ISOTONIC RULES FOR SELECTING GOOD EXPONENTIAL POPULATIONS

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ABSTRACT

The problem of selecting exponential populations better than a control under a simple ordering prior is investigated. Based on some prior information, it is appropriate to set lower bounds for the concerned parameters. The information about the lower bounds of the concerned parameters is taken into account to derive isotonic selection rules for the control known case. An isotonic selection rule for the control unknown case is also proposed. A criterion is proposed to evaluate the performance of the selection rules. Simulation comparisons among the performances of several selection rules are carried out. The simulation results indicate that for the control known case, the new proposed selection rules perform better than some earlier existing selection rules.

1. INTRODUCTION

The problem of selecting populations better than a control under a simple ordering prior has been studied by Gupta and Yang (1984) for the normal means problem, by Gupta and Huang(1983) for the binomial parameters problem and by Gupta and Leu (1986) for the case of two-parameter exponential populations. Huang (1984) has considered the problem in a nonparametric setup. Recently, Liang and Panchapakesan (1987) has studied the problem via a Bayesian approach. In the present paper, under a simple ordering prior, we study the problem of selecting populations better than a control with the underlying populations having exponential distributions.

Let $\pi_1, \ldots, \pi_k$ be k independent populations and population π_i has density function $f(x|\theta_i) = \exp\{-(x-\theta_i)\}I_{(\theta_i,\infty)}(x)$, where $I_A(\cdot)$ denotes the indicator function of the set A. The parameters θ_i, $i = 1, \ldots, k$, are unknown; however, it is known that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$. This is typical, for example, in experiments involving different dose levels of a drug, where the treatment effects will have a known ordering. The k populations are compared with a control π_o, which is characterized by the associated density function $f(x|\theta_o)$. Population π_i is said to be good if $\theta_i \geq \theta_o$ and to be bad otherwise. Our goal is to select all good populations.

Let $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ and let $\Omega = \{\underset{\sim}{\theta}|\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k\}$ be the parameter space. Let $S_i = \{i, i+1, \ldots, k\}$ for $i = 1, \ldots, k$, and let $S_{k+1} = \phi$. S_i can be viewed as an action. If action $S_i (i = 1, \ldots, k)$ is taken, it means that populations $\pi_i, \ldots, \pi_k$ are selected as good populations. Action S_{k+1} corresponds to excluding all the k populations as bad populations. Since θ_i, $i = 1, \ldots, k$ are ordered according to a simple ordering prior, it is therefore appropriate to restrict to the action space $\mathcal{A} = \{S_1, S_2, \ldots, S_k, S_{k+1}\}$.

Definition 1.1

a) A selection rule δ is isotonic if it selects population π_i and if $\theta_i < \theta_j$, then it also selects population π_j.

b) A selection rule δ satisfies the P^*-condition if $\inf_{\underset{\sim}{\theta} \in \Omega} P_{\underset{\sim}{\theta}}\{CS|\delta\} \geq P^*$ where $P^* \in (0,1)$ is a prespecified value and where CS denotes the event of the selection of any nontrivial subset which contains all good populations.

We will restrict our attention to isotonic slection rules δ which satisfy the P^*-condition.

Note that the parameter θ_i can be viewed as the guaranteed lifetime. Based on some prior information, we may be able to set a lower bound for θ_i (for example, $\theta_i \geq 0$). Therefore, it is assumed that $\theta_i \geq a_i$ for each $i = 1, \ldots, k-1$, where the constants a_i, $i = 1, \ldots, k-1$, are known and satisfy that $a_1 \leq a_2 \leq \cdots \leq a_{k-1}$. In Section 2, we deal with the control parameter θ_o known case. The information of the lower bounds $\underset{\sim}{a} = (a_1, \ldots, a_{k-1})$ is taken into account to derive isotonic selection rules. Some properties associated with the selection rules are discussed. An isotonic selection rule for the θ_o unknown case is proposed in Section 3. Simulation comparisons between our selection rules and some earlier existing isotonic selection rules are carried out and reported in Section 4.

2. ISOTONIC SELECTION RULES FOR θ_o KNOWN CASE

Let X_{ij}, $j = 1, \ldots, n$, be a sample from population π_i. Define $Y_i = \min(X_{i1}, \ldots, X_{in})$, and $\hat{Y}_{m:i} = \min(Y_m, \ldots, Y_i)$ for each $m = 1, \ldots, i$; $i = 1, \ldots, k$. When $i = k$, for simplicity, $\hat{Y}_{m:k}$ is denoted by $\hat{Y}_m$. Also, write $\underset{\sim}{\hat{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_k)$.

For given constants $b_1 \leq b_2 \leq \cdots \leq b_{k-1} < 0$, define k k-tuple vectors $\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_k$, as follows:

$$\underset{\sim}{b}_1 = (0, \ldots, \ldots, 0),$$

$$\underset{\sim}{b}_2 = (b_1, 0, \ldots, \ldots, 0),$$

$$\underset{\sim}{b}_i = (b_1, \ldots, b_{i-1}, 0, \ldots, 0),$$

$$\vdots$$

$$\underset{\sim}{b}_k = (b_1, b_2, \ldots, b_{k-1}, 0).$$

The following theorem is useful for deriving isotonic selection rules for θ_o known case.

Theorem 2.1.

a) For the given constants $b_1 \le b_2 \le \cdots \le b_{k-1} < 0$ and $P^* \in (0,1)$, there exist positive constants $d_i \equiv d_i(b_1, \ldots, b_{k-1})$, $i = 1, \ldots, k$, such that

(2.1) $$P_{\underset{\sim}{b}_i}\{\hat{Y}_m \ge d_m \text{ for some } m = 1, \ldots i\} = P^*$$

for each $i = 1, \ldots, k$. Also, $0 < d_1 < d_2 < \cdots < d_k < \infty$.

b) Let the constants $d_i(b_1, \ldots, b_{k-1})$, $i = 1, \ldots k$, be implicitly defined in (2.1). Then, for each $i = 2, \ldots, k$, $d_i(b_1, \ldots, b_{k-1})$ is increasing in $b_1, \ldots, b_{i-1}$, and independent of $b_i, \ldots, b_{k-1}$. That is, the increment of b_i has influence only on $d_{i+1}, \ldots, d_k$.

Proof: See the Appendix.

