QUADRATIC ESTIMATORS OF QUADRATIC FUNCTIONS OF NORMAL PARAMETERS

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Quadratic Estimators

Summary

The estimation problem of a quadratic function of normal parameters $\theta = \xi^2 + b\sigma^2$ is considered. A necessary and sufficient condition for the admissibility under quadratic loss of quadratic estimator of the sample mean and sample standard deviation is given.

1. Introduction and Summary

Let $x_1,\ldots,x_n$, $n \geq 2$, be a random normal sample with unknown mean ξ and unknown standard deviation σ. We consider estimating $\theta = \xi^2 + b\theta^2$ where b is a given non-negative number, under quadratic loss. The case $b = 1$ corresponds to the second moment, estimation of which is of interest if square root transformation of data is made (cf. Box and Cox (1964), Patterson (1966, Land (1971), Carroll and Ruppert (1984)). Indeed assume that the original sample $y_1,\ldots,y_n$ is approximately normalized by the transformation $x = y^{\frac{1}{2}}$ which is commonly used in practice. In this situation the mean of the y's observations is $\xi^2 + \sigma^2$. In the case $b = 0$, i.e. when estimating the square of the mean, the determination of the best unbiased estimator is a common exercise in many textbooks (see, for instance, Lehmann (1983), p. 132). The inadmissibility of the latter estimator in a similar problem has been noticed by Perlman and Rasmussen (1975).

The best unbiased estimator for general θ is easily seen to have the form $\delta_U(X,S) = X^2 + S^2[n^{-1} + (b-1)(n-1)^{-1}] = X^2 + S^2(bn-1)[n(n-1)]^{-1}$. Here $X = \sum_1^n x_j/n$, $S^2 = \sum_1^n (x_j - X)^2$. However if $b \neq n^{-1}$, this estimator can be improved upon very easily. We consider the class of estimators of the form

$$\delta(X,S) = X^2 + c\, S^2,$$

c real. Because of the independence of X and S one obtains

$$E(X^2 + cS^2 - \xi^2 - b\sigma^2)^2$$
$$= E(X^2 - \xi^2 - \sigma^2/n + cS^2 - (b-n^{-1})\sigma^2)^2$$
$$= E(X^2 - \xi^2 - \sigma^2/n)^2 + E(cS^2 - (b-n^{-1})\sigma^2)^2 .$$

The best choice of the constant c is

$$c = a = (b-n^{-1})\ E_{\sigma=1}\ S^2/E_{\sigma=1}\ S^4$$
$$= (bn - 1)/[n(n + 1)]$$

which is different from $(bn - 1)/[n(n - 1)]$ unless $b = n^{-1}$. This estimator is also different from the maximum likelihood estimator $\hat{\delta}(X,S) = X^2 + bS^2/n$.

In this paper we study the admissibility of the resulting estimator

$$\delta_0(X,S) = X^2 + a\ S^2$$

for the scaled quadratic loss:

$$L(\xi,\sigma,\delta) = (\theta - \delta)^2\ \sigma^{-4}$$

under which δ_0 has a constant risk. Notice that this estimator is inadmissible if $0 \leq b < n^{-1}$. Indeed in this case θ is nonnegative but with positive probability δ_0 is negative. Therefore $\max(\delta_0, 0)$ is strictly better than δ_0.

In Section 2 we establish the inadmissibility result for a larger region of b-values, namely, for b such that $n^{-1} \leq b < 1 + 2n^{-1}$. In view of this result and other inadmissibility results in estimation problems of σ^2 (see Stein (1964), Brown (1968), Brewster and Zidek (1974), Strawderman (1974)) one may conjecture that δ_0 is inadmissible for all values of b. However this conjecture is false, and we prove the admissibility of δ_0 in Section 4 of this paper in the case $1 + 2n^{-1} \leq b < 2 + 3n^{-1}$. In fact we show in Section 3 that in this case δ_0 is generalized Bayes procedure with respect to a prior which admits a good approximation (in terms of posterior risk) by probability priors. Similar admissibility phenomenon also happens in the estimation problems of other functions of normal or exponential parameters (see Rukhin (1985), (1986a)). The results of Section 3 also suggest the inadmissibility of δ_0 for large values of b, $b \geq 2 + 3n^{-1}$. However the inadmissibility proof of Section 2 does not work in this situation and we prove the inadmissibility of δ_0 in Section 5 by a different method. In short, we show that δ_0 is admissible if and only if $1 + 2/n \leq b < 2 + 3/n$.

