SEMIMARTINGALES AND MEASURE PRESERVING FLOWS

by

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§1. INTRODUCTION.

The fundamental paper on combining helices with stochastic processes on filtered probability spaces by J. de Sam Lazaro and P. A. Meyer [10] was published in 1975, during the prehistory of semimartingales. In this article we use techniques developed to study semimartingales in a Markov process framework (i.e., [3]) to develop systematically a part of the program begun in [10]. This is the content of paragraph three, the chief result being Corollary (3.2).

Processes with stationary and independent increments are semimartingales and they have local characteristics of a very special form, due to the Lévy-Khintchine formula (c.f. [8, p. 92]). With the notation of paragraph five of this article the local characteristics of such a semimartingale are of the form:

$$(1.1) \qquad B_t = bt; \ C_t = ct; \ \tilde{\Gamma} = F(dy) \times dt.$$

In particular the local characteristics are non-random and moreover $\tilde{\Gamma}$ factors with one factor being Lebesgue measure. It is, perhaps, interesting to see how much of this characterization is retained by a semimartingale with stationary, but not necessarily independent, increments. Since these semimartingales can be realized as helices one can use Palm measure techniques and we obtain in paragraph five (i.e., (5.10)) a factorization of the local characteristics roughly analogous to (1.1). Paragraph four develops the necessary theory about random measures used in paragraph five.

§2. PRELIMINARIES.

We use in this article the basic framework and notation developed in the fundamental work of J. de Sam Lazaro and P. A. Meyer [10]; for semimartingales our notation is that of Meyer (c.f. [12], [4], or [11]).

We suppose given a probability space $(\Omega,\mathcal{A},P)$ and a group $(\theta_t)_{t\in\mathbb{R}}$ of measure-preserving transformations from Ω to Ω. (The group action is $\theta_t\theta_s = \theta_{t+s}$.) Moreover we assume there exists a sub σ-field $\mathcal{A}^0$ of $\mathcal{A}$ such that $\mathcal{A}$ is the P-completion of $\mathcal{A}^0$ and such that $(t,\omega) \longrightarrow \theta_t(\omega)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{A}^0/\mathcal{A}^0$ - measurable.

For a given σ-field $\mathcal{F}\subset\mathcal{A}$ such that $\theta_t^{-1}(\mathcal{F})\subset\mathcal{F}$ for $t \leq 0$, we can define a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ by setting

$$\mathcal{F}_0 = \mathcal{F} ; \; \mathcal{F}_t = \theta_t^{-1}(\mathcal{F}_0).$$

Thus $\mathcal{F}_{s+t} = \theta_t^{-1}(\mathcal{F}_s)$. This is an increasing family of σ-fields and we set $\mathcal{F}_{-\infty} = \bigcap_{t\in\mathbb{R}} \mathcal{F}_t$; $\mathcal{F}_\infty = \bigvee_{t\in\mathbb{R}} \mathcal{F}_t$. This filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ is a <u>filtration under the flow</u> $(\theta_t)_{t\in\mathbb{R}}$. Given a σ-field $\mathcal{F}^0$ inducing a filtration under $(\theta_t)_{t\in\mathbb{R}}$, we can set $\mathcal{F}$ to be the completion of $\mathcal{F}^0$ in $\mathcal{A}$. In this case $\mathcal{F}_t = \mathcal{F}_{t+}$; that is, the filtration is right continuous (cf [10, p. 4]). <u>We will always make the assumption that</u> $\mathcal{F}_\infty = \mathcal{A}$ <u>and</u> $\mathcal{F}_{-\infty}$ <u>contains all P-null sets,</u> for any filtration being considered.

A <u>helix</u> will mean a real-valued process $(Z_t)_{t\in\mathbb{R}}$ with right continuous paths, $Z_t \in \mathcal{F}$, $Z_0 = 0$, and such that

$$Z_{t+h} - Z_{s+h} = (Z_t - Z_s)\circ\theta_h \tag{2.1}$$

holds identically for all $s,t,h \in \mathbb{R}$. This is also called a perfect helix. A crude helix is an $\mathcal{F}_t$-adapted process with right continuous paths $(Z_t)_{t\in\mathbb{R}}$ with $Z_0 = 0$ and such that (2.1) holds a.s. for all $s,t,h \in \mathbb{R}$. The exceptional sets can depend on the choice of s,t, and h. The following fundamental result is due to J. de Sam Lazaro and P. A. Meyer, and uses the "perfection" techniques developed by John Walsh:

(2.2) THEOREM. *If Z is a crude helix then there exists a (perfect) helix, $\overline{Z}$, which is indistinguishable from Z.*

Since a process with stationary increments might naturally be interpreted as a crude helix, Theorem (2.2) is especially useful.

(2.3) DEFINITION. A right continuous process $(Z_t)_{t\in\mathbb{R}}$ will be called a semimartingale if: $Z_t \in \mathcal{F}_t$, $t \in \mathbb{R}$; $Z_0 = 0$ a.s.; and if $(Z_t)_{t\geq 0}$ is a semimartingale in the traditional sense (that is, there exist a local martingale $(M_t)_{t\geq 0}$ and a process $(A_t)_{t\geq 0}$ with paths of finite variation on compacts such that $Z_t = M_t + A_t$, $t\geq 0$ (cf [4] or [11])).

We will be interested here in helices that are semimartingales. These can be thought of, essentially, as semimartingales with stationary increments. Processes with stationary and independent increments are well known to be semimartingales and can easily be put into this framework (cf [10, pp. 30-32]). A time homogeneous Markov process with semigroup $(P_t)_{t\geq 0}$ and admitting an invariant measure (i.e., $\alpha P_t = \alpha$) can also be put into the helix framework (cf [10, p. 32]). A helix semimartingale defined within a Markov framework would correspond to the "additive semimartingales" studied in [3] or the generalized additive functionals studied in [13]. Other examples can be obtained from Lazaro's characterization of the space of square-integrable helix martingales under certain hypotheses [10].

Finally, one might wonder whether or not all "nice" helices are semimartingales. This is not the case: Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (B_t)_{t\geq 0}, (P^x)_{x\in\mathbb{R}})$ be the Dynkin representation of standard Brownian motion (cf [2]). One can fix P to be P^0 and extend B to t in $(-\infty,0]$ in the usual way. Then $(B_t)^{1/3}_{t\geq 0}$ is a helix but it is not a semimartingale (cf [3, p. 195]).

It will be convenient to use the idea of the "big shift", originally proposed by M. Sharpe for the study of Markov processes. We adapt it slightly here for our situation:

$$\Theta_h(Z_t - Z_s) = (Z_{t-h} - Z_{s-h}) \circ \theta_h$$

for any process Z, and h,s,t, $\in \mathbb{R}$. We then have the obvious

(2.4) PROPOSITION. *An adapted, right continuous process* Z *with* $Z_0 = 0$ *is a helix if and only if* $\Theta_h(Z_t - Z_s) = Z_t - Z_s$ *holds, all* t,s,h $\in \mathbb{R}$.

We write $\Theta_h Z$ to denote the process $(\Theta_h(Z_t - Z_0))_{t\geq 0}$.

We establish here some notation (Y is always adapted and right continuous):

$\mathcal{S}$ = {Y: $(Y_t)_{t\geq 0}$ is a semimartingale}

$\mathcal{S}_p$ = {Y: $(Y_t)_{t\geq 0}$ is a special semimartingale}

$\mathcal{V}$ = {Y: $(Y_t)_{t\geq 0}$ a.s. has paths of finite variation on finite intervals}

$\mathcal{V} = \{Y: (Y_t)_{t \geq 0}$ has paths of locally integrable variation$\}$

$\mathcal{L} = \{Y: (Y_t)_{t \geq 0}$ is a local martingale$\}$

$\mathcal{P} = \{Y: (Y_t)_{t \geq 0}$ is predictably measurable$\}$

We let L(Y) denote the (linear) space of predictable, Y-integrable processes for a semimartingale Y, and for $H \in L(Y)$, we will let both $H \cdot Y_t$ and $\int_0^t H_s dY_s$, $t \geq 0$, denote the stochastic integral of H with respect to Y. If $Y \in \mathcal{V}$, we let $\tilde{Y}$ denote its dual predictable projection (also known as the "compensation" of Y), and if $Y \in \mathcal{A}$, then Y^c will denote its continuous martingale part, $[Y,Y]$ will denote its quadratic variation process, and if $[Y,Y] \in \mathcal{V}$, then $\widetilde{[Y,Y]}$ will be sometimes denoted $\langle Y,Y \rangle$, to conform with the notation of [10].

