# A RATING SYSTEM AND ITS APPLICATIONS TO RANKING AND SELECTION

by

Takashi Matsui
Purdue University
and
Dokkyo University

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Department of Statistics
Purdue University

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Let s events be given and consider the sequence of experiments which consist of paired comparisons of the events. In order to measure the relative occurences of these events, let us consider a method of quantification--a rating system--based on n such independent experiments. This method allots a number to each event by specified way according to the respective results of experiments, and is essentially the same as the one given by A. Elo for rating chess players.

A purpose of this paper is to propose the rating system and investigate the several behaviors. Applications of the system for selecting the specified events from among s given events are also studied. The results are given for both indifference zone and subset selection approaches.

I. INTRODUCTION

A proportion or relative frequency of the occurences of an event among the given set of events is a broadly used effective measure for discriminating the each event from others. While we may sometimes use the other notion of quantification to measure the differences of the outcomes of the events. The following system of rating the given events seems effective and easy to handle. Let two mutually exclusive events A and B be given. We rate the result of the i-th trial of A and B by using the respective (i-1)th rating values $A_{i-1}$ and $B_{i-1}$. If two rating values are the same, i.e., $A_i = B_i$, then certain score c ($> 0$) is added (or subtracted) according to the occurences of the events A and B. In case two rating values are different, 100 $\gamma$ percent ($\gamma > 0$) of the difference of two rating values (i.e., $|A_i - B_i| \times \gamma$) is used as a handicap value, and an event with higher rating value scores less if it occurs and vice versa. Illustrations here is only for two given events A and B but of course this system is extended to the case where more than two events are involved and compared pairwise at each trial. The idea of this rating system is essentially due to A. Elo and exclusively used, world wide, for rating chess players. We note here that in actual situation of the game, the values 16.0 and 0.04 are used for c and $\gamma$ respectively. Also our system stated above (or treated hereafter) is slightly different from the actually organized one in the point of marginal conditions such as rounding of the handicap value. This system of rating the events is expressed by using the binomial model in the following way. Let two mutually exclusive events A and B be given and consider the random variable of the following type.

$$X_i = \begin{cases} 1 \text{ with probability } p \text{ (A occurs)} \\ -1 \text{ with probability } 1-p \text{ (B occurs)} \end{cases}$$

Then the i-th rating values of the events A and B are expressed as follows, by using the (i-1)-th rating values.

$$A_i = A_{i-1} + X_i \{c - X_i(A_{i-1} - B_{i-1}) \times \gamma\}$$

$$B_i = B_{i-1} - X_i \{c - X_i(A_{i-1} - B_{i-1}) \times \gamma\}$$

$$i = 1, 2, 3, \ldots.$$

where $A_0$ and $B_0$ are the initial rating values of the events A and B. Above relations are also written in the form of difference equations as

$$A_i = (1 - \gamma) A_{i-1} + B_{i-1} + cX_i$$

$$B_i = \gamma A_{i-1} + (1 - \gamma) B_{i-1} - cX_i \qquad (1.1)$$

$$i = 1, 2, 3, \ldots.$$

Batchelder and Bershad [1] investigate the rating system due to A. Elo by using the Thurstonian model. They also suggest the applicability of this system in many areas of psychology. In [5], Matsui treats this rating system through binomial model stated above for the case of two given events (two persons game), and gives the asymptotic distribution of the rating values. Also he gives the estimator of the occurences of each event based on the rating values.

Since this rating system can be used in many fields of applications where the situation of paired comparisons are involved, let us investigate the bahavior of this system for more general case under the binomial model given above. In Section 2, the rating system is investigated for $2^k$ events case (k is a given positive integer) and moments, asymptotic distribution of the rating values are given. Applications to ranking and selection problems are treated in Section 3. The results are given for both indifference zone approach due to Bechhofer [2] and subset selection approach due to Gupta [4],

## 2. BEHAVIOR OF RATING VALUES

### 2.1 Moments of Rating Values

Let us consider the rating system which we stated in the previous section for the case where $2^k$ events are involved (k is a given positive integer). Hereafter we put $s = 2^{k-1}$.

Let $2s(=2^k)$ events $\pi_1, \pi_2, \ldots, \pi_{2s}$ be given. Corresponding rating values based on the i-th trial are written as $Z_{1i}, Z_{2i}, \ldots, Z_{2s,i}$ and the vector of rating values based on the i-th trial is denoted by

$$\underline{Z}_i = (Z_{1i}, Z_{2i}, \ldots, Z_{2s,i})'; \qquad i = 1, 2, 3, \ldots$$

Vectors of random variables and probabilities which express the trials resulted from each pair of events are defined as follows:

$$\underline{X}_i = (X_{\ell m \cdot i},\ \ell = 1, 2,\ldots,2s-1;\ m = \ell + 1,\ \ell + 2,\ \ldots,\ 2s)',$$

$$i = 1, 2, 3,\ldots.;\ s(2s-1) \times 1 \text{ vector}.$$

$$\underline{p} = (p_{\ell m},\ \ell = 1,2,\ldots,2s-1;\ m = \ell + 1,\ \ell + 2,\ldots,2s)',$$
$$s(2s-1) \times 1 \text{ vector}.$$

Random variable $X_{\ell m \cdot i}$ expresses the i-th trial between events $\pi_\ell$ and $\pi_m$ and associate with the probability $p_{\ell m}$ in the following way.

