)H) \cdot H - (Z_-) \cdot Y \\ &= YZ - (Y_-) \cdot Z - (Z_-) \cdot Y \\ &= [Z,Y], \end{aligned}$$

again by integration by parts. Since

$$\lim_n \int H_s^n d[X,Y]_s = \int H_s d[X,Y]_s,$$

combining (8.9) and (8.10) proves (8.8), and thus the theorem is established. □

(8.11) Theorem. *Let X, Y be two semimartingales and let H be cadlag, adapted. Let $\sigma_n: 0 = T_0^n \leq T_1^n \leq \cdots \leq T_i^n \leq \cdots$ be a sequence of random partitions tending to the identity. Then*

$$\sum_i H_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})(Y^{T_{i+1}^n} - Y^{T_i^n})$$

converges in ucp to $\int H_{s-}d[X,Y]_s$ $(H_{0-} = H_0)$.

Proof. The process $[X,Y]$ is cadlag, adapted, and has paths of finite variation on compacts (8.4), and hence is a semimartingale (5.1). The theorem follows as a corollary of Theorem (7.12) and Theorem (8.6). □

9. CHANGE OF VARIABLES FORMULA

Let $(V_t)_{t\geq 0}$ be an adapted, continuous process with paths of finite variation on compacts. If $f: \mathbb{R} \to \mathbb{R}$ is $\mathscr{C}^1$, the Riemann-Stieltjes change of variables formula, integrating path by path, is well known to be:

(9.1) $$f(V_t) = f(V_0) + \int_0^t f'(V_s)dV_s.$$

If $(V_t)_{t\geq 0}$ is as above but its paths are only right continuous, it is less well known but easy to check that the path by path Lebesgue-Stieltjes formula is:

(9.2) $$f(V_t) = f(V_0) + \int_0^t f'(V_{s-})dV_s + \sum_{0<s\leq t}\{f(V_s) - f(V_{s-}) - f'(V_{s-})\Delta V_s\}.$$

If X is a semimartingale, then X need not have paths of finite variation on compacts (e.g., take the Wiener process, a locally square integrable martingale), and one obtains a different formula.

(9.3) *Theorem*. *Let* X *be a semimartingale and let* $f\colon \mathbb{R} \to \mathbb{R}$ *be* $\mathscr{C}^2$. *Then* $f(X)$ *is a semimartingale and the following formula holds*:

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\,dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\,d[X,X]_s + \sum_{0<s\leqq t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \tfrac{1}{2}f''(X_{s-})(\Delta X_s)^2\}. \tag{9.4}$$

Proof. By Taylor's theorem we know:

$$f(y) = f(x) + f'(x)(y-x) + \tfrac{1}{2}f''(x)(y-x)^2 + R(x,y) \tag{9.5}$$

where $R(x,y) \leqq r(|y-x|)(y-x)^2$, and where $r\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $\lim_{u\to 0} r(u) = 0$.

The Continuous Case: We first assume that X is a continuous semimartingale. Fix a $t > 0$, and let $\sigma_n\colon 0 = T_0^n \leqq \cdots \leqq T_i^n \leqq \cdots \leqq T_{k_n}^n = t$ be a sequence of random partitions of $[0,t]$ tending to the identity. Then

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=0}^{k_n}\{f(X_{T_{i+1}^n}) - f(X_{T_i^n})\} \\ &= \sum_i\{f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n})\} \\ &\quad + \frac{1}{2}\sum_i\{f''(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n})^2\} \\ &\quad + \sum_i R(X_{T_i^n}, X_{T_i^{n+1}}). \end{aligned} \tag{9.6}$$