(2.5) COMMENT. Helices are defined for all $t \in \mathbb{R}$, whereas objects such as semimartingales, dual projections, etc. are defined only for $t \geq 0$. When we come across a process Y defined for $t \geq 0$ and satisfying the shift identity (2.1), we can extend the process to $(-\infty, 0]$ by defining $Y_{-t} \equiv Y_0 \circ \theta_{-t}$ (for $t \geq 0$). The process Y now defined for all $t \in \mathbb{R}$ will continue to justify (2.1) and this will be a bona fide helix.

§ 3. SEMIMARTINGALES AND FLOWS

In this paragraph we will consider semimartingales and how they behave with respect to the "flow" operator $(\theta_t)_{t \in \mathbb{R}}$. Our results are analogous to those of paragraph three of [3]. Our main result is Theorem (3.1), while our most useful result is perhaps Corollary (3.2).

(3.1) THEOREM. Let $Y \in \mathcal{S}$, $s \geq 0$, and $Y_t = 0$ if $t \leq 0$. Then:

(i) $\Theta_s Y \in \mathcal{S}$. If $Y \in \mathcal{S}_p$ (respectively $\mathcal{L}$, $\mathcal{V}$, $\mathcal{A}$, $\mathcal{P} \cap \mathcal{V}$), then $\Theta_s Y \in \mathcal{S}_p$ (respectively $\mathcal{L}$, $\mathcal{V}$, $\mathcal{A}$, $\mathcal{P} \cap \mathcal{V}$).

(ii) If $Y \in \mathcal{S}_p$ has canonical decomposition $Y = M + A$, then $\Theta_s Y = \Theta_s M + \Theta_s A$ is the canonical decomposition of $\Theta_s Y$.

(iii) If $Y \in \mathcal{A}$, then $\widetilde{\Theta_s Y} = \Theta_s \tilde{Y}$.

(iv) We have $(\Theta_s Y)^c = \Theta_s(Y^c)$.

(v) We have $[\Theta_s Y, \Theta_s Y] = \Theta_s([Y,Y])$.

(vi) If $H \in L(Y)$ then $\Theta_s H \in L(\Theta_s Y)$ and $(\Theta_s H)\cdot(\Theta_s Y) = \Theta_s(H\cdot Y)$.

(vii) If $Y \in \mathcal{L}$ is square-integrable, then $\langle\Theta_s Y, \Theta_s Y\rangle = \Theta_s\langle Y,Y\rangle$.

The proof of Theorem (3.1) follows the proof of Lemma (3.7). Our next corollary contains some results already well known: for example part (vii), that $\langle M,M\rangle$ is a helix if M is a square-integrable martingale helix, is proved by Lazaro and Meyer in [10, p. 54].

(3.2) COROLLARY. Let $Y \in \mathcal{S}$ be a helix. Then

(i) There exist helices $M \in \mathcal{L}$, $A \in \mathcal{V}$, such that $Y = M + A$.

(ii) If $Y \in \mathcal{S}_p$ with canonical decomposition $Y = M + A$, then both M and A are helices.

(iii) *If* $Y \in \mathcal{S}$, *then the dual predictable projection*, $\widetilde{Y}$, *is a helix*.

(iv) *The continuous local martingale part of* Y, Y^c, *is a helix*.

(v) *The process* $[Y,Y]$ *is a helix*.

(vi) *If* $H \in L(Y)$ *is homogeneous* (i.e., $H_t \circ \theta_s = H_{t+s}$, all t,s) *then there exists a version of the stochastic integral* $H \cdot Y$ *which is a helix*.

(vii) *If* $Y \in \mathcal{L}$ *is a square integrable then* $\langle Y,Y \rangle$ *is also a helix*.

PROOF. We first note that Y being a helix means $Y_t \neq 0$ for all $t \leq 0$ in general, whereas in Theorem (3.1) we assumed $Y_t = 0$ all $t \leq 0$. This does not really pose a problem, however, since $Y_0 = 0$ for all helices Y which implies $(\theta_s Y)_t = Y_t$ for all $s \geq 0$, which is the crucial consequence of the assumption that $Y_t = 0$, $t \leq 0$ used in Theorem (3.1).

We begin the proof of (ii).

Let $Y \in \mathcal{S}_p$ and let $Y = M+A$ be its canonical decomposition. For $s \geq 0$, $\theta_s Y = \theta_s M + \theta_s A$ is the canonical decomposition of $\theta_s Y$ by Theorem (3.1). Moreover since Y is a helix, $(\theta_s Y)_t = \theta_s(Y_t - Y_0) = (Y_{t-s} - Y_{-s}) \circ \theta_s = Y_t - Y_0 = Y_t$; therefore $(\theta_s Y)_t = Y_t = \theta_s M + \theta_s A$; by the uniqueness of the canonical decomposition we have $\theta_s M = M$ and $\theta_s A = A$. Since this holds for all $s \geq 0$, we conclude M and A are helices by Proposition (2.4). The proofs of (iii), (iv), (v) and (vi) are analogous. Statement (vii) is an immediate consequence of (iii) and (v), since $\langle Y,Y \rangle$ is the dual predictable projection of $[Y,Y]$.

It remains to prove (i). Let $Y \in \mathcal{S}$ be a helix and let $\Delta Y_t = Y_t - Y_{t-}$, which is defined up to an evanescent set. We set

$$J_t = \sum_{0<s\leq t} \Delta Y_s \, 1_{\{|\Delta Y_s|>1\}} \, .$$

Then J is a helix in $\mathcal{V}$ as is easily checked, and hence $Y' = Y-J$ is a helix in $\mathcal{S}$. Since $|\Delta Y'|\leq 1$, we know that $Y \in \mathcal{S}_p$. Letting $Y' = M'+A'$ be its canonical decomposition, by (ii) we have that M' and A' are both helices. Thus $Y = M' + \{A' + J\}$ is a decomposition of Y into helices. □

We now present five lemmas which lead to the proof of Theorem (3.1).

(3.3) LEMMA <u>Let</u> $X \in b\mathcal{F}_\infty$, <u>and</u> <u>let</u> ${}^\pi X = E\{X|\mathcal{F}_t\}$, $t \geq 0$, <u>taking</u> <u>the</u> <u>right</u> <u>continuous</u> <u>version</u>; <u>and</u> <u>we</u> <u>set</u> ${}^\pi X_t=0$ <u>for</u> $t\leq 0$. <u>Then</u> <u>for</u> <u>all</u> $t\geq s\geq 0$, <u>we</u> <u>have</u> $(\Theta_s {}^\pi X)_t = {}^\pi(X\circ\theta_s)_t$.

PROOF. Let $t \geq s \geq 0$ and $W \in b\mathcal{F}_{t-s}$. Then

$$E\{W\circ\theta_s (\Theta_s {}^\pi X)_t\}$$

$$= E\{W\circ\theta_s {}^\pi X_{t-s}\circ\theta_s\}$$

$$= E\{W {}^\pi X_{t-s}\}, \text{ since } \theta_s \text{ is measure preserving}$$

$$= E\{WX\}, \text{ since } W \in \mathcal{F}_{t-s},$$

$$= E\{WX\circ\theta_s\} = E\{W\circ\theta_s X\circ\theta_s\}$$

$$= E\{W\circ\theta_s {}^\pi(X\circ\theta_s)_t\}$$

since $W \circ \theta_s \in \mathcal{F}_t$. Moreover since $\mathcal{F}_t = \theta_s^{-1}(\mathcal{F}_{t-s})$, random variables of the form $W \circ \theta_s$, $W \in b\mathcal{F}_{t-s}$, generate $b\mathcal{F}_t$, and the result follows. □

(3.4) LEMMA. Let $Y \in \mathcal{L}$ be such that the jump process ΔY is bounded by a constant c. Let $Y_t = 0$, $t \leq 0$. Then $\theta_s Y \in \mathcal{L}$, every $s \geq 0$.