$$X_{\ell m \cdot i} = \begin{cases} 1 \text{ with probability } p_{\ell m}\ (\pi_\ell \text{ occurs}). \\ -1 \text{ with probability } 1-p_{\ell m}\ (\pi_m \text{ occurs}). \end{cases}$$

$$(1 \leq \ell < m \leq 2s). \qquad (2.1)$$

Among the 2s events, 2s-1 trials are necessary for a given event to have meets with every other different events. We call this sequence of trials which consists of 2s-1 different meets "a cycle". Method of meets in each cycle are defined by the matrix $\underline{R}_i^{(2s)}$ $(i = 1, 2,\ldots,2s-1)$ given below.

$$\underline{R}_i^{(2s)} = \underline{S}_0^{(2s)} + \underline{S}_{i+1}^{(2s)}, \qquad i = 1, 2,\ldots,2s-1 \qquad (2.2)$$

where

$$\underline{S}_i^{(2s)} = \begin{pmatrix} \underline{S}_i^{(s)} & \underline{0}^{(s)} \\ \underline{0}^{(s)} & \underline{S}_i^{(s)} \end{pmatrix}, \qquad i = 0,\ 1,\ 2,\ldots,s.$$

$$\underline{S}_{s+i}^{(2s)} = \begin{pmatrix} \underline{0}^{(s)} & \underline{S}_i^{(s)} \\ \underline{S}_i^{(s)} & \underline{0}^{(s)} \end{pmatrix}, \qquad i = 1,\ 2,\ldots,s.$$

$$\underline{S}_0^{(1)} = (1-\gamma), \quad S_1^{(1)} = (\gamma)$$

and $\underline{0}^{(s)}$ is a zero matrix of order s×s. Note that the notation $\underline{M}^{(s)}$ or $\underline{M}^{(s\times t)}$ is used hereafter for expressing the matrix $\underline{M}$ of order s×s or s×t, in case it is convenient.

Let us give an example of (2.2) for s = 2. In this case, there are four events $Z_1,\ldots,Z_4$ and matrices $S_i^{(4)}$, i = 0, 1,...,4 have the following respective forms.

$$S_0^{(4)} = \begin{pmatrix} S_0^{(2)} & \\ & S_0^{(2)} \end{pmatrix} = \begin{pmatrix} 1-\gamma & & & \\ & 1-\gamma & & \\ & & 1-\gamma & \\ & & & 1-\gamma \end{pmatrix}$$

$$S_1^{(4)} = \begin{pmatrix} S_1^{(2)} & \\ & S_1^{(2)} \end{pmatrix} = \begin{pmatrix} \gamma & & & \\ & \gamma & & \\ & & \gamma & \\ & & & \gamma \end{pmatrix}$$

$$S_2^{(4)} = \begin{pmatrix} S_2^{(2)} & \\ & S_2^{(2)} \end{pmatrix} = \begin{pmatrix} & \gamma & & \\ \gamma & & & \\ & & & \gamma \\ & & \gamma & \end{pmatrix}.$$

$$S_3^{(4)} = \begin{pmatrix} & S_1^{(2)} \\ S_1^{(2)} & \end{pmatrix} = \begin{pmatrix} & & \gamma & \\ & & & \gamma \\ \gamma & & & \\ & \gamma & & \end{pmatrix}.$$

$$S_4^{(4)} = \begin{pmatrix} & S_2^{(2)} \\ S_2^{(2)} & \end{pmatrix} = \begin{pmatrix} & & & \gamma \\ & & \gamma & \\ & \gamma & & \\ \gamma & & & \end{pmatrix}.$$

Thus $R_1^{(4)}$, $R_2^{(4)}$ and $R_3^{(4)}$ are expressed as

$$R_1^{(4)} = S_o^{(4)} + S_2^{(4)} = \begin{pmatrix} 1-\gamma & \gamma & & \\ \gamma & 1-\gamma & & \\ & & 1-\gamma & \gamma \\ & & \gamma & 1-\gamma \end{pmatrix}$$

$$R_2^{(4)} = S_o^{(4)} + S_3^{(4)} = \begin{pmatrix} 1-\gamma & & \gamma & \\ & 1-\gamma & & \gamma \\ \gamma & & 1-\gamma & \\ & \gamma & & 1-\gamma \end{pmatrix}$$

$$R_3^{(4)} = S_o^{(4)} + S_4^{(4)} = \begin{pmatrix} 1-\gamma & & & \gamma \\ & 1-\gamma & \gamma & \\ & \gamma & 1-\gamma & \\ \gamma & & & 1-\gamma \end{pmatrix}$$

Matrix $R_1^{(4)}$ specifies the meets of pairs of events $Z_1$-$Z_2$ and $Z_3$-$Z_4$. In the same way $R_2^{(4)}$ and $R_3^{(4)}$ show the meets $Z_1$-$Z_3$, $Z_2$-$Z_4$ and $Z_1$-$Z_4$, $Z_2$-$Z_3$ respectively. Also these matrices are used to form the rating values of the i-th trial based on the (i-1)th values in such a way as shown in (1.1).

