CONSISTENCY OF MAXIMUM LIKELIHOOD RECURSION IN STOCHASTIC APPROXIMATION

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ABSTRACT

A class of stochastic approximation procedures which generalizes the nonadaptive maximum likelihood recursion procedures of Wu (1985, 1986) and the nonadaptive Robbins-Monro procedures is defined. These procedures are shown to be consistent in quantal response problems and in some other situations.

1. Introduction and Summary

Suppose that for each $x \in \mathbb{R}$, we have a probability distribution on $\mathbb{R}$ with c.d.f. $F(\cdot|x)$ and mean $M(x) \in (-\infty,\infty)$. Each time we choose a "stimulus level" x_n we see a "response" $Y_n \sim F(\cdot|x_n)$. The goal will be to find a scheme for sequentially choosing the stimulus levels x_1, $x_2, \ldots$ so that x_n converges rapidly to the root θ of

$$M(x) = p, \tag{1.1}$$

where $p \in \mathbb{R}$ is given. We will assume that the regression function M satisfies the usual stochastic approximation conditions:

$$M(\theta) = p \text{ for some } \theta \in \mathbb{R} \tag{1.2}$$

$$\inf_{\delta \le |x - \theta| \le \delta^{-1}} \{M(x) - p\} (x - \theta) > 0, \text{ for all } 0 < \delta < 1. \tag{1.3}$$

Under some weak additional conditions on M and $\sigma^2(x) =: \mathrm{var}(Y|x)$, it is well known that various adaptive and nonadaptive Robbins-Monro (RM) procedures cause x_n to converge to θ. The simplest nonadaptive RM rules take the form

$$x_{n+1} = x_n - (Y_n - p)/(nb) \tag{1.4}$$

for some positive constant $b > 0$. Lai and Robbins (1979) remark that the rule (1.4) sets x_{n+1} equal to the MLE for θ under the parametric model

(1.5) $$Y_{i+1} = p + b(x_i - \theta) + \varepsilon_i \qquad i = 1, 2, \ldots$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are assumed by the model to be i.i.d. $N(0, \sigma^2)$. The iterated least-squares procedure of Lai and Robbins (1982) and, to a lesser extent, the adaptive RM rules of Lai and Robbins (1979) may be regarded as repeated MLE rules under the parametric model

(1.6) $$Y_{i+1} = p + \beta(x_i - \theta) + \varepsilon_i \qquad i = 1, 2, \ldots$$

where the ε_i's continue to be i.i.d. $N(0, \sigma^2)$, and $\theta \in \mathbb{R}$ and $\beta > 0$ are the unknown parameters.

The results of Lai and Robbins (1979) suggest that the asymptotic rate of convergence of adaptive RM rules is not subject to general improvement, even when one has considerable knowledge of the situation beyond what was assumed above. Indeed, suppose that $p = \frac{1}{2}$, and that $F(\cdot|x)$ is the c.d.f. for a Bernoulli distribution for which

(1.7) $$P\{Y = 1|x\} = 1 - P\{Y = 0|x\} = \{1 + e^{-\lambda(x - \theta)}\}^{-1} .$$

Thus, we are in a quantal response situation, and the regression function M is a logit curve. If $x_1, x_2, \ldots$ are determined according to an adaptive RM rule, then $n^{\frac{1}{2}}(x_n - \theta)$ converges in distribution to $N(0, 4/\lambda^2)$. (Lai and Robbins (1979) assume "i.i.d. errors", but it seems clear that their results and methods also apply here.) On the other hand, suppose that λ is known and $x_1, x_2, \ldots$ are all set equal to θ, which is where the Fisher information for θ under the model (1.7) is maximized. If $\hat{\theta}_n$ is the MLE for θ under the model (1.7) based on observations $\{(x_i, Y_i)\}_{i=1}^n$, then $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ also converges in distribution to $N(0, 4/\lambda^2)$. Thus, the MLE under the true location model based on maximally informative observations does no better asymptotically (at least to first order) than adaptive RM.

However, Wu (1985, 1986) has suggested that the small and moderate sample size behavior of RM procedures may be improved upon by the use of repeated maximum likelihood estimation (Wu's term in Wu (1986) is <u>maximum likelihood</u> (ML) <u>recursion</u>.) under parametric models more appropriate to the situation in question than the models (1.5) and (1.6). For example, for quantal response problems where one knows that $P\{Y = 0 \text{ or } 1|x\} = 1$ and that $0 \leq M(x) \leq 1$, the models (1.5) and (1.6) are almost ridiculous. A more appropriate model might have the form

$$(1.8) \qquad P\{Y = 1|x\} = 1 - P\{Y = 0|x\} = H(x|\theta), \ \theta \in \Theta \subset \mathbb{R}^d,$$

where, for each $\theta \in \Theta$, H is a strictly increasing and continuous c.d.f. Wu suggests finding the MLE $\hat{\theta}_n$ for θ based on the previous observations $\{(x_i, Y_i)\}_{i=1}^n$, and then choosing x_{n+1} to satisfy $H(x_{n+1}|\hat{\theta}_n) = p$. (Some other choice rule must be used until the MLE $\hat{\theta}_n$ exists and is unique.) Wu has especially considered the use of the location-scale logit model

(1.9) $$H(x|\alpha,\lambda) = \{1 + e^{-\lambda(x-\alpha)}\}^{-1}, \quad \lambda > 0, \ \alpha \in \mathbb{R}.$$

Here, the rule for choosing x_{n+1} takes the form

(1.10) $$x_{n+1} = \hat{\alpha}_n - (\hat{\lambda}_n)^{-1} \ln (p^{-1} - 1).$$

Since the scale parameter λ as well as the location parameter α is being estimated here, (1.10) will be called the adaptive logit ML recursion rule. If λ is assumed to be known, then the formula is

(1.11) $$x_{n+1} = \hat{\alpha}_n - \lambda^{-1} \ln (p^{-1} - 1),$$

which will be called the nonadaptive logit ML recursion rule. Similar rules (adaptive and nonadaptive) can be based on other parametric models such as the probit model.