The first sum on the right side of (9.6) converges to $\int_0^t f'(X_s)dX_s$ by (7.12); the second sum converges to $\frac{1}{2}\int_0^t f''(X_s)d[X,X]_s$ by (8.2). Both converge in probability. The third term on the right side of (9.6) will tend to 0 if $\lim_{n\to\infty}\sup_i r(|X_{T^n_{i+1}} - X_{T^n_i}|) = 0$. For each ω, $s \mapsto X_s(\omega)$ is uniformly continuous on $[0,t]$, and since by hypothesis $\lim \sup|T^n_{i+1} - T^n_i| = 0$, we have that $\lim_n \sum_i R(X_{T^n_i}, X_{T^n_{i+1}}) = 0$ as well. Thus we have established that (9.4) holds for each t a.s. when X is continuous. The continuity of the paths eliminates the dependence of the exceptional set on t.

<u>The General Case</u>: Once again we have a representation as in (9.6), but we need a closer analysis. Since, for $t > 0$, $\sum_{0<s\leqq t}(\Delta X_s)^2 \leqq [X,X]_t < \infty$ a.s., we know that $\sum_{0<s\leqq t}(\Delta X_s)^2$ converges. Given $\varepsilon > 0$, let A be a subset of $\mathbb{R}_+ \times \Omega$ such that $\sum_{s\in A}(\Delta X_s)^2 \leqq \varepsilon^2$, and let $B = \{(s,\omega): (\Delta X_s)^2 > 0, (s,\omega) \notin A\}$. Then we can rewrite (9.6) as:

$$(9.7)\quad f(X_t) - f(X_0) = \sum_i \{f'(X_{T^n_i})(X_{T^n_{i+1}} - X_{T^n_i})\}$$

$$+ \frac{1}{2}\sum_i \{f''(X_{T^n_i})(X_{T^n_{i+1}} - X_{T^n_i})^2\}$$

$$+ \sum_i 1_{\{B\cap]\!] T^n_i, T^n_{i+1}]\!] \neq \emptyset\}}\{f(X_{T^n_{i+1}}) - f(X_{T^n_i}) - f'(X_{T^n_i})(X_{T^n_{i+1}} - X_{T^n_i})$$

$$- \frac{1}{2} f''(X_{T^n_i})(X_{T^n_{i+1}} - X_{T^n_i})^2\}$$

$$+ \sum_i 1_{\{B\cap]\!] T^n_i, T^n_{i+1}]\!] = \emptyset\}} R(X_{T^n_i}, X_{T^n_{i+1}}).$$

As in the continuous case, the first two sums on the right side of (9.7) converge respectively to $\int_0^t f'(X_{s-})dX_s$ and $\frac{1}{2}\int_0^t f''(X_{s-})d[X,X]_s$ by (7.12) and (8.2). The third sum converges to

$$(9.8)\qquad \sum_{\substack{s\in B\\ |\Delta X_s|>0}} \{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s - \tfrac{1}{2} f''(X_{s-})(\Delta X_s)^2\},$$

and it remains to consider the fourth sum on the right side of (9.7). Since $\lim_n \sup_i |T^n_{i+1} - T^n_i| = 0$, for large enough n we have $|X_{T^n_{i+1}} - X_{T^n_i}| \leqq 2\varepsilon$ when $B \cap]]T^n_i, T^n_{i+1}]] = \emptyset$. But then $R(x,y) \leqq r(|y-x|)(y-x)^2$; we can majorize

$$(9.9)\qquad \sum_i 1_{\{B\cap]]T^n_i,T^n_{i+1}]]=\emptyset\}} R(X_{T^n_i}, X_{T^n_{i+1}})$$

by $r(2\varepsilon)\sum_i (X_{T^n_{i+1}} - X_{T^n_i})^2$; since $\sum_i (X_{T^n_{i+1}} - X_{T^n_i})^2$ converges to $[X,X]_t < \infty$, as n tends to ∞ and as ε tends to 0 we have that $r(2\varepsilon)$ tends to 0 and thus the sums (9.9) tend to 0. Moreover the sums (9.8) clearly tend, as ε tends to 0, to:

$$(9.10)\qquad \sum_{0<s\leqq t} \{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s - \tfrac{1}{2} f''(X_{s-})(\Delta X_s)^2\}$$

provided this series is absolutely convergent.