Turning to the general case, the expression of the rating values of the i-th cycle (2s-1 trials are performed in each cycle) is given as follows.

$$\underline{Z}_{(2s-1)(i-1)+\ell} = R_\ell \, \underline{Z}_{(2s-1)(i-1)+\ell-1} + c\, \underline{Q}_\ell \, \underline{X}_i \quad (2.3)$$

$$i = 1, 2, 3, \ldots; \quad \ell = 1, 2, \ldots, 2s-1$$

Here, $\underline{Q}_i$ is a $2s \times s(2s-1)$ matrix, which depends on $R_i$ and defined as follows:

For every meets of $\pi_\ell$ and $\pi_m$ ($\ell < m$) defined by $\underline{R}_i$, elements of $\underline{Q}_i$ with column $\{4s(\ell-1) - \ell(\ell+1) + 2m\}/2$ and raws $\ell$ and m takes the values 1 and -1 respectively. The left of all elements of $\underline{Q}_i$ are zero. (On the properties of $\underline{Q}_i$, see Section 2.3).

At the end of n-th cycle, we have from (2.3)

$$Z_{(2s-1)n} = \underline{P}_s^n \, \underline{Z}_o + \sum_{j=1}^{n} \underline{P}_s^{n-j} \, \underline{T} \, \underline{X}_j . \quad (2.4)$$

where

$$\underline{P}_s = \prod_{i=1}^{2s-1} \underline{R}_i^{(2s)} \quad (2.5)$$

and

$$T = c \sum_{j=2}^{2s-1} \prod_{i=j}^{2s-1} \underline{R}_i \, \underline{Q}_{j-1} + c \, \underline{Q}_{2s-1}. \tag{2.6}$$

On the evaluation of $\underline{P}_s$ and $\underline{P}_s^n$, we have the following lemmas. Note that the following relations hold.

$$\prod_{i=1}^{s-1} \underline{R}_i^{(2s)} = \begin{pmatrix} (1-2\gamma_{s/2}) \, \underline{E}^{(s)} + (2/s) \, \gamma_{s/2} \, \underline{G}^{(s)} & 0^{(s)} \\ 0^{(s)} & (1-2\gamma_{s/2}) \, \underline{E}^{(s)} + (2/s) \, \gamma_{s/2} \, \underline{G}^{(s)} \end{pmatrix}$$

$$\prod_{i=s}^{2s-1} \underline{R}_i^{(2s)} = \begin{pmatrix} (1-2\gamma_{s/2})\underline{E}^{(s)} + (2/s)\gamma_{s/2}^2 G^{(s)} & (\gamma_s/s)\underline{G}^{(s)} \\ (\gamma_s/s)\underline{G}^{(s)} & (1-2\gamma_{s/2})\underline{E}^{(s)} + (2/s)\gamma_{s/2}^2\underline{G}^{(s)} \end{pmatrix}$$

Lemma 2.1

$$\underline{P}_s = \prod_{i=1}^{2s-1} \underline{R}_i^{(2s)} = (1-2\gamma_s) \, \underline{E}^{(2s)} + (\gamma_s/s) \, \underline{G}^{(2s)}, \tag{2.7}$$

where $\underline{E}^{(2s)}$ is a unit matrix of order $2s \times 2s$, $\underline{G}^{(2s)}$ is a $2s \times 2s$ matrix with all elements of which are 1, and

$$\gamma_s = \{1 - (1 - 2\gamma)^s\}/2 \tag{2.8}$$

For given constants a, b and matrices $\underline{E}^{(s)}$ and $\underline{G}^{(s)}$, we have

$$(a\underline{E}^{(s)} + b\underline{G}^{(s)})^n = a^n\underline{E}^{(s)} + (1/s)\{(bs + a)^n - a^n\}\,\underline{G}^{(s)}. \qquad (2.9)$$

Thus we have the next lemma.

Lemma 2.2

$$\underline{P}_s^n = (1 - 2\gamma_{s\cdot n})\,\underline{E}^{(2s)} + (1/s)\,\gamma_{s\cdot n}\,\underline{G}^{(2s)} \quad (\equiv P_{s\cdot n}) \qquad (2.10)$$

where

$$\gamma_{s\cdot n} = \{1 - (1 - 2\gamma)^{sn}\}/2. \qquad (2.11)$$

By using the lemma 2.1 and 2.2, we have finally the last rating vector of the n-th cycle as

$$\underline{Z}_{(2s-1)n} = \underline{P}_{s\cdot n}\,\underline{Z}_0 + \sum_{j=1}^{n} \underline{P}_{s\cdot(n-j)}\,\underline{T}\,\underline{X}_j \qquad (2.12)$$

From this expression, we have the mean vector and variance-covariance matrix of $\underline{Z}_{(2s-1)n}$ as follows. Since $\underline{X}_i$ and $\underline{X}_j$ $(i \neq j)$ are independent for each i and j, we put $E(\underline{X}_j) = \underline{\eta}$ and $V(\underline{X}_j) = \underline{V}$ $(j = 1, 2,\ldots,n)$. Also note that the next relations hold for $\underline{T}$ and $\underline{P}_{s\cdot j}$ defined by (2.6) and (2.10) respectively.

$$\underline{G}^{(2s)}\,\underline{T}^{(2s\times s(2s-1))} = \underline{0}^{(2s\times s(2s-1))}. \qquad (2.13)$$

$$\sum_{j=0}^{n-1} \underline{P}^2_{s\cdot j} = (\gamma_{2s\cdot n}/\gamma_{2s})\ \underline{E}^{(s)} + (1/2s)(n - \gamma_{2s\cdot n}/\gamma_{2s})\ G^{(s)} \qquad (2.14)$$

