Connecting Brownian Paths

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Summary. We study two processes obtained as follows: Take two independent d-dimensional Brownian motions started at points x, y respectively. For the first process, let $d \geq 3$ and condition on $X_t = Y_t$ for some t (a set of probability zero). Run X out to the point of intersection and then run Y in reversed time from this point back to y. For the second process, let $d \geq 5$ and perform the same construction, this time conditioning on $X_s = Y_t$ for some s, t. The first process is shown to be Doob's conditioned (to go from x to y) Brownian motion Z, and the second has distribution absolutely continuous with respect to that of Z, the Radon-Nikodym density being a constant times the time Z takes to travel from x to y. Similar results (including extensions to the critical dimensions $d = 2$ and $d = 4$) are obtained by conditioning the motions to hit before they leave domains. We use the asymptotics of the probability of 'near misses', and results on the weak convergence of h-transforms.

§1. Introduction

If two Brownian paths in $\mathbb{R}^d$ come close to one another, either at the same or at different times, those paths yield a bond connecting the paths' initial points. We study these bonds. Since we wish such 'near misses' to happen only once, we will do so in dimensions high enough for such events to be very unlikely.

Let $x \neq y$ be points in $\mathbb{R}^d$, $d \geq 3$, and let ${}_yZ_t^x$, $0 \leq t < \varsigma(Z)$ be Doob's Brownian motion conditioned to go from x to y; that is, the Markov process which starts at x, has finite lifetime $\varsigma(Z)$ and transition density

$$(1.1) \qquad {}_yp_t(z,w) = (|w-y|/|z-y|)^{d-2} p_t(z,w), \;\; z \neq y$$

where p_t is the transition density of Brownian motion. Note that $\int {}_yp_t(x,w)dw < 1$, the excess being the probability that ${}_yZ^x$ exceeds its lifetime before time t. This process is, for all purposes we can imagine, as tractable as (unconditioned) Brownian motion. See Doob (1984) for much more information about conditioned Brownian motion, and Durrett (1984) for an elementary account.

Let X_t^x, Y_t^y be independent Brownian motions started at x, y respectively. Let $\varepsilon > 0$ and put

$$L_\varepsilon = \sup\{t > 0;\ |X_t - Y_t| \leq \varepsilon\} \ (\sup(\phi) = 0).$$

It is easily checked that $P(0 < L_\varepsilon) = (\varepsilon/|x-y|)^{d-2}$ if $|x-y| \geq \varepsilon$, and $L_\varepsilon < \infty$ a.s.. Let

$$W_t^\varepsilon = \begin{cases} X_t^x, & 0 \leq t < L_\varepsilon \\ Y_{2L_\varepsilon - t}^y, & L_\varepsilon < t < 2L_\varepsilon \\ \Delta, & t \geq 2L_\varepsilon \end{cases}$$

and make W^ε right continuous at L_ε. We also make ${}_yZ_t^x = \Delta$ for $t \geq \varsigma(Z)$, so that Δ functions as a cemetary state.

(1.2) *Theorem. As $\varepsilon \downarrow 0$, the law of W^ε conditioned on $\{L_\varepsilon > 0\}$ converges weakly to that of ${}_yZ^x$.*

The use here of the **last** time our processes are within ε of each other is for convenience only. The analogue of Theorem 1.2 with L_ε replaced by $\inf\{t\colon |X_t - Y_t| \leq \varepsilon\}$, or by any other time our processes are within ε of each other, still holds, and is easily derived from Theorem 1.2 and some of our lemmas on Brownian paths. Weak convergence is with respect to the usual Skorohod topology. This is discussed more fully in §2 and §7. The proof of Theorem 1.2 is not difficult.

Now let $d \geq 5$, and put

$$M_\varepsilon = \sup\{s > 0;\ |X_s^x - Y_t^y| \leq \varepsilon \text{ for some } t\}$$
$$N_\varepsilon = \sup\{t > 0;\ |X_s^x - Y_t^y| \leq \varepsilon \text{ for some } s\}.$$

(Again the precise forms of M_ε, N_ε are not essential, only convenient). Both M_ε and N_ε are finite a.s.. Let

$$V_t^\varepsilon = \begin{cases} X_t^x, & 0 \leq t \leq M_\varepsilon \\ Y_{M_\varepsilon + N_\varepsilon - t}^y, & M_\varepsilon < t < M_\varepsilon + N_\varepsilon \\ \Delta, & t \geq M_\varepsilon + N_\varepsilon. \end{cases}$$

(1.3) *Theorem. As $\varepsilon \downarrow 0$, the law of V^ε conditioned on $\{M_\varepsilon > 0\}$ converges weakly. The limiting law is absolutely continuous with respect to that of ${}_yZ^x$, with Radon-Nikodym density $\varsigma({}_yZ^x)/E\varsigma({}_yZ^x)$.*

Thus in Theorem (1.2) we condition massive particles to collide, and in Theorem (1.3) we condition Wiener sausages to intersect.

The proof will in fact show that the joint law of $(V^\varepsilon, M^\varepsilon)$ converges weakly to that of a pair (V, M). It is also shown that conditional on V, the time M is uniformly distributed on $[0, \varsigma(V)]$. In particular, it is not determined by V and occurs, in some sense, at a typical point along the path (rather than at, say, a place with rapid oscillation). In contrast, Martin Barlow has observed that the intersection of Brownian motion in $\mathbb{R}^2$ with space time Brownian motion occurs at atypical points; as a consequence of Makarov's theorem, harmonic measure in the domain below the space-time Brownian path is carried by a set with Hausdorff dimension strictly less than that of the whole path.

Neither of these two theorems extends to lower dimensions (in dimensions 2 and 4 respectively, the weak limits still exist, but are standard Brownian motions. In even lower dimensions, the weak limits aren't needed to make sense of the analogous results, but unfortunately these results are false). However, we can go down one dimension by considering domains other than all of $\mathbb{R}^d$.

Let D be any domain in $\mathbb{R}^d$, $d \geq 2$, which has a Green function $G(x, y)$. For $x, y \in D$, $x \neq y$, let ${}_yZ^x$ now be Doob's Brownian motion conditioned to go from x to y before leaving D (in (1.1), $(|w-y|/z-y|)^{d-2}$ gets replaced by $G(w,y)/G(z,y)$ and p_t by the transition density of Brownian motion killed upon leaving D). For $\varsigma(X^x) = \inf\{s;\ X_s^x \notin D\}$, $\varsigma(Y^y) = \inf\{t;\ Y_t^y \notin D\}$, we let $L_\varepsilon = \sup\{t < \min(\varsigma(X^x),\ \varsigma(Y^y));\ |X_t^x - Y_t^y| < \varepsilon\}$, and then define W^ε as before.

(1.4) *Theorem. As $\varepsilon \downarrow 0$, the law of W^ε conditioned on $\{L_\varepsilon > 0\}$ converges weakly to that of ${}_yZ^x$.*

Now let D be a domain in $\mathbb{R}^d$, $d \geq 4$, and suppose that

$$\int_D G(x,z)G(y,z)dz < \infty. \tag{1.5}$$

This condition always holds if $d \geq 5$, since it is true for $D = \mathbb{R}^d$ and the Green function for D is dominated by the Newtonian one. In contrast the condition fails for $D = \mathbb{R}^4$, so is a real restriction here. It holds if D is bounded or is contained in the complement of any solid cone.

Define V^ε as before, now using $M_\varepsilon = \sup\{s < \varsigma(X^x);\ |X_s^x - Y_t^y| < \varepsilon$ for some $t < \varsigma(Y^y)\}$ and $N_\varepsilon = \sup\{t < \varsigma(Y^y);\ |X_s^x - Y_t^y| < \varepsilon$ for some $s < \varsigma(X^x)\}$.

(1.6) *Theorem. As $\varepsilon \downarrow 0$, the law of V_ε conditioned on $\{M_\varepsilon > 0\}$ converges weakly. The limiting law is absolutely continuous with respect to that of ${}_yZ^x$, with Rodon-Nikodym density $\varsigma({}_yZ^x)/E\varsigma({}_yZ^x)$.*

Note that since $\int_D G(x,z)G(y,z)dz = G(x,y)E\varsigma({}_yZ^x)$, we need (1.5) to state the theorem.

It is very likely that our Brownian motions can be replaced by random walks of step size $|x-y|/n$, and 'coming within ε' by 'intersection', and that limiting results will be the same as $n \to \infty$, but we do not attempt to prove this here (although our original argument was a heuristic nonstandard one, and suggests the above result). A different question is that of whether there are random walk results similar to ours which do not involve taking limits. The exact analogues of the above theorems fail, but the following holds: Let X_m^x, Y_n^y be independent standard random walks on the standard d-dimensional lattice, $d \geq 3$. If $\Sigma(x_i - y_i)$ is even, let $L = \sup\{n; X_n^x = Y_n^y\}$. If it is odd, let $L = \sup\{n; X_n^x = Y_{n+1}^y\}$. Then conditioned on $\{L > 0\}$ (now a set of positive probability), the pasting together of X^x and Y^y is a random walk conditioned to go from x to y. In fact, if γ_d is the probability that $X_n^0 \neq Y_n^0$ for any $n \geq 1$, then it is easily seen that the probability of the above pasting giving any fixed path of length m is $(2d)^{-m}\gamma_d$, and this easily implies our assertion. This argument clearly fails if we restrict our random walks to any proper subset of the lattice, or if we replace last hitting time with first hitting time.

Our proofs of the above results are based on Doob's theory h-transforms. Since in Theorem (1.3) we have two time parameters, we'll sometimes need h-transforms of two parameter processes; see §2 for the definition.

In §7, we prove some results about weak convergence of h-transforms. They are folklore in the one-parameter case, but we don't know a reference. In the two-parameter case they appear to be new. Section 2 will set notation, and the proofs of Theorems (1.2) and (1.3) are given in §3. Section 6 contains some discussion of the condition (1.5), and the estimates needed to make the arguments of §3 work are shown in §4 and §5.

These estimates are close to many others in the literature. The sources Lawler (1982), (1985), LeGall (1986a), (1986b), Aizenman (1985), Felder and Frõhlich (1985), Brydges and Spencer (1985) and especially Erdõs and Taylor (1960a), (1960b) all contain related results, from which we have profited, but whereas most of these papers deal with problems like the asymptotics in ε of $P(|X_s^x - Y_t^y| < \varepsilon$ for some $s,t)$, we are principally concerned with the dependence of these objects on x and y. Moreover, we work in domains, and must deal with the possible presence of pathological boundaries. This is especially true in dimension four (see §6). In this context, it should be pointed out that LeGall (1986a) studies asymptotics like those in §4, but for stopping X_t^x and Y_t^y at some fixed t_0 rather than at $\varsigma(X^x) \wedge \varsigma(Y^y)$.

§2. Notation

We now collect some of the notation used in the remainder of the paper.

- $B(x,r)$ is the ball of radius r centered at x.
- $B(K,r)$ is the r-neighborhood of K.
- Λ_ε will be the lattice εZ^d.
- b will usually be the number $1 + (\sqrt{d}/2)$.
- $u_\varepsilon \sim v_\varepsilon$ will mean that $u_\varepsilon / v_\varepsilon \to 1$.
- The letters X, Y, V, W, Z, L, M, N will keep the meanings given them in §1. In addition
$$S_\varepsilon = \inf\{s;\ |X_s^x - Y_t^y| < \varepsilon \text{ for some } t\}$$
$$T_\varepsilon = \inf\{t;\ |X_t^x - Y_t^y| < \varepsilon\}$$
with similar definitions when dealing with domains.
- h_ε, g_ε, g will be special functions, to be defined in §3, but we will often use h or h_n for a generic excessive function.
- We adopt the conventions that c denotes a generic constant, whose value may change from line to line, and that all functions vanish at Δ.