Derivation of Isotonic Selection Rules

It is assumed that based on some prior information, we are able to set a lower bound for θ_i, say $\theta_i \ge a_i$, for each $i = 1, \ldots, k-1$. Here, the constants $a_1, \ldots, a_{k-1}$ are known and satisfy that $a_1 \le a_2 \le \cdots \le a_{k-1}$. If $a_i \ge \theta_o$ for some i, then by the simple ordering prior, we are sure that populations $\pi_i, \pi_{i+1}, \ldots, \pi_k$ are good populations. Thus, without loss of generality, we assume that $a_{k-1} < \theta_o$. For the known control parameter θ_o and the constants $\underset{\sim}{a} = (a_1, \ldots, a_{k-1})$, define k k-tuple vectors $\underset{\sim}{\theta}^*_{i0}$, $i = 1, \ldots, k$, as follows:

$$\underset{\sim}{\theta}^*_{10} = (0, \ldots, \ldots, \ldots, 0).$$

$$\underset{\sim}{\theta}^*_{i0} = (a_1 - \theta_o, \ldots, a_{i-1} - \theta_o, 0, \ldots, 0),$$

$$\vdots$$

$$\underset{\sim}{\theta}^*_{k0} = (a_1 - \theta_o, a_2 - \theta_o, \ldots, a_{k-1} - \theta_o, 0).$$

For a given $P^* \in (0,1)$, from Theorem 2.1, there exist constants $0 < d_1 < d_2 < \cdots < d_k < \infty$ such that for each $i = 1, \ldots, k$,

(2.2) $$P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_m \ge d_m \text{ for some } m = 1, \ldots, i\} = P^*.$$

Let $A(\hat{\underset{\sim}{Y}}) = \{i | \hat{Y}_i \geq \theta_o + d_i\}$. We propose a selection rule $\delta_{1,\underset{\sim}{a}}$ as follows:

(2.3)
$$\delta_{1,\underset{\sim}{a}}(\hat{\underset{\sim}{Y}}) = \begin{cases} S_{\min A(\hat{\underset{\sim}{Y}})} & \text{if } A(\hat{\underset{\sim}{Y}}) \neq \phi, \\ S_{k+1} & \text{otherwise.} \end{cases}$$

By the fact that $d_1 < d_2 < \cdots < d_k$, we see that $\delta_{1,\underset{\sim}{a}}$ is an isotonic selection rule.

Probability of A Correct Selection

For the given lower bounds $\underset{\sim}{a}$, let

$$\Omega_1(\underset{\sim}{a}) = \{\underset{\sim}{\theta} \in \Omega | \theta_o \leq \theta_1\},$$

$$\Omega_i(\underset{\sim}{a}) = \{\underset{\sim}{\theta} \in \Omega | \theta_{i-1} < \theta_o \leq \theta_i,\ a_j \leq \theta_j,\ j = 1, \ldots, i-1\},\ i = 2, \ldots k,$$

$$\text{and } \Omega_{k+1}(\underset{\sim}{a}) = \{\underset{\sim}{\theta} \in \Omega | \theta_k < \theta_o,\ a_j \leq \theta_j,\ j = 1, \ldots, k-1\}.$$

Let $\Omega(\underset{\sim}{a}) = \cup_{i=1}^{k+1} \Omega_i(\underset{\sim}{a})$. Then $\Omega(\underset{\sim}{a})$ is a restricted parameter space, and $\Omega(\underset{\sim}{a}) = \Omega$ when $a_1 = \cdots = a_{k-1} = -\infty$. Note that

(2.4)
$$\inf_{\underset{\sim}{\theta} \in \Omega(\underset{\sim}{a})} P_{\underset{\sim}{\theta}}\{CS | \delta_{1,\underset{\sim}{a}}\} = \min_{1 \leq i \leq k} \inf_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} P_{\underset{\sim}{\theta}}\{CS | \delta_{1,\underset{\sim}{a}}\}.$$

For $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k) \in \Omega_i(\underset{\sim}{a})$, $P_{\underset{\sim}{\theta}}\{CS | \delta_{1,\underset{\sim}{a}}\} = P_{\underset{\sim}{\theta}}\{\hat{\underset{\sim}{Y}}_m \geq \theta_0 + d_m \text{ for some } m = 1, \ldots i\}$ which is increasing in θ_j for each $j = 1, \ldots, k$. Hence,

(2.5)
$$\inf_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} P_{\underset{\sim}{\theta}}\{CS | \delta_{1,\underset{\sim}{a}}\} = P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, i\} = P^*,$$

where the second equality is obtained from (2.2). Then, by (2.4) and (2.5), we have $\inf_{\underset{\sim}{\theta} \in \Omega(\underset{\sim}{a})} P_{\underset{\sim}{\theta}}\{CS | \delta_{1,\underset{\sim}{a}}\} = P^*$.

Computation of $(d_1, \ldots, d_k)$ Values

First, from (2.2), d_1 is chosen such that

(2.6)
$$P_{\underset{\sim}{\theta}^*_{10}}\{\hat{Y}_1 \geq d_1\} = P^*.$$

while the left-hand-side of (2.6) equals to $\exp(-knd_1)$, therefore,

$$d_1 = (kn)^{-1}\ell n P^{*^{-1}}. \tag{2.7}$$

From Lemma A.2, the constants d_i, $i = 2, \ldots, k$, are determined so that (2.8) holds for each $i = 2, \ldots, k$:

$$P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, i-1, \text{ and } \hat{Y}_i \geq d_i\} = [1 - \exp\big(n(a_{i-1} - \theta_o)\big)]P^*. \tag{2.8}$$

Note that

$$\begin{aligned}
&P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, i-1, \text{ and } \hat{Y}_i \geq d_i\} \\
&= P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1, \ldots, i-1, \text{ and } \hat{Y}_i \geq d_i\} \\
&= P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1, \ldots, i-1\} P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_i \geq d_i\} \\
&= P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1, \ldots, i-1\} \exp\big(-n(k-i+1)d_i\big) \\
&= c_i \exp\big(-n(k-i+1)\big) \quad \text{(say)}.
\end{aligned} \tag{2.9}$$

In (2.9), the first equality is obtained due to the fact that $0 < d_1 < \cdots < d_k < \infty$ and by the definition of $\hat{Y}_{m:j}$ and $\hat{Y}_i$. The second equality is obtained based on the independence property between $(\hat{Y}_{m:i-1},\ m = 1, \ldots, i-1)$ and $\hat{Y}_i$. Note that the probability $c_i \equiv P_{\underset{\sim}{\theta}^*_{i0}}\{\hat{Y}_{m:i-1} < d_m$ for all $m = 1, \ldots, i-1\}$ is independent of the value d_i. Thus,

$$d_i = [n(k-i+1)]^{-1}\ell n\Big[c_i/\{[1 - \exp\big(n(a_{i-1} - \theta_0)\big)]P^*\}\Big]. \tag{2.10}$$

Therefore, from (2.7) and (2.10), the d_i, $(i = 2, \ldots, k)$, values can be obtained iteratively.

Properties Related to the Selection Rules $\delta_{i,\underset{\sim}{a}}$

Property 2.1. (Inclusion Property)

a) Let $\underset{\sim}{\hat{y}}_1 = (\hat{y}_{11}, \ldots \hat{y}_{1k})$ and $\underset{\sim}{\hat{y}}_2 = (\hat{y}_{21}, \ldots, \hat{y}_{2k})$ be two observed vectors such that $\hat{y}_{1j} \leq \hat{y}_{2j}$ for all $j = 1, \ldots, k$. Then, for $\underset{\sim}{a}$ being fixed, $\delta_{1,\underset{\sim}{a}}(\underset{\sim}{\hat{y}}_1) \subseteq \delta_{1,\underset{\sim}{a}}(\underset{\sim}{\hat{y}}_2)$.

b) Let $\underset{\sim}{a}_1 = (a_{11}, \ldots, a_{1,k-1})$ and $\underset{\sim}{a}_2 = (a_{21}, \ldots, a_{2,k-1})$ be two $(k-1)$-tuples such that $a_{1j} \leq a_{2j}$ for all $j = 1, \ldots k-1$. Then, $\delta_{1,\underset{\sim}{a}_1}(\hat{\underset{\sim}{Y}}) \supseteq \delta_{1,\underset{\sim}{a}_2}(\hat{\underset{\sim}{Y}})$.

