A BRANCHING PROCESS APPROACH TO A PROBLEM CONNECTED WITH A T.V. GAME SHOW*

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ABSTRACT

A television game considered by Price and Tenenbein (1976) has been approached from a branching process viewpoint. In so doing the original game considered by these authors has been generalized thereby allowing the use of some of the known results in branching processes in answering some of the questions posed by these authors. Some related computational techniques have been discussed for obtaining the solution of an equation (arising in branching processes) needed for certain results pertaining to these games.

1. INTRODUCTION

This work was inspired by a paper authored by Price and Tenenbein (1976) appear in these communications sometime ago. The problem dealt with by these authors apparently was posed to them by the producers of a television game show that they were developing at the time. In particular, as per these authors, the producers were concerned about the size of a budget for the show and wanted to evaluate the chances of winning and the expected pay off for the game. In this context the model considered by the authors is briefly described as follows:

Consider a sequence of independent trinomial trials where each trial results in either event A = [success] with probability a, or event B = [a win of a run of additional $r \geq 1$ trials] with probability b, or event C = [neither of the two events A and B] with probability c, where $a+b+c=1$. The game begins with a fixed initial number $k \geq 1$ of such trials. Each time the event B occurs within these k trials or at any subsequent trial the player is allowed to have r additional trials over and above the remaining trials he is yet to go. The game is terminated either upon the occurrence of the event A or when the player has used up all the trials allowed to him, whichever of these two possibilities happen to occur first. If the game terminates with A, the player wins a prize, otherwise he loses at the end of the trials allowed to him. Let M denote the total number of trials it takes to end the game. The authors, after showing that $P(M<\infty)=1$, attempt to obtain an approximation for the probability P(L) of the game ending with a loss for the player, along with an error bound for this approximation. Numerical values for the approximation along with their error bounds were given for $r=2$ and various values of b and c. The paper ends with a mention of several remaining issues such as the improvement over their approximation for P(L), obtaining a similar approximation for E(M) and the possible desired objective of the producers of the show who may want to limit the E(M) while retaining a fixed value of P(L). A case is also made for a possible use of the present model for developing a sequential inspection plan for quality control problems where each unit under inspection may either have a major defect (event A) or a minor defect (event B) or be free of defects (event C).

In the next section we adopt for the above game a branching process (Galton-Watson process) approach which leads not only to a generalization of the above model but also allows us to use some known results about these processes to the present problem. In particular it gives the probability P(L) as the solution of a known equation in branching process (see equations (14) and (22)), which can be explicitely solved for the authors' case of $r=2$ (see (31) and (32)). Moreover we also establish a simple linear relation between E(M) and P(L) (see (24)), so that if you know one you can know the other. Unfortunately this relation was not noticeable by following the approach of the above authors. Finally we also discuss some computational aspects of certain needed quantities through the use of iterative procedures suggested by the branching process approach.

More recently, essentially the same branching process has been used by Karlin and Tavaré (1982) (see also Pakes (1983)) for the study of certain mating systems in population genetics. Here the above mentioned success event A is referred to as "killing" by these authors. The reader may refer to Harris (1963) and Athreya and Ney (1972) for a detailed account of the branching processes and some of the related well known results that we shall need.

2. A GENERALIZED GAME BASED ON A BRANCHING PROCESS

In the following we shall view the above game slightly differently but essentially in a manner equivalent to the above. To us each trial constitutes in observing two independent random variables ξ and X, where ξ takes values 0 or 1 according as the prize-winning (success) event A occurs or not with $P(\xi=1)=p$, $0<p<1$, and X, which may take the value zero, represents the additional random number of trials that the player will be allowed to have (over and above the trials that he is yet to go) with the probability generating function (p.g.f.) for each X given by

$$f(s) = E(s^X) = \sum_{i=0}^{\infty} p_i s^i, \quad |s| \leq 1. \tag{1}$$

Throughout we shall assume that $p_0>0$ and $p_0+p_1<1$. Furthermore it is assumed that (ξ_i,X_i)'s corresponding to various trials are independent and identically distributed (I.I.D.) random vectors. In the branching process language if $Z_0=k$ denotes the initial number of trials representing the zeroth generation, the (random) number of additional trials allowed during the run of these k initial trials is given by

$$Z_1 = X_{01} + X_{02} + \ldots + X_{0k}, \tag{2}$$

where X's here and below are I.I.D. with the common p.g.f. given by (1), and Z_1 denotes the number of trials for the first generation. Thus in general having defined Z_n for the nth generation, Z_{n+1} for the next generation is given by

$$Z_{n+1} = \begin{cases} \sum_{i=1}^{Z_n} X_{ni} & \text{if } Z_n \geq 1 \\ 0 & \text{if } Z_n = 0. \end{cases} \tag{3}$$