Wu (1985) has done Monte Carlo simulations to compare the performance of adaptive and nonadaptive RM procedures with the performance of his adaptive logit ML recursion rule for moderate sample sizes (n = 10 to 35). He claims that a modification of his adaptive logit ML recursion method with truncated step sizes generally outperforms RM procedures. He also claims that his method is asymptotically equivalent to adaptive RM *if* it is consistent. However, he has not given a rigorous proof of consistency.

To deal with situations other than just quantal response, Wu (1986) has suggested that maximum likelihood recursion be carried out under generalized linear models with canonical link functions. This amounts to assuming that the distribution of Y, given x, has a density of the form

$$(1.12) \qquad \exp[(x - \alpha)\lambda y - b\{\lambda(x - \alpha)\}]$$

with respect to a fixed measure, where b $(\cdot)$ is a known function and $\lambda > 0$ and $\alpha \in \mathbb{R}$ are unknown parameters. Then

$$M(x) = b'\{\lambda(x - \alpha)\}.$$

Providing that p is in the range of b', (Otherwise, (1.1) has no root according to the model.) we may assume without loss of generality that $b'(0) = p$, so that $x = \alpha$ is the root of (1.1).

The models (1.6) and (1.9) are special cases of (1.12). Another special case of (1.12) is the model which assumes that Y, given x, has a Poisson distribution with mean $e^{\lambda(x - \alpha)}$. (See Wu (1986).) Again, Wu has no proof of consistency.

The author has found it enlightening to compare several nonadaptive ML recursion rules for quantal response problems by considering how the shapes of the corresponding efficient score functions affect their behavior. Details and picutres are presented in Sellke (1986). Such geometrical considerations show, for example, that the nonadaptive logit ML recursion rule usually performs far better than the nonadaptive probit ML recursion rule or the nonadaptive RM rule when the initial observations are taken far from θ.

Section 2 of this paper defines a class of procedures called score function rules. These score function rules incorporate the geometrical properties which seem to be responsible for the fact that ML recursion rules are generally consistent. (Again, see Sellke (1986) for picures.) Section 3 shows that score function rules are indeed consistent for quantal response problems under conditions (1.2) and (1.3). Section 4 shows that the nonadaptive Poisson ML recursion rule of Wu (1986) is also consistent under weak conditions.

2. Score Function Rules

Let $f(\cdot,\cdot)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$ such that, for each $y \in \mathbb{R}$, $f(t,y)$ is a nondecreasing function of t which is continuous at 0 and for which $f(0,y) = y - p$. Let $S^{(0)}(\cdot)$ be a strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$ for which

$$(2.1) \qquad \lim_{t \to -\infty} S^{(0)}(t) < 0 < \lim_{t \to \infty} S^{(0)}(t).$$

If $\lim_{t \to -\infty} f(t,y) > 0$ for any possible value y of Y, then we require also that $\lim_{t \to -\infty} S^{(0)}(t) = -\infty$. If $\lim_{t \to \infty} f(t,y) < 0$ for any possible value y, then we require also that $\lim_{t \to \infty} S^{(0)}(t) = \infty$. The score function rule for finding the root θ of (1.1) operates as follows. Given that $(x_i, Y_i)_{i=1}^{n}$ have been observed, define the n^{th} score function $S^{(n)}$ by

$$(2.2) \qquad S^{(n)}(t) =: S^{(0)}(t) + \sum_{i=1}^{n} f(t - x_i, Y_i).$$

The choice rule for x_{n+1} is

$$(2.3) \qquad x_{n+1} = \inf\{t : S^{(n)}(t) \geq 0\}.$$

Note that our assumptions on f and $S^{(0)}$ guarantee that x_{n+1} is always finite.

The nonadaptive RM rule (1.4) can be obtained as a score function rule. Use

$$S^{(1)}(t) = (Y_1 - p) + b(t - x_1)$$

instead of $S^{(0)}$ as the initial score function, and

$$f(t, y) = y - p + bt.$$

Then (1.4) and (2.3) are equivalent for n = 1, 2, ...

