A BOOK REVIEW OF:

INTRODUCTION TO STOCHASTIC INTEGRATION,
BY K. L. CHUNG AND R. J. WILLIAMS

STOCHASTIC CALCULUS AND APPLICATIONS
BY R. J. ELLIOTT

SEMIMARTINGALES: A COURSE ON STOCHASTIC PROCESSES,
BY M. METIVIER

by

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Technical Report #85-5

Departments of Mathematics and Statistics
Purdue University
April 1985
(Revised July 1985)

A BOOK REVIEW OF THE FOLLOWING BOOKS

Introduction to Stochastic Integration, by K. L. Chung and R. J. Williams, Birkhäuser, Boston, Basel, Stuttgart, 1983, 191 pp.

Stochastic Calculus and Applications, by Robert J. Elliott, Springer-Verlag, New York, Heidelberg, Berlin, 1982, 302 pp.

Semimartingales: A Course on Stochastic Processes, by Michel Métivier, Walter de Gruyter, Berlin, New York, 1982, 287 pp.

What do the terms stochastic integration and stochastic calculus connote? In the past they would refer to the Itô integral, a mathematically rigorous way to make sense of integration with respect to a Brownian motion. Times have changed. Now that the Itô integral has become a familiar if not ubiquitous object, "stochastic integration" has come to refer to an imposing panoply of abstract "French" probability theory, only now becoming accessible to the non-specialist.

The stochastic integration of today of course has its roots in Brownian motion. The Wiener process, the mathematical model of Brownian motion, is indeed the wellspring of much of modern probability theory, perhaps due to its triple role of martingale, strong Markov process, and Gaussian process. It is the interplay of the martingale and Markov process properties that underlie the history of stochastic integration. By developing his integral in 1944 with stochastic processes as integrands, Itô [11] was able to study multidimensional diffusions with purely probabilistic techniques, an improvement over the analytic methods of Feller. Many diffusions can be represented as solutions of systems of stochastic differential equations of the form:

$$X_t = X_0 + \int_0^t \sigma(s,X_s)dB_s + \int_0^t b(s,X_s)ds$$

where B is a Brownian motion (i.e., a standard Wiener process). This is still the primary method of studying multidimensional diffusions.

It was Doob, however, who stressed the martingale nature of the Itô integral [8]. Realizing that the martingale property of Brownian motion was the key in Itô's integral, Doob proposed a general martingale integral. To develop it, however, he needed to be able to decompose submartingales into the sum of a martingale and an increasing process. P. A. Meyer [16] found the right conditions under which this could be done and indicated how this might open the door to general stochastic integration [16, p. 204].

The door was not fully open, however. The rich structure of Brownian motion concealed a subtle distinction. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqq 0}, P)$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leqq t$, $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, and $\mathcal{F}_0$ containing all P-null sets, consider a stochastic process H_t as a function mapping $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$. Define the <u>predictable</u> σ-<u>field</u> $\mathcal{P} \equiv \sigma\{H: H_t \in \mathcal{F}_t$ all t and $t \to H_t$ is left continuous$\}$, and the optional σ-field $\mathcal{O} \equiv \sigma\{H: H_t \in \mathcal{F}_t$ and $t \to H_t(\omega)$ is right continuous$\}$, where $\sigma\{\ \}$ denotes the smallest σ-field (on $\mathbb{R}_+ \times \Omega$) making all such processes measurable. For Brownian motion one has $\mathcal{P} = \mathcal{O}$, which is a rather deep consequence of the fact that Brownian motion is a strong Markov process without jumps (cf [10, p. 36]). Therefore it was hard to see that for a general martingale integral one needed the restriction of integrands to predictable processes. After a doomed attempt to get more than just predictable integrands [2], this limitation was realized as natural by P. A. Meyer [17], who extended to local martingales a beautiful theory of stochastic integration, complete with a generalization of Itô's change of variables formula, developed by H. Kunita and S. Watanabe [13]. (Local martingales had been introduced earlier by K. Itô and S. Watanabe [12] who improved the Doob-Meyer decomposition: *any* submartingale can be decomposed into the sum of a *local* martingale and a predictable increasing process.) The treatments of Kunita and Watanabe, as well as that of Meyer, however, were still tied to Markov process theory and had a technical restriction that the filtration be "quasi left continuous", a condition that is often satisfied by natural filtrations of strong Markov processes, but otherwise is not a very reasonable hypothesis.