Proof: The proof of part a) is straightforward.

For the proof of part b), since $a_{1j} \leq a_{2j}$ for all $j = 1, \ldots, k-1$, by part b) of Theorem 2.1, $d_j(\underset{\sim}{a}_1) \leq d_j(\underset{\sim}{a}_2)$ for all $j = 1, \ldots, k$. Then by the definition of the selection rule $\delta_{1,\underset{\sim}{a}}$, we conclude that $\delta_{1,\underset{\sim}{a}_1}(\hat{\underset{\sim}{Y}}) \supseteq \delta_{1,\underset{\sim}{a}_2}(\hat{\underset{\sim}{Y}})$.

Let $S^{\text{`}}$ denote the random size of bad populations included in the selected subset and let $E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}]$ be the associated expected size applying the selection rule $\delta_{1,\underset{\sim}{a}}$ while $\underset{\sim}{\theta}$ is the true state of nature.

<u>Property 2.2</u> For θ_o being fixed and each $\underset{\sim}{\theta} \in \Omega$,

a) $P_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}}\}$ is decreasing in a_i for each $a_i \in \big(-\infty, \min(\theta_i, \theta_o)\big)$, and

b) $E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}]$ is decreasing in a_i for each $a_i \in \big(-\infty, \min(\theta_i, \theta_0)\big)$,

where $-\infty < a_1 \leq a_2 \leq \cdots \leq a_{k-1}$.

Proof: First note that for $\underset{\sim}{\theta} \in \Omega_i = \{\underset{\sim}{\theta} \in \Omega | \theta_{i-1} < \theta_o \leq \theta_i\}$,

$$P_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}}\} = P_{\underset{\sim}{\theta}}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, i\}, \tag{2.11}$$

and

$$E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}] = \sum_{r=1}^{i-1} P_{\underset{\sim}{\theta}}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\}. \tag{2.12}$$

Next, we see that for any $\underset{\sim}{\theta} \in \Omega$, and for each $j = 1, \ldots, k$, $P_{\underset{\sim}{\theta}}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, j\}$ is decreasing in d_m for all $m = 1, \ldots, j$. By Theorem 2.1 b), d_m is increasing in a_r for $r = 1, \ldots, m-1$, and independent of $a_m, \ldots a_{k-1}$. Thus $P_{\underset{\sim}{\theta}}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, j\}$ is decreasing in a_r for each $r = 1, \ldots, k-1$. The above fact with (2.11) and (2.12) together lead to the results.

Property 2.3. (Least Favorable Configuration on $\Omega_i(\underset{\sim}{a})$)

For $\underset{\sim}{a}$ being fixed, and for each $i = 1, \ldots, k$, we have: $\inf_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} P_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}}\} = P_{\underset{\sim}{\theta}_i^*}\{CS|\delta_{1,\underset{\sim}{a}}\}$ where $\underset{\sim}{\theta}_i^* = (a_1, \ldots, a_{i-1}, \theta_o, \ldots, \theta_o)$.

Proof: This can be obtained directly from the expression of (2.11).

Property 2.4. For $\underset{\sim}{a}$ being fixed, $\sup_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} E[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}] \geq (i-1)P^*$ for all $i = 1, \ldots, k+1$.

Proof: It is trivial for $i = 1$. For $i = 2, \ldots, k+1$, from (2.12), for $\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})$

$$\begin{aligned} E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}] &= \sum_{r=1}^{i-1} P_{\underset{\sim}{\theta}}\left\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\right\} \\ &\leq \sum_{r=1}^{i-1} P_{\underset{\sim}{\theta}^*}\left\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\right\} \qquad (2.13) \\ &= \sum_{r=1}^{i-1} P_{\underset{\sim}{\theta}^*}\left\{\hat{Y}_{m:i-1} \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\right\} \end{aligned}$$

where $\underset{\sim}{\theta}^* = (\theta_1^*, \ldots, \theta_k^*)$, $\theta_j^* = \theta_j$ for $1 \leq j \leq i-1$ and $\theta_j^* = \max(d_k + \theta_0, \theta_j)$ for $i \leq j \leq k$, and therefore $\underset{\sim}{\theta}^* \in \Omega_i(\underset{\sim}{a})$.

Let $\Omega_i(\underset{\sim}{a}, d_k) = \{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a}) | \theta_j \geq d_k + \theta_0 \text{ for all } i \leq j \leq k\}$. Since for each $r = 1, \ldots, i-1$, $P_{\underset{\sim}{\theta}^*}\{\hat{Y}_{m:i-1} \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\}$ is increasing in θ_j^* for all $j = 1, \ldots, i-1$, thus,

$$\begin{aligned} \sup_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}] &= \sup_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a}, d_k)} E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta_{1,\underset{\sim}{a}}] \\ &= \sum_{r=1}^{i-1} P_{\underset{\sim}{\theta}_i(d_k)}\{\hat{Y}_{m:i-1} \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\} \qquad (2.14) \end{aligned}$$

where $\underset{\sim}{\theta}_i(d_k) = (\overbrace{\theta_0, \ldots, \theta_0}^{i-1}, \overbrace{\theta_0 + d_k \ldots, \theta_0 + d_k}^{k-i+1})$. Now for each $r = 1, \ldots, i-1$,

$$\begin{aligned} &P_{\underset{\sim}{\theta}_i(d_k)}\{\hat{Y}_{m:i-1} \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\} \\ &\geq P_{\underset{\sim}{\theta}_o}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\} \qquad (2.15) \end{aligned}$$

$$\geq P_{\underset{\sim}{\theta}_o(r)}\{\hat{Y}_m \geq d_m + \theta_o \text{ for some } m = 1, \ldots, r\}$$

$$= P^*$$

where $\underset{\sim}{\theta}_o = (\theta_o \ldots, \theta_o)$ and $\underset{\sim}{\theta}_o(r) = (a_1, \ldots, a_{r-1}, \theta_o, \ldots, \theta_o)$.

Therefore, (2.14) and (2.15) together imply that $\sup_{\underset{\sim}{\theta} \in \Omega_i(\underset{\sim}{a})} E_{\underset{\sim}{\theta}}[S^{\prime}|\delta_{1,\underset{\sim}{a}}] \geq (i-1)P^*$.