Let $T_k = \inf\{t > 0: |X_t| \geqq k\}$, with $X_0 = 0$. By first establishing (9.4) for $X1_{[[0,T^k[[}$, (which is a semimartingale since it is the product of two semimartingales: cf (8.5)) it suffices to consider semimartingales taking their values in intervals of the form $[-k,k]$. For f restricted to $[-k,k]$ we have $|f(y) - f(x) - (y-x)f'(x)| \leq c(y-x)^2$. Then

$$\sum_{0<s\leq t} |f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s| \leq c \sum_{0<s\leq t} (\Delta X_s)^2 \leq c[X,X]_t < \infty$$

and

$$\sum_{0<s\leq t} |f''(X_{s-})|(\Delta X_s)^2 \leq d \sum_{0<s\leq t} (\Delta X_s)^2 \leq d[X,X]_t < \infty \quad \text{a.s.}$$

This implies the sum (9.10) is absolutely convergent. This completes the proof. □

The change of variables formula is also referred to as "Itô's Lemma" and is traditionally stated in a slightly different way, which is more analogous to the Lebesgue-Stieltjes formula (9.2). Define $[X,X]_t^c = [X,X]_t - \sum_{0<s\leq t} (\Delta X_s)^2$, the path by path continuous part of $[X,X]$.

(9.11) Corollary. *Let* X *be a semimartingale and let* $f: \mathbb{R} \to \mathbb{R}$ *be* $\mathscr{C}^2$. *Then* $f(X)$ *is a semimartingale and the folllwing formula holds:*

$$f(X_t)-f(X_0) = \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]_s^c$$
$$+ \sum_{0<s\leq t}\{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s\}.$$

Proof. One need only observe that

$$\Delta\int f''(X_{s-})d[X,X]_s = f''(X_-)\Delta[X,X]$$
$$= f''(X_-)(\Delta X)^2,$$

and the result follows from Theorem (9.3). □

The stochastic integral calculus, as revealed by Theorem (9.3) and Corollary (9.11), is different from the classical Lebesgue-Stieltjes calculus. By restricting the class of integrands to semimartingales made left continuous (instead of $\mathbb{L}$), one can define a stochastic integral that obeys the traditional rules of Lebesgue-Stieltjes calculus.

(9.12) Definition. Let X, Y be semimartingales. Then the Fisk-Stratonovich integral of Y with respect to X, denoted $\int_0^t Y_{s-} \circ dX_s$, is defined by:

$$\int_0^t Y_{s-} \circ dX_s \equiv \int_0^t Y_{s-}dX_s + \frac{1}{2}[Y,X]_t^c.$$

Note that we have defined the Fisk-Stratonovich integral in terms of the semimartingale integral. With some work one can slightly enlarge the domain of the definition (cf [11, p. 360]).

(9.13) Theorem. Let X be a semimartingale and let f be $\mathscr{C}^3$. Then

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-}) \circ dX_s$$
$$+ \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}$$

Proof. Note that f' is $\mathscr{C}^2$, so that $f'(X)$ is a semimartingale and in the domain of definition of the F - S integral. By (9.11) and the definition, it will suffice to show that $\frac{1}{2}[f'(X),X]^c = \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]_s^c$. However

$$f'(X_t) - f'(X_0) = \int_0^t f''(X_{s-})dX_s$$
$$+ \frac{1}{2}\int_0^t f^{(3)}(X_{s-})d[X,X]_s.$$

Thus

$$(9.14) \quad [f'(X),X]^c = [f''(X_-)\cdot X,X]^c + [f^{(3)}(X_-)\cdot[X,X],X]^c.$$

The first term on the right side of (9.14) is $\int_0^t f''(X_{s-})d[X,X]_s^c$ by (8.7); the second term can easily be seen, as a consequence of (8.2) and the fact that $[X,X]$ has paths of finite variation, to be $(\sum_{0<s\leq \cdot} f^{(3)}(X_{s-})(\Delta X_s)^2)^c$; that is, zero. The theorem is thus proved. □

Observe that if X is a semimartingale with continuous paths, then Theorem (9.13) reduces to the classical Riemann-Stieltjes formula:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) \circ dX_s;$$

this is, of course, the main attraction of the Fisk-Stratonovich stochastic integral.