PROOF. Let $T_n = \inf\{t: |Y_t| > n\}$, and $T_n' = \inf\{t: |(\theta_s Y)_t| > n\}$. Then $T_n' = \inf\{t>0: |Y_{t-s} \circ \theta_s| > n\} = s + T_n \circ \theta_s$, hence $[\theta_s(Y_{T_n \wedge \cdot})_t] = (\theta_s Y)_{T_n' \wedge t}$. Since $(Y_{t \wedge T_n})_{t \geq 0}$ is a martingale bounded by n+c, using the notation of Lemma (3.3) we have:

$$Y_{T_n \wedge t} = {}^{\pi}(Y_{T_n})_t,$$

Therefore $(\theta_s Y)_{T_n' \wedge t} = {}^{\pi}(Y_{T_n} \circ \theta_s)_t$ a.s. if $t \geq s$, and $(\theta_s Y)_t = 0$ if $t \leq s$. Therefore $((\theta_s Y)_{T_n' \wedge t})_{t \geq 0}$ is a martingale, each n. Since $\lim_{n \to \infty} T_n' = \infty$ a.s., we conclude that $(\theta_s Y) \in \mathcal{L}$. □

(3.5) LEMMA. Let Y^n be a sequence of elements of $\mathcal{A}$ such that $Y^{n+1} - Y^n \in \mathcal{A}^+$, nonnegative and increasing. Then $\tilde{Y}^{n+1} - \tilde{Y}^n \in \mathcal{A}^+$. Moreover if $Y = \sup_n Y^n$ and $\tilde{Y}' = \sup_n \tilde{Y}^n$, then $Y \in \mathcal{A}^+$ if and only if $\tilde{Y}' \in \mathcal{A}^+$, in which case $\tilde{Y} = \tilde{Y}'$ a.s.

PROOF. This lemma is taken from [3]. The first statement is clear. For the second, note that the dual predictable projection $\tilde{Z}$ of a process $Z \in \mathcal{A}^+$ is characterized by its predictability and the property that $E\{\tilde{Z}_T\} = E\{Z_T\}$ for every finite stopping time T, and then apply the monotone convergence theorem. □

(3.6) LEMMA. Let $Y \in \mathscr{A}$ with $Y_t = 0$ for $t \leq 0$. Let $\tilde{Y}$ be its dual predictable projection. Then $\Theta_s Y \in \mathscr{A}$ and $\widetilde{\Theta_s Y} = \Theta_s \tilde{Y}$ for every $s \geq 0$.

PROOF. It suffices to prove this when Y is positive and increasing (i.e. $Y \in \mathscr{A}^+$). Let $Y^n = Y \wedge n = \min(Y,n)$. Then $\Delta Y^n = |\Delta Y^n| \leq n$, and of course $Y \in \mathcal{S}_p$. Let $Y^n = M^n + A^n$ be its canonical decomposition ($M^n \in \mathscr{L}$, $A^n \in \mathscr{A}^+$ and predictable). Then $A^n = \tilde{Y}^n$, and $\tilde{Y} = \sup_n \tilde{Y}^n$. Moreover it is well known (e.g. [4]) that $|\Delta Y^n| \leq n$ implies that $|\Delta M^n| \leq 2n$. Then Lemma (3.4) implies $\Theta_s M^n \in \mathscr{L}$. Since $Y^n = M^n + \tilde{Y}^n$, also $\Theta_s Y^n = \Theta_s M^n + \Theta_s \tilde{Y}^n$, and since $\Theta_s M^n \in \mathscr{L}$ we must have that $\Theta_s \tilde{Y}^n$ is the dual predictable projection of $\Theta_s Y^n$.

Since $\tilde{Y} \in \mathscr{V}$ is predictable, it is clear that $\Theta_s \tilde{Y} \in \mathscr{V}$ and predictable as well. This implies $\Theta_s \tilde{Y} \in \mathscr{A}$ (e.g., [8, p. 17]). But $\Theta_s Y = \sup_n \Theta_s Y^n$, and $\Theta_s \tilde{Y} = \sup_n \Theta_s \tilde{Y}^n$. Therefore $\Theta_s \tilde{Y}$ is the dual predictable projection of $\Theta_s Y$ by Lemma (3.5). □

(3.7) LEMMA. If $Y \in \mathscr{L}$ and $Y_t = 0$ for $t \leq 0$, then $\Theta_s Y \in \mathscr{L}$ and $(\Theta_s Y)^c = \Theta_s(Y^c)$.

PROOF. Let $J_t^n = \sum_{0<s\leq t} \Delta Y_s 1_{\{|\Delta Y_s|>1/n\}}$, and let $N^n = J^n - \tilde{J}^n$, where $\tilde{J}^n$ is the dual predictable projection of J^n. Then $Y-N^1 \in \mathscr{L}$ and $|\Delta(Y-N^1)| \leq 2$. Therefore by Lemma (3.4) we have $\Theta_s(Y-N^1) \in \mathscr{L}$. Lemma (3.5) implies $\Theta_s N^1 \in \mathscr{L}$. Therefore $\Theta_s Y \in \mathscr{L}$.

Now that we know $\Theta_s Y \in \mathscr{L}$, we may consider its continuous local martingale part $(\Theta_s Y)^c$. Set $K_t^n = \sum_{0<u\leq t} \Delta(\Theta_s Y)_u 1_{\{|\Delta(\Theta_s Y)_u|>1/n\}}$. Then $\Theta_s J^n = K^n$, and as we have seen from Lemma (3.6), $\tilde{K}^n = \widetilde{\Theta_s J^n} = \Theta_s \tilde{J}^n$. Recall $N^n = J^n - \tilde{J}^n$, and we conclude that $\lim_n(\Theta_s N^n)_t = (\Theta_s Y)_t - (\Theta_s Y)_t^c$ with convergence in probability. Since $\lim_n N_t^n = Y_t - Y_t^c$ (in probability), and since Θ_s is measure preserving, we have that $(\Theta_s Y)_t^c = \Theta_s(Y^c)_t$ a.s. when $t \geqq s$. Since $(\Theta_s Y)_t^c = \Theta_s(Y^c)_t = 0$ for $t<s$, the proof is complete. □

<u>PROOF of Theorem</u> (3.1): Statement (iii) is the content of Lemma (3.6) and thus already established.

If $Y \in \mathcal{V}$ (respectively $\mathcal{P} \cap \mathcal{V}$), then $\Theta_s Y \in \mathcal{V}$ (respectively $\mathcal{P} \cap \mathcal{V}$). But Lemma (3.7) showed that $Y \in \mathcal{L}$ implies $\Theta_s Y \in \mathcal{L}$ and statements (i) and (ii) follow easily.

Concerning (iv), let $Y = M + A$ with $M \in \mathcal{L}$, be a decomposition. Then $Y^c = M^c$, and similarly $(\Theta_s Y)^c = (\Theta_s M)^c$, and thus (iv) follows from Lemma (3.7).