Suppose that $P\{Y = 0 \text{ or } Y = 1 | x\} = 1$ for all x, so that we are in a quantal response situation with

$$(2.4) \qquad P\{Y = 1|x\} = 1 - P\{Y = 0|x\} = M(x)$$

Let G be a c.d.f. with density g for which log G and log (1 - G) are concave, and for which $G(0) = p$. ML recursion under the location model for M given by

$$(2.5) \qquad M(x) = G(x - \theta), \qquad \theta \in \mathbb{R},$$

is easily shown to be a score function rule. In this case, the maximum likelihood estimate $\hat{\theta}_n$ exists and is unique precisely when

$$(2.6) \qquad \sum_1^n Y_i > 0 \text{ and } \sum_1^n (1 - Y_i) > 0.$$

Let $\{(\tilde{x}_i, \tilde{Y}_i)\}_{i=1}^{k_o}$ be "initial" data for which (2.6) holds. This initial data may be the result of observations taken before (2.6) holds, or it may be "fake" data which is thought to reflect prior opinion. If we set $S^{(0)}$ equal to $\{-p(1 - p)/g(0)\}$ times the efficient score function of the initial data, and if we use

$$(2.7) \qquad f(t,y) = \frac{p(1 - p)}{g(0)} \cdot \frac{g(-t)}{G(-t) + y - 1},$$

then the resulting score function rule agrees with ML recursion based on the model (2.5). Nonadaptive logit and probit ML recursion are special cases. See Sellke (1986) for details.

Wu's (1986) ML recursion design based on a generalized linear model with a canonical location link function (Take $\lambda = 1$ in (1.12).) is easily shown to be a score function rule with

(2.8) $$f(t, y) = y - b'(-t).$$

A heuristic argument for how a score function rule will behave is as follows. Assume for simplicity that $S^{(0)}(\cdot)$ and $f(\cdot,y)$, $y \in \mathbb{R}$, are continuous. If $Y_n > p$, then $S^{(n)}(x_n) > S^{(n-1)}(x_n) = 0$, so, by (2.3), $x_{n+1} < x_n$. Likewise if $Y_n < p$, then $x_{n+1} > x_n$. Thus, the change between x_n and x_{n+1} is in "the right direction" in that one moves to the left if Y_n is "too big" and to the right if Y_n is "too small". Furthermore, if $f(t, y)$ is strictly increasing in t for each $y \in \mathbb{R}$, then the score functions $S^{(n)}$ get steeper and steeper, which in turn causes the adjustments $x_{n+1} - x_n$ to get smaller and smaller. (Indeed, in the case of the RM rule (1.4), the score function $S^{(n)}$ is a line of slope (nb) and $S^{(n)}(x_n) = Y_n - p$. Thus, the root x_{n+1} of $S^{(n)}(t) = 0$ satisfies $x_{n+1} - x_n = -(Y_n - p)/(nb)$, which agrees with (1.4).) Furthermore, if x_n converges to an incorrect limit, say $x_\infty > \theta$, then by (1.3) one expects

(2.9) $$\underline{\lim}_{n}\, n^{-1} \sum_{1}^{n} (Y_i - p) > 0.$$

If $f(t, y)$ is uniformly (in y) continuous in t at $t = 0$, then (2.9) and $x_n \to x_\infty$ imply that $S^{(n)}(t)$ diverges to $+\infty$ uniformly for t in a small interval $(x_\infty - \delta, x_\infty + \delta)$. But this contradicts $x_n \to x$. (This argument for why x_n cannot converge to a wrong value appears in Wu (1985, 1986).)

3. Consistency of Score Function Rules for Quantal Response Problems

In quantal response, Y is always equal to either 0 or 1, so that a score function rule will be specified by the functions $S^{(0)}(\cdot)$, $f(\cdot, 0)$ and $f(\cdot, 1)$. The author conjectures that score function rules are consistent in quantal response problems whenever $f(t, 0)$ and $f(t, 1)$ are strictly increasing in t at $t = 0$. However, Theorem 1 below requires a slightly stronger assumption. Let $f'_+(t, y)$ and $f'_-(t, y)$ be the right hand and left hand derivatives with respect to t of $f(t, y)$.

Let $\mathfrak{F}_n$, $n = 0,1,2,\ldots$, be the σ-algebra generated by $S^{(0)}$ and by $Y_1,\ldots,Y_n$. (Recall from Section 2 that it may sometimes be convenient to regard $S^{(0)}$ as random.) We assume that

$$P\{Y_{n+1} = 1|\mathfrak{F}_n\} = 1 - P\{Y_{n+1} = 0|\mathfrak{F}_n\} = M(x_{n+1}).$$

Theorem 1. Suppose that x_1, x_2, ... are chosen according to a score function rule for which $f'_+(0, y)$ and $f'_-(0, y)$ exist and are positive for $y = 0, 1$. Suppose further that $P\{Y = 0 \text{ or } Y = 1|x\} = 1$ for all x. Assume that $M(x) = E(Y|x)$ satisfies (1.2) for some $0 < p < 1$, and that

$$(3.1) \qquad (x - \theta)\{M(x) - p\} \geq 0, \qquad x \in \mathbb{R}.$$

Then x_n converges almost surely to a finite limit. If in addition M satisfies (1.3), then x_n converges almost surely to θ.

The heuristic argument of the previous section suggests that the steps $x_{n+1} - x_n$ tend to be in the right direction and that they tend to get smaller over time. For x_n to converge almost surely to θ, it must be the case that x_n does not "wander around" forever, and that x_n cannot "get stuck" at an incorrect value. Lemma 1, which is a sort of upcrossing inequality, implies that x_n cannot wander around forever and therefore must converge. Condition (1.3) and the continuity of $f(t, 0)$ and $f(t, 1)$ at $t = 0$ will imply that x_n cannot converge to an incorrect value.