3. ISOTONIC SELECTION RULE FOR θ_o UNKNOWN CASE

When θ_o is unknown, sampling from the control population π_o is needed. Let $X_{01}, \ldots, X_{0n}$ be a sample from π_o and let $Y_o = \min(X_{o1}, \ldots, X_{on})$. Since θ_o is unknown, we do not know the values of the differences $a_i - \theta_o$, $i = 1, \ldots, k-1$. It seems not possible to take the advantage of the lower bounds $\underset{\sim}{a}$ to derive selection rules. Thus, a simple isotonic selection rule is proposed as follows.

For the given P^*, for each $i = 1, \ldots, k$, let $d_i^* = -\frac{1}{n}\ell n[(1-P^*)(k-i+2)/(k-i+1)]$. Let $A^*(\hat{\underset{\sim}{Y}}, Y_o) = \{i|\hat{Y}_i \geq Y_o - d_i^*\}$. We propose a selection rule δ_1^* as follows:

$$\delta_1^*(\hat{\underset{\sim}{Y}}, Y_o) = \begin{cases} S_{\min A^*(\hat{\underset{\sim}{Y}}, Y_o)} & \text{if } A^*(\hat{\underset{\sim}{Y}}, Y_o) \neq \phi, \\ S_{k+1} & \text{otherwise.} \end{cases} \tag{3.1}$$

Properties related to the selection rule δ_1^* are given below as Remarks.

Remarks

1. The way to define the selection rule δ_1^* is equivalent to letting $a_1 = a_2 = \cdots = a_{k-1} = -\infty$.
2. Based on the choice of the constants, $d_1^*, \ldots, d_k^*$, it is easy to show that $\inf_{\underset{\sim}{\theta} \in \Omega} P_{\underset{\sim}{\theta}}\{CS|\delta_1^*\} = P^*$.
3. The selection rule δ_1^* is isotonic and has the inclusion property described in Property 2.1a).
4. (Least Favorable Configuration on Ω_i) For each $i = 1, \ldots, k$,

$$\inf_{\underset{\sim}{\theta} \in \Omega_i} P_{\underset{\sim}{\theta}}\{CS|\delta_1^*\} = P_{\underset{\sim}{\theta}_i^*}\{CS|\delta_1^*\}, \text{ where } \underset{\sim}{\theta}_i^* = (\overbrace{-\infty, \ldots, -\infty}^{i-1}, \overbrace{\theta_0, \ldots, \theta_0}^{k-i+1}).$$

where $\Omega_i = \{\underset{\sim}{\theta} \in \Omega | \theta_{i-1} < \theta_0 \leq \theta_i\}$.

5. For θ_o being fixed, though unknown, for each $i = 2, \ldots, k+1$,

$$E_{\underset{\sim}{\theta}_i^*}[S^{\prime}|\delta_1^*] \leq E_{\underset{\sim}{\theta}}[S^{\prime}|\delta_1^*] \leq \lim_{\epsilon \downarrow 0} E_{\underset{\sim}{\theta}_i^{**}(\epsilon)}[S^{\prime}|\delta_1^*] \text{ for all } \underset{\sim}{\theta} \in \Omega_i,$$

where $\underset{\sim}{\theta}_i^* = (\overbrace{-\infty,\ldots,-\infty}^{i-1},\overbrace{\theta_0,\ldots,\theta_0}^{k-i+1})$ and $\underset{\sim}{\theta}_i^{**}(\varepsilon) = (\overbrace{\theta_0-\varepsilon,\ldots,\theta_0-\varepsilon}^{i-1},\ \overbrace{\infty,\ldots,\infty}^{k-i+1}), \varepsilon > 0$.

4. SIMULATION COMPARISON OF SELECTION RULES

Gupta and Leu (1986) have studied several isotonic selection rules for selecting good exponential population with respect to a control. Let $\underset{\sim}{Y}^* = (Y_1^*,\ldots,Y_k^*)$ where

$$(4.1) \qquad Y_i^* = \max_{1\le s\le i} \min\{Y_s, \frac{Y_s+Y_{s+1}}{2},\cdots,\frac{Y_s+\cdots+Y_k}{k-s+1}\}, \quad i=1,\ldots,k.$$

When θ_o is known, let $B(\underset{\sim}{Y}^*) = \{i|Y_i^* \ge \theta_o + c_i\}$, where the constants c_i, $i=1,\ldots,k$, are determined so that $\inf_{\underset{\sim}{\theta}\in\Omega_i} P_{\underset{\sim}{\theta}}\{Y_m^* \ge \theta_o + c_m$ for some $m=1,\ldots,i\} \doteq P^*$ for each $i=1,\ldots,k$. The values of the constants c_i, $1\le i\le k$, can be found from Table I of Gupta and Leu (1986) through some transformation. They proposed an isotonic selection rule, say δ_2, as follows:

$$(4.2) \qquad \delta_2(\underset{\sim}{Y}^*) = \begin{cases} S_{\min B(\underset{\sim}{Y}^*)} & \text{if } B(\underset{\sim}{Y}^*) \ne \phi, \\ S_{k+1} & \text{otherwise.} \end{cases}$$

They also considered another isotonic selection rule, say δ_3, as given below. Let $Y_i^{**} = \max(Y_1,\ldots,Y_i)$, $i=1,\ldots,k$ and $\underset{\sim}{Y}^{**} = (Y_1^{**},\ldots,Y_k^{**})$. Let $C(\underset{\sim}{Y}^{**}) = \{i|Y_i^{**} \ge \theta_o + n^{-1}\ell n {P^*}^{-1}\}$. Then

$$(4.3) \qquad \delta_3(\underset{\sim}{Y}^{**}) = \begin{cases} S_{\min C(\underset{\sim}{Y}^{**})} & \text{if } C(\underset{\sim}{Y}^{**}) \ne \phi, \\ S_{k+1} & \text{otherwise.} \end{cases}$$

Note that both the two isotonic selection rules δ_2 and δ_3 are designed under the situation that there is no information available about the values of a lower bound for the concerned parameters. Hence, it can be imaged that these two selection rules might be conservative in the sense that the associated probability of a correct selection might be quite higher than the required P^* level and there might be more bad populations included in the selected subset.

Gupta and Leu (1986) also proposed some isotonic selection rules for θ_o unknown case which are described as below.

Let $B^*(\underset{\sim}{Y}^*, Y_o) = \{i | Y_i^* \geq Y_o - c_i^*\}$ where the constants c_i^*, $i = 1, \ldots, k$ are determined so that $\inf_{\underset{\sim}{\theta} \in \Omega_i} P_{\underset{\sim}{\theta}}\{Y_m^* \geq Y_o - c_m^* \text{ for some } m = 1, \ldots, i\} = P^*$ for all i. The values of the constants c_i^*, $i = 1, \ldots, k$, are available from Table III of Gupta and Leu (1986) through some transformation. They proposed selection rule, say δ_2^*, as follows:

$$(4.4) \qquad \delta_2^*(\underset{\sim}{Y}^*, Y_o) = \begin{cases} S_{\min B^*(\underset{\sim}{Y}^*, Y_o)} & \text{if } B^*(\underset{\sim}{Y}^*, Y_o) \neq \phi, \\ S_{k+1} & \text{otherwise.} \end{cases}$$

Another selection rule, say δ_3^*, which is analogous to δ_3 and studied by Gupta and Leu (1986), is also given below.