Concerning (v), fix a t and let $I(n,t) = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a subdivision of $[0,t]$ such that $\lim_{n\to\infty}$ mesh $I(n,t) = 0$. Let $V_{I(n,t)} = Y_0^2 + \sum_{i=1}^{n}(Y_{t_i} - Y_{t_{i-1}})^2$. It is well known that $\lim_{n\to\infty} V_{I(n,t)} = [Y,Y]_t$ for any semimartingale Y, with convergence in probability. Thus

$$[Y,Y]_t \circ \theta_s = P - \lim V_{I(n,t)} \circ \theta_s$$

$$= P - \lim[(\Theta_s Y)_s^2 + \sum_{i=1}^{n}((\Theta_s Y)_{s+t_i} - (\Theta_s Y)_{s+t_{i-1}})^2],$$

and since $(\Theta_s Y)_u = 0$ for $u<s$, the above limit is $[\Theta_s Y, \Theta_s Y]_{s+t}$ a.s., and (v) follows.

Statement (vii) follows from (v) and (iii), since $\langle Y,Y\rangle$ is the dual predictable projection for $Y \in \mathcal{L}$ and square integrable.

It remains to prove (vi). Let $Y \in \mathcal{S}$, $H \in L(Y)$. Set $D = \{|\Delta Y| > 1\} \cup \{|H\Delta Y| > 1\}$, and $Y'_t = Y_0 + \sum_{0<s\leq t} \Delta Y_s 1_D(s)$. Let $Y'' = Y-Y'$. Then $Y' \in \mathcal{V}$, $Y'' \in \mathcal{S}_p$, and $|\Delta Y''| \leq 1$. Let $Y'' = M + A$ be the canonical decomposition of Y''. The assumption that $H \in L(Y)$ implies that the Stieltjes integral process $H\cdot Y'$ and $H\cdot A$ exist, and also that the stochastic integral $H\cdot M$ exists (c.f. [8, p. 54]). Hence $H^2\cdot[M,M] \in \mathcal{A}^+$. Since the following are pathwise Stieltjes integrals it is simple to verify:

$$(\Theta_s H)\cdot(\Theta_s Y') = \Theta_s(H\cdot Y')$$

$$(\Theta_s H)\cdot(\Theta_s A) = \Theta_s(H\cdot A)$$

$$(\Theta_s H)^2\cdot[\Theta_s M,\Theta_s M] = \Theta_s(H^2\cdot[M,M])$$

where (v) is used in the last equality above. By (i) the first two processes are in $\mathcal{V}$, and the third is in $\mathcal{A}$. Therefore $\Theta_s H$ is integrable with respect to $\Theta_s Y'$, $\Theta_s A$, and $\Theta_s M$. Hence $\Theta_s H \in L(\Theta_s Y)$.

Now let H be simple predictable: that is, let H be of the form

$$(3.8) \qquad H_t = \sum_{i=1}^{n} H^i 1_{(t_{i-1},t_i]}(t).$$

Then $H\cdot Y_t = \sum_{i=1}^{n} H^i(Y_{t\wedge t_i} - Y_{t\wedge t_{i-1}})$ and a simple computation shows that

$$\Theta_s(H\cdot Y) = (\Theta_s H)\cdot(\Theta_s Y)$$

for $Y \in \mathcal{S}$ and H simple predictable. Let $\mathcal{H}$ be the linear space of all bounded, predictable processes for which (vi) holds. Let (H^n) be a sequence of elements of $\mathcal{H}$ which converge uniformly to a bounded process H. Let $Z_t = H \cdot Y_t$. Then $P - \lim_n (H^n \cdot Y)_t = Z_t$, each $t \geq 0$. Therefore $\lim_n \Theta_s(H^n \cdot Y)_t = (\Theta_s Z)_t$ a.s., as well. Moreover $\Theta_s H^n$ converges in probability to $\Theta_s H$, and hence $\lim_n \Theta_s H^n \cdot \Theta_s Y = \Theta_s H \cdot \Theta_s Y$. Since $H^n \in \mathcal{H}$, we have

$$\Theta_s(H \cdot Y)_t = \lim_n \Theta_s(H^n \cdot Y)_t$$

$$= \lim_n (\Theta_s H^n \cdot \Theta_s Y)_t$$

$$= (\Theta_s H \cdot \Theta_s Y)_t.$$

$\mathcal{H}$ contains all processes of the form (3.8), hence a monotone class argument shows that $\mathcal{H}$ is exactly the set of all bounded, predictable processes.

For general $H \in L(Y)$, set $H^n = H 1_{\{|H| \leq n\}}$. Then $P - \lim (H^n \cdot Y)_t = Z_t$ for $t \geq 0$. Since (vi) holds for each H^n, it will hold for general $H \in L(Y)$ by passing to the limit (with convergence in probability). □

We next prove a Radon-Nikodyn type theorem (Theorem (3.11) for helices. These results are already known (c.f., [7], [10, p. 45] or [9]) with a different method of proof. This result is analogous to Motoo's theorem for Markov processes.

(3.9) LEMMA. Let A be an increasing helix and suppose $dA_t << dt$ a.s. Then there exists a r.v. $\varphi \in \mathcal{F}_0$ such that $A_t = \int_0^t \varphi_s \, ds$ a.s., where $\varphi_s = \varphi \circ \theta_s$.

PROOF. Let $Z_t = \liminf_{s\downarrow 0,\ s\in\mathbb{Q}} (A_{t+s}-A_t)/s$. Then by Lebesgue's derivation theorem we have $A_t(\omega) = \int_0^t Z_s(\omega)ds$ for all $t \geq 0$. That A is a helix implies Z is homogeneous: $Z_t = Z_0 \circ \theta_t$. Z_0 is in $\mathfrak{F}_0$ by the right continuity of the filtration. Taking $\varphi = Z_0$ completes the proof. □

(3.10) THEOREM. <u>Let</u> A, $A' \in \mathcal{V}$ <u>both be helices, continuous, and suppose that for any</u> $\varphi \in \mathfrak{F}_0$ <u>such that</u> $\varphi \cdot A = 0$ $((\varphi \cdot A)_t = \int_0^t \varphi_s \, dA_s$ <u>where</u> $\varphi_s = \varphi \circ \theta_s)$ <u>we have</u> $\varphi \cdot A' = 0$. <u>Then there exists</u> $\psi \in \mathfrak{F}_0$ <u>such that</u> $A' = \psi \cdot A$.

PROOF. Without loss of generality (c.f. (2.5)), we take our processes to be defined only on $\Omega x[0,\infty)$. Let A^+, A^- (respectively A'^+, A'^-) be the positive and negative variation processes of A (respectively A'), and set $C = A^+ + A^-$; $C' = A'^+ + A'^-$. All of these processes are crude helices as is easily verified; we take their perfect helix versions. Next set $F_t = t + C_t + C'_t$, and let $\tau_t = \inf\{s>0: F_s>t\}$, the right continuous inverse of F. Define $\hat{A}^+_t = A^+_{\tau_t}$; $\hat{A}^-_t = A^-_{\tau_t}$; $\hat{A}'^+_t = A'^+_{\tau_t}$; $\hat{A}'^-_t = A'_{\tau_t}$, and $\hat{\theta}_t = \theta_{\tau_t}$. Then all the time changed processes are still "helices" under $\hat{\theta}$ (that is, they satisfy (2.1)). (For the proofs of these claims we refer the reader to Lazaro and Meyer [10, p. 48].) The process $d\hat{A}^+_t << dt$ by Lebesgue's change of time lemma (c.f., e.g. [2, p. 206]), hence by Lemma (3.9) we have $\hat{A}^+_t = \int_0^t h^+_s ds = \int_0^t h^+ \circ \theta_{\tau_s} \, ds$ for a r.v. $h^+ \in \mathfrak{F}_{\tau_0} = \mathfrak{F}_0$. We obtain analogous results for $\hat{A}^-$, $\hat{A}'^+$, $\hat{A}'^-$ which yield, respectively, random variables h^-, h'^+, h'^-. Time changing back yields $A^+_t = \int_0^t h^+_s dF_s$, $A^-_t = \int_0^t h^-_s dF_s$, etc. The hypotheses on A and A^- imply that $\{h^+=h^-\} \subset \{h'^+ = h'^-\}$ up to a set of μ-measure 0 where μ is the measure on $\mathbb{R}_+ x\Omega$ given by $\mu(H) = E\{\int_0^t H_s dF_s\}$.