Theorem 2.1

Mean vector and variance-covariance matrix of the last rating value vector of the n-th cycle are given as

$$E(\underline{Z}_{(2s-1)n}) = \underline{P}_{s\cdot n}\underline{Z}_o + \underline{P}^*_{s\cdot n}\ \underline{T}\ \underline{n} \qquad (2.15)$$

$$V(\underline{Z}_{(2s-1)n}) = (\gamma_{2s\cdot n}/\gamma_{2s})\ \underline{T}\ \underline{V}\ \underline{T}' \qquad (2.16)$$

where

$$P^*_{s\cdot n} = \sum_{j=0}^{n-1} \underline{P}_{s\cdot j} = (\gamma_{s\cdot n}/\gamma_s)\ \underline{E}^{(2s)} + (1/2s)(n - \gamma_{s\cdot n}/\gamma_s)\ \underline{G}^{(2s)} \quad (2.17)$$

The expression (2.15) is also written as

$$E(\underline{Z}_{(2s-1)n}) = (1 - 2\gamma_{s\cdot n})\ \underline{Z}_o + (\gamma_{s\cdot n}/s)c_o\underline{J} + (\gamma_{s\cdot n}/\gamma_s)\ \underline{T}\ \underline{n} \qquad (2.18)$$

where $c_o = \underline{J}^{(2s)'}\ \underline{Z}_o$ and $\underline{J}^{(2s)} = (1, 1, \ldots, 1)'$ (2s $\times$ 1 vector). Especially, if all the initial values are the same, i.e., $\underline{Z}_o = z_o\underline{J}$ for given constant $z_o$, we have

$$E(\underline{Z}_{(2s-1)n}) = z_o\underline{J} + (\gamma_{s\cdot n}/\gamma_s)\ \underline{T}\ \underline{n} \qquad (2.19)$$

Initial values vector $\underline{Z}_o$ have much to do with the n-th rating values $A_n$, $B_n$ or their expectations. In order to set the $\underline{Z}_o$, let us use the criterion which minimizes the expectation $E[(\underline{Z}_{(2s-1)n} - \underline{Z}_o)^2]$. By this criterion, $\underline{Z}_o$ is given as $\underline{Z}_o = E[\underline{Z}_{(2s-1)n}]$, which gives that

$$\underline{Z}_o = (c_o/2s)\ \underline{J}^{(2s)} + (1/2\gamma_s)\ \underline{T}\ \underline{n}. \qquad (2.20)$$

We call this optimum initial value.

## 2.2 Smoothing

In order to make more effective use of the rating value, it is recommended to use the quantity which is obtained by averaging the successive rating values. Thus let us use the folloiwng smoothed quantity $\underline{W}_m$, obtained by averaging the last rating values of (n-m+1)-th through n-th cycles. The smoothed quantity is written as

$$\underline{W}_m = \frac{1}{m} \sum_{j=1}^{m} \underline{Z}_{(2s-1)(n-m+j)}. \qquad (2.21)$$

We also write $\underline{W}_m = (W_1, W_2, \ldots, W_{2s})'$, where $W_i = \frac{1}{m} \sum_{j=1}^{m} Z_{i\cdot(2s-1)(n-m+j)}$, $i = 1, 2, \ldots, 2s$.

We first note that $\underline{W}_m$ is written as a sum of the independent vectors $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ in the following way by using the expression (2.12).

$$\underline{W}_m = \frac{1}{m} [\sum_{j=1}^{m} \underline{P}_{s\cdot(n-m+j)}\ \underline{Z}_o + \sum_{j=1}^{m} \sum_{i=1}^{n-m+j} \underline{P}_{s\cdot(n-m+j-i)}\ \underline{T}\ \underline{X}_i]$$

$$= \frac{1}{m} [\sum_{j=1}^{m} \underline{P}_{s\cdot(n-m+j)} \underline{Z}_o + \sum_{i=1}^{n-m+1} \sum_{j=1}^{m} \underline{P}_{s\cdot(n-m+j-i)} \underline{T} \underline{X}_i$$

$$+ \sum_{i=n-m+2}^{n} \sum_{j=i-n+m}^{m} \underline{P}_{s\cdot(n-m+j-i)} \underline{T} \underline{X}_i], \qquad (m \geq 2) \qquad (2.22)$$

Thus, using the Theorem 2.1, we have the following theorem for mean vector and variance-covariance matrix of smoothed quantity $\underline{W}_m$.

Theorem 2.2

$$E(\underline{W}_m) = \frac{1}{m} [(\underline{P}^*_{s\cdot(n+1)} - \underline{P}^*_{s\cdot(n-m+1)}) \underline{Z}_o + (\underline{P}^{**}_{s\cdot(n+1)} - \underline{P}^{**}_{s\cdot(n-m+1)}) \underline{T}\underline{\eta} \qquad (2.23)$$

$$V(\underline{W}_m) = C \underline{T} \underline{V} \underline{T}', \qquad (2.24)$$

where

$$\underline{P}^{**}_{s\cdot n} = (1/2\gamma_s)(n-\gamma_{s\cdot n}/\gamma_s)\underline{E}^{(2s)} + (1/4s)\{n(n-1)-n/\gamma_s + \gamma_{s\cdot n}/\gamma_s\}\underline{G}^{(2s)} \qquad (2.25)$$

$$C = (1/4\gamma_s^2)\{1/m-2(\gamma_{s\cdot m}/\gamma_s-\gamma_{s\cdot m}/\gamma_{2s} + \gamma_s^2\phi_s^2(n,m)/\gamma_{2s})/m^2\} \qquad (2.26)$$

$$\phi_s(n,m) = (\gamma_{s\cdot(n+1)} - \gamma_{s\cdot(n-m+1)})/\gamma_s \qquad (2.27)$$

and $\underline{P}^*_{s\cdot n}$ is given by (2.17).

