THREE PROBLEMS FROM THE THEORY OF RIGHT PROCESSES

by

Thomas S. Salisbury
Purdue University

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Department of Statistics
Purdue University

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Thomas S. Salisbury

Department of Mathematics
Purdue University
West Lafayette, IN 47907

ABSTRACT

Using the continuum hypothesis, we produce an example answering three problems from the theory of right processes. In particular, we give a nontrivial example of a strong Markov process which is not a right process.

*After July 1985; Department of Mathematics
York University
Downsview Ontario M3J 1P3
CANADA

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We will consider the following three questions

1. Are right processes invariant under change of realization; that is, does the definition of a right process depend only on the finite dimensional distributions of the process?
2. Are all 'reasonable' strong Markov processes necessarily right processes?
3. Are all α-excessive functions for a right process necessarily nearly Borel?

These problems lie at the foundation of the theory of right processes, and have been unresolved for some time (see Sharpe [10]). Their relationship is that a positive answer to either of 3 or 2 would imply one for 1 as well. We will show however that there is a negative answer to each. Though it suffices to show this for 1, it will be convenient to discuss 1 and 2 separately. Roughly speaking, our processes will be Brownian motions in R^n that have been forced to add various pathological functions to their collections of excessive functions.

Essentially, a right process is a normal, right continuous, strong Markov process (X_t), taking values in a U-space E [in our case, it will be a continuous process, and E will be R^n], which possesses the additional measurability property that the α-excessive functions f are nearly optional [that is, $f(X_t)$ is indistinguishable from an optional process, for each initial law]. We refer the reader to Sharpe [10] for the precise definition.

By a 'reasonable' process, in question 2, we mean a process satisfying all the above conditions except the last one. Thus the second question asks whether this technical condition, needed to make the arguments of probabilistic potential theory work, is really necessary.

Write P^x for the law of (X_t) started at x. Then at a minimum, we must assume that $P^x(A)$ is universally measurable in x for each measurable A. It is well known (see (9.4) of Getoor [5]) that if these functions are in fact Borel in x, then our three questions have positive answers. Our interest therefore lies in non Borel processes. These arise naturally; for example, the operation of killing a process on a Borel set preserves the universal measurability of the transition function, but not necessarily its Borel measurability.

Note also that for question 1, it is crucial that we allow our processes to be realized on spaces other than path space (that is, we consider right processes in the sense of Sharpe [10], rather than Getoor [5]). In fact, question 1 is equivalent to that of whether the canonical realization of a right process, on path space, is also a right process. A second realization of a right process will be a right process as well if and only if it is a.s. right continuous in the topology of the Ray Knight compactification $\bar{E}$ of E, given by the original realization. Thus question 1 amounts to asking whether the set of simultaneously E and $\bar{E}$-right continuous paths is a P^μ-measurable subset of the set of $\bar{E}$-right continuous paths, for each μ.

Let Ω consist of all continuous paths on R^n, and let (X_t) be the coordinate process on Ω. Let P_0^x be the law of Brownian motion in R^n started at x, and let ε_x be a point mass concentrated on the function identically equal to x. Our examples will involve the probability measures

$$P_Z^x = \begin{cases} P_0^x , & x \notin Z \\ \varepsilon_x , & x \in Z \end{cases} ,$$

where Z is universally null in R^n [that is, it is universally measurable and receives no mass from any finite continuous Borel measure on R^n] .

The $P_Z^x(A)$ are universally measurable in x. Let $n \geqslant 2$. Then Brownian motion in R^n does not hit points, and hence for any stopping time $T > 0$,

$$P_0^\mu(X_T \in \cdot)$$

is a continuous measure. Thus

$$P_Z^\mu(X_T \in Z, X_0 \notin Z) = 0 ,$$

showing that (X_t) is strong Markov under (P_Z^x). Write

$$\Omega_Z = \{\omega \in \Omega ; \omega(t) \in Z \text{ for some } t > 0\} .$$

Assume the continuum hypothesis and the axiom of choice. We will show

Lemma

(a) For $n \geq 2$, there is a universally null Z in R^n such that every nonconstant path hits Z.

(b) For $n \geq 4$, there is a universally null Z' in R^n such that $\Omega_{Z'}$ has inner measure zero and outer measure one under each P_0^μ.