10. STOCHASTIC INTEGRATION WITH PREDICTABLE INTEGRANDS

Up to this point our treatment of stochastic integration has been relatively non-technical. This has been achieved by limiting the integrands to the space $\mathbb{L}$. This is, of course, sufficient to prove a change of variables formula, and is also sufficient in many applications, such as the study of stochastic differential equations. To extend the integral to more general integrands, however, requires some deep results. We have, nevertheless, managed to combine all of them in Theorem (10.1) which we state without proof. We refer the reader who would like to see a proof, with the technical level of knowledge kept as low as possible, to [13].

Recall that if $X = X_0 + M + A$, with $M_0 = A_0 = 0$, M a locally square integrable martingale and A an adapted, cadlag process with paths of finite variation on compacts, then X is a semimartingale (Corollary (5.5)). It will be convenient to call the process A of <u>locally integrable variation</u> if there exist stopping times T^k increasing to ∞ a.s. such that $E\{\int_0^{T_k} |dAs|\} < \infty$. We will denote the total variation process alternatively as $\int_0^t |dA_s|$ and $|A|_t$. The next theorem is a converse to the above:

(10.1) <u>Theorem</u>. <u>Let</u> X <u>be a cadlag, adapted process. The following are equivalent</u>:

(i) X *is a semimartingale (in the sense of* (3.3));

(ii) X *has a decomposition* $X = X_0 + N + B$, *where* N *is a local martingale and* B *has paths of finite variation on compacts;*

(iii) X *has a decomposition* $X = X_0 + M + A$, *where* M *is a local martingale with bounded jumps and* A *has paths of finite variation on compacts;*

Moreover, if X *has a decomposition as in* (ii) *where* B *is of locally integrable variation, then there exists a unique decomposition* $X = X_0 + \overline{N} + \overline{B}$, *where* $\overline{B}$ *is a predictably measurable process with paths of finite variation on compacts.*

(10.2) *Definition*. If X is a semimartingale such that X has a decomposition $X = X_0 + N + B$ as in (10.1) (ii) where B is of locally integrable variation, X is said to be a *special semimartingale*. The decomposition $X = X_0 + \overline{N} + \overline{B}$ of Theorem (10.1) is called the *canonical decomposition* of X.

Let X be a semimartingale with a decomposition $X = X_0 + N + B$. Define

$$j_2(N,B) = \| [N,N]_\infty^{1/2} + |X_0| + \int_0^\infty |dB_s| \|_{L^2}.$$

If $j_2(N,B) < \infty$ for some decomposition, then X is clearly special. This motivates:

(10.3) *Definition*. Let X be a special semimartingale with canonical decomposition $X = X_0 + \overline{N} + \overline{B}$. Then

$\|X\|_{\mathscr{H}^2} \equiv \| [\bar{N},\bar{N}]_\infty^{1/2}\|_{L^2} + \|\int^\infty |d\bar{B}_s|\|_{L^2}$. <u>The space of</u> $\mathscr{H}^2$ <u>semimartingales</u> is the space of special semimartingales with finite $\mathscr{H}^2$ (pseudo) norm.

(10.4) <u>Theorem</u>. <u>The space of</u> $\mathscr{H}^2$ <u>semimartingales is a Banach space</u>.