The admissibility of δ_0 within the class of quadratic estimators in a more general setting has been studied by Ping, Wu and Li (1983).

2. Inadmissibility Result for Small b.

In this Section we establish the inadmissibility of δ_0 in the case $n^{-1} \leq b < 1 + 2n^{-1}$, or $0 \leq an < 1$. We prove this result by exhibiting a class of estimators improving upon δ_0. These estimators have the form

$$\delta(X,S) = \delta_0(X,S) - 2n^{-1}S^2h(n^{\frac{1}{2}}|X|/S)$$

where h is a measurable function. In other terms δ is a scale equivariant estimator.

Theorem 1. For $0 \leq an < 1$ estimator of the form (2.1) has smaller quadratic risk than δ_0 if

$$0 \geq h(z) \geq (1-an)\min[0,(n+1)(n+2)^{-1} - (1+z^2)^{-1}]. \tag{2.2}$$

Proof. The risk function of any estimator (2.1) depends only on $|\xi|/\sigma$. Denote

$$\eta = n^{\frac{1}{2}}|\xi|/\sigma,\ d_k = \int_0^\infty \exp(-y^2/2)y^k dy = 2^{(k-1)/2}\Gamma((k+1)/2),$$

so that $d_{k+2} = (k+1)d_k$ for all positive k and put $C = 4/[d_{n-2}n(2\pi)^{\frac{1}{2}}]$. One has

$$E_{\xi\sigma}[(\delta_0 - \theta)^2 - (\delta - \theta)^2]\sigma^{-4}$$

$$= E[(\delta_0 - \eta^2/n - b)^2 - (\delta_0 - n^{-1}\ S^2h - \eta^2/n - b)^2]$$

$$= 4n^{-1}E_{\eta 1}S^2h(n^{\frac{1}{2}}|X|/S)(X^2 + aS^2 - n^{-1}\ S^2h - \eta^2/n - b)$$

$$= C \int_{-\infty}^{\infty} \int_{0}^{\infty} s^n\, h(x/s)\, \exp\{-(x-\eta)^2/2 - s^2/2\}$$

$$[x^2 + ans^2 - s^2 h(x/s) - \eta^2 - bn]\, dxds$$

$$= C \exp\,(-\eta^2/2) \int_{-\infty}^{\infty} \int_{0}^{\infty} s^{n+1} h(z)\, \exp\{-s^2(1+z^2)/2 + zs\eta\}$$

$$[z^2 s^2 + ans^2 - s^2 h(z) - \eta^2 - bn]\, dzds$$

$$= 2C \exp(-\eta^2/2) \sum_{k=0}^{\infty} \eta^{2k}/(2k)!$$

$$[d_{n+3+2k} \int_0^{\infty} z^{2k}(1 + z^2)^{-(n+4+2k)/2} h(z)\, [z^2 + an - h(z)]dz$$

$$- d_{n+1+2k}(\eta^2 + bn) \int_0^{\infty} z^{2k}\, (1 + z^2)^{-(n+2+2k)/2}\, h(z)dz]$$

$$= 2C \exp(-\eta^2/2) \sum_{k=0}^{\infty} [\eta^{2k}/(2k)!]\, d_{n+1+2k}$$

$$\int_0^{\infty} z^{2k}\, (1+z^2)^{-(n+2k+2)/2} h(z)$$

$$[-bn + (z^2 + an - h(z))\, (n+2+2k)\, (1 + z^2)^{-1}$$

$$- 2k(2k - 1)\, (1 + z^2)/[z^2(n + 2k)]]dz.$$

It follows that δ with a nonpositive function h improves upon δ_0 if for all $k = 0, 1, \ldots$

$$-bn + (z^2 + an - h(z))\ (n+2+2k)/(1+z^2) \leq 2k(2k-1)t^2/(n+2k),$$

$$t^2 = (1+z^2)/z^2.$$

This inequality holds for a nonpositive h if

$$(1-an+h(z))/(1+z^2) \geq -bn/(n+2+2k) + 1 - 2k(2k-1)t^2[(n+2k)\ (n+2k+2)]] \quad (2.3)$$

which can be valid on the set of values of z such that

$$(1-an)/(1+z^2) \geq 1 - \inf_{k \geq 0}[bn/(n+2+2k) + 2k(2k-1)t^2/[(n+2k)\ (n+2k+2)]]. \quad (2.4)$$

It is easy to check that since $t^2 > 1$ the infinum in (2.4) is attained at $k = 0$, so that the desired values z are such that

$$(1-an)/(1+z^2) \geq 1 - bn/(n+2) = (1-an)\ (n+1)/(n+2).$$

It follows now from (2.3) that any nonpositive function h under condition (2.2) produces an estimator (2.1) which is better than δ_0.