Note that this result reduces to the Theorem 2.1 when m=1. By (2.13), we have $\underline{G}\ \underline{T}\ \underline{\eta} = \underline{0}$, so $E(\underline{W}_m)$ is also written as

$$E(\underline{W}_m) = \frac{1}{m} [\phi_s(n,m)\underline{Z}_0 + (c_0/2s)(m-\phi_s(n,m))\underline{J} + (1/2\gamma_s)(m-\phi_s(n,m)\underline{T}\ \underline{\eta}] \quad (2.28)$$

If all the initial values are identical, i.e., $\underline{Z}_0 = z_0 J$, then

$$E(\underline{W}_m) = z_0\underline{J} + (1/2m\gamma_s)(m-\phi_s(n,m))\ \underline{T}\ \underline{\eta}, \qquad (2.29)$$

and if $\underline{Z}_0$ is optimum in the sense as stated in Section 2.1, it is given as $\underline{Z}_0 = (c_0/2s)\underline{J} + (1/2\gamma_s)\ \underline{T}\ \underline{\eta}$ and

$$E(\underline{W}_m) = (c_0/2s)\ \underline{J} + (1/2\gamma_s)\ \underline{T}\ \underline{\eta} \qquad (2.30)$$

Since $\underline{W}_m$ is written as sum of the independent vectors $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ as shown in (2.22), by the same inference as we give in [5], we have the following theorem for the asymptotic distribution of $\underline{W}_m$.

Theorem 2.3

If we put $m = kn$ for fixed k ($0 < k \leq 1$), the smoothed rating value $\underline{W}_m$ follows asymptotically normal with mean vector (2.23) and variance-covariance matrix (2.24).

## 2.3. $\underline{T}\ \underline{\eta}$ and $\underline{T}\ \underline{V}\ \underline{T}'$

Let us consider the problem under the more restricted condition on probability vector $\underline{p}$ given in section 2.1. Let us assume here in this section that for given constant $p_0$ such that $0 \leq p_0 \leq 1$,

$$\underline{p} = p_0\ \underline{J}^{(s(2s-1))} \qquad (2.31)$$

Now, we try to evaluate the mean vector (2.23), variance-covariance matrix (2.24) of $\underline{W}_m$ under the restriction (2.31).

First, we give lemmas concerning the properties of matrices $\underline{Q}_i$ and $\underline{T}$ defined in Section 2.1. From the definition of $\underline{Q}_i$, we have

$$\underline{Q}_i \underline{Q}_i' = \frac{1}{\gamma} (\underline{E}^{(2s)} - \underline{R}_i^{(2s)}), \qquad i=1,2,\ldots,2s-1 \tag{2.32}$$

Also according to the definition of $\underline{Q}_i$, i=1,2,...,2s-1, we obtain the follwoing lemma.

Lemma 2.3.

$$\underline{Q}_i \underline{Q}_j' = \begin{cases} \underline{O}^{(2s)} & \text{for } i\neq j. \\ (\underline{E}^{(2s)}-\underline{R}_i^{(2s)}\underline{R}_i^{(2s)'})/2\gamma(1-\gamma) & \text{for } i = j \end{cases} \tag{2.33}$$

For each $\ell = 0,1,2,\ldots,k-1$ and for such i which satisfies $2^\ell \leqq i \leqq 2^{\ell+1} - 1$, we have

$$\underline{Q}_i \underline{J}^{(s(2s-1))} = \underline{J}_{2^\ell}^{(s(2s-1))} \tag{2.34}$$

where

$$\underline{J}_{2^\ell}^{(s(2s-1))} = (\underline{J}^{(2^\ell)}, -\underline{J}^{(2^\ell)},\ldots, \underline{J}^{(2^\ell)}, -\underline{J}^{(2^\ell)})' \tag{2.35}$$

Further we have the following.

Lemma 2.4

$$\underline{T}\ \underline{J} = c \sum_{\ell=1}^{k} \frac{1}{\gamma} (\gamma_{2^{k-1}} - \gamma_{2^{k-1}-2^{\ell-1}})\ \underline{J}_{2^{\ell-1}} \tag{2.36}$$

Proof

We have by (2.34), for any $\ell$ such that $0 \leqq \ell \leqq k-1$, $\underline{Q}_i\ \underline{J} = \underline{J}_{2^\ell}$ for $2^\ell \leqq i \leqq 2^{\ell+1} - 1$. So, for any i such that $2^\ell - 1 \leqq i \leqq 2^{\ell-1}$, we have

$$\underline{R}_i\ \underline{J}_{2^{\ell-1}} = (1-2\gamma)\ \underline{J}$$

For $m \leqq k-2$, $\underline{R}_i\ \underline{J}_{2^m}$ takes a value $\underline{J}$ or $(1-2\gamma)\ \underline{J}$ in turn. Thus

$$\underline{R}_{2^\ell-1}\ \underline{R}_{2^\ell-2} \cdots \underline{R}_{2^{\ell-1}}\ \underline{J}_{2^m} = \begin{cases} (1-2\gamma)^{2^{\ell-1}} & \text{for } m = \ell-1, \\ (1-2\gamma)^{2^{\ell-2}} & \text{for } m \leqq \ell-2 \end{cases}$$

By applying this to the respective sums which compose the matrix $\underline{T}$, we have the assertion.