- We move x and y inside and outside expectations at will. Thus the following are taken to be tautologies

$$
\begin{aligned}
&{}_yE^x[f(Z)] = E[f({}_yZ^x)], \ {}_yP^x(Z \in A) = P({}_yZ^x \in A) \\
&E^{x,y}[f(X,Y)] = E[f(X^x, Y^y)], \ P^{x,y}((X,Y) \in A) = P((X^x, Y^y) \in A).
\end{aligned}
$$

Similarly, when dealing with general h-transforms, we write

$$
Ef({}_hX^x) = {}_hE^x[f(X)]
$$

etc.

- When dealing with domains, we use the same letters $(P^{x,y}, G, p_t, \ldots)$ when dealing with objects killed upon leaving D, as we did before, for the corresponding unkilled objects. This should cause no confusion, as we are consistent within sections. To talk about the unkilled object in a domain section we just add a '0'. (e.g. $P_0^{x,y}, G_0, p_t^0, \ldots$).

We now describe the form of weak convergence used in the theorems of §1. We must allow jumps, since our processes usually have two, one from the 'seam' between X and Y, and another at ς. Let $\overline{D}$ be the one-point compactification of D, and let Δ be a point isolated from $\overline{D}$. Let Ω be the space of paths with values in $\overline{D} \cup \{\Delta\}$ which are right continuous with left limits, and stay at Δ forever once they reach it. Endow Ω with the Skorokhod topology (see Billingsley (1968) and Lindvall (1973)). Weak convergence will always be that of probabilities on Ω (or $\Omega \times \Omega$; see §7).

To go along with Ω, we have other standard notation; for $\omega \in \Omega$ we write $\varsigma(\omega) = \inf\{t;\, \omega(t) = \Delta\}$, $(\theta_t\omega)(s) = \omega(t+s)$. (Respectively the lifetime, and shift operator). We use the σ-fields $\mathcal{F}_t$ generated by the evaluation maps $w \to w(s)$, for $s \le t$. Thus, since $X, Y, \ldots$ will take values in Ω, we can write random variables depending on X_s for $s \le t$ in the form $\tau(X)$ for $\tau \in \mathcal{F}_t$. Similarly, the lifetime of X is $\varsigma(X)$.

Recall that if h is excessive (i.e. superharmonic) we say that ${}_hX$ is an **h-transform** of X if for each positive $\tau \in \mathcal{F}_t$,

$$
E[\tau({}_hX), \varsigma({}_hX) > t] \;=\; E[\tau(X)h(X_t)/h(X_0), \varsigma(X) > t].
$$

It will be convenient to have a similar definition for two-parameter processes as well.

A **lower layer** is a set $\varsigma \subset [0,\infty) \times [0,\infty)$ such that

(a) If $(s,t) \in \varsigma$ and $s' \le s$, $t' \le t$, then $(s',t') \in \varsigma$;

(b) If $s_n \downarrow s$, $t_n \downarrow t$ and $(s_n,t_n) \notin \varsigma$ for any n, then $(s,t) \notin \varsigma$.

A **bipath** will be a function of the form

$$
\omega(s,t) = \begin{cases} (\omega^1(s), \omega^2(t)), & (s,t) \in \varsigma \\ (\Delta, \Delta), & (s,t) \notin \varsigma \end{cases}
$$

where $\omega^1, \omega^2 \in \Omega$, ς is a lower layer, and $\varsigma(\omega^1) = \inf\{s;\, (s,0) \notin \varsigma\}$, $\varsigma(\omega^2) = \inf\{t;\, (0,t) \notin \varsigma\}$ (so that ω^1 is the first component of $\omega(\cdot,0)$ and ω^2 is the second component of $\omega(0,\cdot)$).

A **biprocess** is a random bipath U such that each $U_{s,t}$ is measurable. We write U_s^1 and U_t^2 for the first and second components of $U_{s,0}$ and $U_{0,t}$ respectively, and $\varsigma(U)$ for its associated lower layer.

The simplest biprocess is called bi-Brownian motion; we let X and Y be independent Brownian motions and set $\varsigma(U) = [0,\varsigma(X)) \times [0,\varsigma(Y))$, $U^1 = X$, $U^2 = Y$ (unless we are working in domains, clearly $\varsigma(U) = [0,\infty) \times [0,\infty)$). Thus $U_{s,t} = (X_s, Y_t)$ while both X and Y are alive.

Let $U_{s,t}$ be a bi-Brownian motion, and let $h(x,y)$ be **biexcessive** (that is, $h(x,\cdot)$ and $h(\cdot,y)$ are each excessive). A biprocess ${}_hU_{s,t}$ is called an **h-bitransform** of $U_{s,t}$, if for every $s,t \geq 0$ and positive $\sigma \in \mathcal{F}_s$, $\tau \in \mathcal{F}_t$, we have that

$$E[\sigma({}_hU^1)\tau({}_hU^2),\ (s,t) \in \varsigma({}_hU)]$$
$$= E[\sigma(U^1)\tau(U^2)h(U_{s,t})/h(U_{0,0}),\ (s,t) \in \varsigma(U)].$$

Not much is known about these objects, but see for example Walsh (1981) and Cairoli (1968). All we will need will be the following fact: let A be an open subset of $D \times D$, and let $h(x,y) = P((X_s^x, Y_t^y) \in A$ for some $s,t > 0)$. Kill bi-Brownian motion at the "last exit" from A; that is, set

$${}_hU_{s,t} = \begin{cases} (X_s, Y_t), & \text{if there is some } s' > s \text{ and } t' > t \text{ such that } (X_{s'}, Y_{t'}) \in A \\ (\Delta,\Delta), & \text{otherwise.} \end{cases}$$

Now condition on $\Gamma = \{(X_s^x, Y_t^y) \in A$ for some $s,t\}$ (that is, on $\{{}_hU_{0,0} \neq (\Delta,\Delta)\}$).

(2.1) *Proposition. When conditioned on Γ, ${}_hU_{s,t}$ is an h-bitransform of bi-Brownian motion.*

The proof is identical to the corresponding well known result in one parameter – see p.568 of Doob (1984).

We will be concerned with the above only when A is the event $|X_s - Y_t| < \varepsilon$ for some s,t, so that we are killing bi-Brownian motion at the last near miss. The set ς for this process is just $\{(s,t)\colon\ |X_{s'} - X_{t'}| < \varepsilon$ for some $s' \geq s$ and $t' \geq t\}$. This set is contained in $[0, \sup\{s\colon\ |X_s - Y_t| < \varepsilon$ for some $t\} \times [0, \sup\{t\colon\ |X_s - Y_t| < \varepsilon$ for some $s\}$, but is not in general equal to this rectangle, since X_t can be within ε of the Y path several times, and the supremum of the t for which this happens can correspond to a smaller time for the Y process than for other t for which it happens. Part of what we prove is that the set ς becomes more rectangular as $\varepsilon \to 0$.

§3. Outlines of the Proofs of Theorems (1.2) and (1.3)

We start with a lemma on time reversal. Let $Z = {}_yZ^x$. Since Z has finite lifetime we may define its reverse;

$$\hat{Z}_t = \begin{cases} y, & t = 0 \\ Z_{\varsigma(Z)-t}, & 0 < t < \varsigma(Z) \\ \Delta, & t \geq \varsigma(Z). \end{cases}$$

In general, $\hat{}$ will be used to denote reversal of a process with finite lifetime.

(3.1) *Lemma. Let σ, $\tau \in \mathcal{F}_t$ be positive. Then*

$$ {}_yE^x[\sigma(Z)\tau(\hat{Z}), \varsigma(Z) > 2t] = E^{x,y}[\sigma(X)\tau(Y)G(X_t, Y_t)/G(x,y)] $$

Proof:

$$ \begin{aligned} &{}_yE^x[\sigma(Z)\tau(\hat{Z}), \varsigma(Z) > 2t] \\ &\quad = {}_yE^x[\sigma(Z)\,{}_yE^{Z_t}[\tau(\hat{Z}), \varsigma(Z) > t], \varsigma(Z) > t] \\ &\quad = {}_yE^x[\sigma(Z)\,{}_{Z_t}E^y[\tau(Z), \varsigma(Z) > t], \varsigma(Z) > t] \\ &\quad = \frac{1}{G(x,y)}E^x[\sigma(X)\,{}_{X_t}E^y[\tau(Z), \varsigma(Z) > t]G(X_t, y)] \\ &\quad = \frac{1}{G(x,y)}E^x[\sigma(X)\frac{1}{G(y, X_t)}E^y[\tau(Y)G(Y_t, X_t)]G(X_t, y)] \\ &\quad = E^{x,y}[\sigma(X)\tau(Y)G(X_t, Y_t)/G(x,y)] \end{aligned} $$

by symmetry of Brownian motion under time reversal. □

The following result may be found in the appendix (Proposition (7.6)), generalized to domains;

(3.2) *Lemma. Let h_n and h be strictly positive and superharmonic on $\mathbb{R}^d$ with $h_n \to h$ a.e. and $h_n(x) \to h(x) < \infty$. Then ${}_{h_n}P^x \to {}_hP^x$ weakly.*

Proof of Theorem (1.2). Let

$$ \begin{aligned} h_\varepsilon(x,y) &= P(|X_t^x - Y_t^y| < \varepsilon \text{ for some } t) \\ &= P(|X_{2t}^{x-y}| < \varepsilon \text{ for some } t) \\ &= C_d\varepsilon^{d-2}G(x-y, 0) = C_d\varepsilon^{d-2}G(x,y), \text{ whenever } |x-y| \geq \varepsilon. \end{aligned} $$

Let $X^{\varepsilon,x}$ and $Y^{\varepsilon,y}$ be X^x and Y^y killed at L_ε. Then (see Doob (1984)) conditioned on $L_\varepsilon > 0$, the $\mathbb{R}^{2d}$-valued process $(X_t^{\varepsilon,x}, Y_t^{\varepsilon,y})$ is an h_ε-transform of (X_t^x, Y_t^y), so that by Lemma (3.2) its law converges weakly to that of a $G(\cdot,\cdot)$ transform. We have that $W^\varepsilon = \Phi(X^{\varepsilon,x}, Y^{\varepsilon,y})$, where

$$ \Phi(\omega, \omega')(s) = \begin{cases} \omega(s), & s < \varsigma(\omega) \\ \omega'([\varsigma(\omega) + \varsigma(\omega') - s]-), & \varsigma(\omega) \leq s < \varsigma(\omega) + \varsigma(\omega') \\ \Delta, & s \geq \varsigma(\omega) + \varsigma(\omega') \end{cases} $$

(defined on $\Omega_0 = \{(\omega, \omega');\ \varsigma(\omega') < \infty\}$). Since Φ is continuous on Ω_0, we'll have that the law of W^ε converges weakly, once we know that $G(\cdot,\cdot)$-transforms have finite lifetimes a.s..

This is easily seen; either by a direct calculation that $G(\cdot,\cdot)$ is a potential in $\mathbb{R}^{2d}$, or by the observation that the difference of the first and second components of a $G(\cdot,\cdot)$-transform is itself a transform by $G(0,\cdot)$ in $\mathbb{R}^d$. This identification of the limit law of W^ε as that of ${}_yZ^x$ follows immediately from Lemma (3.1). □

We'll follow a similar approach to Theorem (1.3). The following generalization of Lemma (3.1) has a similar proof and will be omitted.