Therefore we set

$$\psi = [(h'^{+} - h'^{-})/(h^{+} - h^{-})]\, 1_{\{h^{+} \neq h^{-}\}},$$

and we obtain

$$A'_t = A'^{+}_t - A'^{-}_t = \int_0^t h'^{+}_s dF_s - \int_0^t h'^{-}_s dF_s$$

$$= \int_0^t (h'^{+}_s - h'^{-}_s) dF_s$$

$$= \int_0^t \psi_s (h^{+}_s - h^{-}_s) dF_s$$

$$= \int_0^t \psi_s dA_s . \qquad \square$$

The following theorem covers the discontinuous case (again, we are taking our process to be defined only on $\Omega \times [0,\infty)$).

(3.11) THEOREM. Let $A, A' \in \mathcal{V}$ be helices such that for φ a homogeneous process $\int_0^t \varphi_s dA_s = 0$ implies $\int_0^t \varphi_s dA'_s = 0$ a.s. Then there exists a homogeneous process ψ_s such that $A'_t = \int_0^t \psi_s dA_s$.

PROOF. Set $A^c_t = A_t - A_0 - \sum_{0<s\leq t} \Delta A_s$, and define A'^c analogously. Then A^c and A'^c verify the hypotheses of Theorem (3.10) whence $A'^c = \int_0^t \gamma_s dA^c_s$. Let $D = \{\Delta A = 0\}$. Then 1_D is a homogeneous process and $1_D \cdot A = 0$. Hence by hypothesis $1_D \cdot A' = 0$ a.s. as well. Hence up to an evanescent set $\{\Delta A' \neq 0\} \subset \{\Delta A = 0\}$. Next set

$$\psi_s = \gamma_s 1_D + \frac{\Delta A'}{\Delta A} 1_{D^c} \qquad \text{and}$$

the result follows. $\square$

COMMENT. In theorems (3.10) and (3.11) the only time we used that (θ_t) are measure preserving was when we used perfect versions of the helices. Thus these results essentially hold in any such "shift" framework.

§4. RANDOM MEASURES AND FLOWS

In this paragraph G will denote a Borel subset of a compact metric space (i.e., a Lusin space), and $\mathcal{G}$ will denote its Borel σ-field. We let $\hat{\mathcal{P}}$ denote the σ-field $\mathcal{P}\otimes\mathcal{G}$ on $\Omega \times \mathbb{R} \times G$, where $\mathcal{P}$ (resp. $\mathcal{O}$) is the smallest σ-field on $\Omega\times\mathbb{R}$ containing the adapted processes whose paths are left (resp. right) continuous with compact support. We state the following well known lemma without proof (c.f. [6]). Let $(E,\mathcal{E})$ denote a measurable space and suppose $\mathcal{M}$ is an arbitrary family of positive σ-finite measures on it.

(4.1) LEMMA. Let $\{f(B);\ B\in\mathcal{G}\}$ be a family of functions in $\mathcal{E}$ such that $f(\cup B_n) = \Sigma f(B_n)$, m - a.e., for every $m\in\mathcal{M}$, and every sequence (B_n) of pairwise disjoint sets in $\mathcal{G}$. Then there exists a positive kernel $K(a,dy)$ from $(E,\mathcal{E})$ into $(G,\mathcal{G})$ such that $K(\cdot,B) = f(B)$ m - a.e., for all $m\in\mathcal{M}$ and $B\in\mathcal{G}$

If $\Gamma(\omega;\, dt,\, dy)$ is a positive kernel from $(\Omega,\mathcal{F})$ into $(\mathbb{R}\times G,\ \mathcal{B}\otimes\mathcal{G})$ (where $\mathcal{B}$ denotes the Borel sets of $\mathbb{R}$), for any measurable W on $\hat{\Omega} = \Omega\times\mathbb{R}\times G$ we set:

$$W * \Gamma_t(\omega) = \int W(\omega,s,y)1_{(-\infty,t]}(s)\Gamma(\omega;ds,dy)$$

whenever this integral makes sense.

(4.2) DEFINITION. Γ as above is called a <u>random</u> <u>measure</u> if $W*\Gamma$ is $\mathcal{O}$-measurable (i.e., optional) whenever W, positive is $\mathcal{O}\otimes\mathcal{G}$-measurable on $\hat{\Omega}$. We denote

$$\hat{\mathcal{J}}_\sigma = \{\Gamma: \ \Gamma \text{ a random measure and such that there exists a } \hat{\mathcal{P}}\text{-measurable partition } D_n \text{ of } \hat{\Omega} \text{ such that } 1_{D_n}*\Gamma \in \mathcal{J}, \text{ every } n\}$$

$$\hat{\mathcal{P}} \cap \hat{\mathcal{J}}_\sigma = \{\Gamma \in \hat{\mathcal{J}}_\sigma : \ W*\Gamma \text{ is predictably measurable for } W \in b\hat{\mathcal{P}}^+\}.$$

Note that $\Gamma(\omega;dt,\{0\}) = dY_t(\omega)$ defines a random measure if $Y \in \hat{\mathcal{J}}^+$ by taking $G = \{0\}$. Random measures play many roles analogous to those of processes in $\mathcal{J}^+$. In particular for $\Gamma \in \hat{\mathcal{J}}_\sigma$, we restrict Γ to $\Omega \times \mathbb{R}_+ \times G$ by taking

$$\Lambda(\omega;dt,dy) = \Gamma(\omega;dt,dy)1_{\Omega\times[0,\infty)\times G}$$

as the restriction of Γ. We then know that there exists a random measure, denoted $\tilde{\Gamma}$, on $\Omega \times \mathbb{R} \times G$, which is the "dual predictable projection" of Λ, the restriction of Γ. We refer the reader to Jacod [8] for all facts about random measures.

We next extend the concept of the "big shift" to $\hat{\Omega}$. We define

(4.3) $$\Theta_h\{W(\omega,t,y) - W(\omega,s,y)\}$$

$$= W(\theta_h\omega,t-h,y) - W(\theta_h\omega,s-h,y)$$

and $\Theta_h W(\omega,t,y)$ denotes $\Theta_h[W(\omega,t,y)-W(\omega,0,y)]$.

(4.4) $$\Theta_h\Gamma(\omega;du,dy) = \Gamma(\theta_h\omega;du-h,dy).$$

A simple computation yields:

(4.5) PROPOSITION. Let W be positive and either $W(\omega,t,y) = 0$ for $t \leq 0$, or $W(\omega,t,y)$ is a helix for fixed y. Then $\Theta_h(W*\Gamma)_t = (\Theta_h W*\Theta_h\Gamma)_t$

(4.6) THEOREM. Let $\Gamma \in \hat{\mathcal{A}}_\sigma$, and let $\tilde{\Gamma}$ be the dual predictable projection of the restriction of Γ to $\Omega \times R_+ \times G$. Then $\Theta_s\Gamma \in \hat{\mathcal{A}}_\sigma$ and $\Theta_s\tilde{\Gamma}$ is a version of the dual predictable projection of $\Theta_s\Gamma$.