According to this Lemma, the elements of $\underline{T}\ \underline{J}$ $(\equiv(\tau_1, \tau_2, \ldots, \tau_{2s})')$ are given as

$$\tau_i = c \sum_{\ell=1}^{k} \frac{1}{\gamma} (\gamma_{2^{k-1}} - \gamma_{2^{k-1}-2^{\ell-1}})(-1)^{[(i-1)/2^{\ell-1}]} \tag{2.37}$$

where [x] is the greatest integer not greater than x. Next, $\underline{T}\ \underline{T}'$ is evaluated in the following way. By the first part of the lemma 2.3, we can write

$$\underline{T}\,\underline{T}' = c^2\,[\sum (\pi \underline{R}_i)\, \underline{Q}_{j-1}\, \underline{Q}_{j-1}'\, (\pi R_i)' + \underline{Q}_{2s-1}\, \underline{Q}_{2s-1}']$$

Again by applying the lemma 2.3, we have

$$\underline{T}\,\underline{T}' = (c^2/\gamma_2)\,\{-(\prod_{i=1}^{2s-1} \underline{R}_i)\,(\prod_{i=1}^{2s-1} \underline{R}_i)' + \underline{E}\}$$

Thus, by using the lemma 2.1, we have the following.

Lemma 3.5

$$\underline{T}\,\underline{T}' = (2c^2\gamma_{2s}/\gamma_2)(\underline{E}^{(2s)} - (1/2s)\underline{G}^{(2s)}) \qquad (2.38)$$

By using the lemma 2.3-2.5, we find that $\underline{T}\,\underline{\eta}$ and $\underline{T}\,\underline{V}\,\underline{T}'$ appearing in Theorem 2.2 reduce to the following respective forms.

Theorem 2.4

Under the assumption (2.31), we have

$$\underline{T}\,\underline{\eta} = (c/\gamma)(2p_o-1) \sum_{\ell=1}^{k} (\gamma_{2^{k-1}} - \gamma_{2^{k-1}-2^{\ell-1}})\, \underline{J}_{2^{\ell-1}}^{(2s)} \qquad (2.39)$$

$$\underline{T}\,\underline{V}\,\underline{T}' = (4c^2\gamma_{2s}/s\gamma_2)p_o(1-p_o)(2s\underline{E}^{(2s)} - \underline{G}^{(2s)}) \qquad (2.40)$$

# 3. APPLICATIONS TO RANKING AND SELECTION

## 3.1. Preliminaries

The rating values can be used in various situations as stated in Section 1, for measuring the occurrences of each event (such as the ability or strength of a player) during the sequence of trials. We now apply the results of previous sections to the ranking and selection problems and try to select the specified events by using the rating values.

We use the smoothed quantity $\underline{W}_m$ defined by (2.21), as a statistic for the selection. Since rating values are fully dependent on the initial values vector $\underline{Z}_o$, we set the initial values of all players to be equal, that is $\underline{Z}_o = z_o\, \underline{J}^{(2s)}$. We assume here that the probability vector $\underline{p}$ given in section 2.1 has the following form.

$$\underline{p} = p_o\, \underline{J}^{(s(2s-1))}, \qquad 1/2 \leq p_o \leq 1 \tag{3.1}$$

Asymptotic distribution of $\underline{W}_m$ is normal as given in theorem 2.3. We write for convenience that

$$E(\underline{W}_m) = \underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_{2s})', \tag{3.2}$$

$$V(\underline{W}_m) = \underline{\Lambda} = (\lambda_{ij}), \qquad i,j = 1, 2, \ldots, 2s \tag{3.3}$$

$\mu_i$ and $\lambda_{ij}$ has the following forms.

$$\mu_i = z_o + c(2p_o - 1)(m - \phi_s(n,m))\tau_i / 2m\gamma_s \tag{3.4}$$

$$i = 1, 2, \ldots, 2s$$

$$\lambda_{ij} = \begin{cases} C\cdot 4c^2(2s-1)\gamma_{2s}p_o(1-p_o)/s\gamma_2, & i=j \\ -C\cdot 4c^2\gamma_{2s}p_o(1-p_o)/s\gamma_2, & i\neq j \end{cases} \tag{3.5}$$

$$(i,j = 1, 2, \ldots, 2s)$$

where C is given by (2.26) and $\tau_i$ is an element of $\underline{T}\ \underline{J}$, given by (2.36). Let us give a lemma for $\tau_i$'s which is verified by using lemma 2.4.

Lemma 3.1

For the elements $\tau_1, \tau_2, \ldots, \tau_{2s}$ of a vector $\underline{T}\ \underline{J}$, we have the following relations.

$$\tau_i - \tau_{i+1} = \begin{cases} 2\alpha_1, \text{ if } i=2m+1, \quad m=0,1,2,\ldots,2^{k-1}-1. \\ 2\alpha_{m+1} - 2\sum_{j=1}^{m} \alpha_j, \text{ if } i=2^m\cdot\ell,\ \ell=\text{odd}, \end{cases} \tag{3.6}$$

$$m \in (1,2,\ldots,k-1) \text{ and } 2 \leq i \leq 2^k-1.$$

where

$$\alpha_\ell = (c/\gamma)(\gamma_{2^{k-1}} - \gamma_{2^{k-1}-2^{\ell-1}}).$$

From this lemma, we can show the following.