Let $Z, \hat{Z}$ be as in Lemma (3.1). Set

$$Z_r^{s,t} = \begin{cases} Z_{s+r}, & s+r+t < \varsigma(Z) \\ \Delta, & \text{otherwise.} \end{cases}$$

(3.3) *Lemma. Let $\sigma \in \mathcal{F}_s$, $\tau \in \mathcal{F}_t$, $\xi \in \mathcal{F}_\infty$ all be positive. Then*

$$\begin{aligned} &{}_yE^x[\sigma(Z)\tau(\hat{Z})\xi(Z^{s,t}), \varsigma(Z) > s+t] \\ &\quad = E^{x,y}[\sigma(X)\tau(Y)_{Y_t}E^{X_s}[\xi(Z)]G(X_s,Y_t)/G(x,y)]. \end{aligned}$$

The easy proof of this lemma is omitted.

Let

$$g_\varepsilon(x,y) = P^{x,y}(|X_s - Y_t| < \varepsilon \text{ for some } s,t).$$

As before, we must determine the asymptotic behaviour of g_ε. Since this is tricker than before, we'll merely record the result, postponing the proof to §5A. Let $g(x,y) = \int G(x,z)G(y,z)dz$, which, by scaling, is $c|x-z|^{-(d-4)}$ for a constant c not depending on x or y.

(3.4) *Lemma.* $g_\varepsilon(x',y')/g_\varepsilon(x,y) \to g(x',y')/g(x,y)$ *as $\varepsilon \downarrow 0$, for any $x' \neq y'$, $x \neq y$.*

We'll show the following in §5A as well.

(3.5) *Lemma. For each $s > 0$ and $x \neq y$,*

$$P^{x,y}(S_\varepsilon < s < M_\varepsilon)/g_\varepsilon(x,y) \to 0 \text{ as } \varepsilon \downarrow 0.$$

Proof of Theorem (1.3).

Let

$$U_{s,t}^\varepsilon = \begin{cases} (X_s^x, Y_t^y), & \text{if } \exists s' > s,\ t' > t \text{ s.t. } |X_{s'}^x - Y_{t'}^y| < \varepsilon \\ (\Delta,\Delta), & \text{otherwise.} \end{cases}$$

Condition on $\{M_\varepsilon > 0\}$. By Proposition (2.1), U^ε is a g_ε-bitransform (see §2 for the definition). In §7 we prove a weak convergence result for such bitransforms, analogous to

Lemma (3.2) above. To apply it, we must verify condition (7.7), but Lemma (3.5) does precisely this. The conclusion we obtain is exactly that the joint law of $(U^{1,\varepsilon}, U^{2,\varepsilon})$ on $\Omega \times \Omega$, conditioned on $\{M_\varepsilon > 0\}$, converges weakly to that of some pair (U^1, U^2), where

$$U_s^{1,\varepsilon} = \begin{cases} X_s^x, & s \le M^\varepsilon \\ \Delta, & s > M^\varepsilon \end{cases}$$

$$U_t^{2,\varepsilon} = \begin{cases} Y_t^y, & t \le N^\varepsilon \\ \Delta, & t > N^\varepsilon. \end{cases}$$

Since $V^\varepsilon = \Phi(U^{1,\varepsilon}, U^{2,\varepsilon})$ we have as before that the law of V^ε converges weakly to that of $V = \Phi(U^1, U^2)$.

To identify the law of V, let I be uniform on $[0,1]$, and independent of ${}_yZ^x$. Let

$$Z_s^1 = \begin{cases} Z_s, & s < I\varsigma(Z) \\ \Delta, & \text{otherwise} \end{cases}$$

$$Z_t^2 = \begin{cases} \hat{Z}_t, & t < (1-I)\varsigma(Z) \\ \Delta, & \text{otherwise.} \end{cases}$$

Since $V = \Phi(U^1, U^2)$ and $Z = \Phi(Z^1, Z^2)$, we need to show that the joint law of (U^1, U^2) is absolutely continuous with respect to that of (Z^1, Z^2), with Radon Nikodym derivative $\varsigma(Z)/{}_yE^x\varsigma(Z)$. Thus let $\sigma \in \mathcal{F}_s$ and $\tau \in \mathcal{F}_t$ be positive. Recall that $g(x,y) = G(x,y)_yE^x\varsigma(z)$. By Lemma (3.3),

$$\begin{aligned}
&E[\sigma(U^1)\tau(U^2), \varsigma(U^1) > s \text{ and } \varsigma(U^2) > t] \\
&= E^{x,y}[\sigma(X)\tau(Y)g(X_s, Y_t)/g(x,y)] \\
&= E^{x,y}\Big[\sigma(X)\tau(Y)\frac{G(X_s, Y_t)_{Y_t}E^{X_s}\varsigma(Z)}{G(x,y)_yE^x\varsigma(Z)}\Big] \\
&= {}_yE^x[\sigma(Z)\tau(\hat{Z})(\varsigma(Z) - s - t)/{}_yE^x\varsigma(Z), \varsigma(Z) > s + t] \\
&= {}_yE^x[\sigma(Z)\tau(\hat{Z})\varsigma(Z)/{}_yE^x\varsigma(Z), I\varsigma(Z) \in (s, \varsigma(Z) - t)] \\
&= {}_yE^x[\sigma(Z^1)\tau(Z^2)\varsigma(Z)/{}_yE^x\varsigma(Z), \varsigma(Z^1) > s \text{ and } \varsigma(Z^2) > t]
\end{aligned}$$

as required. □

It is now clear how to modify the argument above to obtain:

(3.6) *Corollary As $\varepsilon \downarrow 0$, the joint law of $(V_\varepsilon, M_\varepsilon)$ conditioned on $\{M_\varepsilon > 0\}$ converges weakly to that of a pair (V, M). $M/\varsigma(V)$ is uniform on the interval $[0,1]$ and is independent of V.*

§4. Asymptotics I

This section covers the proof of Theorem (1.4). The arguments given in §3 apply equally well in this context, once we show the following; [recall that X^x and Y^y are now Brownian motions, killed upon leaving a domain $D \subset \mathbb{R}^d$, and that $h_\varepsilon(x,y) = P(|X_t^x - Y_t^y| < \varepsilon$ for some $t < \varsigma(X^x) \wedge \varsigma(Y^y))$].

(4.1) *Proposition. Let $x, y \in D$, $x \neq y$.*

(a) If $d \geq 3$ then $h_\varepsilon(x,y) \sim C_d G(x,y)\varepsilon^{d-2}$.

(b) If $d = 2$ then $h_\varepsilon(x,y) \sim C_2 G(x,y)/\log(1/\varepsilon)$ provided D has a Green function.

These statements are proved in §4A and §4B respectively.

First we sketch the proof for $d \geq 3$. The analog of (a) with h_ε replaced by $\tilde{h}_\varepsilon(x,y) = P(G(X_t^x, Y_t^y) = \varepsilon^{-(d-2)}$ for some $t < \varsigma(X^x) \wedge \varsigma(Y^y))$ are easily proved. Observe that h_ε is harmonic on $D \times D - U_\varepsilon$, where U_ε is all (z_1, z_2) satisfying $|z_2 - z_1| \leq \varepsilon$, with boundary values 1 on U_ε and 0 elsewhere, while $\tilde{h}_\varepsilon$ differs only in that U_ε is replaced by a different neighborhood V_ε of the diagonal in $D \times D$, which is asymptotically the same except at the boundary of $D \times D$ (see figure 1). We must show that the parts near the boundary don't count.

Figure 1.

Formally, let $T = T_\varepsilon$ be $\inf\{t > 0;\ |X_t^x - Y_t^y| < \varepsilon,\ t < \varsigma(X^x) \wedge \varsigma(Y^y)\}$ and let D_n be relatively compact domains, $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. We prove the following.

(4.2) *Lemma. Under the conditions of Proposition (4.1),*

$$\lim_{n\to\infty} \limsup_{\varepsilon \downarrow 0} P^{x,y}(T_\varepsilon < \infty,\ X_{T_\varepsilon} \notin D_n)/P^{x,y}(T_\varepsilon < \infty) = 0.$$

To prove this we essentially write down a density for $X^x_{T_\varepsilon}$.

§4A $D \subset \mathbb{R}^d$, $d \geq 3$.

Warning: recall that P, G, etc. ... refer to BM killed upon leaving D. If we need (unkilled) BM on all of $\mathbb{R}^d$, we write P_0, G_0, etc. ...

Before proving Lemma (4.2) in dimensions greater than 2, we use it.

Proof of Proposition (4.1)(a).

Since $G_0 - G$ is continuous on $D \times D$, letting $\varepsilon \downarrow 0$ and then $n \to \infty$ we have, by Lemma (4.2), (writing $T = T_\varepsilon$)

$$\begin{aligned}
&E[G(X^x_T, Y^y_T), X^x_T \in D_n, T < \infty] \\
&\quad \sim E[G_0(X^x_T, Y^y_T), X^x_T \in D_n, T < \infty] \\
&\quad = C_d \varepsilon^{-(d-2)} P^{x,y}(X_T \in D_n, T < \infty) \\
&\quad C_d \sim \varepsilon^{-(d-2)} P^{x,y}(T < \infty) = C_d \varepsilon^{-(d-2)} h_\varepsilon(x,y).
\end{aligned}$$

That is, given $\theta > 0$ we can find $\varepsilon_0(\theta) = \varepsilon_0 > 0$ and $n(\varepsilon)$ such that if $\varepsilon < \varepsilon_0$ and $n > n(\varepsilon)$ then the ratio of the first to the last terms in this expression is within θ of 1. Also,

$$\begin{aligned}
&G(x,y) - E[G(X^x_T, Y^y_T), X^x_T \in D_n, T < \infty] \\
&\quad = E[G(X^x_T, Y^y_T), X^x_T \notin D_n, T < \infty] \\
&\quad \leq C_d \varepsilon^{-(d-2)} P^{x,y}(T < \infty)[P^{x,y}(X_T \notin D_n, T < \infty)/P^{x,y}(T < \infty)] \\
&\quad \leq G(x,y) P^{x,y}(X_T \notin D_n, T < \infty)/P^{x,y}(T < \infty) \\
&\quad \to 0 \text{ as } n \to \infty
\end{aligned}$$

by Lemma (4.2) again, showing the result. □

Proof of Lemma (4.2), $d \geq 3$

Write

$$\Theta(x,y;z) = \int_0^\infty p_t(x,z) p_t(y,z)\, dt$$

$$\Theta_0(x,y;z) = \int_0^\infty p^0_t(x,z) p^0_t(y,z)\, dt.$$

Since $p^0_t(x,z)p^0_t(y,z)$ is the transition density at (z,z) of Brownian motion in $\mathbb{R}^{2d}$ started at (x,y), $\Theta_0(\cdot,\cdot;z)$ is harmonic in (x,y) off any neighbourhood of (z,z). Thus

$$\Theta_0(x,y;z) = E^{x,y}[\Theta_0(X_T, Y_T; z)]. \tag{4A.1}$$

Let $|\gamma| = 1$. Then

$$\int_{\mathbb{R}^d} \Theta_0(0,\gamma;z)dz = \int_0^\infty p_{2t}(0,\gamma)dt = \frac{1}{2}G_0(0,\gamma) < \infty$$

so

$$\int_{|z|<k} \Theta_0(0,\gamma;z)dz \;/\; \int_{\mathbb{R}^d} \Theta_0(0,\gamma;z)dz \to 1 \text{ as } k \to \infty.$$

By scaling,

$$(4A.2) \qquad \int_{|z|<\delta} \Theta_0(0,\varepsilon\gamma;z)dz \;/\; \int_{\mathbb{R}^d} \Theta_0(0,\varepsilon\gamma;z)dz \to 1$$

as $\varepsilon \downarrow 0$, for each fixed $\delta > 0$.