PROOF. Let $(D_n)_{n\geq 1}$ be a $\hat{\mathcal{P}}$-measurable partition of $\hat{\Omega}$ such that $1_{D_n} * \Gamma \in \mathcal{A}$ for every n. Set $D_0' = \Omega \times (-\infty,s) \times G$, and $D_n' = \{(\omega,t,y): \Theta_s 1_{D_n}(\omega,t,y) = 1\}$, for $n\geq 1$. Then $(D_n')_{n\geq 0}$ is a $\hat{\mathcal{P}}$-measurable partition of $\hat{\Omega}$. Moreover $1_{D_0'} * (\Theta_s\Gamma)=0$. For $n\geq 1$, $1_{D_n'} * (\Theta_s\Gamma)=\Theta_s(1_{D_n}*\Gamma)$ by Proposition (4.5), and these processes belong to $\mathcal{A}$ by Theorem (3.1). Therefore $\Theta_s\Gamma \in \hat{\mathcal{A}}_\sigma$, and we let $\Theta_s\Gamma$ denote the dual predictable projection of its restriction to $\Omega \times \mathbb{R}_+ \times G$, which we now know exists. Analogously we know $\Theta_s\tilde{\Gamma}$ is in $\hat{\mathcal{P}} \cap \hat{\mathcal{J}}$. Then $n\geq 1$ and $B \in \mathcal{G}$, we have:

$$(1_B 1_{D_n'}) * \widetilde{\Theta_s\Gamma} = (1_B 1_{D_n'}*\Theta_s\Gamma)^\sim$$

$$= (\Theta_s 1_B 1_{D_n}*(\Theta_s\Gamma))^\sim$$

$$= (\Theta_s(1_B 1_{D_n}*\Gamma))^\sim$$

with the last equality by Proposition (4.5). On the other hand:

$$1_B 1_{D'_n} * \Theta_s \tilde{\Gamma} = \Theta_s 1_B 1_{D_n} * \Theta_s \tilde{\Gamma}$$

$$= \Theta_s (1_B 1_{D_n} * \tilde{\Gamma})$$

$$= \Theta_s [(1_B 1_{D_n} * \Gamma)^{\sim}]$$

$$= [\Theta_s (1_B 1_{D_n} * \Gamma)]^{\sim}$$

with the second equality by Proposition (4.5) and the last equality by Theorem (3.1). The result now follows. □

(4.7) DEFINITION. A random measure Γ is called <u>integer valued</u> if it has the form:

$$\Gamma(\omega;dt,dy) = \sum_{s \geq 0} 1_{\Lambda} (\omega,s) \, \varepsilon_{(s,Z_s(\omega))}(dt,dy)$$

where Λ is an optional set and where Z is a G-valued optional process. Here ε_a denotes the Dirac point-mass measure at a point a. We will write $\hat{\mathcal{J}}^1_\sigma$ for those random measures in $\hat{\mathcal{J}}_\sigma$ which are integer valued.

(4.8) DEFINITION. A random measure Γ will be called <u>additive</u> if

(i) $\Gamma(\cdot,\{0\} \times G) = 0$ a.s.

(ii) $(\Theta_s \Gamma)(\cdot,dt,dy) = \Gamma(\cdot,dt,dy)$

for all $s \in \mathbb{R}$, a.s.

In what follows, since we will be dealing with dual projections and increasing processes and hence be interested only in $\mathbb{R}_+$, we will freely abuse the word "helix" and apply it to processes defined on $[0,\infty)$ and satisfying (2.1). These can easily be extended to be true helices as discussed in Comment (2.5).

(4.9) THEOREM. Let Γ be an integer valued measure in $\hat{\mathcal{J}}^1$ such that there exists a $\hat{\mathcal{P}}$-measurable partition $(D_n)_{n\geq 1}$ of $\Omega \times \mathbb{R}_+ \times G$ where $C^n = 1_{D^n} * \Gamma \in \mathcal{J}$ is a helix for each n. Then there exists an increasing, predictable helix F and a positive kernel $K(\omega,t;dy)$ from $(\Omega \times \mathbb{R}_+, \mathcal{O})$ into $(G, \mathcal{G})$ such that

$$\tilde{\Gamma}(\omega;dt,dy) = dF_t(\omega)K(\omega,t;dy), \quad t\geq 0,$$

is a version of $\tilde{\Gamma} \in \hat{\mathcal{P}} \cap \hat{\mathcal{J}}_\sigma$ of the dual predictable projection of the restriction of Γ to $(\Omega \times \mathbb{R}_+ \times G)$.

PROOF. Let $a_n = E\{\int_0^\infty e^{-|s|} dC_s^n\} < \infty$. Choose b_n such that $\sum_{n\geq 1} a_n b_n < \infty$. Let $H = \sum_{n\geq 1} b_n C^n$. Then $H \in \mathcal{J}^+$ is a helix, and we let F denote its dual predictable projection, which is also a helix by Corollary (3.2).

For $B \in \mathcal{G}$, the process $Y(n,B) = (1_B 1_{D_n}) * \Gamma \in \mathcal{J}$; call its dual predictable projection $\tilde{Y}(n,B)$. Since $Y(n,B)_t << dC_t^n << dH_t$, we have $d\tilde{Y}(n,B)_t << dF_t$ a.s. By Theorem (3.11) there exists a homogeneous process $f(n,B)$ such that $\tilde{Y}(n,B) = f(n,B) \cdot F$ a.s.

We next apply Lemma (4.1) where $(E, \mathcal{E}) = (\Omega \times \mathbb{R}_+, \mathcal{O})$ and $\mathcal{M} = \{P(d\omega) \times dC_t^n(\omega)\}$. For every pairwise sequence (B_q) we have $Y(n, \cup B_q) = \sum_q Y(n,B_q)$, and hence

$\tilde{\Upsilon}(n,\cup B_q) = \sum_q \tilde{\Upsilon}(n,B_q)$ up to an evanescent set. Therefore $f(n,\cup B_q) = \Sigma f(n,B_q)$ m - a.e., each $m \in \mathcal{M}$. Therefore by Lemma (4.1) there exists a positive kernel $K^n(\omega,t;dy)$ such that $K^n(\cdot,B) = f(n,B)$ m - a.e., all $m \in \mathcal{M}$ and $B \in \mathcal{G}$. We now set:

$$(4.10) \qquad \tilde{\Gamma}^n(\omega;dt,dy) = dF_t(\omega)K^n(\omega,t;dy).$$

Since $\tilde{\Gamma} = \sum_{n\geq 1} \tilde{\Gamma}^n$ is then the dual predictable projection of $\tilde{\Gamma}$, and since the same F appears in (4.10) for each n, if we set $K = \sum_{n\geq 1} K^n$, we obtain the desired result.

§5. LOCAL CHARACTERISTICS OF HELIX SEMIMARTINGALES

Let $Y = (Y^i)_{i\leq m}$ be an m-dimensional semimartingale. Let $J_t^i = \sum_{0<s\leq t} \Delta Y_s^i 1_{\{|\Delta Y_s^i|>1\}}$, and $J = (J^i)_{i\leq m}$. Then $Y - Y_0 - J$ is an m-dimensional special semimartingale. We let $Y - Y_0 - J = M + B$ be its canonical decomposition. The <u>jump measure</u> Γ of Y is defined by:

$$(5.1) \qquad \Gamma(\omega;dt,dy) = \sum_{s>0} 1_{\{\Delta_s Y(\omega)\neq 0\}} \varepsilon_{(s,\Delta Y_s(\omega))}(dt,dy).$$

Note that Γ is an additive random measure on $\Omega \times \mathbb{R} \times G$ if Y is a helix. (Once again, we abuse the word "helix" to apply to processes defined on $\Omega \times \mathbb{R}_+$ but extendable to $\Omega \times \mathbb{R}$; c.f. comment (2.5).)

(5.2) DEFINITION. The <u>local characteristics</u> of Y consist of the triplet $(B,C,\tilde{\Gamma})$ defined as follows:

(i) $B = (B^i)_{i \leq m}$

(ii) $C = (C^{ij})_{i,j \leq m}$, where

$$C^{ij} = [(Y^i)^c, (Y^j)^c]$$

(iii) $\tilde{\Gamma}$ is the dual predictable projection of the integer-valued random measure of (5.1).

See Jacod [8, pp. 88-97] for all facts concerning local characteristics.

(5.3) THEOREM. *Let* Y *be an m-dimensional semimartingale.*

(i) *If* Y *is a helix or if* $Y_t = 0$ *for* $t \leq 0$, *then* $(\Theta_s B, \Theta_s C, \Theta_s \tilde{\Gamma})$ *is a version of the local characteristics of* $\Theta_s Y$, *any* $s \geq 0$.