This immediately yields

Theorem

(a) For $n \geq 2$ there is a universally null Z in R^n such that no realization of (X_t, P_Z^x) is a right process

(b) For $n \geq 4$ there is a universally null set Z' in R^n such that $(X_t, P_{Z'}^x)$ has two realizations, one of which isn't a right process and the other of which is.

Proof: Take Z to be the set given by (a) of the lemma. Let $(\bar{X}_t, \bar{P}_Z^x)$ be a right process which is a realization of (X_t, P_Z^x). Then by (9.4) of Getoor [5], $f(\bar{X}_t)$ is a.s. right continuous, for each excessive f. Take $f = 1_{R^n/Z}$. Since $f(\bar{X}_t) = f(x)$ $\bar{P}_Z^x$-a.s. for each t, we have that f is excessive. Thus $f(\bar{X}_t) = f(x)$ for each t, $\bar{P}_Z^x$-a.s., and hence $\bar{X}_t$ a.s. never hits Z when started outside Z, a contradiction.

The realizations in (b) are (X_t) on Ω and $(\Omega \backslash \Omega_{Z'}) \cup \{\text{constant functions}\}$ respectively, where Z' is the set given by (b) of the lemma. [Note that, though we have not mentioned them, shift operators θ_t are usually assumed to come with the machinery of right processes. With this in mind, note that $\Omega\backslash\Omega_{Z'}$ is preserved by all shifts]. The α-excessive functions for $(P^x_{Z'})$ are

$$f = g \cdot 1_{\mathbf{R}^n/Z'} + h \cdot 1_{Z'}$$

where g is α-excessive for (P^x_0) and $h \geq 0$ is arbitrary (such f are universally measurable since Z' is universally null). $f(X_t)$ is a.s. right continuous on $\Omega\backslash\Omega_{Z'}$, since it equals $g(X_t)$ there. As in part (a), $f(X_t)$ is in general not right continuous on $\Omega_{Z'}$, hence is not a.s. right continuous on Ω. □

The theorem answers questions 1 and 2. Since $3 \Rightarrow 1$ (see Sharpe [10]), the α-excessive functions for the process of (b) cannot be nearly Borel. We can see this directly, as if $B \supset Z'$ is Borel then by (b) of the lemma, $P^x_0(X \text{ hits } B) = 1$ for each x, so that $1_{\mathbf{R}^n\backslash Z'}$ cannot be nearly Borel.

Proof of Lemma :(a): Let $N' \subset \mathbf{R}$ be uncountable and universally null (see, for example, Sierpinski and Szpilrajn [11]). Then

$$N = N' + Q$$

is as well. In addition, $N\backslash C$ is dense, for any countable C (as $y+Q$ intersects C for only countably many $y \in N'$).

Write λ for the first uncountable ordinal (the usual notation Ω being unacceptable to a probabilist!)

Let Ω_j be the set of $\omega \in \Omega$ whose projections onto the jth component are not constant. Since Ω is separable, it has cardinality $\aleph_1$ (by CH), so we can well order Ω_1 as

$$(\omega_\beta)_{\beta<\lambda} .$$

Well order N as well, as

$$(y_\alpha)_{\alpha<\lambda} .$$

Since N is dense, there is some $\alpha(0)$ such that ω_0 hits $\{y_{\alpha(0)}\} \times \mathbf{R}^{n-1}$. Let the first such hit be at

$$x_0 = (y_{\alpha(0)}, z_0) , \; z_0 \in \mathbf{R}^{n-1} .$$

Let $\alpha(1)$ be the first $\alpha > \alpha(0)$ such that ω_1 hits $\{y_\alpha\} \times \mathbf{R}^{n-1}$ [we may find such an α since by choice of N ,

$$N \setminus \{y_\alpha \; ; \; \alpha \leqslant \alpha(0)\} \text{ is dense }],$$

and let the first such hit be at $x_1 = (y_{\alpha(1)}, z_1)$. Iterate by transfinite induction, to obtain a set

$$Z_1 = \{x_\beta ; \beta<\lambda\}$$

which is hit by every $\omega \in \Omega_1$.