Now fix n and let $\delta > 0$ be so small that $B(D_n, \delta) \subset D_{n+1}$. By (4A.1) we have

$$(4A.3) \qquad \int_{\mathbb{R}^d} \Theta_0(x,y;z)dz = [\int_{\mathbb{R}^d} \Theta_0(0,\varepsilon\gamma;z)dz]P_0^{x,y}(T<\infty), \text{ and}$$

$$\begin{aligned}\int_{D\setminus D_n} \Theta_0(x,y;z)dz &= E_0^{x,y}[\int_{D\setminus D_n} \Theta_0(X_T,Y_T;z)dz] \\ &\geq E_0^{x,y}[X_T \in D\setminus D_{n+1}, \int_{B(X_T,\delta)} \Theta_0(X_T,Y_T;z)dz] \\ &= [\int_{|z|<\delta} \Theta_0(0,\varepsilon\gamma;z)dz]P_0^{x,y}(X_T \in D\setminus D_{n+1}).\end{aligned}$$

Thus by (4A.2),

$$(4A.4) \qquad \lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} P_0^{x,y}(T<\infty, X_T \in D\setminus D_{n+1})/P_0^{x,y}(T<\infty) = 0.$$

To complete the proof, we need to show that

$$\liminf_{\varepsilon\downarrow 0} P^{x,y}(T<\varsigma)/P_0^{x,y}(T<\infty) > 0.$$

But

$$\begin{aligned}\frac{1}{2}G(x,y) &= \int_D \Theta(x,y;z)dz \\ &= E^{x,y}[\int_D \Theta(X_T,Y_T;z)dz, T<\varsigma] \\ &\leq (\int_{\mathbb{R}^d} \Theta_0(0,\varepsilon\gamma;z)dz)P^{x,y}(T<\varsigma),\end{aligned}$$

so by (4A.3)

$$P^{x,y}(T<\varsigma)/P_0^{x,y}(T<\infty) \geq G(x,y)/G_0(x,y) > 0. \qquad \square$$

Note. In (4A.4) we are not claiming that

$$\lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} P_0(|X_t^x - Y_t^y| < \varepsilon \text{ for some } t \text{ at which } X_t^x \in D\backslash D_{n+1})/P_0^{x,y}(T<\infty) = 0.$$

In fact, if $|\partial D| > 0$ this is in general false.

§4B $D \subset \mathbb{R}^2$

First, note that the proof of Lemma (4.2) in the last section works for $d = 2$, provided D is bounded. [If $D \subset B(0, r)$ say, then replace p_t^0, G_0, Θ_0 etc. by the transition density etc. for Brownian motion killed upon leaving $B(0, 2r)$. The main fact then is that

$$\frac{\inf\{\int_{B(x,\delta)} \Theta_0(x,y;z)dz; |x-y| \le \varepsilon, |x| \le r\}}{\sup\{\int_{B(0,2r)} \Theta_0(x,y;z)dz; |x-y| \le \varepsilon, |x| \le r\}} \longrightarrow 1$$

as $\varepsilon \downarrow 0$, for each $\delta > 0$].

This is the only honest ingredient in this section; we'll bootstrap our way from it to the proof of Proposition (4.1(b). We first handle that part of D near ∞ ((4B.5) below) and then, more generally, the part near ∂D.

Let $D \subset \mathbb{R}^2$ have a Green function G. Then

$$G(x,y) \ge C_2 \log^+(\delta/|x-y|) \text{ if } d(x,\partial D) > \delta,$$

so that for $T = T_\varepsilon$,

$$(4B.1) \qquad \begin{aligned} G(x,y) &= E[G(X_T^x, Y_T^y), T < \infty] \\ &\ge C_2 \log^+(\delta/\varepsilon) P^{x,y}(d(X_T, \partial D) > \delta, T < \infty). \end{aligned}$$

Let D_n be relatively compact domains, $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. In general, let σ_A be the first exit time from A, and let G_A be the Green function of A. Then by the first remark of this section,

$$(4B.2) \qquad G_{D_n}(x,y) = \lim_{\varepsilon\downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}(T < \sigma_{D_n}).$$

Let $\delta_n = d(D_n, \partial D) > 0$. We have that

$$(4B.3) \qquad \begin{aligned} G(x,y) &\ge \lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}(T<\infty, d(X_T, \partial D) > \delta) \\ &\ge \lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}(T < \infty, T < \sigma_{D_n}) \\ &= G(x,y). \end{aligned}$$

The first statement is by (4B.1), the second by the implication that $T < \sigma_{D_n} \Rightarrow d(X_T, \partial D) > \delta_n$, and the third by (4B.2) and the fact that $G = \lim G_{D_n}$.

Thus we have equality throughout, and hence also

$$(4B.4) \qquad \lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \log(1/\varepsilon) P^{x,y}(T<\infty, d(X_T,\partial D) > \delta_n, T \geq \sigma_{D_n}) = 0.$$

Since D has a Green function, there are two disjoint closed balls B_1 and B_2 so that $H_k = D \cup B_k$ has a Green function too, $k=1,2$. Apply (4B.4) to H_1 and H_2 rather than to D, to find $\delta < d(B_1,B_2)/2$ and $r>0$ such that

$$\limsup_{\varepsilon\downarrow 0} \log(1/\varepsilon) P^{x,y}(T<\infty, T\geq \sigma_{B(0,r)}, d(X_T < B_1) > \delta \text{ or } d(X_T,B_2)>\delta) < \theta.$$

Since $d(z,B_1) \mathrm{v} d(z,B_2) > \delta$ for every $z \in \mathbb{R}^2$ we have in fact that

$$(4B.5) \qquad \limsup_{\varepsilon\downarrow 0} \log(1/\varepsilon) P^{x,y}(T<\infty, T \geq \sigma_{B(0,r)}) < \theta.$$

Then apply our first remark once more, to find $\delta > 0$ such that

$$(4B.6) \qquad \limsup_{\varepsilon\downarrow 0} \log(1/\varepsilon) P^{x,y}(T<\infty, T<\sigma_{B(0,r)} \text{ but } d(X_T,\partial D) < \delta) < \theta.$$

Combining (4B.5) and (4B.6), we have that

$$(4B.7) \qquad \limsup_{\varepsilon\downarrow 0} \log(1/\varepsilon) P^{x,y}(T<\infty, X_T \in B(\partial D,\delta)) < 2\theta.$$

Together with (4B.3), this yields both Theorem (4.1)(b), and Lemma (4.2) (for $d=2$). □

§5. Asymptotics II

To complete the proof of Theorem (1.3), we must establish Lemmas (3.4) and (3.5). We'll do this in §5A, and will show the analogous results for domains in §5B and §5C. Since the arguments of §3 apply equally well to domains, this will also show Theorem (1.6).

We will restate the results to be shown, and, for ease of reference, separate the hypotheses into three cases;

(5.1) Proposition

For $x, x', y, y' \in D$, $x \neq y$, $x' \neq y'$ we have that $g_\varepsilon(x',y')/g_\varepsilon(x,y) \to g(x',y')/g(x,y)$ as $\varepsilon \downarrow 0$, provided

(a) $D = \mathbb{R}^d$, $d \geq 5$; or

(b) D is a domain in $\mathbb{R}^d$, $d \geq 5$; or

(c) D is a domain in $\mathbb{R}^4$ and $g(x,y) < \infty$.

Recall that S_ε and M_ε are respectively the first and last times s such that $|X_s^x - Y_t^y| < \varepsilon$ for some t.

(5.2) *Lemma. Fix* $x, y \in D$, $x \neq y$, $s > 0$. *Then* $P^{x,y}(S_\varepsilon < s < M_\varepsilon)/g_\varepsilon(x,y) \to 0$ *as* $\varepsilon \downarrow 0$, *provided*

(a) $D = \mathbb{R}^d$, $d \geq 5$; *or*

(b) D *is a domain in* $\mathbb{R}^d$, $d \geq 5$; *or*

(c) D *is a domain in* $\mathbb{R}^4$ *and* $g(x,y) < \infty$.

Our approach is as before. The basic proof is that of (a). It requires minor modification in case (b), with a new lemma (Lemma 5B.1) needed, to the effect that the part of D near ∂D doesn't matter. In case (c) we must also worry about that part of D near ∞ ((a) of Lemma 5C.4)).

Though we have not stated them above (as we did in Proposition (4.1)), the decay rates of $g_\varepsilon(x,y)$ will emerge in the course of the proof. They are ε^{d-4}, ε^{d-4}, $1/\log(1/\varepsilon)$ respectively. With a little more work involving a scaling argument, we can show in cases (a) and (b) that $g_\varepsilon(x,y) \sim c(d)\varepsilon^{d-4}g(x,y)$, where the constants do not depend on the domain. Since we do not use this result we omit the proof. We conjecture that in case (c), $g_\varepsilon(x,y) \sim cg(x,y)/\log(1/\varepsilon)$. We also have a completely different approach to these questions, which uses approximate Laplacians to compute the potential part of the superharmonic function $g_\varepsilon(\cdot, y)$, but it is significantly longer than the one given here.

§5A. $D = \mathbb{R}^d$, $d \geq 5$

(5A.1) *Lemma. Fix* $x \neq y$.

(a) $\varepsilon^{-(d-4)}g_\varepsilon(x,y)$ *is bounded in* ε, *above and away from zero.*

(b) *There is a constant c such that for each closed* K,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-(d-4)} P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in K) \leq c \int_K G(x,z)G(y,z)\,dz.$$

(c) *For each* $\rho > 0$,

$$P(\text{diameter}\{z; z = X_s^x \text{ for some } s, |z - Y_t^y| < \varepsilon \text{ for some } t\} > \rho) = o(\varepsilon^{d-4}) \text{ as } \varepsilon \downarrow 0.$$

Proof.

(a) Consider first the upper bound on g_ε. Let Λ_ε be the lattice εZ^d, and let $Q_\varepsilon = (-\varepsilon/2, \varepsilon/2)^d$. Set $b = 1 + \frac{\sqrt{d}}{2}$. We assume $2b\varepsilon < |x - y|$. Then

$$
\begin{aligned}
g_\varepsilon(x,y) &\le \sum_{z\in\Lambda\varepsilon} P(X^x \text{ and } Y^y \text{ hit } B(z,b\varepsilon)) \\
&\le c\varepsilon^{d-2} + \sum_{\substack{z\in\Lambda_\varepsilon \\ |z-x|,|z-y|>2b\varepsilon}} [b\varepsilon/|x-z|]^{d-2}[b\varepsilon/|x-z|]^{d-2} \\
&\le c\varepsilon^{d-2} + c\varepsilon^{d-4} \sum_{\substack{z\in\Lambda_\varepsilon \\ |z-x|,|z-y|>2b\varepsilon}} \int_{z+Q_\varepsilon} |x-z|^{-(d-2)}|y-z|^{-(d-2)}dz \\
&\le c\varepsilon^{d-2} + c\varepsilon^{d-4} g(x,y)
\end{aligned}
$$

giving the upper bound.