Lemma 1. Suppose that all the assumptions of Theorem 1 except possibly (1.3) hold. Then there exists a function $U : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, 1]$ such that $\lim_{A \to \infty} U(\eta, A) = 0$ for each $\eta > 0$ and such that the following holds. If L is a number for which $L \geq \theta$, $S^{(m-1)}(L) \geq 0$, and

$$S^{(m-1)}(L + \eta) - S^{(m-1)}(L) > e^A + 1, \tag{3.2}$$

then

$$P\{\sup_{n \geq m} x_n > L + 2\eta \mid \mathfrak{F}_{m-1}\} \leq U(\eta, A). \tag{3.3}$$

As Jeff Wu has pointed out, the following proof of Lemma 1 is a sort of probabilistic Zeno's paradox. One shows that, with high probability, it takes a long time for the x_n sequence to exceed $L + \eta$. Given that it took a long time to exceed $L + \eta$, then with even higher probability the x_n sequence takes an even longer time to exceed $L + (3/2)\eta$. Continuing in this way, one shows that x_m cannot exceed $L + 2\eta$ in finite time except on a set of small probability.

Proof of Lemma 1

Since $f'_+(0, 0)$ and $f'_+(0, 1)$ exist and are positive, there exist $\varepsilon > 0$ and $\nu > 0$ such that

$$(3.4) \qquad \frac{f(t_2, y) - f(t_1, y)}{t_2 - t_1} > \varepsilon, \qquad y = 0, 1$$

when $0 \leq t_1 < t_2 < 2\nu$, and $t_2 - t_1 > t_1/2$. It will suffice to prove that Lemma 1 holds for $\eta \leq \nu$. Note for future reference in Section 4 that the only properties of $F(\cdot|x)$ that are really used in the following proof are that $E(Y^2|x)$ is bounded and $M(x) \geq p$ for $x_m \leq x \leq x_m + 2\eta$, and that Y is always bounded below.

For each j = 0, 1, ..., let $\tilde{Y}(1, j)$, $\tilde{Y}(2, j)$, ... be the (perhaps finite) subsequence of Y_m, Y_{m+1}, ... obtained by deleting all Y_n's for which $x_n < L + (2 - 2^{1-j})\eta$. For k, j = 0, 1, ..., let $\mathcal{B}(k, j) = \mathfrak{F}_n$ if $\tilde{Y}(k + 1, j)$ corresponds to Y_{n+1}. Thus, $\mathcal{B}(k, j)$ is generated by the past of the original process just before $\tilde{Y}(k + 1, j)$ is observed. Let N_j equal the total number of $\tilde{Y}(k, j)$'s. If $N_j < \infty$, set $\tilde{Y}(N_j + i + 1, j) = p$ and $\mathcal{B}(N_j + i, j) = \mathfrak{F}_\infty$ for i = 0, 1, ... Again for each j = 0, 1, ..., let $\{T(k, j)\}_{k=0}^{\infty}$ be the random walk generated by the $\tilde{Y}(k, j) - p$ sequence, reflected downward at 0. Thus, $T(0, j) = 0$ and

$$(3.5) \qquad T(k+1, j) = \min\{T(k, j) + \tilde{Y}(k+1, j) - p, 0\}.$$

Since $E\{\tilde{Y}(k+1, j) - p|\mathscr{B}(k, j)\} \geq 0$ and $E[\{\tilde{Y}(k+1, j) - p\}^2|\mathscr{B}(k, j)] \leq 1$, it is easy to show that

$$(3.6) \qquad W(k, j) = T(k, j)^2 - k, \qquad k = 0, 1, \ldots$$

is a $\mathscr{B}(k, j)$-supermartingale in k for each j. Let a and be positive numbers. A trivial stopping time argument shows that

$$(3.7) \qquad P\{\inf_{k \leq b} T(k, j) \leq -a|\mathfrak{F}_{m-1}\} \leq b/a^2.$$

I now want to show that, if A is sufficiently large, then with high probability it takes the $T(k, j)$ process more than

$$(3.8) \qquad b_j =: (\eta\, \varepsilon)^{-1}\, 2^{j+2} \exp\{(\sqrt{2})^{j+1} A\}$$

steps to cross below the level

$$(3.9) \qquad -a_j =: -\exp\{(\sqrt{2})^j A\}$$

for <u>every</u> j. But by (3.7)

$$(3.10) \quad P\{\inf_{k \leq b_j} T(k,j) \leq -a_j|\mathfrak{F}_{m-1}\} \leq (\eta\varepsilon)^{-1}\, 2^{j+2} \exp\{(1-\sqrt{2})(\sqrt{2})^{j+1} A\}.$$

Set

$$(3.11) \qquad U(\eta, A) = \sum_{j=0}^{\infty} (\eta\varepsilon)^{-1}\, 2^{j+2} \exp\{(1-\sqrt{2})(\sqrt{2})^{j+1} A\},$$

and note that $\lim_{A \to \infty} U(\eta, A) = 0$. Define the event E_A by

$$(3.12) \qquad E_A = :\{ \inf_{k \leq b_j} T(k, j) > -a_j \text{ for all } j = 0, 1, \ldots\}.$$

By (3.10) and Borel-Cantelli, $P\{E_A^c | \mathfrak{F}_{m-1}\} \leq U(\eta, A)$.

The rest of the argument is geometry. The point is that x_n will never again exceed $L + 2\eta$ if E_A occurs. Assume that E_A occurs.