Now let ν be a continuous finite Borel measure on $\mathbf{R}^n$. Take a disintegration

$$\nu(dx) = \eta(y,dz)\mu(dy) ,$$

where μ and each $\eta(y,\cdot)$ are finite Borel measures. Then $\mu = \mu^d+\mu^c$ where μ^d is atomic and μ^c continuous. Let Λ be a countable set on which μ^d is concentrated, and let $B \supset N$ be Borel, with $\mu^c(B) = 0$. Then

$$\nu^*(Z_1) \leq \nu(Z_1 \cap (\Lambda\times R^{n-1})) + \int_{B\setminus\Lambda} \eta(y,R^{n-1})\mu(dy) = 0 ,$$

so that Z_1 is universally null. Similarly, we obtain universally null Z_j which are hit by all elements of Ω_j , $j=2\ldots n$. Then $Z= \cup Z_j$ works for part (a).

(b): Now let

$$\Omega' = \{\omega\in\Omega : \omega \text{ is non constant, and the capacity of its range is zero}\} .$$

Brownian paths in R^n , $n \geq 4$ have capacity zero (and hence no self-intersections), so $P_0^x(\Omega') = 1$ for each x . The set of compact subsets of Ω has cardinality $\aleph_1$, since Ω has a countable base. Well order the set of compact subsets of Ω , that have positive P_0^x measure for some x , as

$$(K_\alpha)_{\alpha<\lambda} .$$

Then we may choose $\varphi_0 \in K_0 \cap \Omega'$. Since $\text{Range}(\varphi_0)$ is of capacity 0 , we may choose ψ_0 in $K_0 \cap \Omega'$ not to hit it at any $t > 0$. Choose $x_0 \in Z \cap \text{Range}(\psi_0)$. Now suppose $\varphi_\beta \in \Omega'$, $x_\beta \in Z$ are given, $\beta < \alpha$. Choose $\varphi_\alpha \in K_\alpha \cap \Omega'$ not to hit the countable set $\{x_\beta ; \beta<\alpha\}$. The capacity of

$$\bigcup_{\beta<\alpha} \text{Range}(\varphi_\beta)$$

is zero, so we may choose $\Psi_\alpha \in K_\alpha \cap \Omega'$ not to hit it at any $t > 0$. Choose $x_\alpha \in Z \cap \mathrm{Range}(\Psi_\alpha)$. By transfinite induction we obtain

$$Z' = \{x_\alpha ; \alpha < \lambda\} \subset Z$$

such that for each α, both $\Omega_{Z'}$, and $\Omega \backslash \Omega_{Z'}$, intersect K_α. By the tightness of probability measures on the Polish space Ω, we obtain the conclusion of part (b). □

In view of the discussion of question 1 in terms of the Ray-Knight compactification $\bar{E}$, it is instructive to write down the compactification of the process given in (b) of the theorem. We must first specify a preliminary compactification of $E = \mathbf{R}^n$. We of course choose the 1-point compactification S^n. Then it is easily verified that $\bar{E}$ is the space $S^n \times \{0,1\}$, in which E embeds as $(S^n \backslash Z') \times \{0\} \cup Z' \times \{1\}$. The associated Ray process on $\bar{E}$ performs a Brownian motion on $\mathbf{R}^n \subset S^n \times \{0\}$. The points of $S^n \times \{1\}$ are eternal holding points.

It should be remarked that a straightforward modification of our arguments shows the same results under the weaker axiom system ZFC+ Martin's Axiom - see Martin and Solovay [8] and Laver [7].

Also note that Z will not be capacitable (for the Newtonian capacity). Under Gödel's axiom of constructability, a standard argument (using 8.F.7 of Moschovakis [9]) shows that Z may be taken to be a PCA set. As far as capacitability goes, we may do better. Under Gödel's axiom there is a function $\mathbf{R} \to \mathbf{R}$

whose graph G is coanalytic yet contains no perfect subset (see 5.A.6 of Moschovakis [9]). The latter property implies that G has inner capacity zero, while, since G is a graph, a variant of Hall's Lemma (see the argument in the appendix to Davis and Lewis [2]) shows that G has positive outer capacity. See also 33.1 of Choquet [1].

Some of the arguments we have used are well known in the set theoretic literature; see in particular Sierpinsky and Szpilrajn [11], and Hausdorff [6]. Also relevant are Talagrand [12] and Erdös, Kunen and Mauldin [4].

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