For the lower bound we assume, without loss of generality, that $x = (0,\dots,0)$, $y = (1,0,\dots,0)$ and ε is of the form 2^{-n} (so that we have $x, y \in \Lambda_\varepsilon$). Then for each $\beta \in (0,1/2)$, it follows as above that

$$
\begin{aligned}
g_\varepsilon(x,y) &\ge P(\cup_{z\in\Lambda_\varepsilon\setminus\{x,y\}}\{X^x \text{ and } Y^y \text{ hit } B(z,\beta\varepsilon)\}) \\
&\ge \sum_{z\in\Lambda_\varepsilon\setminus\{x,y\}} P(X^x \text{ and } Y^h \text{ hit } B(z,\beta\varepsilon)) \\
&- 1/2 \sum_{\substack{z,z'\in\Lambda_\varepsilon\setminus\{x,y\} \\ z\neq z'}} P(X^x \text{ and } Y^y \text{ hit both } B(z,\beta\varepsilon) \text{ and } B(z',\beta\varepsilon))
\end{aligned}
$$

Now $P(X^x$ and Y^y hit both $B(z,\beta\varepsilon)$ and $B(z',\beta\varepsilon))$ is majorized by $P_{zz}+P_{zz'}+P_{z'z}+P_{z'z'}$ where for example $P_{zz'}$ is the probability that X^x hits first $B(z,\beta\varepsilon)$ and then $B(z',\beta\varepsilon)$ and that Y^y hits first $B(z',\beta\varepsilon)$ and then $B(z,\beta\varepsilon)$. Each of these four probabilities is easily bounded, since the strong Markov property can be used. This, together with the above inequality, yields

$$
\begin{aligned}
g_\varepsilon(x,y) &\ge (\beta\varepsilon)^{2(d-2)} \sum_{z\in\Lambda_\varepsilon\setminus\{x,y\}} |x-z|^{-(d-2)}|y-z|^{-(d-2)} \\
&- \sum_{\substack{z,z'\in\Lambda_\varepsilon\setminus\{x,y\} \\ z\neq z'}} (\beta\varepsilon)^{4(d-2)}|x-z|^{-(d-2)}|z-z'|^{-2(d-2)}[(y-z)^{-(d-2)} \\
&\quad + |y-z'|^{-(d-2)}] \\
&\ge c\beta^{2(d-2)}\varepsilon^{d-4}\int_{\mathbb{R}^d} |x-z|^{-(d-2)}|y-z|^{-(d-2)}dz \\
&- c\beta^{4(d-2)}\varepsilon^{2d-8}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B(z,\varepsilon/2)} |x-z|^{-(d-2)}|z-z'|^{-2(d-2)}[|y-z|^{-(d-2)} \\
&+ |y-z'|^{-(d-2)}]dz'dz
\end{aligned}
$$

The second integral is $O(\varepsilon^{-(d-4)})$, so that $g_\varepsilon(x,y) \ge \varepsilon^{d-4}(c\beta^{2(d-2)}$ - $c\beta^{4(d-2)})$, where the two (different) constants c do not depend on β. Thus we may choose β so small that the second factor is strictly positive, giving the lower bound.

(b) Let $F(K,\varepsilon) = F = \{z \in \Lambda_\varepsilon;\ z + \overline{Q}_\varepsilon \cap K \neq \phi\}$. Then as above,

$$\begin{aligned}
P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in K) &\leq \sum_{z\in F} P(X^x \text{ and } Y^y \text{ hit } B(z,b\varepsilon)) \\
&\leq c\varepsilon^{d-2} \\
&+ c\varepsilon^{d-4} \sum_{\substack{z\in F \\ |z-x|,|z-y|>2b\varepsilon}} \int_{z+Q_\varepsilon} |x-w|^{-(d-2)}|y-w|^{-(d-2)}dw \\
&\leq c\varepsilon^{d-2} + c\varepsilon^{d-4}\int_{B(K,b\varepsilon)} |x-z|^{-(d-2)}|y-z|^{-(d-2)}dz
\end{aligned}$$

from which the result follows.

(c) For ε small, the probability in question is bounded by

$$(5A.2) \qquad \sum_{\substack{z,w\in\Lambda_\varepsilon \\ |z-w|\geq\rho/2}} P(X^x \text{ and } Y^y \text{ hit both } B(z,b\varepsilon) \text{ and } B(w,b\varepsilon))$$

and as in part (a), this is easily seen to be $o(\varepsilon^{d-4})$. □

Proof of Proposition (5.1)(a)

We will approximate $g_\varepsilon(x,y)$ by $\sum_{z\in\Lambda_\delta} P(|X^x_s - Y^y_t| < \varepsilon$ for some s,t at which $X^x_s \in z + Q_{\delta-\rho})$ for suitable δ, ρ.

Recall that $b = 1 + \sqrt{d}/2$, and set μ to be the uniform distribution on $\partial B(0,b)$. For $\varepsilon > 0$ let $\eta(\varepsilon) = P(|X^\mu_s - Y^\mu_t| < \varepsilon$ for some s,t at which $X^\mu_s \in Q_1)$.

The argument for the upper bound in (a) of Lemma (5A.1) shows that

$$(5A.3) \qquad \eta(\varepsilon) = O(\varepsilon^{d-4}) \text{ as } \varepsilon \downarrow 0.$$

Let $\theta > 0$ and choose $\lambda > 0$ so small that

$$(5A.4) \qquad \int_{B(x,2\lambda)\cup B(y,2\lambda)} |x-z|^{-(d-2)}|y-z|^{-(d-2)}dz \leq \theta.$$

Choose δ so that if $|z-x| > \lambda$ and $\delta' \leq \delta$ then the hitting density of X^x on $\partial B(z, b\delta')$ (with respect to normalized surface area) lies between $(1-\theta)(b\delta'/|x-z|)^{d-2}$ and $(1+\theta)(b\delta'/|x-z|)^{d-2}$. The same is of course true for Y^y if $|z-y| > \lambda$ as well, so that in this case

$$\begin{aligned}
(5A.5) \qquad &(1-\theta)^2(b\delta'/|x-z|)^{d-2}(b\delta'/|y-z|)^{d-2}\eta(\varepsilon/\delta') \\
&\leq P(|X^x_s - Y^y_t| < \varepsilon \text{ for some } s,t \text{ at which } X^x_s \in z + Q_{\delta'}) \\
&\leq (1+\theta)^2(b\delta'/|x-z|)^{d-2}(b\delta'/|y-z|)^{d-2}\eta(\varepsilon/\delta')
\end{aligned}$$

(for ε small). By choosing δ possibly even smaller, we can also guarantee that if $|z-x|$, $|z-y|>\lambda$ then

$$(5A.6)\qquad \begin{aligned}&(1-\theta)\delta^d|x-z|^{-(d-2)}|y-z|^{-(d-2)}\\ &\leq \int_{z+Q_\delta}|x-w|^{-(d-2)}|y-w|^{-(d-2)}dw\\ &\leq (1+\theta)\delta^d|x-z|^{-(d-2)}|y-z|^{-(d-2)}.\end{aligned}$$

For $\rho\in(0,\delta/2)$, let $K=\cup_{z\in\Lambda_\delta}z+(Q_\delta\backslash Q_{\delta-\rho})$. Choose ρ so small that

$$(5A.7)\qquad \int_K|x-w|^{-(d-2)}|y-w|^{-(d-2)}dw\leq\theta.$$

Write $A_z^\varepsilon=\{|X_s^x-Y_t^y|<\varepsilon \text{ for some } s,t \text{ at which } X_s\in z+Q_{\delta-\rho}\}$. Then

$$(5A.8)\qquad \begin{aligned}&\sum_{\substack{z\in\Lambda_\delta\\ |z-x|,|z-y|>\lambda}} P^{x,y}(A_z^\varepsilon)-\sum_{\substack{z,w\in\Lambda_\delta\\ z\neq w\\ |z-x|,|z-y|,\\ |w-x|,|w-y|>\lambda}} P^{x,y}(A_z^\varepsilon\cap A_w^\varepsilon)\\ &\leq g_\varepsilon(x,y)\\ &\leq \sum_{\substack{z\in\Lambda_\delta\\ |z-x|,|z-y|>\lambda}} P^{x,y}(A_z^\varepsilon)+P^{x,y}(S_\varepsilon<\infty,X_{S_\varepsilon}\in K)\\ &+P^{x,y}(S_\varepsilon<\infty,X_{S_\varepsilon}\in B(x,2\lambda)\cup B(y,2\lambda)).\end{aligned}$$

Looking first at the upper bound, we use (b) of Lemma (5A.1) [applied with (5A.4) and (5A.7)], then (5A.5) [with $\delta'=\delta-\rho$], (5A.6), and (5A.3) to see that

$$\begin{aligned}g_\varepsilon(x,y)&\leq \frac{(1+\theta)^2}{1-\theta}\frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)\cdot\\ &\quad\cdot\sum_{\substack{z\in\Lambda_\delta\\ |z-x|,|z-y|>\lambda}}\int_{z+Q_\delta}|x-w|^{-(d-2)}|y-w|^{-(d-2)}dw+c\theta\varepsilon^{d-4}\\ &\leq \frac{(1+\theta)^2}{1-\theta}\frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)g(x,y)+c\theta\varepsilon^{d-4}\\ &\leq \frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)g(x,y)+c\theta\varepsilon^{d-4}.\end{aligned}$$

Observe that the second sum in the left side of (5A.8) is bounded by a sum such as (5A.2), hence is $o(\varepsilon^{d-4})$. Thus we obtain a lower bound

$$g_\varepsilon(x,y)\geq(b(\delta-\rho))^{2(d-2)}\delta^{-d}\eta(\varepsilon/(\delta-\rho))g(x,y)-c\theta\varepsilon^{d-4},$$

now using (5A.5), (5A.6), (5A.4), (5A.7) and (5A.3). If $x'\neq y'$ we may choose δ and ρ to give the same conclusions for x' and y' as well. Thus, for small enough ε,

$$\left|\frac{g_\varepsilon(x,y)}{g(x,y)}-\frac{g_\varepsilon(x',y')}{g(x',y')}\right|\leq c\theta\varepsilon^{d-4}.$$

The conclusion of the proposition now follows, using the lower bound in (a) of Lemma (5A.1), and letting ε and then θ tend to zero. □

Proof of Lemma (5.2)(a)

Fix $x \neq y$ and let $\alpha > 0$. For $\delta > 0$, let σ_δ be the first time X^x leaves $B(x, \delta)$. Then (b) of Lemma (5A.1) lets us find δ so small that

$$(5A.9) \qquad P^{x,y}(S_\varepsilon \leq \sigma_\delta) \leq \alpha \varepsilon^{d-4} \text{ as } \varepsilon \downarrow 0.$$

Let β be a bound on the density of σ_δ. For any $\epsilon > 0$, it will also be a bound on the density of $\sigma_\delta + S_\varepsilon \circ \theta_{\sigma_\delta}$, conditioned on $S_\varepsilon \circ \theta_{\sigma_\delta} < \infty$. Thus, if we choose $\gamma < \alpha/\beta$, we obtain

$$(5A.10) \qquad P^{x,y}(\sigma_\delta < S_\varepsilon, S_\varepsilon \in (s-\gamma, s)) < \alpha P^{x,y}(S_\varepsilon < \infty) \leq c\alpha\varepsilon^{d-4} \text{ as } \varepsilon \downarrow 0.$$

By the strong Markov property of X at S_ε we can find ρ so small that

$$(5A.11) \qquad P^{x,y}(S_\varepsilon < \infty, |X_{S_\varepsilon + t} - X_{S_\varepsilon}| < \rho \text{ for some } t \geq \gamma) \leq \alpha\varepsilon^{d-4} \text{ as } \varepsilon \downarrow 0.$$

Finally, (c) of Lemma (5A.1) shows that for ε sufficiently small,

$$(5A.12) \qquad P^{x,y}(S_\varepsilon < \infty, |X_{S_\varepsilon} - X_{M_\varepsilon}| > \rho) \leq \alpha\varepsilon^{d-4}.$$

Putting (5A.9) - (5A.12) together shows that

$$P^{x,y}(S_\varepsilon < s < M_\varepsilon) \leq \alpha(3+c)\varepsilon^{d-4}$$

as required. □

§5B. $D \subset \mathbb{R}^d$, $d \geq 5$

Observe that (b) of Lemma (5.2) follows from (a) of that result (note that (a) of (5A.1) easily generalizes), so that only (b) of Proposition (5.1) need concern us here. Recall once more that $P_0^{x,y}$ refers to Brownian motion on all of $\mathbb{R}^d$, the notation $P^{x,y}$ now being reserved for Brownian motion killed upon leaving D.