Let $C^*(\underset{\sim}{Y}^{**}, Y_o) = \{i | Y_i^{**} \geq Y_o - e_i^*\}$ where $e_i^* = n^{-1}\ell n(2P^*)$ if $P^* \leq \frac{1}{2}$ and $e_i^* = -n^{-1}\ell n[2(1 - P^*)]$ if $P^* > \frac{1}{2}$ for all i. Then,

$$(4.5) \qquad \delta_3^*(\underset{\sim}{Y}^{**}, Y_o) = \begin{cases} S_{\min C^*(\underset{\sim}{Y}^{**}, Y_o)} & \text{if } C^*(\underset{\sim}{Y}^{**}, Y_o) \neq \phi, \\ S_{k+1} & \text{otherwise.} \end{cases}$$

For evaluating the performance of a selection rule δ, we consider the ratio $R(\delta, \underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta] / P_{\underset{\sim}{\theta}}\{CS|\delta\}$. For a selection rule $\delta$, we always desire that $P_{\underset{\sim}{\theta}}\{CS|\delta\}$ is large while $E_{\underset{\sim}{\theta}}[S^{\text{`}}|\delta]$ is small. Hence, for two selection rules α_1 and α_2, we say that α_1 is better than α_2 at $\underset{\sim}{\theta}$ if $R(\alpha_1, \underset{\sim}{\theta}) < R(\alpha_2, \underset{\sim}{\theta})$, and α_1 is better than α_2 over $\Omega^* \subset \Omega$ if $R(\alpha_1, \underset{\sim}{\theta}) \leq R(\alpha_2, \underset{\sim}{\theta})$ holds for all $\underset{\sim}{\theta} \in \Omega^*$ and the strict inequality holds for some $\underset{\sim}{\theta} \in \Omega^*$.

Simulation Study

In the following, some simulation studies are carried out to compare the performance of the selection rules $\delta_{1,\underset{\sim}{a}}$ δ_2, δ_3 and of δ_j^*, $j = 1, 2, 3$, according to the magnitude of the ratio $R(\delta, \underset{\sim}{\theta})$. When θ_0 is known, two cases have been investigated according to whether some prior information about the lower bounds of the concerned parameters is available or not. When there is no information available, we let $a_1 = \cdots = a_{k-1} = -\infty$. This is the situation under which the two selection rules δ_2 and δ_3 are designed. When θ_0 is unknown, the three selection rules δ_j^*, $j = 1, 2, 3$, are designed under the same situation where $a_1 = \cdots = a_{k-1} = -\infty$.

The simulation process was repeated 1000 times. The relative frequency of a correct selection is used as an approximation to the probability of a correct selection. The relative frequency of the number of bad populations included in the selected subset is treated as an approximation to the expected size of bad populations included in the selected subset. The ratios $R(\alpha, \underset{\sim}{\theta})$ is approximated by the ratio of the above two relative frequencies.

The Monte Carlo simulation has been carried out for the case $k = 4$. The common sample size n is chosen to be seven and $P^* = 0.95$. We also chose $\theta_4 = \theta_0$ and $\theta_i < \theta_0$ for $i = 1, 2, 3$. Thus, there are three bad populations. For each case, all the considered selection rules are applied to the same data. The simulation results are tabulated in Table I and Table II. The numbers in the parentheses are the standard error of the corresponding estimates.

Discussion of the Tables

Let $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta\}$, $\hat{E}_{\underset{\sim}{\theta}}[S'|\delta]$ and $\hat{R}(\delta, \underset{\sim}{\theta})$ denote estimates of $P_{\underset{\sim}{\theta}}\{CS|\delta\}$, $E_{\underset{\sim}{\theta}}[S'|\delta]$ and $R(\delta, \underset{\sim}{\theta})$, respectively. For θ_0 known case, from Table I, simulation results indicate the following evidences.

1. $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}_1}\} \leq \hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}_2}\} \leq \hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_2\} \leq \hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_3\}$ for all $\underset{\sim}{\theta}$ under the study. Also, except for selection rule $\delta_{1,\underset{\sim}{a}_1}$, for each of the other three selection rules, the corresponding $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta\}$ are larger than the prespecified level $P^* = 0.95$. When the values of parameters θ_1, θ_2, and θ_3 are close to the control θ_0, the value of $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_{1,\underset{\sim}{a}_1}\}$ is still close to P^*, while $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_2\}$ and $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta_3\}$ are quite higher than P^*. This evidence indicates selection rules δ_2 and δ_3 are conservative.
2. $\hat{E}_{\underset{\sim}{\theta}}[S'|\delta_{1,\underset{\sim}{a}_1}] \leq \hat{E}_{\underset{\sim}{\theta}}[S'|\delta_{1,\underset{\sim}{a}_2}] \leq \hat{E}_{\underset{\sim}{\theta}}[S'|\delta_2] \leq \hat{E}_{\underset{\sim}{\theta}}[S'|\delta_3]$ for all $\underset{\sim}{\theta}$ under the study. When the values of the parameters θ_1, θ_2 and θ_3 are far from the control θ_0, the estimated $\hat{E}_{\underset{\sim}{\theta}}[S'|\delta]$ are small for all selection rules under study. However, when the values of θ_1, θ_2 and θ_3 are close to θ_0, the values of $\hat{E}_{\underset{\sim}{\theta}}[S'|\delta_{1,\underset{\sim}{a}_j}]$, $j = 1, 2$ are still small while the value of $\hat{E}_{\underset{\sim}{\theta}}[S'|\delta_3]$ becomes large.

3. Except for the configuration where $\underset{\sim}{\theta} = (0.2, 0.2, 0.2, 1)$, $\hat{R}(\delta_{1,\underset{\sim}{a}_j}, \underset{\sim}{\theta}) \leq \hat{R}(\delta_2, \underset{\sim}{\theta}) \leq \hat{R}(\delta_3, \theta)$ for all $\underset{\sim}{\theta}$ under study.

Note that the selection rule $\delta_{1,\underset{\sim}{a}_2}$ is designed under the situation where no information is available about the values of a lower bound of the concerned parameters. This situation is the same as that under which both the two selection rules δ_2 and δ_3 are designed. The simulation results of Table I indicates that for all $\underset{\sim}{\theta}$ under study, the performance of $\delta_{1,\underset{\sim}{a}_2}$ is better than that of δ_2 and δ_3.

For θ_0 unknown case, from Table II, the simulation results indicate that when the values of θ_1, θ_2 and θ_3 are far from the control parameter θ_0, the performance of δ_1^* is better than the others. However, in other situations, either δ_2^* performs best or δ_3^* performs best, depending on the values of the concerned parameters.

It is also interesting to find that, from Table I and Table II, for any parameter configurations under study, the performance of δ_j is better than that of δ_j^*, for $i = 1, 2, 3$. This result may be due to the fact whether θ_0 is known or not.

Table I.