<u>Proof</u>. The space is clearly a normed linear space. Since $\|\bar{N}_\infty\|_{L^2} = \| [\bar{N},\bar{N}]_\infty^{1/2}\|_{L^2}$, it follows from Doob's maximal quadratic inequality that the space of square integrable martingales is complete. As for $\bar{B}$, let $(\bar{B}^n)$ be a sequence such that $\sum_n \| |\bar{B}^n|_\infty\|_{L^2} < \infty$. Then the series $\sum_n \bar{B}^n$ converges to a limit $\bar{B}$, and $\lim_{m\to\infty} \sum_{n\geq m} \int_0^\infty |d\bar{B}_s^n| = 0$ in L^1 and is dominated in L^2 by $\sum \int_0^\infty |d\bar{B}_s^n|$, hence tends to 0 in L^2 as well. Thus $\sum \bar{B}^n$ converges to $\bar{B}$ in $L^2(dP)$. Completeness then follows. □

For simplicity, we <u>henceforth assume that all semimartingales</u> X <u>have the property</u> $X_0 = 0$.

Let $b\mathbb{L}$ denote the bounded processes in $\mathbb{L}$. For $H \in b\mathbb{L}$ and $X \in \mathscr{H}^2$, then $H\cdot X$ is also in $\mathscr{H}^2$. Moreover if $X = \bar{N} + \bar{B}$, then the canonical decomposition of $H\cdot X$ is $H\cdot\bar{N} + H\cdot\bar{B}$, and

$$(10.5)\quad \|H\cdot X\|_{\mathscr{H}^2} = \| [H\cdot\bar{N},H\cdot\bar{N}]^{1/2}\|_{L^2} + \|\int_0^\infty |d(H\cdot\bar{B})_s|\|_{L^2}$$

$$= \| (\int_0^\infty H_s^2 d[\bar{N},\bar{N}]_s)^{1/2}\|_{L^2} + \|\int_0^\infty |H_s| |d\bar{B}_s|\|_{L^2}.$$

Since the integrals $\int H_s d[\bar{N},\bar{N}]_s$ and $\int H_s d\bar{B}_s$ make sense as Lebesgue-Stieltjes integrals path by path for any $H \in b\mathcal{P}$ (the bounded predictable processes), we use the property (10.5) to extend our class of integrands.

(10.6) Definition. Given an $\mathcal{H}^2$-semimartingale X with canonical decomposition $X = \bar{N} + \bar{B}$, and processes $H,J \in b\mathcal{P}$, define $d_X(H,J) = \|(\int_0^\infty (H_s - J_s)^2 d[\bar{N},\bar{N}]_s)^{1/2}\|_{L^2} + \|\int_0^\infty |H_s - J_s||d\bar{B}_s|\|_{L^2}$.

(10.7) Theorem. $b\mathbb{L}$ is dense in $b\mathcal{P}$ under the "distance" $d_X(\cdot,\cdot)$.

Proof. Let $\mathcal{M} = \{H \in b\mathcal{P}$: for any $\varepsilon > 0$, there exists a $J \in b\mathbb{L}$ such that $d_X(H,J) < \varepsilon\}$.

Then $\mathcal{M}$ contains $b\mathbb{L}$ and the constants. Moreover if $H^n \in \mathcal{M}$ and increasing to H with H bounded, then by the dominated convergence theorem for $n > N$, $d_X(H,H^n) < \delta$. Since $H^n \in \mathcal{M}$, there exists a $J^n \in b\mathbb{L}$ such that $d_X(H^n,J^n) < \gamma$. Therefore for $n > N$, there exists a $J^n \in b\mathbb{L}$ such that $d_X(H,J^n) < \varepsilon$ by appropriate choices of δ and γ. An application of the monotone class theorem yields the result. □

(10.8) Theorem. Given a semimartingale X in $\mathcal{H}^2$ and $H^n \in b\mathbb{L}$ such that H^n is Cauchy under d_X, then $H^n \cdot X$ is Cauchy in $\mathcal{H}^2$.