Lemma 3.2

For means of the rating values given by (3.4), we have

$$\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_k.$$

Equality holds if and only if $p_o = 1/2$.

### 3.2. Indifference Zone Approach

We consider the selection of the t-best events, which are associated with the events with mean rating values $\mu_1, \mu_2, \ldots \mu_t$ as shown by lemma 3.2. Our selection procedure $R_1$ is stated as follows, see Bechhofer [2].

$R_1$: Select the events associated with the t-best sample values $W_{[1]}, W_{[2]}, \ldots, W_{[t]}$, where $W_{[1]} \geqq W_{[2]} \geqq \cdots \geqq W_{[2s]}$.

The selection criterion using the procedure $R_1$ is stated as follows:

$$P(CS|R_1) \geqq P^* \text{ whenever } \mu_t - \mu_{t+1} \geqq \delta^* \tag{3.7}$$

where $P^*$, $\delta^* > 0$ is a given constants specified. Since $\mu_1, \mu_2, \ldots, \mu_t$ are associate with events having the t-best rating values, as shown by lemma 3.2, a probability of a correct selection using the procedure $R_1$ (denoted by $P(CS|R_1)$) is given as follows, where $W_{(i)}$ is a sample rating value associate with the event with $\mu_i$ ($i=1,2,\ldots,2s$).

$$P(CS|R_1) = P(W_{(1)}, W_{(2)}, \ldots, W_{(t)} \geqq W_{(j)},\ j = t+1,\ldots,2s)$$

$$= P(\underline{A}_j\ \underline{W}_m \geqq \underline{0},\ j=1,2,\ldots,t)$$

$$= P(\underline{A}\ \underline{W} \geqq \underline{0}) \tag{3.8}$$

where

$$\underline{A}_j = (\underline{0}^{((2s-t)\times(j-1))}, \underline{J}^{(2s-t)}, \underline{0}^{((2s-t)\times(t-j))}, -\underline{E}^{((2s-t)\times(2s-t))}), j=1,2,\ldots,t$$

$$\underline{A} = \begin{pmatrix} \underline{A}_1 & & \\ & \ddots & \\ & & \underline{A}_t \end{pmatrix}_{(t(2s-t)\times t(2s-t))}$$

and

$$\underline{W} = (\underline{W}_m, \ldots, \underline{W}_m)' \quad (2st\times 1).$$

From theorem 2.3, $\underline{W}_m$ follows asymptotically normal with mean vector (3.2) and variance-covariance matrix (3.3), thus we know that

$$\underline{A}_j \ \underline{\mu} = C^* \ (2p_o-1) \ \underline{T}_j$$

$$\underline{A}_j \ \underline{\Delta} \ \underline{A}_j' = C^{**}p_o(1-p_o)(\underline{E}^{(2s-t)} + \underline{G}^{(2s-t)}), \ j=1,2,\ldots,t$$

$$\underline{A}_i \ \underline{\Delta} \ \underline{A}_j' = C^{**}p_o(1-p_o) \ \underline{E}^{(2s-t)}, \quad i,j=1,2,\ldots,t; \quad i \neq j$$

where

$$C^* = c(m-\phi_s(n,m))/2m\gamma_s \qquad (3.9)$$

$$C^{**} = C\cdot 8c^2(2s-1)\gamma_{2s}/\gamma_2 \qquad (3.10)$$

and

$$\underline{T}_j = \underline{A}_j \; \underline{T} \; \underline{J} = (\tau_j-\tau_{t+1}, \tau_j-\tau_{t+2},\ldots,\tau_j-\tau_{2s})', \quad j=1,2,\ldots,t.$$

Thus we find that $\underline{A}\;\underline{W}$ follows asymptotically normal

$$\underline{A}\;\underline{W} \sim N(\underline{A}\;\; \underline{\mu}^*, \underline{\Lambda}^*) \tag{3.11}$$

where

$$\underline{A}\;\underline{\mu}^* = \begin{pmatrix} \underline{A}_1\;\underline{\mu} \\ \vdots \\ \underline{A}_t\;\underline{\mu} \end{pmatrix} = C^*(2p_o-1)\begin{pmatrix} \underline{T}_1 \\ \vdots \\ \underline{T}_t \end{pmatrix} \equiv C^*(2p_o-1)\;\underline{T} \tag{3.12}$$

and

$$\Lambda^* = C^{**}p_o(1-p_o)\begin{pmatrix} E+G & E \ldots\ldots E \\ E & \ddots \quad \vdots \\ \vdots & \ddots \quad E \\ E \ldots\ldots E & E+G \end{pmatrix} \equiv C^{**}p_o(1-p_o)\;\underline{\Lambda}^{**}. \tag{3.13}$$

And (3.8) is expressed as

$$P(CS|R_1) = p(\underline{U} \geq \frac{-\underline{A}\;\underline{\mu}^*}{\sqrt{C^{**}p_o(1-p_o)}})$$

$$= P(\underline{U} \leq \frac{C^*(2p_o-1)}{\sqrt{C^{**}p_o(1-p_o)}}\;\underline{T}) \tag{3.14}$$

where

$$\underline{U} \equiv \frac{\underline{A}\,\underline{W} - \underline{A}\,\underline{\mu}^*}{\sqrt{C^{**}p_o(1-p_o)}} \sim N(\underline{0}, \underline{\Lambda}^{**}). \qquad (3.15)$$