(see Kendall (1951) and Neuts (1969) for a similar branching process interpretation in connection with an M/G/1 queue.) It is clear that as soon as a Z_n becomes zero, the additional trials having run their course the game ends. Instead the game may also end at the trial when the event A occurs for the first time. Let for $n \geq 0$,

$$Y_n = \sum_{i=0}^{n} Z_i, \quad Y = \sum_{i=0}^{\infty} Z_i \tag{4}$$

Then

$$Y_n \nearrow Y, \text{ a.s.}, \tag{5}$$

where Y is generally referred to as the "total progeny" of the branching process $\{Z_n\}$. Thus it is easy to see that

$$M = \min(Y,N); \quad P(\text{loss}) = P(M=N), \tag{6}$$

where M is the number of trials it takes to end the game and N is the number of trials it takes for the event A to occur for the first time, so that

$$P(N=j) = p^{j-1}q, \quad j = 1,2,\ldots, \tag{7}$$

with $q=1-p$. The game considered by Price and Tenenbein (1976) is now a special case of the above with

$$\begin{cases} a = p; \quad q = b+c; \quad p_i = 0 \text{ for } i \neq 0,r; \\ p_0 = c(b+c)^{-1}; \quad p_r = b(b+c)^{-1} \end{cases} \tag{8}$$

where the events B and C are observed through the random variable X depending upon if $X\geq 1$ or $X=0$ respectively. It is worth noting that here no generality is lost in observing two independent random variables ξ and X at each trial instead of observing a single trinomial trial as was considered by the above authors. Also since $P(N<\infty)=1$, it is immediate that $P(M<\infty)=1$; in fact since N is a geometric random variable, M must have all its moments finite.

The random variable Y has been dealt with among others by Otter (1949), Bharucha-Reid (1960), Harris (1963), Prabhu (1965), Mullikin (1968), Feller (1969), Dwass (1969) and Pakes (1971). Again because of the mutual independence built into the branching process, it is easy to see that if $Z_0=k$,

$$Y = \eta_1 + \eta_2 + \ldots + \eta_k \tag{9}$$

where η_i's are I.I.D. with distribution same as that of Y corresponding to the case with $Z_0=1$. Let

$$g_n(s) = E(s^{Y_n}),\ n \geq 0; \quad g(s) = E(s^Y); \quad |s| \leq 1 \tag{10}$$

Then it is well known that $P(Z_n\to 0)$, the so called probability of extinction, satisfies the relation

$$P(Z_n\to 0) = P(Y<\infty) = \beta^{Z_0} \tag{11}$$

where β is the smallest root lying between 0 and 1 of the equation $f(s)=s$ and is equal to one if $E(X)\leq 1$ and is strictly less than one if $E(X)>1$. Thus in the latter case Y is an improper random variable with

$$P(Y=\infty) = 1 - g(1-) = 1 - \beta^{Z_0}. \tag{12}$$

In the case $Z_0=1$, it is well known (see Harris (1963), Prabhu (1965), Dwass (1969) and Pakes (1971)) that $g_n(s)$, $n\geq 0$, satisfy the relations

$$g_0(s) = s, \quad g_{n+1}(s) = sf(g_n(s)), \; n \geq 0, \tag{13}$$

and that their limit $g(s)$ equals to the unique regular solution $F(s)$ for t within the open unit disc of the equation

$$t = sf(t), \quad |s| < 1, \tag{14}$$

with $0<g(1-)=\beta\leq 1$. For the case with $Z_0=k\geq 1$, it follows from (9) that

$$g(s) = [F(s)]^k, \quad |s| \leq 1. \tag{15}$$

Some authors in defining Y_n and Y (see Mullikin (1968), Karlin and Tavaré (1982)), unlike (4), do not include Z_0 so that equations (13) and (14) differ slightly in their case.