Proof. We have $\|H^n \cdot X - H^m \cdot X\|_{\mathcal{H}^2} = d_X(H^n,H^m)$. □

(10.9) Theorem. Let X be a semimartingale in $\mathscr{H}^2$ and let $H \in b\mathscr{P}$. If $H^n \in b\mathbb{L}$ and $J^n \in b\mathbb{L}$ such that $\lim_n d_X(H^n,H) = \lim_n d_X(J^n,H) = 0$, then $H^n \cdot X$ and $J^n \cdot H$ tend to the same limit in $\mathscr{H}^2$.

Proof. Let $Y = \lim H^n \cdot X$ and let $Z = \lim J^n \cdot X$, in $\mathscr{H}^2$. Then

$$\begin{aligned}\|Y-Z\|_{\mathscr{H}^2} &\leqq \|Y - H^n \cdot X\|_{\mathscr{H}^2} + \|H^n \cdot X - J^n \cdot X\|_{\mathscr{H}^2} + \|J^n \cdot X - Z\|_{\mathscr{H}^2} \\ &\leqq 2\varepsilon + \|H^n \cdot X - J^n \cdot X\|_{\mathscr{H}^2} \quad (n \geqq N_\varepsilon) \\ &\leqq 2\varepsilon + d_X(H^n,J^n) \\ &\leqq 2\varepsilon + d_X(H^n,H) + d_X(H,J^n) \\ &\leqq 4\varepsilon, \text{ and the result follows.}\end{aligned}$$

□

We can now make the:

(10.10) Definition. Let X be a semimartingale in $\mathscr{H}^2$ and let $H \in b\mathscr{P}$. Let $H^n \in b\mathbb{L}$ such that $\lim d_X(H^n,H) = 0$. The stochastic integral $H \cdot X$ is the (unique) semimartingale Y in $\mathscr{H}^2$ given by $\lim H^n \cdot X = Y = H \cdot X$, with convergence in $\mathscr{H}^2$.

(10.11) Theorem. Let X be a semimartingale in $\mathscr{H}^2$. Then $E\{(\sup_t |X_t|)^2\} \leqq C\|X\|^2_{\mathscr{H}^2}$.

Proof. Let $X^*_\infty = \sup_t |X_t|$. Then $X^*_\infty \leqq \overline{N}^*_\infty + \int_0^\infty |d\overline{B}_s|$, and by Doob's maximal quadratic inequality,

$$E((\bar{N}^*_\infty)^2) \leqq 4E(\bar{N}^2_\infty) = 4E([\bar{N},\bar{N}]_\infty).$$

Thus $$E\{(X^*_\infty)^2\} \leqq 2E\{(\bar{N}^*_\infty)^2\} + 2E\{(\int_0^\infty |d\bar{B}_s|)^2\}$$

$$\leqq 8\| [\bar{N},\bar{N}]^{1/2}_\infty \|^2_{L^2} + 2\|\int_0^\infty |d\bar{B}_s|\|^2_{L^2}$$

and the result follows. □

(10.12) Corollary. *If X^n is a sequence of semimartingales converging to X in $\mathscr{H}^2$, then there exists a subsequence n_k such that $\lim_{n_k \to \infty} (X^{n_k} - X)^*_\infty = 0$ a.s..*

Proof. Since $(X^n - X)^*$ converges to 0 in L^2 by (10.11), there exists a subsequence converging a.s.. □

We are now in a position to investigate some of the properties of this more general stochastic integral. The bilinearity is evident, and we state it without proof.