In order to have some x_n, $n > m$, exceed $L + \eta$, it is necessary to bring the value of $S^{(n)}(L + \eta)$ down from above $e^A + 1$ to below zero. If $x_n \leq L + \eta$, then, by (2.2) and the fact that $f(t, y)$ is increasing in t,

$$(3.13) \qquad S^{(n)}(L + \eta) - S^{(n-1)}(L + \eta) \geq Y_n - p.$$

Thus, since $Y_n - p > -1$, $S^{(n)}(L + \eta)$ can decrease by at most 1 for each observation until x_n exceeds $L + \eta$. (This is where the fact that Y is bounded below is used.) Hence $S^{(n)}(L + \eta)$ must hit the interval $[e^A, e^A + 1)$ on its way down to zero if it ever gets down below zero. Furthermore, $S^{(n)}(L + \eta) < e^A + 1$, $n > m$, implies that $x_{n+1} \geq L$, since the score function gets steeper as more observations are made. Hence, the (x_{n+1}, Y_{n+1}) observations which take $S^{(n)}(L + \eta)$ down to zero after its last previous visit to $[e^A, e^A + 1)$ all have $x_{n+1} \geq L$. By (3.13), the sum of the $(Y_{n+1} - p)$ values for these observations must be less than $-e^A$. Thus, $S^{(n)}(L + \eta)$ cannot drop below zero before $T(k, 0)$ crosses below $-e^A$. But the event E_A (with $j = 0$) implies that one takes at least b_0 observations with $L \leq x_n \leq L + \eta$ before $T(k, 0)$ crosses below $-e^A = -a_0$.

By (2.2) and (3.4), each observation (x_n, Y_n) with $L \leq x_n \leq L + \eta$ causes the difference

$$S^{(n)}\{L + (3/2)\eta\} - S^{(n)}\{L + \eta\} \tag{3.14}$$

to increase by more than $\varepsilon\eta/2$. (Recall we assume $\eta \leq \nu$.) Hence, the difference (3.14) exceeds

$$b_0\varepsilon\eta/2 = 2\exp\{\sqrt{2}A\} > \exp\{\sqrt{2}A\} + 1 \tag{3.15}$$

before $S^{(n)}(L + \eta)$ drops down below zero.

Now we iterate the argument. The geometry of the score functions implies that $S^{(n)}\{L + (3/2)\eta\}$ cannot drop below zero before $T(k, 1)$ crosses below $-\exp\{\sqrt{2}A\} = -a_1$. The event E_A with $j = 1$ implies that at least b_1 observations with $L + \eta \leq x_n \leq L + (3/2)\eta$ are taken before this occurs. Finally, (2.2) and (3.4) imply that the difference

$$S^{(n)}\{L + (7/4)\eta\} - S^{(n)}\{L + (3/2)\eta\}$$

exceeds

$$b_1 \varepsilon \eta/4 = 2 \exp\{(\sqrt{2})^2 A\} > \exp\{(\sqrt{2})^2 A\} + 1$$

before $S^{(n)}\{L + (3/2)\eta\}$ drops below zero, and we are ready for the next iteration.

Thus, by induction we get that the event E_A implies that, for each j, at least b_j observations must be taken before x_n can exceed $L + 2\eta$. Since $b_j \to \infty$ as $j \to \infty$, it follows that x_n can never again exceed $L + 2\eta$.

Proof of Theorem 1

Continue to suppose that all the assumptions of Theorem 1 except possibly (1.3) hold. If x_n does not converge to an extended real number, then there must be a pair of rational numbers $a < b$ such that the x_n sequence crosses the interval [a, b] infinitely often. We may assume without loss of generality that $\theta \leq a < b$. However, if the x_n sequence crosses [a, b] infinitely often, then it is not hard to show that

$$S^{(n)}(b) - S^{(n)}(a) \to \infty$$

By Lemma 1, the x_n sequence cannot cross any such interval [a, b] infinitely often with positive probability. Hence, x_n must converge to an extended real number.

I now want to show that $x_n \to \infty$ is impossible. I want to be able to assume here that $S^{(n)}(x_{n+1}) = 0$, but this may not be true if $S^{(n)}$ has jumps. However, one can at each stage simply set $S^{(n)}(x_{n+1}) = 0$, leaving $S^{(n)}$ unchanged elsewhere. This has absolutely no effect on the behavior of the score function rule.

Let $\tilde{x}_n = \max_{i\leq n} x_i$. Fix $A > 0$, and let $m = m_A$ be the first n for which $\tilde{x}_n > \theta$ and for which

$$S^{(n-1)}(x_n + 1) - S^{(n-1)}(x_n) > e^A + 1. \tag{3.16}$$

Since $S^{(m-1)}(\tilde{x}_m) > 0$ if $x_m < \tilde{x}_m$, and $S^{(m-1)}(\tilde{x}_m) = 0$ if $x_m = \tilde{x}_m$, in either case Lemma 1 implies

$$P\{\sup_{n \geq m} x_n > \tilde{x}_m + 1 | F_{m-1}\} < U(1, A). \tag{3.17}$$

Thus, the conditional probability that $x_n \to \infty$, given that m_A is finite, is less than $U(1, A)$. It will now be shown that m_A is necessarily finite if $x_n \to \infty$, so that $P\{x_n \to \infty\} \leq \inf_A U(1, A) = 0$.