Now, the least favourable configuration (L.F.C.) for the procedure $R_1$ is given as follows. By (3.4), since

$$\mu_t - \mu_{t+1} = C^*(2p_o - 1)(\tau_t - \tau_{t+1}),$$

a condition $\mu_t - \mu_{t+1} \geqq \delta^*$ in the criterion (3.7) is equivalent to

$$p_o \geqq 1/2 + \delta^*/\{2C^*(\tau_t - \tau_{t+1})\} \qquad (3.16)$$

Further, $\underline{T} \geqq \underline{0}$ and $(2p_o-1)/\sqrt{p_o(1-p_o)}$ is increasing in $p_o$ for $p_o \geqq 1/2$, infimum of the probability (3.14) is attained at $p_o^*$ in the following manner.

Theorem 3.1

The L.F.C. of the procedure $R_1$ is attained at

$$p_o = 1/2 + \delta^*/\{2C^*(\tau_t - \tau_{t+1})\} \qquad (\equiv p_o^*) \qquad (3.17)$$

i.e., $\mu_t - \mu_{t+1} = \delta^*$.

In order to evaluate the probability (3.14), following lemma is useful. See Gupta [3].

Lemma 3.3

If n-dimensional vector $\underline{X}$ follows normal with mean vector $\underline{0}$ and variance-covariance matrix $\Sigma = (\sigma_{ij})$, where $\sigma_{ii} = 1 + a_i^2$ and $\sigma_{ij} = a_i a_j$ $(i,j = 1,2,\ldots,n;\ i \neq j)$. Then

$$P(\underline{X} \leq \underline{W}) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi(a_i x + W_i)\ \phi(x)dx$$

where $\underline{W} = (W_1, W_2,\ldots,W_n)'$ and $\Phi(x)$, $\phi(x)$ are c.d.f. and p.d.f. of standard normal.

By using the lemma 3.3, we can evaluate the $P(CS|R_1)$ for $t = 1$. When $t = 1$, we can see

$$\underline{U} \sim N(\underline{0}^{(2s-1)}, \underline{E}^{(2s-1)} + \underline{G}^{(2s-1)}) \qquad (3.18)$$

Thus putting

$$\kappa = \frac{C^*(2p_o^*-1)}{\sqrt{C^{**}p_o^*(1-p_o^*)}} \qquad (3.19)$$

we have the following.

Theorem 3.2

$$P(CS|R_1)_{LFC} = \int_{-\infty}^{\infty} \prod_{\ell=1}^{2s-1} \Phi(x + \kappa(\tau_1 - \tau_{\ell+1}))\phi(x)dx \qquad (3.20)$$

## 3.3 Subset Selection Approach

In this section, let us consider a problem of selecting a subset of events which contain the t-best rating mean values, i.e., the events associate with events having the rating values $\mu_1, \mu_2, \ldots, \mu_t$. In this case our procedure $R_2$ is stated as follows, see Gupta and Panchapakesan [4].

$R_2$: Select the event i if and only if $W_i \geqq W_{[t]} - d$, where $W_{[i]}$ is the ordered $W_i$, such that $W_{[1]} \geqq W_{[2]} \geqq \cdots \geqq W_{[2s]}$.

Then the probability of a correct selection, using $R_2$ is given by

$$P(CS|R_2) = P(W_{(1)}, W_{(2)}, \ldots, W_{(t)} \geqq W_{(j)} - d, \quad j=t+1, \ldots, 2s)$$

where, as before, $W_{(i)}$ is an unknown sample rating value associate with the event having $\mu_i$ $(i=1,2,\ldots,2s)$. Then, in the same way as we considered in Section 3.2., we have

$$P(CS|R_2) = P(\underline{A}_j \underline{W}_m \geqq -d \, \underline{J}^{(2s-t)}, \quad j=1,2,\ldots,t)$$

$$= P(\underline{A} \, \underline{W} \geqq -d \, \underline{J}^{(t(2s-t))})$$

$$= P\left(\underline{U} \leqq \frac{d \, \underline{J}^{(t(2s-t))} + C^*(2p_o - 1) \, \underline{T}}{\sqrt{C^* p_o (1-p_o)}}\right) \qquad (3.21)$$

where all notations are the same as those we used in Section 3.2. Since $\underline{T} \geqq \underline{0}$, the right hand side of the bracket in (3.21) is increasing in $p_o$

for $p_o \geqq 1/2$, and we have the following.

Theorem 3.3

The L.F.C. by using the procedure $R_2$ is attained at $p_o = 1/2$ (i.e., $\mu_1 = \mu_2 = \ldots = \mu_{2s}$ from lemma 3.2), and $P(CS|R_2)$ under this configuration is given by

$$P(CS|R_2)_{LFC} = P(\underline{U} \leqq 2d/\sqrt{C^{**}}) \tag{3.22}$$

where $\underline{U} \sim N(\underline{0},\Lambda^{**})$.

Especially, if t = 1, then we have

$$P(CS|R_2)_{LFC} = \int_{-\infty}^{\infty} \Phi^{2s-1}(x + 2d/\sqrt{C^{**}})\phi(x)dx \tag{3.23}$$

## References


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