This method can also be used to prove the inadmissibility of δ_0 when $bn > (n+2)^2$. In this case the improving function h is positive and

$$h \leq \max\{0, z^2 + an - (1 + z^2) \max[bn/(n+2), 1 + z^{-2}]\}.$$

However, the proof fails when $2n + 3 \leq bn \leq (n+2)^2$ and δ_0 is in fact inadmissible.

3. Generalized Bayes Estimators of θ.

Let $\lambda(\xi,\sigma)$ be the density of the (generalized) prior distribution over $\{(\xi,\sigma), \sigma > 0\}$ with respect to the uniform measure $d\xi d\sigma/\sigma$. Notice that the latter is traditionally used since it corresponds to the right Haar measure over the group of linear transformations of the real line.

The Bayes estimator δ_B has the form

$$\delta_B(x,s) = \frac{\int_{-\infty}^{\infty}\int_{0}^{\infty} [\xi^2 + b\sigma^2]\sigma^{-n-4} \exp\{-[n(x-\xi)^2 + s^2]/2\sigma^2\}\lambda(\xi,\sigma)d\xi d\sigma/\sigma}{\int_{-\infty}^{\infty}\int_{0}^{\infty} \sigma^{-n-4} \exp\{-[n(x-\xi)^2 + s^2]/2\sigma^2\}\lambda(\xi,\sigma)d\xi d\sigma/\sigma} .$$

Assume that the following integrations by parts are legitimate:

$$[s^2 + n(x-\xi)^2] \int_0^{\infty} \lambda\, \sigma^{-n-5} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}d\sigma$$

$$= \int_0^{\infty} [(n+2)\lambda - \sigma\lambda_\sigma]\sigma^{-n-3} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\sigma, \quad (3.1)$$

$$\int_{-\infty}^{\infty} (\xi-x)\xi\lambda \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\xi$$

$$= \sigma^2 n^{-1} \int_{-\infty}^{\infty} [\lambda + \xi\lambda_\xi] \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\xi \qquad (3.2)$$

and

$$\int_{-\infty}^{\infty} (\xi-x)^2\lambda \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\xi$$

$$= \sigma^2 n^{-1} \int_{-\infty}^{\infty} [\lambda + \sigma^2 n^{-1} \lambda_{\xi\xi}] \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\, d\xi. \qquad (3.3)$$

Combining these formulae we derive the following representation of the Bayes estimator

$$\delta_B(x,s) - x^2 - as^2 = \int_{-\infty}^{\infty}\int_0^{\infty} [b\sigma^2 + (an-1)\ (\xi-x)^2 + 2\xi(\xi-x) - a(s^2+n(x-\xi)^2)]$$

$$\times \lambda\ \sigma^{-n-3} \exp\{-[n(x - \xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma$$

$$/ \int_{-\infty}^{\infty}\int_0^{\infty} \lambda\ \sigma^{-n-5} \exp\{-[n(x - \xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma$$

$$= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} [\mathcal{D}\lambda]\sigma^{-n-3} \exp\{-[n(x - \xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma$$

$$/\; n \int_{-\infty}^{\infty}\int_{0}^{\infty} \lambda\, \sigma^{-n-5} \exp\{-[n(x - \xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma \tag{3.4}$$

where

$$\mathcal{D}\lambda = (a - n^{-1})\, \sigma^2\, \lambda_{\xi\xi} + 2\xi\lambda_{\xi} + an\sigma\lambda_{\sigma} + 2\lambda .$$

It follows from (3.4) that $\delta_B = \delta_0$ if and only if the prior density λ solves the parabolic differential equation

$$\mathcal{D}\,\lambda = 0\,. \tag{3.5}$$

For instance, the conjugate density $\lambda(\xi,\sigma) = \sigma^{-2/an}$ is a solution of (3.5). However (3.5) admits many other solutions. One of them is $\lambda = \tilde{\lambda}$

$$\tilde{\lambda}\,(\xi,\sigma) = \exp\{-n(2 - an)\xi^2/[2(an - 1)\sigma^2]\}/\sigma\,. \tag{3.6}$$

Notice that in the case $1 < an < 2$,

$$\int_{-\infty}^{\infty} \tilde{\lambda}\,(\xi,\sigma)\, d\xi < \infty$$

for all σ.