Since $f(0, y) = y - p$ and $f'_+(0, y) > 0$, $y = 0, 1$, it follows that there exists an $\varepsilon > 0$ such that $f(t, y) > y - p + \varepsilon$ for $y = 0, 1$ and for all $t \geq 1$. Thus, by (2.2) and the fact that $\tilde{x}_{n+1} \geq \tilde{x}_n$,

$$S^{(n)}(\tilde{x}_{n+1} + 1) - S^{(n-1)}(\tilde{x}_n + 1) > Y_{n+1} - p + \varepsilon. \tag{3.18}$$

If $x_n > \theta$ for all sufficiently large n, then the martingale SLLN stated in the Appendix implies

$$\sum_{1}^{n} (Y_i - p + \varepsilon) \to \infty.$$

(Recall that $E(Y_n - p|x_n) \geq 0$ when $x_n > \theta$.) Thus, $x_n \to \infty$ implies $S^{(n-1)}(\tilde{x}_n + 1) \to \infty$. Also $x_n \to \infty$ implies that $x_n = \tilde{x}_n$ and $S^{(n-1)}(\tilde{x}_n) = 0$ for infinitely many n. Hence, the difference in (3.16) eventually exceeds any positive number if $x_n \to \infty$, so that m_A is almost surely finite.

Likewise, $x_n \to -\infty$ is also impossible, so that x_n must converge to a finite limit x_∞.

Now suppose that (1.3) holds and that $P\{x_\infty > \theta\} > 0$. Then $P\{x_\infty \in (a, b)\} > 0$ for some $b > a > \theta$. By (1.3) there exists an $\varepsilon > 0$ such that

$$\inf_{x \in (a,b)} \{M(x) - p\} > 3\varepsilon. \tag{3.19}$$

Since f(t, 0) and f(t, 1) are continuous in t at t = 0, there exists a $\delta > 0$ such that

$$f(-\delta, y) > y - p - \varepsilon, \qquad y = 0, 1. \tag{3.20}$$

Let $(c, d) \subset (a, b)$ be such that $d - c < \delta$ and $P\{x_\infty \in (c, d)\} > 0$.

Let

$$n_1 = \sup\{n : x_n \notin (c, d)\}. \tag{3.21}$$

If $n_1 < \infty$, then (3.19) and the martingale SLLN imply that

$$k^{-1} \sum_{i=1}^{k} (Y_{n_{1+i}} - p) > 2\varepsilon \text{ for k sufficiently large.} \tag{3.22}$$

But then

$$S^{(n_1+k)}(c) = S^{(n_1)}(c) + \sum_{i=1}^{k} f(c - x_{n_1+i}, Y_{n_1+i})$$

$$\geq S^{(n_1)}(c) + \sum_{i=1}^{k} f(-\delta, Y_{n_1+i})$$

$$\geq S^{(n_1)}(c) + \sum_{i=1}^{k} (Y_{n_1+i} - p - \varepsilon) \text{ by (3.20).}$$

$$\geq S^{(n_1)}(c) + k\varepsilon \text{ for sufficeintly large k, by (3.22)}$$

$$> 0 \text{ for sufficiently large k.}$$

But $S^{(n_1+k)}(c) > 0$ implies $x_{n_1+k} \leq c$, which contradicts the definition

(3.21) of n_1. This contradiction shows that $P\{n_1 = \infty\} = 1$, and thus, that $P\{x_\infty \in (c, d)\} = 0$. Hence, $P\{x_\infty > \theta\} = 0$. Likewise, $P\{x_\infty < \theta\} = 0$, so $P\{x_\infty = \theta\} = 1$.

4. Consistency of Poisson ML Recursion

Wu's (1986) nonadaptive Poisson ML recursion method is a score function rule. If it is desired to stochastically approximate the root of (1.1) for $p > 0$, then the parametric model used is that Y, given x, has a Poisson distribution with mean $p\ e^{x-\theta}$. (One can, of course, change the scale and use mean $p\ e^{\lambda(x-\theta)}$ for known λ.) The score function rule then has

$$f(t, y) = y - p\ e^{-t}. \tag{4.1}$$

The initial score function $S^{(0)}$ based on "pre-rule" data $\{(\tilde{x}_i, \tilde{Y}_i)\}_{i=1}^{k_0}$ with $\sum_1^{k_0} \tilde{Y}_i > 0$ will be given by

$$S^{(0)}(t) = \sum_{i=1}^{k} \tilde{Y}_i - p\ e^{-(t-\tilde{x}_i)}.$$

Suppose now more generally that $g : \mathbb{R} \to \mathbb{R}$ is a concave strictly increasing function for which $g(0) = 0$. Let $f(t, y)$ be given by

$$f(t, y) = y - p + g(t). \tag{4.2}$$

Clearly, (4.1) is a special case of (4.2). Let $\sigma^2(x) =: \mathrm{var}\,(Y|x)$. For $K \in \mathbb{R}$, let

$$M_K(x) =: E(Y \wedge K|x),$$

where $\wedge$ denotes the minimum.

<u>Theorem 2</u>. Suppose that the following conditions hold.

(4.3) For some constant $B > 0$, $P(Y \geq -B|x) = 1$ for all $x \in \mathbb{R}$.

(4.4) M satisfies (1.2) and (1.3).