(10.13) Theorem. *Let X, Y be $\mathscr{H}^2$ semimartingales and $H,K \in b\mathscr{P}$. Then $(H+K)\cdot X = H\cdot X + K\cdot X$, and $H\cdot(X+Y) = H\cdot X + H\cdot Y$.*

(10.14) Theorem. *Let X be a square-integrable martingale and let $H \in b\mathscr{P}$. Then $H\cdot X$ is a square integrable martingale.*

Proof. Clearly X is a semimartingale in $\mathscr{H}^2$. Let $H^n \in b\mathbb{L}$ such that $\lim_{n\to\infty} d_X(H^n,H) = 0$. Then $H^n \cdot X$ is a square integrable martingale by (7.10) for each n. The theorem follows by L^2-convergence. □

(10.15) Theorem. Let X be an $\mathcal{H}^2$-semimartingale with paths of finite variation on compacts. Let $H \in b\mathcal{P}$. Then $H \cdot X$ agrees with a path by path Lebesgue-Stieltjes integral.

Proof. Let $H^n \in b\mathbb{L}$ such that $\lim d_X(H^n,H) = 0$. Then $H^n \cdot X$ is a Lebesgue-Stieltjes integral for each n, and the result follows by passing to the limit. □

(10.16) Theorem. Let X be an $\mathcal{H}^2$-semimartingale and $H \in b\mathcal{P}$. Then $\Delta(H \cdot X) = H(\Delta X)$.

Proof. Let $H^n \in b\mathbb{L}$ such that $\lim_{n\to\infty} d_X(H^n,H) = 0$. Then there exists a subsequence n_k such that $\lim_{n_k\to\infty} (H^{n_k} \cdot X - H \cdot X)^*_\infty = 0$ a.s., by (10.12). This implies $\Delta(H^{n_k} \cdot X) \to \Delta(H \cdot X)$ outside of an evanscent set. However since $H^{n_k} \in \mathbb{L}$, we know $\Delta(H^{n_k} \cdot X) = H^{n_k}(\Delta X)$. Therefore $\lim_{n\to\infty} H^n_t(\omega) = \dfrac{\Delta(H \cdot X)_t}{\Delta X_t}$, on $\{\Delta X_t \neq 0\}$, hence the limit exists. If

$$\Lambda = \{\omega\colon \text{there exists } t > 0 \text{ such that } \lim_n H^n_t(\omega) \neq H_t(\omega) \text{ and } \Delta X_t \neq 0\},$$

and if $P(\Lambda) > 0$, then we would contradict that $\lim_{n\to\infty} d_X(H^n,H) = 0$, since

$$\lim_n d_X(H^n,H) \geqq \lim_n \| 1_\Lambda \left(\int (H^n_s - H_s)^2 d(\Delta\bar{N}_s)^2\right)^{1/2} + 1_\Lambda \int |H^n_s - H_s|\, d|\Delta\bar{B}_s| \|_{L^2},$$

and if $\Delta X_s \neq 0$, then $|\Delta \bar{N}_s| + |\Delta \bar{B}_s| > 0$. Thus $P(\Lambda) = 0$, and we have

$$\Delta(H\cdot X)_t = \lim_{n_k} H_t^{n_k}\Delta X_t = H_t \Delta X_t. \qquad \square$$

(10.17) Theorem. Let X be an $\mathscr{H}^2$-semimartingale, and let $H,K \in b\mathscr{P}$. Then $H\cdot(K\cdot X) = (HK)\cdot X$.

Proof. This follows from the result for $H,K \in b\mathbb{L}$ (7.9), and then by taking limits. $\square$

(10.18) Theorem. Let X, Y be $\mathscr{H}^2$-semimartingales and let $H,K \in b\mathscr{P}$. Then

$$[H\cdot X, K\cdot Y]_t = \int_0^t H_s K_s d[X,Y]_s$$

and, in particular,

$$[H\cdot X, H\cdot X]_t = \int_0^t H_s^2 d[X,X]_s.$$

Proof. As in the proof of Theorem (8.7), it suffices to show $[H\cdot X, Y]_t = \int_0^t H_s d[X,Y]_s$. Let $H^n \in b\mathbb{L}$ such that $d_X(H^n,H) \to 0$. By stopping we can also assume $Y_- \in b\mathbb{L}$, and it is then easy to check that $d_X(H^n Y_-, HY_-)$ also tends to 0. We have, by (8.7),