If an = 1, then the general solution of (3.5) has the form

$$\lambda(\xi,\sigma) = \sigma^{-2} f(\xi/\sigma^2) \tag{3.7}$$

where f is arbitrary smooth function.

These facts suggest the admissibility of δ_0 if $1 \leq an < 2$.

The following gives a heuristic argument in favor of inadmissibility of δ_0 when an > 2.

Put $\lambda(\xi,\sigma) = Z(\xi/\sigma^\beta, \sigma^{1-\beta})/\sigma^\beta$, where $\beta = 2/(an)$, $\beta > 1$. If $u = \xi/\sigma^\beta$, $v = \sigma^{1-\beta}$, then

$$\mathcal{D}\lambda = (a - n^{-1})\ v^{-2/(an-2)}\ [v^2 Z_{uu} + qvZ_v]$$

where $q = (an - 2)n/(an - 1)$.

The equation

$$v^2\ Z_{uu} + qvZ_v = 0 \tag{3.8}$$

for q > 0 is closely related to the adjoint heat equation general solutions of which are known (see for example Widder (1975)).

However none of these solutions corresponds to a prior density which admits "good" approximation by proper prior densities in terms of posterior risk. (Exactly this condition is known to be responsible for the admissibiilty of generalized Bayes estimators, see Farrell (1968)).

Indeed in the estimation problem of σ^2 (which formally corresponds to $a \to \infty$ in (3.4)) we see that if $\delta_1(x,s)$ is generalized Bayes estimator of σ^2 under loss $(\delta - \delta^2)^2\sigma^{-4}$ and smooth prior density λ then

$$\delta_1(x,s) - s^2/(n+1)$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} [\sigma^2\lambda_{\xi\xi} + n\sigma\lambda_{\sigma}]\sigma^{-n-3} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma$$

$$/n \int_{-\infty}^{\infty}\int_{0}^{\infty} \lambda\sigma^{-n-5} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\}\, d\xi d\sigma.$$

Since $s^2/(n+1)$ is an inadmissible estimator of σ^2 if follows that the equation

$$\sigma^2\, \lambda_{\xi\xi} + n\sigma\, \lambda_{\sigma} = 0$$

(which essentially coincides with (3.8)) cannot have solutions approximable by proper prior densities. Therefore (3.8) cannot have such solutions either. A similar argument was used in Rukhin (1986b) to prove the inadmissibility of the traditional estimator of a linear function of the mean and the variance.

Theorem 2. Assume that the prior density λ is differentiable and the integrations by parts in (3.1)-(3.3) are legitimate. Then the generalized Bayes estimator δ_B has form (3.4) and $\delta_B = \delta_0$ if and only if (3.5) holds.

4. Admissibility Result

We prove here

Theorem 3. The estimator δ_0 is an admissible estimator of $\theta = \xi^2 + b\sigma^2$ under quadratic loss if $1 + 2n^{-1} \leq b < 2 + 3n^{-1}$, i.e. if $1 \leq an < 2$.

Proof. To prove the admissibility in the case $1 < an < 2$ we construct a sequence $h_m = h_m(\sigma)$, $m = 1, 2, \ldots$ such that $h_m(\sigma) \to 1$, $h_m > 0$,

$$\int_{\infty}^{\infty}\int_{0}^{\infty} \tilde{\lambda}_m (\xi,\sigma)\, d\xi d\sigma/\sigma < \infty \tag{4.1}$$

and as $m \to \infty$

$$\rho_m = \int_{-\infty}^{\infty}\int_{0}^{\infty} [E_{\xi\sigma}(\delta_0 - \theta)^2 - E_{\xi\sigma}(\delta_m - \theta)^2]\, \tilde{\lambda}_m(\xi,\sigma) d\xi d\sigma/\sigma \to 0. \tag{4.2}$$

Here δ_m denotes the Bayes estimator with respect to prior density $\tilde{\lambda}_m(\xi,\sigma) = \tilde{\lambda}(\xi,\sigma)h_m(\sigma)$ where $\tilde{\lambda}$ is defined by (3.6).