(ii) *If* Y *is a helix then* B *and* C *are helices and* $\tilde{\Gamma}$ *is additive.*

PROOF. (i) The jump measure of $\Theta_s Y$ is easily seen to be $\Theta_s \Gamma$, and $\Theta_s J$ is the corresponding J-process for $\Theta_s Y$. The result then follows from Theorem (3.1) and Theorem (4.6). (ii) Γ is additive because Y is a helix. Therefore $\tilde{\Gamma}$ is additive as a consequence of Theorem (4.9). B and C are helices by Corollary (3.2). □

(5.4) THEOREM. *Let* Y *be a helix and an m-dimensional semimartingale. Then there exists:*

(i) *a predictable, increasing "helix"* F *on* $\mathbb{R}_+ \times \Omega$;

(ii) *a homogeneous process* $b = (b^i)_{i \leq m}$;

(iii) a homogeneous process $c = (c^{ij})_{k,j\leq m}$ with values in the set of all symmetric nonnegative matrices;

(iv) a positive kernel $K(\omega,t;dy)$ from $(\Omega \times \mathbb{R}_+, \mathcal{O})$ into $(\mathbb{R}^m, \mathcal{B}^m)$ such that the local characteristics of Y are given by:

$$B = b \cdot F; \quad C = c \cdot F$$

$$\tilde{\Gamma}(\omega;dt,dy) = dF_t(\omega)K(\omega,t;dy)$$

where $b \cdot F$ denotes $\int_0^t b_s dF_s$.

COMMENTS. Although F is only defined on $\mathbb{R}_+ \times \Omega$ one can extend F to a helix on $\mathbb{R} \times \Omega$ (cf Comment (2.5)). The fact that c can take its values in the space of all nonnegative symmetric matrices is part of the established theory of local characteristics (c.f. [8]). Also, it is known that one can take the kernel K such that $K(\{0\}) = 0$ and $\int \min(1,|y|^2)K(dy) < \infty$.

PROOF. By Theorem (5.3) we know that B and C are helices and that Γ and hence $\tilde{\Gamma}$ are additive random measures, where Γ is as defined in (5.1): the jump measure of Y. Let $D_0 = \Omega \times \mathbb{R}_+ \times \{0\}$, and for $n \geq 1$ set

$$D_n = \{(\omega,t,y): \omega \in \Omega,\ t \geq 0,\ y \in \mathbb{R}^m,\ \ y \in [\tfrac{1}{n},\tfrac{1}{n-1})\}.$$

Then $1_{D_n} * \Gamma \in \mathcal{A}$, hence $\Gamma \in \hat{\mathcal{A}}^1_\sigma$. Theorem (4.9) then guarantees the existence of F' and K' such that

$$\tilde{\Gamma}(\omega;dt,dy) = dF'_t(\omega)K'(\omega,t;dy).$$

Next set

$$F_t = F'_t + \sum_{i\leq m} \int_0^t |dB_s^i| + \sum_{i\leq m} C_t^{i,i},$$

where $\int_0^t |dB_s|$ denotes the total variation process. Then $dF_t << dF'_t$ and hence $\tilde{\Gamma}$ admits a second factorization with a new kernel K such that:

$$\tilde{\Gamma}(\omega;dt,dy) = dF_t(\omega)K(\omega,t;dy).$$

Since $dB_t^i << dF_t$ and $dC_t^{i,j} << dF_t$, by Theorem (3.11) there exist homogeneous processes b and c such that $B = b\cdot F$ and $C = c\cdot F$. □

We next record a result which is fundamental to the theory of helices. It is due to Mecke, and we present it here as interpreted by Lazaro and Meyer [10] (c.f. also Geman and Horowitz [5]). Let Z be an increasing helix such that $Z_\infty = +\infty$ and $Z_{-\infty} = -\infty$ for all ω. Such a helix we will call a _total helix_.

(5.5) THEOREM. _Let_ Z _be a total helix. Then there exists a_ σ_-finite measure_ μ _on_ $(\Omega, \mathcal{F}^0)$ _such that one has, for all positive_ f $\mathcal{F}^0 \otimes \mathcal{B}$_-measurable,_

$$\int P(d\omega) \int f(\theta_t\omega,t)dZ_t(\omega) = \iint f(\omega,t)\mu(d\omega)dt.$$

Moreover, μ _is given by_

$$\mu(A) = \int P(d\omega) \int_0^1 1_A(\theta_s\omega)dZ_s(\omega).$$

We refer the reader to [10, p.43] for the relatively simple proof. The measure μ is called the Palm measure of the helix Z. Next we combine Theorem (5.4) with Theorem (5.5) to obtain:

(5.6) LEMMA. Let Y be a helix semimartingale with local characteristics B, C, and $\tilde{\Gamma}$. Then there exist $\mathfrak{F}_0$-measurable random variables b and c, a positive kernel $K(\omega,t;dy)$ and a σ-finite measure μ on $(\Omega, \mathfrak{F}^0)$ such that for any $\mathfrak{F}\otimes\mathcal{B}$-positive H:

(i) $\int\int H(\theta_s\omega,s)dB_s(\omega)P(d\omega) = \int\int H(\omega,s)b(\omega)\mu(d\omega)ds$

(ii) $\int\int H(\theta_s\omega,s)dC_s(\omega)P(d\omega) = \int\int H(\omega,s)c(\omega)\mu(d\omega)ds$

(iii) for any $\mathfrak{F}\otimes\mathcal{G}\otimes\mathcal{B}^+$-measurable positive W:

$$\int\int W(\omega,t,y)\tilde{\Gamma}(\omega;dt,dy)P(d\omega)$$

$$= \int\int W(\theta_{-t}\omega,t,y)K(\theta_{-t}\omega,t;dy)\mu(d\omega)dt.$$

PROOF. By Theorem (5.4) we know that there exists an increasing "helix" on $\Omega \times \mathbb{R}_+$ such that $B_t(\omega) = \int_0^t \hat{b}_s(\omega)1_{[0,t]}(s)d\hat{F}_s(\omega)$. Without loss of generality we can extend $\hat{F}$ to be a helix on $\Omega \times \mathbb{R}$ (c.f. (2.5)), and we can then replace $\hat{F}$ with $F_t = \hat{F}_t + t$, so that F is a total helix. We can then write

$$B_t(\omega) = \int b(\theta_s\omega)\, 1_{[0,t]}(s)dF_s(\omega),$$

and hence by Theorem (5.5):

$$\iint H(\theta_s\omega,s)dB_s dP = \iint H(\theta_s\omega,s)b(\theta_s\omega)1_{[0,t]}(s)dF_0(\omega)$$

$$= \iint H(\omega,s)1_{[0,t]}(s)b(\omega)\mu(d\omega)ds.$$

The proof for C is analogous.

As for (iii), take W to be positive and $\mathfrak{F}\otimes\mathcal{G}\otimes\mathcal{B}^+$ measurable. Then

$$\int P(d\omega) \int W(\omega,t,y)\tilde{\Gamma}(\omega;dt,dy)$$

$$= \int P(d\omega)\int W(\omega,t,y)K(\omega,t;dy)dF_t(\omega),$$

and letting $\hat{W}(\omega,t,y) = W(\theta_{-t}\omega,t,y)$ and $\hat{K}(\omega,t;dy) = K(\theta_{-t}\omega,t;dy)$ we have by Lemma (5.6):

$$\int P(d\omega)\int \hat{W}(\theta_t\omega,t,y)\hat{K}(\theta_t\omega,t;dy)dF_t(\omega)$$

$$= \iint \hat{W}(\omega,t,y)\hat{K}(\omega,t;dy)\mu(d\omega)dt$$

$$= \iint W(\theta_{-t}\omega,t,y)K(\theta_{-t}\omega,t;dy)\mu(d\omega)dt.$$

□

The following lemma is quite simple and we omit the proof. (See [10, p. 42] for an analogous lemma.)