As in §4 we must handle the part of D near ∂D. Let D_n be relatively compact domains, $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. If ∂D has Lebesgue measure zero then the following is a consequence of (b) of Lemma (5A.1).

(5B.1) *Lemma*

$$\lim_{n\to\infty} \limsup_{\varepsilon \downarrow 0} \varepsilon^{-(d-4)} P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n) = 0.$$

Proof: Part (b) of Lemma (5A.1) shows that

$$\lim_{r\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(d-4)} P_0^{x,y}(S_\varepsilon < \infty, |X_{S_\varepsilon}| \geq r) = 0.$$

Thus, it will suffice to show, for $r > 0$ fixed, that

$$(5B.2) \qquad \lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(d-4)} P_0^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in B(0,r) \cap D \backslash D_n) = 0.$$

Without loss of generality, we assume $B(x,3) \cup B(y,3) \subset D_n$ for each n. Let $\mu(dz, dw)$ be the probability on $B(x,1) \times B(y,1)$ under which z is uniform on $B(x,1)$ and $w = z + y - x$. Write

$$P_0^\mu = \int P_0^{z,w} \mu(dz, dw).$$

Let σ_ρ be the first time X leaves $B(x,\rho)$, and let τ_ρ be the first time Y leaves $B(y,\rho)$. In addition to

$$S_\varepsilon = \inf\{s > 0; |X_s - Y_t| < \varepsilon \text{ for some } t > 0\}$$

we will consider

$$\Sigma_\varepsilon = \inf\{s > \sigma_2; |X_s - Y_t| < \varepsilon \text{ for some } t > \tau_2\}.$$

We will compare the measures

$$\begin{aligned} a_\varepsilon(dz) &= P_0^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in dz) \\ b_\varepsilon(dz) &= P_0^\mu(S_\varepsilon < \infty, X_{S_\varepsilon} \in dz) \\ \alpha_\varepsilon(dz) &= P_0^{x,y}(\Sigma_\varepsilon < \infty, X_{\Sigma_\varepsilon} \in dz) \\ \beta_\varepsilon(dz) &= P_0^\mu(\Sigma_\varepsilon < \infty, X_{\Sigma_\varepsilon} \in dz). \end{aligned}$$

Let A be the event $\{|X_s - Y_t| < \varepsilon$ for some $X_s \in B(x,2) \cup B(y,2)$ and also for some $X_s \notin B(x,3) \cup B(y,3)\}$. By (c) of Lemma (5A.1), $P_0^\mu(A) = o(\varepsilon^{d-4})$. If B is any set which is disjoint from $B(x,3) \cup B(y,3)$, then clearly (for $\varepsilon < 1$),

$$\begin{aligned} a_\varepsilon(B) &\leq \alpha_\varepsilon(B) \leq a_\varepsilon(B) + P_0^{x,y}(A) \\ b_\varepsilon(B) &\leq \beta_\varepsilon(B) \leq b_\varepsilon(B) + P_0^\mu(A). \end{aligned}$$

The law of $(X_{\sigma_3}, Y_{\tau_3})$ under $P_0^{x,y}$ has a bounded density with respect to its law under P_0^μ. If C is such a bound, and B is a set as above, we therefore get an inequality

$$\alpha_\varepsilon(B) \leq C\beta_\varepsilon(B).$$

Finally, notice that the law of X_{S_ε} under $P_0^{x+z,y+z}$ is just a translate of its law under $P_0^{x,y}$. Thus b_ε is a convolution of a_ε with the uniform distribution on $B(x,1)$. Since $a_\varepsilon(\mathbb{R}^d) = O(\varepsilon^{d-4})$, we see that b_ε has a density which is bounded by $c\varepsilon^{d-4}$. Putting all this together, we see that for B as above,

$$\begin{aligned} a_\varepsilon(B) \leq \alpha_\varepsilon(B) \leq C\beta_\varepsilon(B) &\leq C(b_\varepsilon(B) + P_0^\mu(A)) \\ &\leq C|B|\varepsilon^{d-4} + o(\varepsilon^{d-4}) (|B| \text{ being the Lebesgue measure of } B). \end{aligned}$$

In particular, since $B(0,r)\cap D\backslash D_n \downarrow \emptyset$, we have $|B(0,r)\cap D\backslash D_n| \downarrow 0$, and (5B.2) follows. $\square$

The proof of Proposition (5.1)(b) now proceeds as in §5A, with only minor modifications: We choose n so large that $P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n) \le \theta\varepsilon^{d-4}$ and then approximate $g_\varepsilon(x,y)$ by

$$\sum_{\substack{z\in\Lambda_\delta\cap D_{n+1}\\ |z-x|,|z-y|>\lambda}} P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s,t \text{ at which } X_s^x \in z+Q_{\delta-\rho})$$

for λ, δ and ρ sufficiently small. As long as we restrict z to be in D_{n+1}, all the approximations of §5A carry over (now of course using the Green function of D rather than the Newtonian one). For example, in the left hand side of (5A.5) we need to replace $\eta(\varepsilon/\delta')$ by $\eta(\varepsilon/\delta', r/\delta')$ where

$\eta(\varepsilon,r) = P_0(|X_s^\mu - Y_t^\mu| < \varepsilon$ for some s,t such that $X_s^\mu \in Q_1$, and X^μ and Y^μ stay inside $B(0,r)$ before times s and t respectively)

and the particular choice of r is $d(D_{n+1}, \partial D)$. Then, to approximate $\eta(\varepsilon/\delta', r/\delta')$ by $\eta(\varepsilon/\delta')$ we need to choose δ so small that $r/\delta \ge r_0$, where

$$\eta(\varepsilon, r_0) \ge (1-\theta)\eta(\varepsilon) \text{ as } \varepsilon \downarrow 0.$$

§5C $D \subset \mathbb{R}^d$, $d=4$

In contrast to §5A and §5B, we must first spend some time deriving the order of magnitude of g_ε. Once that is settled, we'll prove a version of Lemmas (5A.1) and (5B.1) suitable for dimension four, after which the argument settles into the pattern set in the preceding two sections.

Let $B = B(0,1)$. The following uses an argument of Erdõs and Taylor (1960b) for the lower bound, and one of Lawler (1982) for the upper.

(5C.1) *Lemma. There are constants c, c' such that if ε is small and if $|x|,|y| \le 100$ and $|x-y| \ge 1/100$ then*

$$c \le (\log 1/\varepsilon) P_0(|X_s^x - Y_t^y| < \varepsilon \textit{ for some } s \textit{ at which } X_s^x \in B) \le c'.$$

Proof.

Lower bound: Set $\delta(\varepsilon) = \varepsilon(\log 1/\varepsilon)^{1/4}$ and let $\beta < 1/2$ remain to be chosen later. Then

$$\begin{aligned} &P_0(|X^x_s - Y^y_t| < \varepsilon \text{ for some } s \text{ at which } X^x_s \in B) \\ &\geq \sum_{z \in \Lambda_{\delta(\varepsilon)} \cap B(0,1/2)} P_0(X^x \text{ and } Y^y \text{ hit } B(z, \beta\varepsilon)) \\ &- \sum_{\substack{z,w \in \Lambda_{\delta(\varepsilon)} \cap B(0,1/2) \\ z \neq w}} P_0(X^x \text{ and } Y^y \text{ hit both} \\ &\qquad B(z,\beta\varepsilon) \text{ and } B(w,\beta\varepsilon)). \end{aligned}$$

Arguing as in Lemma (5A.1), the first sum is bounded below by $c(\beta\varepsilon/\delta(\varepsilon))^4 = c\beta^4 / \log(1/\varepsilon)$, and the second is bounded above by

$$\begin{aligned} &c(\beta\varepsilon/\delta(\varepsilon))^8 \int_B \int_{B \setminus B(0,\delta(\varepsilon)/2)} G_0(x,z)G_0(z,z')^2[G_0(y,z) + G_0(y,z')]dz'dz \\ &\leq c(\beta\varepsilon/\delta(\varepsilon))^8 \log(1/\delta(\varepsilon)) \leq c\beta^8/\log(1/\varepsilon). \end{aligned}$$

The two constants c may differ, but they don't depend on β, so that choosing β small enough yields the desired lower bound.

Upper bound: Let $|\gamma| = 1$ and consider

$$\Gamma_r = \int_0^\infty 1_{B(0,r)}(Y^\gamma_t)G(0, Y^\gamma_t)dt.$$

Then it is easily seen that

$$\begin{aligned} E_0\Gamma_r &= \int_{B(0,r)} G_0(0,z)G_0(\gamma,z)dz = 2\pi^2 \log r + O(1) \\ E_0\Gamma_r^2 &= 2\int_{B(0,r)} \int_{B(0,r)} G_0(0,z)G_0(0,w)G_0(\gamma,z)G_0(z,w)dwdz \\ &= (2\pi^2 \log r)^2 + O(\log r) \text{ as } r \to \infty. \end{aligned}$$

Thus $\mathrm{Var}(\Gamma_r) = O(\log r)$. Let

$$\Phi(\varepsilon) = \cup\{B(z, 2\varepsilon/3); Y^y \text{ hits } B(z,\varepsilon/3) \cap B, z \in \Lambda_{\varepsilon/3}\}.$$

Let σ be the first time X^x hits $\Phi(\varepsilon)$, and if it does so in $B(z, 2\varepsilon/3)$, let τ be the first time Y^y hits $B(z, \varepsilon/3)$. Then

$$\begin{aligned} \int_B G_0(x,z)G_0(y,z)dz &= E_0[\int_0^\infty 1_B(Y^y_t)G_0(x, Y^y_t)dt] \\ &= E_0[\sigma < \infty, \int_0^\infty 1_B(Y^y_t)G_0(X^x_\sigma, Y^y_t)dt] \\ &\geq \frac{\pi}{2}^2 (\log 1/\varepsilon)[P_0^{x,y}(\sigma < \infty) \\ &- P_0(\sigma < \infty, \int_\tau^\infty 1_B(Y^y_t)G_0(X^x_\sigma, Y^y_t)dt \leq \frac{\pi}{2}^2 \log 1/\varepsilon)]. \end{aligned}$$

Thus

$$
\begin{aligned}
&P_0(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s,t \text{ at which } Y_t^y \in B) \\
&\quad \le P_0^{x,y}(\sigma < \infty) \\
&\quad \le c[\log 1/\varepsilon]^{-1} + P_0\Big(\sigma < \infty, \int_\tau^\infty 1_B(Y_t^y) G_0(X_\sigma^x, Y_t^y)\,dt \le \frac{\pi}{2}^2 \log 1/\varepsilon\Big).
\end{aligned}
$$

The latter probability is majorized by

$$
\begin{aligned}
(5C.2) \qquad & O(\varepsilon^2) + \sum_{\substack{z \in \Lambda_{\varepsilon/3} \cap B(0,2) \\ |x-z|,|y-z| > 2\varepsilon}} [P_0(X^x \text{ hits } B(z, 2\varepsilon/3), Y^y \text{ hits } B(z,\varepsilon/3)] \cdot \\
& \cdot \sup_{\substack{x' \in \partial B(z,2\varepsilon/3) \\ y' \in \partial B(z,\varepsilon/3)}} P_0\Big(\int_0^\infty 1_B(Y_t^{y'}) G_0(x', Y_t^{y'})\,dt \le c \log 1/\varepsilon\Big)].
\end{aligned}
$$

As usual, we must worry about the boundary. Since the first factor in the above sum is $O(\varepsilon^4)$, we may peel off a strip about the boundary, of thickness $2\varepsilon^{1/2}$, and get that (5C.2) is

$$
O(\varepsilon^{1/2}) + O\Big(P_0\Big(\int_0^\infty 1_{B(x',\varepsilon^{1/2})}(Y_t^{y'}) G_0(x', Y_t^{y'})\,dt \le c \log 1/\varepsilon\Big)\Big)
$$

where x' and y' are fixed, $|x'-y'| = \varepsilon$. By scaling, this is

$$
O(\varepsilon^{1/2}) + O\Big(P_0\Big(\int_0^\infty 1_{B(0,\varepsilon^{-1/2})}(Y_t^\gamma) G_0(0, Y_t^\gamma)\,dt \le c \log(\varepsilon^{-1/2})\Big)\Big),
$$

and by Chebyshev's inequality and our computations involving Γ_r, this is $O(1/\log(1/\varepsilon))$ as required. □

For Γ open write G_Γ for the Green function of Γ, and τ_Γ for the time of first exit from Γ. We take the convention that $G_\Gamma(\cdot,\cdot)$ vanishes off $\Gamma \times \Gamma$. Let $B' = B(0,2)$, $B = B(0,4)$ and set $\eta = \inf\{G_B(x,z);\ x,z \in B'\} > 0$.