Remark: As mentioned earlier a model (based on a branching process) similar to our game-model has recently been studied by Karlin and Tavaré (1982) (see also Pakes (1983)), where our success event A is referred to as a "killing" event. However there is one minor but basic difference namely that in their case if such a killing event occurs to any individual belonging to a given generation, the whole generation is considered as killed and the process thus stops at such a generation. Instead in our case the process stops by trials rather than by generations. Moreover, the offsprings being trials, it is visualized that the trials corresponding to a given generation are run one by one in a certain order so that as soon as the event A occurs the process stops at that trial. Thus the trials (belonging to this last generation) that have already been run will be counted as per our definition of M given by (6).

3. SOME CHARACTERISTICS OF A GENERALIZED GAME

In this section we shall derive expressions for certain useful quantities relating to a generalized game, such as E(M), P(loss), distribution of M, the cost involved for a game, etc. As we shall see below, most of these quantities are easy to derive once we look upon the original game in our modified manner, allowing us to express M as in (6). We assume that $Z_0=k\geq 1$. Note that

$$P(\text{event A never occurs}|Y) = \begin{cases} p^Y & \text{if } Y < \infty \\ 0 & \text{if } Y = \infty, \end{cases} \quad (16)$$

$$P(\text{event A occurs at nth trial}|n\leq Y<\infty) = p^{n-1}q, \quad (17)$$

so that for $|s|\leq 1$, using (11), (12) and (15), we have

$$E(s^M;\ \text{winning},\ Y<\infty) = \sum_{i=k}^{\infty} P(Y=i) \sum_{\ell=1}^{i} p^{\ell-1}q\ s^{\ell}$$

$$= \frac{qs}{(1-ps)} (\beta^k-[F(ps)]^k, \quad (18)$$

$$E(s^M;\ \text{winning},\ Y=\infty) = (1-\beta^k) \sum_{\ell=1}^{\infty} p^{\ell-1}q\ s^{\ell}$$

$$= (1-\beta^k)\ sq(1-ps)^{-1}, \quad (19)$$

and

$$E(s^M;\ \text{loss}) = E(s^M;\ Y<\infty,\ \text{loss})$$

$$= \sum_{i=k}^{\infty} P(Y=i)(ps)^i = [F(ps)]^k, \quad (20)$$

where F(s) is the unique regular root for t (within the open unit disc) of the equation (14). The above expressions in turn yield

$$E(s^M) = \{(qs) + (1-s)\ [F(ps)]^k\}\ (1-ps)^{-1}, \quad (21)$$

$$P(\text{winning}) = 1 - [F(p)]^k;\quad P(\text{loss}) = [F(p)]^k \quad (22)$$

In particular using (6) and (7) (or (21)) we have

$$E(M) = \frac{1}{q}(1-E(p^Y)) = \frac{1-[F(p)]^k}{q}, \tag{23}$$

so that

$$P(\text{winning}) = q\,E(M). \tag{24}$$

It is this relation which got overlooked by the approach adopted by Price and Tenenbein (1976). Thus the producers of the show could control one of the two P(winning) and E(M), by controlling the other and fixing the value of q appropriately. Again using (21) and after some lengthy standard calculations one also obtains

$$\text{Var}(M) = \frac{p}{q^2} + \frac{[F(p)]^k}{q}\{1 - \frac{[F(p)]^k}{q} - 2k[1-pf'(F(p))]^{-1}\}, \tag{25}$$

where a prime indicates the derivative of a function. Here it is known and can be easily shown that $sf'(F(s))<1$, for all $0\leq s\leq 1$, except for the case where $s=1$ and $f'(1)=1$, where it is equal to one. Again in obtaining (25) we have also used the fact that for $0<s<1$

$$F'(s) = f(F(s))\,[1-sf'(F(s))]^{-1}, \tag{26}$$

which can be established by using (14) of which F(s) is the unique regular solution. It is interesting to note that, since the expression within {} of (25) is negative, Var(M) is strictly less than $\text{Var}(N)=p/q^2$.