For some positive constants a, b, C, and K,

(4.5) $$M^2(x) + \sigma^2(x) < C \quad \text{for} \quad \theta - a \leq x \leq \theta + b$$

and

(4.6) $$\inf_{0 \leq h < \delta^{-1}} \{M_K(x + b + h) - p\} > 0, \text{ for all } \delta > 0.$$

Then any score function rule using an f of the form (4.2) with a concave initial score function $S^{(0)}$ causes x_n to converge almost surely to θ.

Remark. Theorem 2 shows that Wu's nonadaptive Poisson ML recursion rule is consistent under weak conditions. Note that, although the Poisson model implies that Y takes on only integer values, Theorem 2 assumes only that the set of possible values of Y is bounded below. It is tacitly assumed in Theorem 2 that $S^{(0)}(\infty) = \infty$ holds if this is necessary to insure that (2.3) always yields a finite value for x_{n+1}. For example, $S^{(0)}(\infty) = \infty$ may be necessary if $\lim_{t \to \infty} g(t) < B+p$, where B is the constant in (4.3).

Proof of Theorem 2

The proof of Theorem 1 applies almost without change to show that $(x_n - \theta)^+$, the positive part of $(x_n - \theta)$, converges to zero almost surely. This argument does not apply to show that $(x_n - \theta)^-$ converges to zero, in part because Y is not necessarily bounded above, but also because no assumptions have been made concerning $\sigma(x)$ for $x < \theta - a$. Thus, different techniques are called for.

First, let us show that x_n cannot converge to an incorrect finite value $x_\infty < \theta$. The argument used in the proof of Theorem 1 applies here provided that we can prove (4.8) below, which is a weaker analog of (3.22). If $c < d < \theta$, where

$$\sup_{x \in (c,d)} \{M(x) - p\} < -3\varepsilon, \tag{4.7}$$

for some $\varepsilon > 0$, and if $x_n \in (c,d)$ for all n greater than a constant N, then I claim

(4.8) $$\underline{\lim}_{k} k^{-1} \sum_{i=1}^{k} (Y_{N+i} - p) < -2\varepsilon.$$

Note that, by (4.7), S_k defined by

(4.9) $$S_k = \sum_{i=1}^{k} (Y_{N+i} - p + 3\varepsilon)$$

is a supermartingale with respect to the filtration $\mathcal{G}_k = \mathcal{F}_{N+k}$. Furthermore, the increments $(Y_{N+i} - p + 3\varepsilon)$ are, by (4.3), bounded below by $(-B - p + 3\varepsilon)$. Thus, the desired conclusion (4.8) follows from Corollary 1 below of Lemma 2.

<u>Lemma 2</u>. Let

$$S_n = \sum_{i=1}^{n} X_i, \qquad n = 0, 1, \ldots$$

be a supermartingale with respect to $\{\mathcal{G}_n\}_{n=0}^{\infty}$. If $P\{X_n \geq -1\} = 1$ for all n, then either $S_\infty =: \lim S_n$ exists and is finite, or $\underline{\lim}_{n\to\infty} S_n = -\infty$.

Proof of Lemma 2

Let C be an arbitrary constant, and let k be an arbitrary positive integer. It will suffice to prove that either S_∞ exists and is finite, or that $\inf_{n \geq k} S_n \leq C$. If $S_k \leq C$, the second alternative holds. If $S_k > C$, define a (perhaps infinite) stopping time t by

$$t =: \inf \{n \geq k : S_n \leq C\}$$

Then $(S_{n \wedge t} - C + 1)$ is a positive supermartingale for $n \geq k$ and must therefore by the martingale convergence theorem converge almost surely to a finite limit. If $t = \infty$, then S_∞ exists and is finite. If $t < \infty$, then $\inf_{n \geq k} S_n \leq C$.

Corollary 1. If $\{S_n\}_{n=0}^{\infty}$ is as in Lemma 2, and if $h(n) \to 0$ as $n \to \infty$, then $\underline{\lim}_n \{h_{(n)} S_n\} \leq 0$.

Proof of Theorem 2 (continued)

In the remainder of the proof, it will be convenient to assume that $p = 0$. Since $S^{(0)}$ and g are both assumed to be concave, all of the score functions $S^{(n)}$ will be concave and will have a finite left-hand derivative and a finite right-hand derivative at each point. Let $d_n > 0$ equal the right-hand derivative of $S^{(n)}$ at x_n, and note that d_n is $\mathfrak{F}_{n-1}$ measurable. Then it follows from the concavity of $S^{(n)}$ that

(4.10) $$X_{n+1} - X_n \geq Y_n/d_n.$$

Suppose now that $x_m < \theta$ for some positive integer m. It will now be shown that $\sup_{n \geq m} x_m \geq \theta$ with probability one. Let n_0 be the $\mathfrak{F}_n$-stopping time defined by

(4.11) $$n_0 =: \inf\{n \geq m : x_{n_0} + 1 \geq \theta\}.$$

(Recall that x_{n+1} is determined by $S^{(0)}$ and $Y_1, \ldots, Y_n$.). Define W_n for $n \geq m$ by