Table of the estimated $\hat{P}_{\theta}\{CS|\delta\}$, $\hat{E}_{\theta}[S^{\text{`}}|\delta]$ and $\hat{R}(\delta, \underset{\sim}{\theta})$ for θ_0 known case where $\underset{\sim}{a}_1 = (0.2, 0.2, 0.2)$, $\underset{\sim}{a}_2 = (-\infty, -\infty, -\infty)$, and $\theta_0 = \theta_4 = 1$.

	$\delta_{1,\underset{\sim}{a}_1}$			$\delta_{1,\underset{\sim}{a}_2}$			δ_2			δ_3		
$(\theta_1, \theta_2, \theta_3)$	$\hat{P}\{CS\}$	$\hat{E}[S^{\text{`}}]$	$\hat{R}$	$\hat{P}\{CS\}$	$\hat{E}[S^{\text{`}}]$	$\hat{R}$	$\hat{P}\{CS\}$	$\hat{E}[S^{\text{`}}]$	$\hat{R}$	$\hat{P}\{CS\}$	$\hat{E}[S^{\text{`}}]$	$\hat{R}$
(0.2, 0.2, 0.2)	0.9460 (0.0072)	0.0030 (0.0017)	0.00317	0.9520 (0.0068)	0.0030 (0.0017)	0.00315	0.9520 (0.0068)	0.0030 (0.0017)	0.00315	0.9520 (0.0068)	0.0150 (0.0061)	0.01576
(0.2, 0.2, 0.5)	0.9470 (0.0071)	0.0350 (0.0058)	0.03696	0.9530 (0.0067)	0.0350 (0.0058)	0.03673	0.9550 (0.0066)	0.0370 (0.0060)	0.03874	0.9550 (0.0066)	0.0490 (0.0083)	0.05131
(0.2, 0.2, 0.8)	0.9510 (0.0068)	0.2260 (0.0132)	0.23765	0.9590 (0.0063)	0.2280 (0.0133)	0.23775	0.9650 (0.0058)	0.2320 (0.0136)	0.24042	0.9670 (0.0057)	0.2420 (0.0143)	0.25026
(0.2, 0.5, 0.5)	0.9470 (0.0071)	0.0360 (0.0061)	0.03802	0.9530 (0.0067)	0.0360 (0.0061)	0.03778	0.9550 (0.0066)	0.0430 (0.0072)	0.04503	0.9550 (0.0066)	0.0820 (0.0115)	0.08586
(0.2, 0.5, 0.8)	0.9510 (0.0068)	0.2320 (0.0138)	0.24400	0.9590 (0.0063)	0.2340 (0.0138)	0.24400	0.9650 (0.0058)	0.2550 (0.0150)	0.26425	0.9670 (0.0057)	0.2710 (0.0159)	0.28025
(0.2, 0.8, 0.8)	0.9510 (0.0068)	0.2650 (0.0165)	0.27865	0.9600 (0.0062)	0.2690 (0.0166)	0.28021	0.9710 (0.0053)	0.4180 (0.0218)	0.43048	0.9760 (0.0048)	0.6070 (0.0258)	0.62193
(0.5, 0.5, 0.5)	0.9470 (0.0071)	0.0360 (0.0061)	0.03802	0.9530 (0.0067)	0.0360 (0.0061)	0.03778	0.9550 (0.0066)	0.0430 (0.0072)	0.04503	0.9560 (0.0065)	0.1740 (0.0197)	0.18201
(0.5, 0.5, 0.8)	0.9510 (0.0068)	0.2320 (0.0138)	0.24395	0.9550 (0.0063)	0.2340 (0.0138)	0.24400	0.9650 (0.0058)	0.2550 (0.0150)	0.26425	0.9680 (0.0056)	0.3550 (0.0217)	0.36674
(0.5, 0.8, 0.8)	0.9510 (0.0068)	0.2670 (0.0168)	0.28076	0.9600 (0.0062)	0.2710 (0.0169)	0.28229	0.9720 (0.0052)	0.4390 (0.0232)	0.45165	0.9760 (0.0048)	0.6790 (0.0286)	0.69570
(0.8, 0.8, 0.8)	0.9510 (0.0068)	0.2720 (0.0174)	0.28602	0.9600 (0.0062)	0.2760 (0.0176)	0.28750	0.9740 (0.0050)	0.6150 (0.0307)	0.63142	0.9830 (0.0041)	1.0940 (0.0382)	1.11292

Table II.

Table of the estimated $\hat{P}_{\underset{\sim}{\theta}}\{CS|\delta\}$, $\hat{E}_{\underset{\sim}{\theta}}[S^{‘}|\delta]$ and $\hat{R}(\delta, \underset{\sim}{\theta})$ for θ_0 unknown case where $\theta_0 = \theta_1 = 1$,

	δ_1^*			δ_2^*			δ_3^*		
$(\theta_1, \theta_2, \theta_3)$	$\hat{P}\{CS\}$	$\hat{E}[S^{‘}]$	$\hat{R}$	$\hat{P}\{CS\}$	$\hat{E}[S^{‘}]$	$\hat{R}$	$\hat{P}\{CS\}$	$\hat{E}[S^{‘}]$	$\hat{R}$
(0.2, 0.2, 0.2)	0.9550 (0.0066)	0.0310 (0.0058)	0.03246	0.9550 (0.0066)	0.0330 (0.0062)	0.03456	0.9550 (0.0066)	0.1250 (0.0165)	0.13889
(0.2, 0.2, 0.5)	0.9550 (0.0066)	0.2010 (0.0134)	0.21047	0.9560 (0.0065)	0.1720 (0.0128)	0.17992	0.9560 (0.0065)	0.2360 (0.0185)	0.24686
(0.2, 0.2, 0.8)	0.9570 (0.0064)	0.8570 (0.0134)	0.89561	0.9580 (0.0063)	0.8190 (0.0139)	0.85491	0.9580 (0.0063)	0.8360 (0.0173)	0.87765
(0.2, 0.5, 0.5)	0.9550 (0.0066)	0.2740 (0.0191)	0.28691	0.9560 (0.0065)	0.3170 (0.0213)	0.33159	0.9560 (0.0065)	0.4470 (0.0254)	0.46757
(0.2, 0.5, 0.8)	0.9570 (0.0064)	1.0650 (0.0205)	1.11285	0.9580 (0.0063)	0.9930 (0.0203)	1.03634	0.9580 (0.0063)	0.9600 (0.0209)	1.00209
(0.2, 0.8, 0.8)	0.9580 (0.0063)	1.6900 (0.0241)	1.76409	0.9580 (0.0063)	1.6600 (0.0241)	1.73278	0.9600 (0.0062)	1.6580 (0.0219)	1.72708
(0.5, 0.5, 0.5)	0.9550 (0.0066)	0.2960 (0.0214)	0.30995	0.9560 (0.0065)	0.4460 (0.0285)	0.46653	0.9560 (0.0065)	0.7880 (0.0369)	0.82427
(0.5, 0.5, 0.8)	0.9570 (0.0064)	1.1440 (0.0251)	1.19540	0.9580 (0.0063)	1.1310 (0.0267)	1.18058	0.9580 (0.0063)	1.2230 (0.0304)	1.27662
(0.5, 0.8, 0.8)	0.9580 (0.0063)	1.8860 (0.0294)	1.96868	0.9580 (0.0063)	1.8240 (0.0286)	1.90397	0.9600 (0.0062)	1.8110 (0.0280)	1.88646
(0.8, 0.8, 0.8)	0.9580 (0.0063)	2.4800 (0.0344)	2.58873	0.9590 (0.0063)	2.4750 (0.0337)	2.58081	0.9600 (0.0062)	2.4990 (0.0323)	2.60313

APPENDIX

The proof of Theorem 2.1 can be completed through the considerations of the following lemmas.