Suppose now that each trial takes a random length of time, where these times, denoted by $\tau_1,\tau_2,\tau_3,\ldots\ldots$ for various trials, are I.I.D. random variables with a common cumulative distribution function (c.d.f.) $H(\cdot)$, and are independent of everything else. Then the cost of a game is given by

$$\mathcal{C} = \begin{cases} \lambda \sum_{i=1}^{M} \tau_i + \rho, & \text{if } M = N \\ \lambda \sum_{i=1}^{M} \tau_i, & \text{if } M = Y, \end{cases} \tag{27}$$

where λ is the cost per unit time and ρ is the amount for the prize pay-off when the player wins the game. Thus the Laplace-

Stieltjes transform (L.S.T.) of $\tilde{C}$ is given by

$$\begin{aligned}\psi_{\tilde{C}}(\theta) &\equiv E(\exp[-\theta\tilde{C}]) \\ &= \exp(-\theta\rho)E([H^*(\lambda\theta)]^M;\ \text{winning}) \\ &\quad + E([H^*(\lambda\theta)]^M;\ \text{loss}),\end{aligned} \tag{28}$$

for $Re(\theta)\geq 0$, where $H^*(\theta)$ is the L.S.T. of the c.d.f. $H(\cdot)$. Using (18)-(20) in (28) we find

$$\begin{aligned}\psi_{\tilde{C}}(\theta) &= \exp(-\theta\rho)\ qH^*(\lambda\theta)\{1-[F(pH^*(\lambda\theta))]^k\} \\ &\quad \cdot [1-pH^*(\lambda\theta)]^{-1} + [F(pH^*(\lambda\theta))]^k.\end{aligned} \tag{29}$$

Finally from this and (23) we obtain the expected cost of a game given by

$$E(\tilde{C}) = [\lambda E(\tau) + q\,\rho]\ E(M), \tag{30}$$

which is not unexpected in view of the relation (24).

We note that in most of the above expressions the function $F(s)$ appears predominantly. To obtain $F(s)$ involves solving the equation (14) for t for given values of s. In the next section we shall discuss some iterative techniques of solving (14) arising in the context of branching processes. But first we deal with two special cases for $f(\cdot)$, where (14) is explicitly solvable for F.

The first one was considered by Price and Tenebein (1976) which corresponds to the case with

$$f(s) = p_0 + (1-p_0)s^2. \tag{31}$$

For this it is easy to solve (14) explicitly yielding

$$F(s) = \{1-[1-4s^2p_0(1-p_0)]^{1/2}\}\ (2s(1-p_0))^{-1},\ 0<s\leq 1, \tag{32}$$

so that when $Z_0=1$, the probability of extinction is given by

$$F(1-) = \begin{cases} 1 & \text{if } p_0 \geq \frac{1}{2} \\ p_0(1-p_0)^{-1} & \text{if } p_0 < \frac{1}{2} \end{cases} . \tag{33}$$

The expression (32) also turns out to be the p.g.f. of the time it takes for a random walk on integers to travel from 0 to 1 with

(33) being the probability that this time will be finite (see Dwass (1968), (1969)). For the present case (32) when used with s=p in (22) yields an explicit expression for P(loss), which was previously obtained approximately by Price and Tennebein (1976).

Our second example corresponds to the case of so called linear fractional p.g.f. for $f(\cdot)$, with

$$f(s)=(1-\alpha_0-\alpha_1)(1-\alpha_0)^{-1}+\alpha_1 s(1-\alpha_0 s)^{-1},$$

$$0<\alpha_0,\alpha_1,\alpha_0+\alpha_1<1. \qquad (34)$$

Again for this case too (14) can be explicitly solved yielding

$$F(s) = \left\{(1+\alpha_0 s - \frac{\alpha_1 s}{1-\alpha_0}) - [(1+\alpha_0 s - \frac{\alpha_1 s}{1-\alpha_0})^2 - 4\alpha_0 s\ (1 - \frac{\alpha_1}{1-\alpha_0})]^{1/2}\right\} (2\alpha_0)^{-1}. \qquad (35)$$

