Stochastic Search in a Convex Region[1]

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SUMMARY

A stochastic search strategy is proposed for locating a possibly mobile target in a bounded, convex region of the plane. The strategy is asymptotically minimax as $\varepsilon \to 0$ with respect to the time required to get within ε of the target. The proof involves the study of first passages to time-dependent boundaries by a certain semi-Markov process.

1. Introduction

"Princess and Monster" [7] is a two-person game with two players who are restricted to a bounded, connected, two-dimensional region Ω. The Monster M has maximum speed 1, the Princess P has maximum speed $v < 1$. Neither player obtains any information about the position of the other until the distance between the two is $\leq \varepsilon$; at this time M captures P and the game ends. The payoff to P is the time elapsed before capture.

This game is a crude model for naval operations involving a surface ship M attempting to locate a submarine P with active sonar. Here the parameter 2ε (the sweep width) is typically small relative to the dimensions of Ω.

The P and M game is too complex to admit simple minimax strategies. Even if the continuum Ω is replaced by a finite set of points, and even if P's strategy is known to M,

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M's optimal strategy can only be determined approximately by a dynamic programming algorithm [2]. Nevertheless, for convex Ω Gal [4] [5] and Fitzgerald [3] have exhibited strategies for both players that are asymptotically minimax as $\varepsilon \to 0$, in the sense that the ratio of the expected payoff to the minimax value approaches 1 uniformly over the opponents' strategies. They have also shown that the minimax value $V(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} 2\varepsilon\, V(\varepsilon) = |\Omega|,$$

where $|\Omega|$ denotes the area of Ω.

P's strategy is easily described. Let $Q_1, Q_2, \ldots$ be an i.i.d. sequence of random points uniformly distributed in Ω. P starts at Q_1, stays there T time units, moves to Q_2 at full speed, stays there T time units, and so on. The parameter $T \to \infty$ as $\varepsilon \to 0$, but $\varepsilon T \to 0$ (e.g., $T = \varepsilon^{-1/2}$). It is not difficult to show that no matter what strategy M uses, the expected time to capture is at least $|\Omega|/2\varepsilon$ (approximately) when ε is small.

M's strategy is more complicated. The region Ω is partitioned into long, narrow (width $\varepsilon^{1/2}$) rectangles. M searches in one of these rectangles for a long time T, and so on (cf. [3] or [5] for details).

Despite its asymptotically minimax character, this strategy for M has the defect that when ε is small, M is confined to small subregions of Ω for very long periods of time. If the rules of the game were changed to allow P a small amount of partial information; e.g., if P were informed of M's position about once every $\varepsilon^{-1/3}$ time units, then she could elude it indefinitely. Thus, Gal's strategy for M is not robust to changes in the rules which might be relevant in naval operations.

The purpose of this paper is to present an alternative strategy for M that is (nearly) asymptotically minimax but does not suffer from the localization of Gal's strategy. Our plan is nearly independent of ε, and is robust to changes in the rules that allow P to have occasional partial information.

Let Ω be a compact, convex region in $\mathbb{R}^2$ with smooth (C^∞) boundary $\partial\Omega$, and let ν be the normalized arc-length measure on $\partial\Omega$ (i.e., $\int_{\partial\Omega} d\nu = 1$). Assume that Ω is strictly convex in the sense that any line tangent to $\partial\Omega$ meets Ω in only one point. Let $\Theta_1, \Theta_2, \ldots$ be i.i.d. random variables with distribution

$$Pr\{\Theta_i \in d\theta\} = \frac{1}{2}\sin\theta\, d\theta, \quad 0 \leq \theta \leq \pi. \tag{1.1}$$

Define a sequence of random points $P_0, P_1, P_2, \ldots$ on $\partial\Omega$ as follows. Let P_0 have distribution ν. Having defined P_i, draw the chord in Ω from P_i that makes an angle Θ_i with the tangent to $\partial\Omega$ at P_i, and define P_{i+1} to be the second point of intersection of this chord with $\partial\Omega$. The trajectory of M is obtained by following the chords $P_0P_1, P_1P_2, \ldots$ at unit speed. Let $X(t)$ denote the position of M at time $t \geq 0$.

PROPOSITION 1: The stochastic process $P_0, P_1, \ldots$ is a stationary, Harris recurrent Markov chain on $\partial\Omega$ with stationary distribution ν. For every continuous $f : \partial\Omega \to \mathbb{R}$

$$\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n} f(P_i) = \int_{\partial\Omega} f(p)\nu(dp) \quad \text{a.s.} \tag{1.2}$$

and

$$\lim_{n\to\infty} E\big(f(P_n)|P_0 = p_0\big) = \int_{\partial\Omega} f(p)\nu(dp) \tag{1.3}$$

for every $p_0 \in \partial\Omega$.