(5.7) LEMMA. *Let* λ *be a positive measure on* $\Omega \times \mathbb{R} \times G$ *such that one has for every positive* W *which is* $\mathcal{F}^0 \otimes \mathcal{B} \otimes \mathcal{G}$ *measurable:*

$$\int W(\omega,t,y)\lambda(d\omega,dt,dy) = \int W(\omega,t-u,y)\lambda(d\omega,dt,dy),$$

for any $u \in \mathbb{R}$. *If the measure* τ *on* $\Omega \times G$, $\tau(A) = \lambda(A\times]0,1])$ *is* σ*-finite, then one has* $\lambda(d\omega,dt,dy) = \tau(d\omega,dy) \times dt$.

We now come to our principal result which describes the local characteristics of a helix semimartingale in a way roughly analogous to Jacod's description of the local characteristics of a process with stationary and independent increments [8, p. 92].

(5.8) THEOREM. *Let* Y *be a helix semimartingale with local characteristics* B, C, *and* $\tilde{\Gamma}$. *Then there exist* $\mathcal{F}_0$*-measurable random variables* b *and* c, *a positive kernel* $K(\omega;dy)$ *from* $(\Omega,\mathcal{F})$ *into* $(G,\mathcal{G})$, *and a* σ*-finite measure* μ *on* $(\Omega,\mathcal{F}^0)$ *such that for any positive* $\mathcal{F}\otimes\mathcal{B}$ *positive* H:

(i) $\iint H(\theta_s\omega,s)dB_s(\omega)P(d\omega) = \iint H(\omega,s)b(\omega)\mu(d\omega)ds$

(ii) $\iint H(\theta_s\omega,s)dC_s(\omega)P(d\omega) = \iint H(\omega,s)c(\omega)\mu(d\omega)ds$

(iii) *for any* $\mathcal{F} \otimes \mathcal{G}\otimes\mathcal{B}$*-measurable positive* r.v. W:

$$\iint W(\theta_s\omega,s,y)\tilde{\Gamma}(\omega;ds,dy)P(d\omega) = \int W(\omega,s,y)1_{[0,\infty)}(s)K(\omega;dy)\mu(d\omega)ds$$

(5.9) COMMENT. One can summarize Theorem (5.8) in shorthand by saying that under the bijections $f(\omega,t) = (\theta_t\omega,t)$ and $g(\omega,t,y) = (\theta_t\omega,t,y)$, the local characteristics of a helix semimartingale are given by:

$$f^{-1} \circ dB_s(\omega)P(d\omega) = b(\omega)\mu(d\omega)ds$$

$$f^{-1} \circ dC_s(\omega)P(d\omega) = c(\omega)\mu(d\omega)ds \tag{5.10}$$

$$g^{-1} \circ \tilde{\Gamma}(\omega;ds,dy)P(d\omega) = 1_{[0,\infty)}(s)K(\omega;dy)\mu(d\omega)ds$$

where the σ-finite measure μ can be taken to be the same in all three equations, and where "ds" denotes Lebesgue measure.

PROOF. The statements (i) and (ii) are the same as in Lemma (5.6) and hence already proven. Consider then (iii): define the measure Λ by

$$\Lambda(d\omega,dt,dy) = \tilde{\Gamma}(\omega;dt,dy)P(d\omega)$$

on $\Omega \times [0,\infty) \times G$, and

$$\Lambda(d\omega,dt,dy) = \tilde{\Gamma}(\omega,-dt,dy)P(d\omega)$$

on $\Omega \times (-\infty,0] \times G$. Since $\tilde{\Gamma}$ is additive and θ_u are automorphisms under P we have

$$\int W(\omega,t,y)\,\Lambda(d\omega,dt,dy) = \int W(\theta_u\omega,t-u,y)\Lambda(d\omega,dt,dy). \tag{5.11}$$

We next define a new measure λ by

$$\lambda(W) = \int W(\theta_t\omega,t,y)\tilde{\Gamma}(\omega;dt,dy)P(d\omega).$$

Thus λ is the image of Λ under the bijection $(\omega,t,y) \longrightarrow (\theta_t\omega,t,y)$. Then (5.11) implies

$$\int W(\omega,t,y)\lambda(d\omega,dt,dy) = \int W(\omega,t-u,y)\lambda(d\omega,dt,dy).$$

Lemma (5.7) then implies

$$(5.12) \qquad \int\int W(\theta_t\omega,t,y)\tilde{\Gamma}(\omega;dt,dy)P(d\omega) = \int\int W(\omega,t,y)\tau(d\omega,dy)dt.$$

On the other hand, by Lemma (5.6) we have

$$(5.13) \qquad \int\int W(\theta_t\omega,t,y)\tilde{\Gamma}(\omega;dt,dy)P(d\omega) = \int\int W(\omega,t,y)\hat{K}(\theta_{-t}\omega,t;dy)\mu(d\omega)dt.$$

The equalities (5.12) and (5.13) together imply $\tau(d\omega,dy)dt = \hat{K}(\theta_{-t}\omega,t;dy)\ \mu(d\omega)dt$ which means there exists a kernel $K(\omega;dy)$ such that

$$\hat{K}(\theta_{-t}\omega,t;dy)\mu(d\omega) = K(\omega;dy)\mu(d\omega)$$

a.e. (dt). □

REFERENCES

1. A. Benveniste, Processus stationnaires et mesures de Palm du flot special sous une function, pp. 97-153 Sém. de Probabilités IX, Springer Lecture Notes in Math 465, 1975.

2. R. M. Blumenthal, R. K. Getoor, Markov Processes and Potential Theory. Academic Press. New York, 1968.

3. E. Çinlar, J. Jacod, P. Protter, M. J. Sharpe, Semimartingales and Markov Processes, Z. Wahrsch. Verw. Gebiete 54, 1980, p. 161-219.

4. C. Dellacherie, P.A. Meyer, Probabilities and Potential B. North Holland. Amsterdam, 1982.

5. D. Geman, J. Horowitz, Remarks on Palm Measures, Ann. Inst. Henri Poincaré, 9, 1973, p. 215-232.

6. R. K. Getoor, On the construction of kernels, p. 443-463, Sém. de Proba. IX, Springer Lecture Notes in Math. 465, 1975.

7. J. Horowitz, Short proof of a 'Radon-Nikodym theorem' for additive functionals, Z. Wahrsch. Verw. Gebiete, 25, 1973, p. 159-162.

8. J. Jacod, Calcul Stochastique et Problèmes de Martingales. Springer Lect. Notes in Math. 714, 1979.

9. J. de Sam Lazaro, P. A. Meyer, Méthodes de martingales et théorie des flots, Z. Wahrsch. Verw. Gebiete, 18, 1971, p. 116-140.

10. J. de Sam Lazaro, P. A. Meyer, Questions de théorie des flots, p. 1-96, Sém. de Proba. IX, Springer Lecture Notes in Math. 465, 1975.

11. M. Metivier, Semimartingales. De Gruyter. Berlin, New York, 1982.

12. P. A. Meyer, Un cours sur les intégrales stochastiques, p. 245-400, Sém. de Proba. X, Springer Lecture Notes in Math. 511, 1976.

13. P. A. Meyer, Martingales locales fonctionelles additives I, p. 775-785, Sém. de Proba. XII, Springer Lect. Notes in Math. 649, 1978.

RÉSUMÉ.-- On étudie des semimartingales dans un cadre de la théorie ergodigue d'après J. de Sam Lazaro et P. A. Meyer. On montre que les caracteristiques locales d'une semimartingale - hélice possèdent une forme à peu près analogue à ceux des processus à acroissements indépendents et stationnaires.

SUMMARY.-- We study semimartingales within an ergodic theory framework as pioneered by J. de Sam Lazaro and P. A. Meyer. We show that the local characteristics of a helix semimartingale have a form roughly analogous to those of a process with stationary and independent increments.