4. COMPUTATION OF THE FUNCTION F(p)

We shall briefly discuss now the problem of obtaining the smallest solution F(p) of the equation

$$t = p\ f(t), \qquad (36)$$

for t within the unit (real) interval, for a given value of p with $0<p\leq 1$, where we also allow here the value p=1 unless otherwise mentioned. We exclude however the case with p=1 and $f'(1)\leq 1$, where the root of (36) is known to be one. One procedure is suggested by the way the above question arose namely via a system of iterative equations (similar to (13)) given by

$$G_0(s) = s, \quad G_{n+1}(s) = p\ f(G_n(s)), \quad n \geq 0, \qquad (37)$$

where $G_n(s)$ can be shown to converge monotonically to F(p) as $n\to\infty$. Thus starting with a given value of s as $G_0(s)$, we iteratively compute $G_n(s)$ using (37) for a few values of n. As it turns out, typically after only a few iterations the obtained value of $G_n(s)$ is already fairly close to the desired solution F(p) of (36). In fact, as also pointed out by Karlin and Tavaré (1982) (Their analog of equation (36) is given by (40) below and is only slightly different from ours because of the fact that they do not include Z_0 in their definition of Y_n and Y (see (42) below)), using a modification of the proof provided by Athreya and Ney (1972, pp. 38-41) it is easy to establish the existence of the limit

$$\lim_{n\to\infty} \gamma^{-n}(G_n(s) - F(p)) \equiv p\, Q(s),\ 0 \leq s < s_1, \tag{38}$$

where

$$\gamma = p\, f'(ps_0), \tag{39}$$

s_0 and s_1 are the two real roots (both functions of p) with $0<s_0<1\leq s_1$, of the equation

$$t = f(pt), \tag{40}$$

and Q(s) satisfies the functional equation

$$Q(f(ps)) = \gamma Q(s), \tag{41}$$

subject to $Q(s_0) = 0$, $Q'(s_0) = 1$ and $Q'(s)>0$ for every $s\in[0,s_1)$. Furthermore as it turns out, the convergence in (38) is <u>uniform</u> for s over $[0,\delta]$ for every $0<\delta<s_1$. As noted earlier γ defined by (39) is strictly less than one except for the case when p=1, f'(1)=1, which we have already excluded. Also comparing equations (40) and (14) with s=p, it is easily seen that

$$F(p) = ps_0, \tag{42}$$

so that

$$\gamma = p\, f'(F(p)). \tag{43}$$

Thus it follows from (38) that the sequence $G_n(s)$ defined by (37) converges exponentially fast to the desired solution F(p) of (36).

Again by taking $s=pp_0$ in (37), since $G_1(pp_0)>pf(0)=pp_0=G_0(pp_0)$, it is easily seen by an induction argument that $G_n(pp_0)\uparrow F(p)$ as $n\to\infty$. Similarly if instead we take s=p in (37) when p<1, since $G_1(p)<p=G_0(p)$, we have via an induction argument, $G_n(p)\downarrow F(p)$ as $n\to\infty$. Thus since $G_n(p)-G_n(pp_0)\downarrow 0$ as $n\to\infty$, by computing both $G_n(p)$ and $G_n(pp_0)$ not only would we obtain F(p) approximately for large enough n, but also an error bound in the form of the difference $G_n(p)-G_n(pp_0)$.

However a better approach is the following. The equation (36) can be rewritten as

$$t = pp_0[1-p(f(t)-p_0)t^{-1}]^{-1}, \tag{44}$$

which suggests an alternative way of defining the iterating sequence as

$$H_0(s) = s; \; H_{n+1}(s) = \phi(H_n(s)), \; n \geq 0, \tag{45}$$

where

$$\phi(s) = pp_0 s[s+pp_0-pf(s)]^{-1}, \tag{46}$$

which is an improper p.g.f. with

$$\phi(1) = pp_0(q+pp_0)^{-1} < 1. \tag{47}$$

Again for $0<s\leq 1$, it can be easily seen that

$$s \lesseqgtr \phi(s) \Leftrightarrow s \lesseqgtr pf(s) \Leftrightarrow s \lesseqgtr F(p), \tag{48}$$