(5C.3) *Lemma. Given $\theta > 0$ there is an $\alpha = \alpha(\theta) > 0$ such that if Γ is an open subset of B, μ is a probability measure on $\overline{B}' \cap \Gamma$, and $G_\Gamma \mu(z) \le \eta/2$ on a subset of B' of Lebesgue measure at least θ then*

$$
P_0^\mu(\tau_\Gamma < \tau_B) \ge \alpha.
$$

Proof. Let A be a Borel subset of B. Then

$$
\begin{aligned}
\int_A G_B\mu(z)\,dz &= \int\Big(\int_A G(x,z)\,dz\Big)\mu(dx) \\
&= P_0^\mu\Big(\int_0^{\tau_B} 1_A(X_t)\,dt\Big) \\
&= P_0^\mu\Big(\int_0^{\tau_\Gamma} 1_A(X_t)\,dt\Big) + P_0^\mu\Big(\int_{\tau_\Gamma}^{\tau_B} 1_A(X_t)\,dt\Big) \\
&= \int_A G_\Gamma\mu(z)\,dz + \int_A G_B\nu(z)\,dz,
\end{aligned}
$$

where

$$\nu(dx) = P_0^{\mu}(\tau_\Gamma < \tau_B, X_{\tau_\Gamma} \in dx).$$

Thus, $G_B\mu(z) = G_\Gamma\mu(z) + G_B\nu(z)$. For $dz - a.a.\ z \in B$ (recall our convention that $G_\Gamma\mu = 0$ off Γ). In particular,

$$G_B\nu(z) \geq \eta - G_\Gamma\mu(z)$$

for $dz - a.a.\ z \in B'$. By our hypothesis on μ, we therefore have the $G_B\nu \geq \eta/2$ on a set of Lebesgue measure at least θ. Set

$$c = \sup_{x \in B} \int_B G_B(x,z)dz < \infty.$$

Then

$$c|\nu| \geq \int_B G_B\nu(z)dz \geq \eta\theta/2.$$

Taking α to be $\eta\theta/2c$, we get $|\nu| \geq \alpha$, as required. □

Now let D be a domain in $\mathbb{R}^4$ with $\int_D G(x,z)G(y,z)dz < \infty$ for some (and hence every) $x \neq y$. Aside from a new argument in part (a), the following result amounts to reproving Lemmas (5A.1) and (5B.1). Recall that we take relatively compact domains D_n with $\overline{D}_n \subset D_{n+1}$ and $D_n \uparrow D$.

(5C.4) *Lemma*

(a) $\lim_{r\to\infty} \limsup_{\varepsilon\downarrow 0} (\log 1/\varepsilon)\ P(|X_s^x - Y_t^y| < \varepsilon$ *for some* s,t *at which* $|X_s^x| > r) = 0$.

(b) For each $x \neq y$, $(\log 1/\varepsilon)g_\varepsilon(x,y)$ *is bounded in* ε, *above and away from zero.*

(c) There is a constant c *such that for each compact* $K \subset D$,

$$\limsup_{\varepsilon\downarrow 0}(\log 1/\varepsilon)P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in K) \leq c\int_K G(x,z)G(y,z)dz.$$

(d) For each $\rho > 0$,

$$P(\textit{diameter}\ \{z; z = X_s^x \textit{ for some } s, |z - Y_t^y| < \varepsilon \textit{ for some } t\} > \rho)$$
$$= o(1/(\log 1/\varepsilon)) \textit{ as } \varepsilon \downarrow 0.$$

(e) $\lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} (\log 1/\varepsilon)P^{x,y}\ (S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n) = 0$.

Proof.

(a)

$$\begin{aligned}
&P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s,t \text{ at which } |X_s^x| > r) \\
&\quad \le \sum_{\substack{z\in\Lambda_1 \\ |z|>r/2}} P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s,t \text{ at which } X_s^x \in B(z,1)) \\
&\quad \le c(\log 1/\varepsilon)^{-1} \sum_{\substack{z\in\Lambda_1 \\ |z|>r/2}} P(X^x \text{ and } Y^y \text{ hit } B(z,2))
\end{aligned}$$

the latter by Lemma (5C.1) and the strong Markov properties of X^x and Y^y, say at the first times they hit $B(z,1)$ and $B(z,2)$ respectively. Thus it will suffice to show that

$$\sum_{z\in\Lambda_1} P(X^x \text{ and } Y^y \text{ hit } B(z,2)) < \infty.$$

Let η be the constant of (5C.3). If μ and ν are measures on $B(z,2)\cap D$, and

$$\int_{(z+Q_1)} G_{B(z,4)\cap D}\mu(z') G_{B(z,4)\cap D}\nu(z') dz' \le \eta^2/8$$

(with the convention of (5C.3), that $G_\Gamma \mu = 0$ off Γ) then at least one of $G_{B(z,4)\cap D}\mu$ or $G_{B(z,4)\cap D}\nu$ is less than $\eta/2$ on a set of $z' \in z + Q_1$ of Lebesgue measure bigger than $\theta = 1/4$. Thus by Lemma (5C.3),

P_0^μ (X leaves D before it leaves $B(z,4)) \ge \alpha$ or

P_0^ν (Y leaves D before it leaves $B(z,4)) \ge \alpha$.

If we condition X^x and Y^y to hit $B(z,2)$ before leaving D, and take μ and ν to be their respective hitting distributions, then we see that for each $z \in \Lambda_1$, at least one of the following holds:

$\int_{(z+Q_1)\cap D} G(x,z')G(y,z')dz' \ge \frac{\eta^2}{8} P(X^x \text{ and } Y^y \text{ hit } B(z,2))$;

$P(X^x \text{ leaves } D \text{ in } B(z,4)) \ge \alpha P(X^x \text{ hits } B(z,2))$;

$P(Y^y \text{ leaves } D \text{ in } B(z,4)) \ge \alpha P(Y^y \text{ hits } B(z,2))$.

Let I, II, III respectively be the sets of $z \in \Lambda_1$ such that these conditions hold. Then

$$\begin{aligned}
&\sum_{z\in\Lambda_1} P(X^x \text{ and } Y^y \text{ hit } B(z,2)) \\
&\quad \le \sum_{z\in I} P(X^x \text{ and } Y^y \text{ hit } B(z,2)) + \sum_{z\in II} P(X^x \text{ hits } B(z,2)) \\
&\quad + \sum_{z\in III} P(Y^y \text{ hits } B(z,2)) \\
&\quad \le \frac{8}{\eta^2}\int_D G(x,z')G(y,z')dz' + 2c'/\alpha < \infty,
\end{aligned}$$

where $c' = \#\{z \in \Lambda_1;\ B(z,4) \text{ intersects } B(0,4)\}$.

(b) The upper bound in (b) has essentially just been shown. The lower bound follows from the argument of Lemma (5C.1), with trivial modifications.

(c) For $0 < \delta < d(K, \partial D)/2$, let $\kappa = \{z \in \Lambda_\delta;\ z + \overline{Q}_\delta \cap K \neq \emptyset\}$. Then

$$\begin{aligned} P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in K) &\leq \sum_{z \in \kappa} P(|X^x_s - Y^y_t| < \varepsilon \text{ for some } s \text{ at which } X^x_s \in B(z,\delta)) \\ &\leq c(\log \delta/\varepsilon)^{-1} \sum_{z \in \kappa} P(X^x \text{ and } Y^y \text{ hit } B(z, 2\delta)) \end{aligned}$$

by Lemma (5C.1) and the strong Markov property (as in part (a)). The conclusion of (c) now follows as in Lemma (5A.1), letting $\varepsilon \to 0$ and then $\delta \to 0$.

(d) For ε small, the probability in question is bounded by

$$\sum_{\substack{z,w \in \Lambda_{\rho'} \\ |z-w) > 2\rho'}} P(|X^x_s - Y^y_t| < \varepsilon \text{ for some } s,t \text{ at which } X^x_s \in B(z,\rho') \text{ and for some } s,t \text{ at which } X^x_s \in B(w,\rho'))$$

where $\rho' = \rho/5$ and we handle this as above.

(e) The argument is the same as that in Lemma (5B.1), except that instead of appealing to (b) of Lemma (5A.1), we use part (a) of the present lemma. □

At this point we have obtained four dimensional versions of all the lemmas used in the proof of (a) and (b) of Proposition (5.1) and Lemma (5.2). It should be clear that (c) now follows as well, with only minor modifications of the old arguments.

§6 Remarks

In dimension four, there are several forms of our condition

$$\int G(x,z)G(y,z)\,dz < \infty. \tag{6.1}$$

It is clearly equivalent to having

$$_yE^x\varsigma(z) < \infty \tag{6.2}$$

(and we used this in §3). Moreover it is not hard to see that it is equivalent to

$$E^{x,y}\Big[\int_0^{\varsigma(X)} \int_0^{\varsigma(Y)} 1_{B(0,\varepsilon)}(X_s - Y_t)\,ds\,dt\Big] < \infty \tag{6.3}$$

for some $\varepsilon > 0$ (in one direction, use the argument of (a) of Lemma (5C.1)). We have another proof of the dimension four result, that involves looking at X only at the times

$$\sigma_1 = \inf\{s > 0;\ |X_s| \in \{2^k;\ k \geq 0\}\}, \ldots$$
$$\sigma_{n+1} = \inf\{s > \sigma_n;\ |X_s| \in \{2^k;\ k \geq 0\} \backslash |X_{\sigma_n}|\}$$

(and similarly for Y). This argument shows that for $m_k = \#\{n;\ |X_{\sigma_n}| = 2^k\}$, the above conditions are equivalent to

$$\sum_{k=1}^{\infty} E^x[m_k^2] < \infty. \tag{6.4}$$

An interesting condition that is not equivalent is that $\varsigma(X^x) < \infty$ a.s.. Since $P_0(|X_s^x - Y_t^y| < \varepsilon$ for some $s, t = 1)$ (see for example LeGall (1986a)), and (6.1) implies that $g_\varepsilon(x,y) \to 0$ as $\varepsilon \downarrow 0$, this condition is implied by any of the above (and moreover is equivalent to $g_\varepsilon(x,y) \to 0$). The converse is not true; for example, consider a domain obtained by taking $\mathbb{R}^4$ and deleting concentric half-spheres whose radii increase so rapidly that X and Y expect to spend a long time within $1/n$ of each other during a transition between the n^{th} radii.