First note that the constant $d_1(b_1, \ldots, b_{k-1})$ can be determined as follows:

$$P^* = P_{\underset{\sim}{b}_1}\{\hat{Y}_1 \geq d_1\} = [1 - G(d_1)]^k \tag{A.1}$$

where $G(x) = (1 - \exp\{-nx\})I_{(0,\infty)}(x)$. Hence, the determination of the value of d_1 is independent of the parameters $b_1, \ldots, b_{k-1}$.

Lemma A.1. $d_i(b_1, \ldots, b_{k-1}) > d_1$ for each $i = 2, \ldots, k$.

Proof: Suppose that for some $i \geq 2$, $d_i \leq d_1$. Then, from (2.1) and (A.1),

$$\begin{aligned} P^* &= P_{\underset{\sim}{b}_i}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, 2, \ldots, i\} \\ &\geq P_{\underset{\sim}{b}_i}\{\hat{Y}_i \geq d_i\} \\ &= [1 - G(d_i)]^{k-i+1} \\ &> [1 - G(d_1)]^k = P^* \end{aligned}$$

which is a contradiction. So, $d_i > d_1$ for all $i = 2, \ldots, k$.

Lemma A.2. Suppose that for some $i (2 \leq i \leq k)$, there exist constants $0 < d_1 < d_2 < \cdots < d_i < \infty$ such that

$$P_{\underset{\sim}{b}_j}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, 2, \ldots, j\} = P^*$$

for each $j = 1, 2, \ldots, i$. Let

$$A_j = P_{\underset{\sim}{b}_j}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, j-1\} \text{ and} \tag{A.2}$$

$$B_j = P_{\underset{\sim}{b}_j}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, j-1 \text{ and } \hat{Y}_j \geq d_j\}. \tag{A.3}$$

Then, $A_j = \exp\{nb_{j-1}\}P^*$ and $B_j = (1 - \exp\{nb_{j-1}\})P^*$.

Proof: By the increasing property of the constants $d_1, \ldots, d_i$,

$$\begin{aligned} A_j &= P_{\underset{\sim}{b}_j}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, j-1\} \\ &= \sum_{r=1}^{j-1} P_{\underset{\sim}{b}_j}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, r-1,\ \hat{Y}_r \geq d_r\}, \end{aligned}$$

where

$$P_{\underset{\sim}{b}_j}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, r-1,\ \hat{Y}_r \geq d_r\}$$

$$= P_{\underset{\sim}{b}_j}\{\hat{Y}_{m:r-1} < d_m \text{ for all } m = 1, \ldots, r-1, \quad \hat{Y}_r \geq d_r\}$$

$$= P_{\underset{\sim}{b}_j}\{\hat{Y}_{m:r-1} < d_m \text{ for all } m = 1, \ldots, r-1\} \cdot [\prod_{m=r}^{j-1} (1 - G(d_r - b_m))][1 - G(d_r)]^{k-j+1}$$

$$= P_{\underset{\sim}{b}_{j-1}}\{\hat{Y}_{m:r-1} < d_m \text{ for all } m = 1, \ldots, r-1\} \cdot [\prod_{m=r}^{j-1} (1 - G(d_r - b_m))][1 - G(d_r)]^{k-j+1}$$

$$= P_{\underset{\sim}{b}_{j-1}}\{\hat{Y}_{m:r-1} < d_m \text{ for all } m = 1, \ldots, r-1,\ \hat{Y}_r \geq d_r\} \cdot [1 - G(d_r - b_{m-1})]/[1 - G(d_r)]$$

$$= P_{b_{j-1}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, r-1, \hat{Y}_r \geq d_r\} \cdot \exp\{nb_{j-1}\}.$$

Hence,

$$A_j = \exp\{nb_{j-1}\} \sum_{r=1}^{j-1} P_{\underset{\sim}{b}_{j-1}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, r-1,\ \hat{Y}_r \geq d_r\}$$

$$= \exp\{nb_{j-1}\} P_{\underset{\sim}{b}_{j-1}}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, j-1\}$$

$$= \exp\{nb_{j-1}\} P^*$$

where the last equality is obtained by the definition of $d_1, \ldots, d_i$. Therefore, $B_j = P^* - A_j = (1 - \exp\{nb_{j-1}\})P^*$.

Now, for fixed $b_1, \ldots, b_{i-1}$ and $d_1, \ldots, d_i$, define

$$(A.4) \qquad A^*_{i+1}(b_i) = P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, i\},$$

and

$$(A.5) \qquad B^*_{i+1}(b_i) = P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, i \text{ and } \hat{Y}_{i+1} \geq d_i\},$$

where $b_{i-1} \leq b_i < 0$. If $A^*_{i+1}(b_i) + B^*_{i+1}(b_i) > P^*$, then there exists some constant $d_{i+1} > d_i$ such that $P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_m \geq d_m \text{ for some } m = 1, \ldots, i+1\} = P^*$. Hence, to claim that $d_{i+1}(b_1, \ldots, b_{k-1}) > d_i(b_1, \ldots, b_{k-1})$, it suffices to show that $A^*_{i+1}(b_i) + B^*_{i+1}(b_i) > P^*$ for all $0 > b_i \gtreqless b_{i-1}$. Let

$$(A.6) \qquad h(b_i) = A^*_{i+1}(b_i) + B^*_{i+1}(b_i) - P^*$$

$$= P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_m \geq d^*_m \text{ for some } m = 1, \ldots, i+1\} - P^*$$

where $d^*_m = d_m$ for each $m = 1, \ldots, i$ and $d^*_{i+1} = d_i$. It is easy to see that $h(b_i)$ is increasing in b_i. Hence, it suffices to show that $h(b_{i-1}) > 0$. By applying a discussion similar to that used for the proof of Lemma A.2, we have $A^*_{i+1}(b_{i-1}) = \exp\{nb_{i-1}\}P^*$. Hence,

$$\begin{aligned} h(b_{i-1}) &= \left(\exp\{nb_{i-1}\}P^* + B^*_{i+1}(b_{i-1})\right) - P^* \\ &= \left(\exp\{nb_{i-1}\}P^* + B^*_{i+1}(b_{i-1})\right) - (A_i + B_i) \\ &= B^*_{i+1}(b_{i-1}) - B_i. \end{aligned} \tag{A.7}$$