See [12] for the definition of Harris recurrence. Observe that the Markov chain P_n has a transition kernel $k(p, p')$ that is jointly continuous in p, p',

$$Pr\{P_1 \in dp' | P_0 = p\} = k(p, p')\nu(dp'). \tag{1.4}$$

Using this and the compactness of $\partial\Omega$ it can be shown that (1.3) holds uniformly in p_0.

PROPOSITION 2: The stochastic process $X(t)$, $t \geq 0$, is an ergodic semi-Markov process whose stationary distribution is the uniform distribution on Ω. In particular, if $f : \Omega \to \mathbb{R}$ is continuous, then

$$\lim_{t \to \infty} t^{-1} \int_0^t f\big(X(t)\big)dt = \int_\Omega f(\underset{\sim}{x})d\underset{\sim}{x}/|\Omega| \quad \text{a.s.} \tag{1.5}$$

and

$$\lim_{t \to \infty} E\Big(f\big(X(t)\big)|X(0),\ \Theta_1\Big) = \int_\Omega f(\underset{\sim}{x})d\underset{\sim}{x}/|\Omega| \tag{1.6}$$

In fact, (1.5) holds uniformly over all values of $X(0)$ and Θ_1. We shall not prove this, however.

For $\delta > 0$ let $\mathcal{F}_\delta$ be the set of continuously differentiable functions $y : \mathbb{R} \to \Omega$ such that $|y'(t)| \leq v$ and dist $\big(y(t), \partial\Omega\big) \geq \delta$ for all $t \in \mathbb{R}$. For $q \in \Omega$ let $y_q(t) \equiv q$. For $y \in \mathcal{F}_\delta$ define

$$\tau_\varepsilon(y) = \inf\{t \geq 0 :\ \text{dist}\ \big(y(t), X(t)\big) \leq \varepsilon\}.$$

THEOREM 1: For any $q \in \Omega \backslash \partial\Omega$ and $t \geq 0$,

$$\lim_{\varepsilon \to 0} 2\varepsilon|\Omega|^{-1} E\tau_\varepsilon(y_q) = 1 \tag{1.7}$$

and

$$\lim_{\varepsilon \to 0} Pr\left\{2\varepsilon|\Omega|^{-1}\tau_\varepsilon(y_q) \geq t\right\} = e^{-t}. \tag{1.8}$$

THEOREM 2: For any $\delta > 0$

$$\lim_{\varepsilon \to 0} \sup_{y \in \mathcal{F}_\delta} 2\varepsilon |\Omega|^{-1} E\tau_\varepsilon(y) = 1. \tag{1.9}$$

This shows that the strategy for M of following the random trajectory $X(t)$ is almost asymptotically minimax. Since, according to the rules of the game, P is not required to stay at least δ away from $\partial\Omega$, M should actually follow the trajectory $X(t)$ for a convex region containing Ω in its interior, with some reasonable modification at $\partial\Omega$ (there is no point in searching outside Ω). It is clear that one may construct an asymptotically minimax family of strategies for M by choosing convex regions that shrink to Ω as $\varepsilon \to 0$.

Our strategy does not suffer from the localization defect of Gal's strategy. Even if P is given the position and direction of M from time to time, she will not be able to predict its course for very long, in view of Proposition 2. Therefore, the strategy for M that we have described is not only fully efficient in the minimax sense, but also robust to partial information.

We shall assume throughout that the arc-length of $\partial\Omega$ is 1. We prove Propositions 1, 2 in sections 2–3; the rest of the paper is devoted to proofs of Theorems 1, 2.

The results we have stated are true without the assumption that Ω is convex; it is only necessary to assume that Ω is connected and compact, and that $\partial\Omega$ consists of finitely many smooth closed curves. The only substantial modifications needed for the non-convex case in the arguments below are in proving the irreducibility of P_n (Section 2) and the existence of the density f_2 (Section 6, Lemma 6).

The results of this paper were announced in [9]. A related problem was studied in [10].

2. Ergodic Properties of P_n

In this section we shall prove that $\{P_n\}_{n\geq 0}$ is an aperiodic, ν-irreducible, Harris recurrent Markov chain with stationary probability measure ν. Standard ergodic theorems ([12], Th. 4.3.6, Th. 6.2.8) then imply (1.2) and (1.3).