A straightforward calculation shows that with a generic constant A independent of m

$$\rho_m = A \int_{-\infty}^{\infty} dx \int_{0}^{\infty} s^{n-2} ds \Big[\int_{-\infty}^{\infty}\int_{0}^{\infty} \tilde{\lambda} h_m' \sigma^{-n-2} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\} d\xi d\sigma\Big]^2$$

$$/ \int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{\lambda}\, h_n \sigma^{-n-5} \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\} d\xi d\sigma$$

$$\leq A \int_{-\infty}^{\infty} dx \int_{0}^{\infty} s^{n-2}\, ds \int_{-\infty}^{\infty}\int_{0}^{\infty} \tilde{\lambda}\, [h_m']^2/h_m \exp\{-[n(x-\xi)^2 + s^2]/(2\sigma^2)\} d\xi d\sigma$$

$$= A \int_{0}^{\infty} \frac{[h_m'(\sigma)]^2}{h_m(\sigma)}\, \sigma d\sigma.$$

Condition (4.1) means that

$$\int_{0}^{\infty} h_m(\sigma)\, d\sigma/\sigma < \infty$$

so that if, for example,

$$h_m(\sigma) = (1 + (\log \sigma)^2/m)^{-1}$$

then (4.2) holds.

Similar argument (with $\tilde{\lambda}$ being replaced by (3.7)) proves the admissibility of δ_0 in the case an = 1. Notice that our proof essentially follows that given by Blyth (1951) (see also Farrell (1964)).

5. Inadmissibility Result for Large b.

In this section we prove the inadmissibility of δ_0 by obtaining an unbiased estimator of the risk of any procedure of the form (2.1) and by solving the resulting integro-differential inequality.

Theorem 4. The estimator δ_0 is inadmissible if $b \geq 2 + 3n^{-1}$ and $n > 3$.

Proof. Let δ be of the form (2.1) with bounded differentiable function h. Then

$$\Delta = E_{\xi\sigma} [(\delta_0 - \theta)^2 - (\delta - \theta)^2] \sigma^{-4}$$

$$= C \int_{-\infty}^{\infty} \int_{0}^{\infty} [z^2 s^2 + ans^2 - s^2 h - \eta^2 - bn] hs^{n+1}$$

$$\times \exp\{-[(sz - \eta)^2 + s^2]/2\} dzds \qquad (5.1)$$

One has

$$\int_{\infty}^{\infty} (z^2 s^2 - \eta^2) h \exp\{-(zs - \eta)^2/2\} dz$$

$$= \int_{-\infty}^{\infty} [(z + \eta/s)h' + h] \exp\{-(zs - \eta)^2/2\} dz$$

$$= \int_{-\infty}^{\infty} [h + 2zh' - h''/s^2] \exp\{-(zs - \eta)^2/2\} dz. \qquad (5.2)$$

Also for any bounded smooth function $f(z)$ and positive ν

$$\int_{\infty}^{\infty}\int_{0}^{\infty} f\, s^{\nu+1} \exp\{-[(zs - \eta)^2 + s^2]/2\, dzds$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} [\nu s^{\nu-1} - z(zs - \eta)s^{\nu}]\, f \exp\{-[(zs - \eta)^2 + s^2]/2\}dzds$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} [(\nu - 1)\, f - z\, f']\, s^{\nu-1} \exp\{-[(zs - \eta)^2 + s^2]/2\}dzds\,. \quad (5.3)$$

Because of (5.2) and (5.3) applied to $f = (an - h)h$, $\nu = n + 2$ and to $(\nu - 1)f - zf' = h''$, $\nu = n$ we see that (5.1) can be written as

$$\Delta = C \int_{-\infty}^{\infty}\int_{0}^{\infty} [\mathscr{D}h]s^{n+1} \exp\{-[(sz - \eta)^2 + s^2]/2\}dzds$$

where

$$\mathscr{D}h = -(n + 1)h^2 + 2zh'h - (an - 2)zh' - z^{n-1}\int_{z}^{\infty} h''(t)t^{-n}dt. \quad (5.4)$$