<u>Lemma A.3.</u> Suppose that $0 < d_1 < d_2 < \cdots < d_i$ be chosen so that (2.1) is true for each $j = 1, \ldots, i$. Let B_i and $B^*_{i+1}(b_i)$ be defined in (A.3) and (A.5), respectively. Then

$$B^*_{i+1}(b_i) = [1 - G(d_i)]^{k-i}\{G(d_1 - b_i) + \exp(nb_i)\sum_{j=2}^{i} Q_{i,j}[G(d_j) - G(d_{j-1})]\},$$

and

$$B_i = [1 - G(d_i)]^{k-i+1}\{G(d_1 - b_{i-1}) + \exp(nb_{i-1})\sum_{j=2}^{i-1} Q_{i-1,j}[G(d_j) - G(d_{j-1})]\},$$

where

$$Q_{ij} = P_{\underset{\sim}{b}_i}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1, \ldots, j-1\}, \quad 2 \le j \le i. \tag{A.8}$$

Proof:

$$\begin{aligned} B^*_{i+1}(b_i) &= P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_m < d_m \text{ for all } m = 1, \ldots, i,\ \hat{Y}_{i+1} \ge d_i\} \\ &= P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i} < d_m \text{ for all } m = 1, \ldots, i,\ \hat{Y}_{i+1} \ge d_i\} \\ &\quad (\text{ since } 0 < d_1 < \cdots < d_i) \\ &= [1 - G(d_i)]^{k-i}\ P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i} < d_m \text{ for all } m = 1, \ldots, i\}, \end{aligned}$$

where

$$P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i} < d_m \text{ for all } m = 1,\ldots,i\}$$

$$= \int_{y_i=b_i}^{d_i} P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i} < d_m \text{ for all } m = 1,\ldots,i,\ Y_i = y_i\} dG(y_i - b_i)$$

$$= G(d_1 - b_i) + \sum_{j=2}^{i} \int_{d_{j-1}}^{d_j} P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1,\ Y_i = y_i\} dG(y_i - b_i)$$

$$= G(d_1 - b_i) + \sum_{j=2}^{i} P_{\underset{\sim}{b}_{i+1}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1\} \cdot [G(d_j - b_i) - G(d_{j-1} - b_i)]$$

$$= G(d_1 - b_i) + \exp\{nb_i\} \sum_{j=2}^{i} P_{\underset{\sim}{b}_i}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1\} \cdot [G(d_j) - G(d_{j-1})].$$

Hence,

$$B^*_{i+1}(b_i) = [1 - G(d_i)]^{k-i}\{G(d_1 - b_i) + \exp(nb_i) \sum_{j=2}^{i} Q_{ij}[G(d_j) - G(d_{j-1})]\}.$$

The proof for B_i is analogous to that for $B^*_{i+1}(b_i)$ and hence the detail is omitted here.

<u>*Lemma A.4.*</u> $h(b_{i-1}) > 0$.

Proof: It is equivalent to showing that $B^*_{i+1}(b_{i-1}) - B_i > 0$. By the definition of $\hat{Y}_{m:j}$, for each $j = 2, \ldots, i-1$,

$$\{\hat{Y}_{m:i-2} < d_m \text{ for all } m = 1,\ldots,j-1\} \subset \{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1\}.$$

Therefore,

$$\begin{aligned} & P_{\underset{\sim}{b}_{i-1}}\{\hat{Y}_{m:i-2} < d_m \text{ for all } m = 1,\ldots,j-1\} \\ \text{(A.9)} \qquad & \leq P_{\underset{\sim}{b}_{i-1}}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1\} \\ & \leq P_{\underset{\sim}{b}_i}\{\hat{Y}_{m:i-1} < d_m \text{ for all } m = 1,\ldots,j-1\}. \\ & = Q_{ij}. \end{aligned}$$

In (A.9), the last inequality is obtained due to the fact that for $j < i$, $Q_{ij} \equiv P_{\underset{\sim}{b}_i}\{\hat{Y}_{m:i-1} < d_m$ for all $m = 1, \ldots, j-1\}$ is decreasing in b_{i-1} and $b_{i-1} < 0$. So,

$$h(b_{i-1}) \geq [1 - G(d_i)]^{k-i} G(d_i)\{G(d_1 - b_{i-1}) + \exp\{nb_{i-1}\} \sum_{j=2}^{i-1} Q_{ij}[G(d_j) - G(d_{j-1})]\}$$

$$> 0.$$

Proof of Theorem 2.1.

By Lemmas A.1, A.4 and induction method, the proof of part a) is completed.

Proof of part b). For the way the value of the constant d_j is determined, we can find that $d_j(b_1, \ldots, b_{k-1})$ depends only on $b_1, \ldots, b_{j-1}$.

Now, for each j, consider the two j-tubles $(b_1^0, \ldots, b_j^0)$ and $(b_1^*, \ldots, b_j^*)$ where $b_r^0 = b_r^*$ for all $r = 1, \ldots, j-1$, but $b_j^0 < b_j^*$, and $b_1^0 \leq b_2^0 \leq \cdots \leq b_j^0$. For $b_1^0, \ldots, b_{j-1}^0$ being fixed and constants c_m, $1 \leq m \leq j+1$ satisfying $0 < c_1 < c_2 < \cdots < c_{j+1}$, the probability $P_{\underset{\sim}{b}_{j+1}^0}\{\hat{Y}_m \geq c_m \text{ for some } m = 1, \ldots, j+1\}$ is an increasing function of b_j^0 where $\underset{\sim}{b}_{j+1}^0 = (b_1^0, \ldots, b_{j-1}^0, b_j^0, 0, \ldots, 0)$ (k-tuples). Therefore, in order to achieve the P^* value, we must have

$$(A.10) \qquad d_{j+1}(b_1^0, \ldots, b_j^0) < d_{j+1}(b_1^*, \ldots, b_j^*).$$

In general, consider the two $(k-1)$-tuples $(b_1^0 \leq b_2^0 \leq \cdots \leq b_{k-1}^0 < 0)$ and $(b_1^* \leq b_2^* \leq \cdots \leq b_{k-1}^* < 0)$ satisfying $b_j^0 \leq b_j^*$ for all $j = 1, \ldots, k-1$. Let

$$\underset{\sim}{b}^1 = (b_1^0, \ldots, b_{k-1}^0),$$

$$\underset{\sim}{b}^2 = (b_1^0, \ldots, b_{k-2}^0, b_{k-1}^*)$$

$$\vdots$$

$$\underset{\sim}{b}^i = (b_1^0, \ldots, b_{k-i}^0, b_{k-i+1}^*, \ldots, b_{k-1}^*)$$

$$\vdots$$

$$\underset{\sim}{b}^k = (b_1^*, \ldots, b_{k-1}^*).$$

By the result of (A.10), we have $d_j(\underset{\sim}{b}^i) \leq d_j(\underset{\sim}{b}^{i+1})$ for each $i = 1, \ldots, k-1$; $j = 1, \ldots, k$. Hence, the proof of part b) is completed.

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