so that by standard induction arguments it follows that as $n\to\infty$, $H_n(s)\uparrow F(p)$ for $s<pf(s)$ and $H_n(s)\downarrow F(p)$ for $s>pf(s)$. A similar behavior holds for the sequence $G_n(s)$. In particular, since $s<pf(s)$ holds for $s=pp_0$, we have $H_n(pp_0)\uparrow F(p)$ as $n\to\infty$. Again note that since for $s>0$

$$H_1(s) > G_1(s) \Leftrightarrow \phi(s) > pf(s) \Leftrightarrow pf(s) > s, \tag{49}$$

and $H_0(s) = G_0(s) = s$, it follows by an induction argument that for all s satisfying $0<s<pf(s)$, we have

$$H_{n+1}(s) = \phi(H_n(s))>\phi(G_n(s))>pf(G_n(s))=G_{n+1}(s), \; n\geq 1 \tag{50}$$

so that for $s=pp_0$ we have

$$F(p) > H_n(pp_0) > G_n(pp_0), \; n \geq 1. \tag{51}$$

Consequently the sequence $H_n(pp_0)$ appears to perform much better than $G_n(pp_0)$ in converging to $F(p)$ as $n\to\infty$. Likewise for $s>pf(s)$, it can be shown that

$$\begin{cases} H_1(s) < G_1(s) \\ H_{n+1}(s)=\phi(H_n(s))<\phi(G_n(s))<pf(G_n(s))=G_{n+1}(s), \; n\geq 1, \end{cases} \tag{52}$$

Holds, so that

$$F(p) < H_n(s) < G_n(s), \quad n \geq 1. \tag{53}$$

Thus the sequence $H_n(s)$ beats the sequence $G_n(s)$ again in its approach towards $F(p)$. In conclusion when $p<1$ one may instead compute $H_n(p)$ and $H_n(pp_0)$ which satisfy

$$H_n(pp_0) < F(p) < H_n(p), \quad n \geq 0, \tag{54}$$

and use the differences $H_n(p)-H_n(pp_0)$ for the error bounds for either of the two, $H_n(p)$ and $H_n(pp_0)$, taken as an approximation to $F(p)$ for large enough n.

5. CONCLUDING REMARKS

(a) As also pointed out by Price and Tenenbein (1976) the above models can be used for developing a sequential inspection plan for quality control problems (see Wetherill (1982)), where each unit under inspection (independent of the other units) may either have a major defect (event A), a minor defect (event B) or be free of defects (event C). If an event A is observed we stop the process and try to remedy the problem which may be causing the major defects in the items. If on the other hand an event B is observed we go for an additional positive random number of items to be inspected over and above those we have yet to examine. Finally if an event C is observed we pass on to the next item for inspection if there are yet more to go. One sequential inspection plan would be to stop not only when an event A is observed but also when the number of times the event B has occurred exceeds a given number or alternatively when the total number of items to be inspected (trials) i.e. the random variable M (see (6)) exceeds a given number. This given number and also the distribution of the additional random number of items to be desired whenever B occurs, are chosen depending upon a suitable loss function involving the cost of inspection, etc. and upon how frequently the occurrence of the event B is considered acceptable in a given situation. Some of the results found in Puri (1969) or their analog for the discrete time case may be useful for the distribution problems involved here. However a detailed treatment of these problems based upon our generalized branching process model will have to wait for another communication.

(b) It is evident from (50) and (52) that for all starting values of $0<s<1$, the sequence $H_n(s)$ performs better than the sequence $G_n(s)$. In view of (38) it also follows that the sequence $H_n(s)$ converges to $F(p)$ at least as fast as $G_n(s)$, which converges at an exponential rate of $-\ln\gamma$ given by (39). In fact it appears that the corresponding exponential rate constant for the sequence $H_n(s)$ may be much larger than $-\ln\gamma$. These considerations will have to be now investigated separately.

(c) Finally the above class of models can be generalized to the case where the probability p for the event A may be allowed to vary (in a suitable manner) from trial to trial. Equally appropriate may be for some situations is to allow $f(s)$ and in particular the probability p_0 to vary from trial to trial.

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