Therefore the inadmissibility of δ_0 will be established if one produces a solution to the inequality

$$\mathscr{D}h > 0.$$

In the case an > 2 we put

$$h(z) = \varepsilon(1 + \alpha^2 z^2)^{-2}.$$

Inequality (5.4) holds if

$$\varepsilon[-(n+1)\ (1+z^2)^{-4} - 8\ z^2\ (1+z^2)^{-5}] + 4(an-2)\ z^2(1+z^2)^{-3}$$

$$\geq 4\alpha^2 \int_1^\infty [5z^2t^2]\ (1 + z^2t^2)^{-4}\ t^{-n}\ dt.$$

For this inequality to be valid it suffices to choose ε and α so that

$$4\alpha^2 \int_1^\infty (1 + z^2t^2)^{-4}\ t^{-n}\ dt \geq \varepsilon(n + 1)\ (1 + z^2)^{-4} \tag{5.5}$$

and

$$(an-2)\ (1+z^2)^{-3} > 2\varepsilon(1+z^2)^{-5} + 5\alpha^2 \int_1^\infty (1+z^2t^2)^{-4}\ t^{2-n}\ dt. \tag{5.6}$$

Inequality (5.5) holds if

$$4\alpha^2/(n + 7) = \varepsilon(n + 1)$$

and (5.6) is met if

$$an - 2 \geq \varepsilon[2 + 5(n+1)\ (n+7)/(n-3)].$$

In the case an = 2 we put

$$h(z) = \varepsilon(1 - rz^2)\ (1 + z^2)^{-2} \quad \varepsilon,\ r > 0. \tag{5.7}$$

The inequality (5.4) in this case means that

$$\varepsilon[-(n+1)(1-rz^2)^2 + 4z^2(rz^2-r-2)/(1+z^2)](1+z^2)^{-4}$$

$$\geq \int_1^\infty \frac{[-6rz^4t^4 + 4z^2t^2(5 + 4r) - 2r - 4]}{(1 + z^2t^2)^4\ t^n}\, dt$$

which is implied by two inequalities

$$2(13r^2 + 34r + 25) \int_1^\infty (1 + z^2t^2)^{-4}\ t^{-r}\ dt$$

$$\leq \varepsilon(1+z^2)^{-4}\ [(2r(n+2) + n + 7)^2 - (n+1)\ (n+5)]/(n+5).$$

and

$$\varepsilon r\ (n+5)\ (1+z^2)^{-4}\ [z^2 - (2r(n+2) + n + 7)/(r(n+5))]^2$$

$$\leq 6 \int_1^{\infty} (z^2 t^2 - (4r+5)/3r)^2\ (1+z^2 t^2)^{-4}\ t^{-n}\ dt.$$

As in (5.5) the first of these inequalities can be used to determine ε, and the second is seen to hold for sufficiently small positive r.

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The following constitutes my response to A. Rukhin's paper "Quadratic estimators of quadratic functions of normal parameters."

With just a few cosmetic changes I feel that this manuscript is ready for publication. The proof of Theorem 1 has been corrected and the latter portion of the paper is much more readable because of the elimination of some unneeded notation and bulk of material.

Let (p,l) denote page p, line l (with negative l to mean l lines from the bottom).

Typo:

(3,6): Again, sigma sub zero should be delta sub zero.

Suggested grammatical changes:

Summary: "Necessary and sufficient. . ." to "A necessary and . . ."

(1,3): "We consider statistical . . ." to "We consider estimating theta=. . ., where b is a given nonnegative number , under quadratic loss."

(1,10-11): "In this situation the mean . . ." to "In this situation the mean of the y observations is . . ."

Before the last sentence of section 1: Might want to insert "In short, we show that delta sub zero is admissible if and only if $1+2/n \leq b < 2+3/n$."

(7,5): "Let lambda . . ." to "Let lambda(,) be the density of the (generalized) . . ."

(9,-6): "For instance conjugate" to "For instance, the conjugate. . ."

(11,7): Since s2/(n+1) is inadmissible" to "Since s2/(n+1) is an admissible"

(11,-3): "and integrations by parts" to "and the integrations by parts"

(13,-6): "Similar argument" to "A similar argument"

Closing question for the author: Any speculations as to the statement of Theorem 4 when n=2 or n=3?