(10.19) $$[H^n\cdot X, Y] = \int H^n d[X,Y] \to \int H d[X,Y];$$

Let $Z^n = H^n\cdot X$, and by (10.12) we know there is a subsequence n_k such that $\lim_{n_k\to\infty}(Z^{n_k} - Z)_\infty^* = 0$ a.s., where $Z = H\cdot X$. Then

$$[Z^{n_k},Y] = Z^{n_k}Y - (Y_-)\cdot Z^{n_k} - (Z_-^{n_k})\cdot Y$$
$$= Z^{n_k}Y - (Y_-H^{n_k})\cdot X - (Z_-^{n_k})\cdot Y,$$

by integration by parts and by (10.17). Taking limits we have

$$\lim_{n_k\to\infty} [Z^{n_k},Y] = ZY - Y_-\cdot(H\cdot X) - Z_-\cdot Y$$
$$= ZY - Y_-\cdot(Z) - Z_-\cdot Y$$
$$= [Z,Y] = [H\cdot X,Y].$$

Combining this with (10.19) yields the result. □

Now let X be <u>any</u> semimartingale with $X_0 = 0$ for simplicity. Let $X = M + A$ be a decomposition where the local martingale has bounded jumps. Define

$$T_n = \inf\{t > 0 : |M_t| > n \text{ or } \int_0^t |dA_s| > n\}.$$

Then

$$X^{T_n-} = \begin{cases} X_t & \text{if } T_n(\omega) < t \\ X_{T_n-} & \text{if } T_n(\omega) \geqq t, \end{cases}$$

and $X^{T_n-} = M^{T_n} + A^{T_n-} - (\Delta M_{T_n})1_{[[T_n,\infty[[}$; if c is a bound for the jumps of M, then $|X^{T_n-}| \leqq 2n + 2c$; that is, it is bounded. Note that X^{T_n-} is a semimartingale and is in $\mathscr{H}^2$. This allows the extension of the stochastic integral to arbitrary semimartingales.

(10.20) <u>Definition</u>. Let X be a semimartingale and $H \in b\mathcal{P}$. Let T^n be stopping times increasing to ∞ such that $X^{T_n -}$ is in $\mathcal{H}^2$. Define $H \cdot X$ to be $H \cdot (X^{T_n -})$ on $[[0, T_n[[$ for each n, and call $H \cdot X$ <u>the stochastic integral</u>.

Note that if $T_m > T_n$ in Definition (10.20), then if $H^n \in b\mathbb{L}$ converge to $H \in b\mathcal{P}$ in $d_{(X^{T_m -})}(\cdot,\cdot)$, then they converge as well in $d_{(X^{T_n -})}(\cdot,\cdot)$, so the integral is well defined. We can further extend the class of integrands. A process $H \in \mathcal{P}$ is said to be <u>locally bounded</u> if there exist stopping times T^k increasing to ∞ a.s. such that $(H - H_0)^{T^k}$ is in $b\mathcal{P}$ for each k.

(10.21) <u>Definition</u>. Let X be a semimartingale and let $H \in \mathcal{P}$ be locally bounded. The <u>stochastic integral</u> $H \cdot X$ is defined to be $H_0 X_0 + (H - H_0)^{T^k} \cdot X$ on $[[0, T^k]]$.

It is now a simple matter to check that all the properties (10.13) through (10.18) still hold for this mild extension.

These techniques can be carried further, but we do not do so here. By developing the semimartingale topology, which is closely tied to the $\mathcal{H}^2$ norm, one can extend the stochastic integral to the space of predictable, <u>integrable</u> processes. We refer the interested reader to [3] and [15].

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