C. Doléans-Dade and P. A. Meyer [6] removed the restriction of quasi left continuity and purged the theory of its Markov process connections, making it a purely martingale theory. The subject then lay dormant for six years until 1976 when P. A. Meyer published his seminal "course" on stochastic integration [18] proving many new properties of the integrals and stimulating a virtual explosion of interest in the subject which continues to this day.

Meyer emphasized in his "course" the centrality of semimartingales: a *semimartingale* is a process Z that can be decomposed as Z = M + A, where M is a local martingale and A is the difference of two increasing processes. Semimartingales arose in an *ad hoc* manner as the most general differentials for which one has an integral: a local martingale integral for M and a path by path Lebesgue-Stieltjes integral for A. Therefore the theorem of C. Dellacherie [3] and K. Bichteler [1] (discovered independently), contains that rare satisfaction of an *a posteriori* justification of a definition: they showed that if a right continuous process Z is a linear differential that satisfies an extremely weak dominated convergence theorem, then Z is a fortiori a semimartingale.

Thus the theory of stochastic integration is now a mature one. The key objects (such as semimartingales and the predictable σ-field) have been isolated, extraneous theory (such as Markov process theory) has been removed, and also proofs have been simplified to the extent that an excellent pedagogic treatment is possible.

The three books under review are some of the first attempts at such a pedagogic treatment. The first book, by K. L. Chung and R. J. Williams, is the most elementary. It begins with a treatment of general right continuous martingales, but then to treat stochastic integration the authors consider only the case of local martingales with continuous paths. (They actually develop the theory for continuous semimartingales, but curiously do not name the processes semimartingales.) The restriction to continuous paths permits the use of an idea of M. Sharpe [20], that allows one to avoid invoking the Doob-Meyer decomposition theorem mentioned earlier. This approach also avoids the technical difficulties of studying martingale integrals with jumps. One especially powerful tool in the study of continuous semimartingales is the use of local time. It is, therefore, disappointing that after having developed stochastic integration for continuous semimartingales, Chung and Williams present local time only for Brownian motion and not for continuous semimartingales.

The other two books under review do consider stochastic integration for general semimartingales. The theory can be made much simpler than one might imagine because of an elementary consequence of an all too obscure result of C. Doléans-Dade [5] and K. A. Yen [19]: a semimartingale Z has a decomposition Z = M + A where M is a locally square integrable local martingale, and A is a process with paths of finite variation on compacts. This result allows one to reduce the study of martingale integrals to L^2 martingale integrals, where elementary Hilbert space techniques simplify the theory. Although R. J. Elliott develops the machinery necessary to prove this "fundamental theorem of local martingales" (as P. A. Meyer has dubbed it), he proves only its precursor (p. 117). Thus his treatment, while traditional, is perhaps more complicated than it need be. Attempts to simplify, however, can also go astray. In Lemma 6.18 (p. 51) Elliott gives an "alternative proof" that the debut of an optional set is a stopping time, using a monotone class argument. This argument is both wrong and misleadingly simple, as the known proofs use capacity theory. A direct proof can be found in the excellent article by C. Dellacherie [4].

The book by M. Métivier is a good choice for the reader who wants a thorough yet pedagogic treatment. Métivier uses a traditional approach while also including the vector-valued measure approach pioneered by himself and J. Pellaumail (cf. [1], [14], [15]). The latter approach is especially valuable for infinite dimensional spaces, and Métivier's book is the only one of the three to present the Métivier-Pellaumail-Doob inequality for martingales, one of the deepest results to emerge from stochastic integration theory. This result is key to Métivier's treatment. While the book is well written, it does have too many misprints.

None of the three books treats the theorem of Bichteler and Dellacherie mentioned earlier which says, loosely speaking, that semimartingales are the most general reasonable stochastic differentials possible. Métivier does, however, discuss the result in his historical notes, and he provides references. Indeed, the bibliography in Métivier's book is superior to those of the other two books, and more of the type one might perhaps expect from a pedagogic book written at this level.

While K. L. Chung and R. J. Williams regret the omission of stochastic differential equations in their book, R. J. Elliott and M. Métivier both include them. Elliott follows the "traditional" approach of C. Doléans-Dade and P. A. Meyer [7], although a treatment such as the one by M. Emery [9] might have been more elegant. Métivier uses the approach he developed together with J. Pellaumail, based on the inequality mentioned earlier. Given this deep inequality, Métivier's approach is simpler. The connections with Markov process theory are touched on by Elliott, while Métivier pursues a few questions in a different direction.

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April, 1985

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