(4.12) $$W_n =: x_{(n \wedge n_0)} + 1$$

By (4.10) and the fact that

$$E(Y_{n+1} | \mathfrak{F}_n) = M(x_n) < 0$$

whenever $x_n < \theta$, it follows that W_n is an $\mathfrak{F}_n$-submartingale for $n \geq m$. Furthermore, the conditions on g and $S^{(0)}$ which insure that $S^{(n)}$ always has a finite root will usually be enough to insure that W_n is bounded above. If W_n is not bounded above, one can simply redefine W_n for $n \geq n_0$ by

$$W_n = W_{n_0-1} + Y_{n_0}/d_{n_0}, \qquad n \geq n_0.$$

Then W_n will still be an $\mathfrak{F}_n$-submartingale, and (4.3) now implies that W_n is bounded above. (Note that d_{n_0} will be bounded below by the right-hand derivative of $S^{(0)}$ at θ.) The martingale convergence theorem implies that W_n must converge to a finite limit. From this and the fact demonstrated above that x_n cannot converge to a finite limit $x_\infty < \theta$, it follows that $\sup_{n \geq m} x_n \geq \theta$.

We are now in a position to use the almost-supermartingale convergence theorem of Robbins and Siegmund (1971) (See appendix for a statement.) to show that $(x_n - \theta)^-$ must converge to zero. Define Z_n by

$$Z_n =: \{(x_{n+1} - \theta)^-\}^2 \wedge a^2, \tag{4.13}$$

where $a > 0$ is the constant in (4.5). Note that

$$\begin{aligned} E(Z_n|\mathfrak{F}_{n-1}) &= E[\{(x_{n+1} - x_n + x_n - \theta)^-\}^2 \wedge a^2|\mathfrak{F}_{n-1}] \\ &\leq E[\{(Y_n/d_n + x_n - \theta)^-\}^2 \wedge a^2|\mathfrak{F}_{n-1}], \text{ by (4.10)} \\ &\leq Z_{n-1} + a^2 I_{\{x_n > \theta + b\}} \\ &\quad + E\{(Y_n/d_n)^2|\mathfrak{F}_{n-1}\} I_{\{\theta - a \leq x_n \leq \theta + b\}} \\ &\quad + 2E\{(Y_n/d_n)(x_n - \theta)|\mathfrak{F}_{n-1}\} I_{\{\theta - a \leq x_n \leq \theta\}}. \end{aligned} \tag{4.14}$$

$$\leq Z_{n-1} + a^2 I_{\{x_n > \theta + b\}}$$

$$+ (d_n)^{-2} E\{(Y_n)^2 | \mathfrak{F}_{n-1}\} I_{\{\theta - a \leq x_n \leq \theta + b\}}.$$

But we know that $(x_n - \theta)^+ \to 0$, a.s., so that

(4.15) $$\sum_1^\infty a^2 I_{\{x_n > \theta + b\}} < \infty, \text{ a. s.}$$

By (4.5),

(4.16) $$E\{(Y_n)^2 | \mathfrak{F}_{n-1}\} I_{\{\theta - a \leq x_n \leq \theta + b\}} \leq C.$$

Let $\tilde{d}$ be the right-hand derivative of g at (a + b). Then

(4.17) $$d_n I_{\{\theta-a\leq x_n\leq\theta+b\}} \geq \tilde{d} I_{\{\theta-a\leq x_n\leq\theta+b\}} \sum_{i=1}^n I_{\{\theta-a\leq x_i\leq\theta+b\}}$$

It follows from (4.16) and (4.17) that

(4.18) $$\sum_1^\infty (d_n)^{-2} E\{(Y_n)^2 | \mathfrak{F}_{n-1}\} I_{\{\theta-a\leq x_n\leq\theta+b\}} \leq (\tilde{d})^{-2} C \sum_{k=1}^\infty k^{-2} < \infty.$$

By (4.15) and (4.18), Z_n satisfies

(4.19) $$E(Z_n | \mathfrak{F}_{n-1}) \leq Z_{n-1} + b_{n-1}, \qquad n = 1, 2, \ldots,$$

where Z_n and b_n are nonnegative and $\mathfrak{F}_n$-measurable, and $\sum b_n < \infty$, a.s. The almost-supermartingale theorem of Robbins and Siegmund (1971) implies that Z_n converges to a finite limit. But Z_n visits any interval $[0, \varepsilon)$, $\varepsilon > 0$, infinitely often, since $\sup_{n \geq m} x_n \geq \theta$ for every m. Thus, $Z_n \to 0$, a.s., which in turn implies $(x_n - \theta)^- \to 0$, a.s.

Appendix

The following result is a special case of a theorem found on page 148 of Neveu (1965). Neveu's result is rederived as Application 1 in Robbins and Siegmund (1971).

A Martingale SLLN Let $X_1, X_2, \ldots$ be a martingale difference sequence with respect to a filtration $\{\mathfrak{F}_n\}_{n=0}^{\infty}$. If $\sum n^{-2} E(X_n^2 | \mathfrak{F}_{n-1}) < \infty$, a.s., then $\lim_{n\to\infty} n^{-1} \sum X_n = 0$, a.s.

The following convergence theorem for non-negative almost supermartingales is a special case of Theorem 1 of Robbins and Siegmund (1971).

Theorem. (Robbins and Siegmund)

Let $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ be a filtration, and let $\{Z_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two adapted sequences of non-negative random variables such that

$$E(Z_n | \mathfrak{F}_{n-1}) \leq Z_{n-1} + b_{n-1}$$

for $n \geq 2$. Then $\lim_{n \to \infty} Z_n$ exists and is finite, a.s., on the set where $\sum b_n < \infty$.

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