(A) *Recurrence*

It is apparent from (1.4) that the transition probabilities are absolutely continuous relative to ν; i.e.,

$$Pr\{P_n \in dp' | P_0 = p\} = k_n(p, p')\nu(dp'),$$

where $k_n(p,p')$ is jointly continuous in p, p'. Since $\partial\Omega$ is compact, for each $p \in \partial\Omega$ the sequence $k_n(p,p')\nu(dp')$ has a weakly convergent subsequence. Consequently, the potential kernel $\sum_{n=0}^{\infty} k_n(p,p')\nu(dp')$ is improper. It follows that if $\{P_n\}$ is ν-irreducible, then it is Harris recurrent ([12], Th. 3.2.6, Th. 3.2.7).

(B) *ν-Irreducibility and Aperiodicity*

The strict convexity of Ω and (1.1) guarantee that $k(p,p') > 0$ provided that $p \neq p'$. Therefore, if $p \in \partial\Omega$ and $F \subset \partial\Omega$ is a Borel measurable set with $\nu(F) > 0$, then $Pr\{P_1 \in F | P_0 = p\} > 0$. It follows that $\{P_n\}$ is ν-irreducible and aperiodic ([12], Defs. 3.2.1, 6.1.4).

(C) *Invariance of ν*

Let $\varphi \in \mathbb{K}$, and $p \in \partial\Omega$. ($\mathbb{K}$ denotes the circle group $\{\theta : 0 \leq \theta \leq 2\pi\}$). Consider the ray emanating from p at angle φ to the horizontal (horizontal means parallel to the x-axis);

define p' to be the second point of intersection with $\partial\Omega$, if there is one. Let ψ (respectively, $-\psi'$) be the angle between the tangent to $\partial\Omega$ at p (resp., p') and the horizontal. Let dp be an infinitesimal arc on $\partial\Omega$ centered at p, dp' the corresponding infinitesimal arc at p', and $\nu_\varphi(dp)$ the length of the projection of dp (or dp') onto a line perpendicular to the segment pp' (Figure 1). Let $d\varphi$ be an infinitesimal segment of $\mathbb{K}$ centered at φ. Finally, let Φ_i be the angle between the horizontal and the line $P_{i-1}P_i$.

(Figure 1 here)

LEMMA 1: For each $p \in \partial\Omega$ and $\varphi \in \mathbb{K}$ such that p' is defined

$$\begin{aligned} &Pr\{P_0 \in dp; \Phi_1 \in d\varphi\} \\ =&Pr\{P_0 \in dp'; \Phi_1 \in -d\varphi\} \\ =&\frac{1}{2}\nu_\varphi(dp)\,d\varphi \end{aligned} \tag{2.1}$$

PROOF: This follows immediately from (1.1) and the fact that

$$\begin{aligned} &\sin(\varphi+\psi)\nu(dp) \\ =&\sin(-\varphi+\psi')\nu(dp') \\ =&\nu_\varphi(dp). \end{aligned}$$ □

We argue now that P_1 has the same distribution as P_0; since the distribution of P_0 is ν, this proves that ν is invariant. If $P_0 \in dp'$ and $\Phi_1 = -\varphi$ then $P_1 \in dp$. Consequently, Lemma 1 implies that

$$Pr\{P_0 \in dp;\ \Phi_1 \in d\varphi\} = Pr\{P_1 \in dp;\ \Phi_1 \in -d\varphi\},$$

from which it follows that $Pr\{P_0 \in dp\} = Pr\{P_1 \in dp\}$.

3. Ergodic Properties of $X(t)$

Let T_i be the length of $P_{i-1}P_i$, $i = 1, 2, \ldots$; this is the time the trajectory $X(t)$ takes to traverse $P_{i-1}P_i$.

LEMMA 2: For each continuous function $f : \Omega \to \mathbb{R}$,

$$E \int_0^{T_1} f\big(X(t)\big)\,dt = \pi \int_\Omega f(\underset{\sim}{x})\,d\underset{\sim}{x}. \tag{3.1}$$

Before proving this we indicate how it implies (1.5) and (1.6). Define

$$\begin{aligned} \overline{T}_n =& T_1 + T_2 + \cdots + T_n, \quad n \geq 1, \\ \overline{T}_0 =& 0, \\ Y_n =& \int_{\overline{T}_{n-1}}^{\overline{T}_n} f\big(X(t)\big)\,dt, \quad n \geq 1. \end{aligned}$$