In fact, in terms of the m_k, the condition that $\varsigma(X^x) < \infty$ a.s. amounts to having only $E^x m_k \to 0$ as $k \to \infty$. In this case, the decay rate of $g_\varepsilon(x,y)$ need not be $1/\log(1/\varepsilon)$, as in general $\liminf(\log 1/\varepsilon) g_\varepsilon(x,y) \geq c \int G(x,z)G(y,z)dz$. Thus another condition equivalent to (6.1) is that

$$g_\varepsilon(x,y) = O(1/\log(1/\varepsilon)) \text{ for each } x, y. \tag{6.5}$$

Moreover, if (6.1) does hot hold and we condition on $S_\varepsilon < \infty$, then $|X_{S_\varepsilon}|$ gets increasingly large as $\varepsilon \downarrow 0$. That is, the conditioning forces the paths to leave compact sets before dying. Since we worked hard in §5 to control exactly this kind of behaviour, it is reassuring that this is where the result breaks down in general.

§7 Appendix: Weak Convergence of h-transforms

We turn to the weak convergence results needed in the foregoing. Let $D \subset \mathbb{R}^d$ be a domain with a Green function $G(x,y)$. Recall that Ω is the Skorokhod space based on $\overline{D} \cup \{\Delta\}$, where $\overline{D} = D \cup \{\partial\}$ is the one-point compactification of D and Δ is an additional isolated point. Weak convergence is that of probabilities on Ω or $\Omega \times \Omega$.

In the following result there are two obstacles; $\varsigma \to 0$ (hypothesized away by (7.2)), and oscillatory discontinuities at ς. The latter is eliminated by our choice of compactification [we haven't tried to verify the result for the Martin compactification, but it certainly fails for the Euclidean one (viz. Littlewood's crocodile)].

(7.1) *Lemma. Let (h_n) be strictly positive and superharmonic on D, and let $x \in D$. Then ${}_{h_n}P^x$ are tight provided $h_n(x) < \infty \forall n$, and*

$$\inf\{h_n(y)/h_n(x);\ y \in B(x,\delta)\} \to 1 \tag{7.2}$$

uniformly in n as $\delta \downarrow 0$.

Proof. Let ρ be a metric on $\overline{D} \cup \{\Delta\}$, compatible with its topology, and satisfying $\rho(y,z) \geq |y-z|$ for $y, z \in D$. Write $|\Delta - y| = \infty$ for $y \in D$. Fix ε, and R. For $\delta > 0$ let $K = \{y;\ \rho(y, \partial) > \varepsilon\}$, and define

$$\tau = \inf\{t;\ |X_s - X_t| > \varepsilon \text{ for some } s \in (t-\delta, t)\} \wedge R,$$

$$\tau' = \inf\{t > \tau;\ X_t \in K\} \wedge R.$$

If $\tau > \delta$, we may divide $[0, \tau)$ into equal intervals of length $\geq \delta/2$ on which the Euclidean increments of X are $\leq \varepsilon$. By enlarging the last such interval, and only requiring the ρ-increments to be $\leq 2\varepsilon$, the same is true on $[0, \tau')$ (now assuming only that $\tau' > \delta$). Thus we will have tightness (see Billingsley (1968) and Lindvall (1973)) provided the following hold (again, for x, ε, R fixed):

$$(7.3) \qquad {}_{h_n}P^x(\tau' < \varsigma \wedge R) \to 0 \text{ uniformly in } n, \text{ as } \delta \downarrow 0$$

$$(7.4) \qquad {}_{h_n}P^x(\varsigma \leq \delta) \to 0 \text{ uniformly in } n, \text{ as } \delta \downarrow 0.$$

To show (7.3), let h_n' be the réduite of h_n on K. Without loss of generality we assume $x \in K$. We have that

$$\begin{aligned} {}_{h_n}P^x(\tau' < \varsigma \wedge R) &= E^x[h_n(X_{\tau'})/h_n(x),\ \tau' < \varsigma \wedge R] \\ &= E^x[h_n'(X_{\tau'})/h_n'(x),\ \tau' < \varsigma \wedge R] \\ &= E^x[h_n'(X_\tau)/h_n'(x),\ \tau < \varsigma \wedge R]. \end{aligned}$$

Since $P^x(\tau < \varsigma \wedge R) \to 0$, we need only show that the $h_n'(X_\tau)/h_n'(x)$ are uniformly integrable (in n, δ).

First let $d \geq 3$. We build up to this by noting that if U is uniform on the unit sphere in $\mathbb{R}^d$, the following are uniformly integrable:

$$|U - z|^{-(d-2)}, z \in B(0,2),$$

[approximate $\partial B(0,1)$ by a disk, to get

$$E[|U-z|^{(d-2)}, |U-z|^{(d-2)} \geq n] \leq c \int_{B^{d-1}(0,1)} |y|^{-(d-2)} 1_{\{|y|^{-(d-2)} \geq cn\}} dy$$

which $\to 0$ as $n \to \infty$]

$$|x + rU - z|^{-(d-2)}/|x-z|^{-(d-2)}, z \in \mathbb{R}^d \text{ and } r > 0,$$

[translation, scaling, and then boundedness as $|z| \to \infty$]

$$|X_\tau^x - z|^{(d-2)}/|x-z|^{-(d-2)},\ z \in \mathbb{R}^d \text{ and } \varepsilon > 0, \text{ under } P_0^x$$

[condition on $|X_\tau^x - x|$, as X_τ^x is spherically symmetric about x under P_0^x]

$$G(X_\tau^x, z)/G(x,z), \ z \in K \text{ and } \delta > 0, \quad \text{under } P^x$$

[as $G(x,z) \geq c|x-z|^{-(d-2)}$ for $z \in K$, and $G(y,z) \leq |y-z|^{-(d-2)}$ for $y, z \in D$], and

$$h_n'(X_\tau^x)/h_n'(x), \ n > 0 \text{ and } \varepsilon > 0, \text{ under } P^x$$

[as $h_n'(y)/h_n'(x) = \int_K G(y,z)/G(x,z)\nu(dz)$ for some probability ν], showing (7.3). The same argument works for $d = 2$, replacing $|y-z|^{-(d-2)}$ by $k + \log^+(1/|y-z|)$ where k is so large that this dominates $G(y,z)$ for $y \in D$, $z \in K$.

To show (7.4), let $\sigma_r = \inf\{t;\ |X_t - x| \geq r\}$. It is enough to show that

(7.5) $$_{h_n}P^x(\sigma_r < \varsigma) \to 1 \text{ uniformly in } n \text{ as } r \downarrow 0,$$

and this follows easily from (7.2) since

$$_{h_n}P^x(\sigma_r < \varsigma) = \frac{1}{h_n(x)} E^x[h_n(X_{\sigma_r})].$$ □

(7.6) *Proposition. Let h_n and h be strictly positive and superharmonic on D, with $h_n \to h$ a.e., and $h_n(x) \to h(x) < \infty$. Then $_{h_n}P^x \to {}_hP^x$ weakly.*

Proof. Let $\hat{h}_n$ be the superharmonic regularization of $\inf\{h_m;\ m \geq n\}$. Then $\lim \hat{h}_n = \lim h_n = h$ ae, so since the $\hat{h}_n$ are increasing, in fact $\hat{h}_n \to h$ everywhere. Since $\hat{h}_n$ is lsc. Thus for every $\varepsilon > 0$ and for every n_0, we may find $\delta > 0$ such that

$$h_n(y) \geq \hat{h}_n(y) \geq h_{n_0}(x) - \varepsilon \text{ for each } y \in B(x,\delta) \text{ and } n \geq n_0.$$

since $h_n(x) \to h(x)$, and $\hat{h}_n(x) \to h(x)$ (7.2) follows immediately, showing tightness.

Thus we need only show convergence of the finite dimensional distributions. Let $\xi \in \mathcal{F}_t$. Then

$$_{h_n}E^x[\xi, \varsigma > t] = E^x[\xi h_n(X_t)/h_n(x)].$$

Without loss of generality, $\xi \leq 1_K(X_t)$ for some compact $K \subset D$, so that it suffices to show the uniform integrability of $1_K(X_t)h_n(X_t)/h_n(x)$ in n (t, K fixed). This follows as in Lemma (7.1). □

Recall from §2 that the 'paths' of (two parameter) h-bitransforms $_hU_{s,t}$ have the form

$$\omega_{s,t} = \begin{cases} (\omega_s^1, \omega_t^2), & (s,t) \in \varsigma \\ (\Delta, \Delta), & (s,t) \notin \varsigma \end{cases}$$

where $\varsigma \subset \mathbb{R}_t^2$ is a lower layer. We are interested in 'limits' with ς a rectangle $[0,\varsigma^1) \times [0,\varsigma^2)$, but we need to approach such objects by h-transforms with non-rectangular ς. Our approach will be to look at weak convergence (over $\Omega \times \Omega$) of the law of the pair $({}_hU^1, {}_hU^2)$ where ${}_hU_s^1$ is the first component of ${}_hU_{s,0}$ and ${}_hU_t^2$ is the second component of ${}_hU_{0,t}$. We'll do so under the condition that

$$(7.7) \qquad {}_{h_n}P^{x,y}(U_s^1 \neq \Delta,\ U_t^2 \neq \Delta \text{ but } (s,t) \notin \varsigma) \ \to\ 0 \text{ as } n \to \infty, \text{ for each } s,t > 0.$$

(7.8) *Proposition. Let $(h_n), h$ be strictly positive biexcessive functions on $D \times D$. Let ${}_{h_n}U^{x,y}$ and ${}_hU^{x,y}$ be bitransforms of bi-Brownian motion by h_n and h respectively. Assume (7.7). If $h_n \to h$ a.e. and $h_n(x,y) \to h(x,y)$ then the law of $({}_{h_n}U^1, {}_{h_n}U^2)$ converges weakly to that of $({}_hU^1, {}_hU^2)$.*

Proof. For tightness, it suffices to show separately that the laws of ${}_{h_n}U^1$ and ${}_{h_n}U^2$ are tight. Since they are (one parameter) transforms by $h_n(\cdot,y)$ and $h_n(x,\cdot)$, we need only check (7.2) for these. But biexcessive functions are excessive (Avanissian (1961)) so as in Proposition (7.6), in fact (7.2) holds for $h_n(\cdot,\cdot)$ (which is even stronger).

Turning to the finite dimensional joint distributions, let $\sigma \in \mathcal{F}_s$, $\tau \in \mathcal{F}t$. Then

$$\begin{aligned} &{}_{h_n}E^{x,y}[\sigma(U^1)\tau(U^2),\ U_s^1 \neq \Delta, U_t^2 \neq \Delta] \\ &\quad \geq {}_{h_n}E^{x,y}[\sigma(U^1)\tau(U^2),\ (s,t) \in \varsigma] \\ &\quad \geq {}_{h_n}E^{x,y}[\sigma(U^1)\tau(U^2),\ U_s^1 \neq \Delta, U_t^2 \neq \Delta] \\ &\quad - {}_{h_n}P^{x,y}[U_s^1 \neq \Delta, U_t^2 \neq \Delta \text{ but } (s,t) \notin \varsigma]. \end{aligned}$$

Moreover,

$$\begin{aligned} &{}_{h_n}E^{x,y}[\sigma(U^1)\tau(U^2), (s,t) \in \varsigma] \\ &\quad = E^{x,y}[\sigma(X)\tau(Y)h_n(X_s,Y_t)/h_n(x,y)] \end{aligned}$$

by definition. The integrand converges a.e. by assumption, and the appropriate tightness is shown as before (using Avanissian's theorem again). Combined with (7.7), this establishes the result. □

Note that some 'path' condition such as (7.7) is required. In fact, though this need not concern us here, simple examples show that h need not uniquely determine the law of an h-bitransform. If the h_n-bitransforms in question were known to have rectangular ς's then it turns out their laws would be determined, but in this case (7.7) would be trivially satisfied as well.

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