Observe that T_n and Y_n are functions of (P_{n-1}, P_n); by (3.1) and the stationarity of $\{P_n\}$

$$\begin{aligned} ET_n =& \pi|\Omega| \qquad \text{and} \\ EY_n =& \pi \int_\Omega f(\underset{\sim}{x})\,d\underset{\sim}{x}. \end{aligned}$$

The pointwise ergodic theorem for Harris recurrent Markov chains ([12], Th. 4.3.6) implies that

$$\begin{aligned} \overline{T}_n/n \to \pi|\Omega| \quad \text{a.s.} \quad \text{and} \\ \sum_{i=1}^n Y_i/n \to \pi \int_\Omega f(\underset{\sim}{x})\,d\underset{\sim}{x} \quad \text{a.s.} \end{aligned}$$

Therefore,

$$\int_0^{\overline{T}_n} f\big(X(t)\big)\,dt/\overline{T}_n \to \int_\Omega f(\underset{\sim}{x})\,d\underset{\sim}{x}\ /|\Omega| \quad \text{a.s. ;}$$

(1.5) follows routinely. The weak convergence (1.6) is an instance of a standard ergodic theorem for semi-Markov processes ([11], Th. 2; [1], Th. 3.1). It must be checked that T_1 has a nonarithmetic distribution ([1], Th. 3.1(iii)). For this it suffices to show that for each $\varepsilon > 0$, $Pr\{0 < T_1 = |P_0P_1| < \varepsilon\} > 0$: but this is obvious from the construction of $\{P_n\}$. This proves (1.6).

PROOF of Lemma 2: For each convex subset A of Ω define $L(A)$ to be the length of the line segment $P_0P_1 \cap A$, and define $\lambda(A) = EL(A)$. Observe that λ extends to a positive Borel measure. To prove (3.1) it suffices to show that λ is $\pi\times$ Lebesgue measure.

Recall that Φ_1 is the angle between the horizontal and P_0P_1. We shall prove that for each $\varphi \in \mathbb{K}$, $E\big(L(A)|\Phi_1 = \varphi\big)$ is a constant multiple of Lebesgue measure. By Lemma 1, for all $p \in \partial\Omega$ and $\varphi \in \mathbb{K}$ such that the ray $R(p,\varphi)$ emanating from p at angle φ to the horizontal points into Ω,

$$Pr\{P_0 \in dp|\Phi_1 = \varphi\} = (1/C_\varphi)\nu_\varphi(dp) \tag{3.2}$$

where C_φ is the length of the orthogonal projection of Ω onto a line perpendicular to $R(p,\varphi)$ (Figure 2). Let A be a square of side β contained in Ω with two sides parallel to $R(p,\varphi)$; define

$$B = \{p \in \partial\Omega : R(p,\varphi) \cap A \neq \phi\}$$

(Figure 2). If $\Phi_1 = \varphi$ and $P_0 \in B$ then $L(A) = \beta$, but if $\Phi_1 = \varphi$ and $P_0 \notin B$ then $L(A) = 0$. Hence, by (3.2)

$$E\big(L(A)|\Phi_1 = \varphi\big) = C_\varphi^{-1}\beta^2.$$

Since the collection of squares contained in Ω with two sides parallel to $R(p,\varphi)$ generates the Borel σ-algebra, it follows that

$$E\big(L(A)|\Phi_1 = \varphi\big) = C_\varphi^{-1}|A|$$

for all Borel sets $A \subset \Omega$.

Now $\lambda(A)$ may be obtained by integrating over $\varphi \in \mathbb{K}$. Observe that (2.1) and (3.2) imply

$$Pr\{\Phi_1 \in d\varphi\} = (C_\varphi/2)d\varphi, \varphi \in \mathbb{K}.$$

Consequently,

$$\lambda(A) = EL(A) = 2\pi|A|/2 = \pi|A|. \qquad \square$$

(Figure 2 here)

4. An Upper Bound on the Probability of a Hit

One expects that as $\varepsilon \to 0$, $\tau_\varepsilon(y) \to \infty$ in probability for $y \in \mathcal{F}_\delta$. The following lemma provides an estimate for the rate of convergence.

LEMMA 3: For each $\delta > 0$ and $C > 0$ there exists $K < \infty$ such that for every $y \in \mathcal{F}_\delta$, $p_0 \in \partial\Omega$, and $\varepsilon > 0$,

$$Pr\{\tau_\varepsilon(y) \leq C|P_0 = p_0\} \leq K\varepsilon. \tag{4.1}$$

PROOF: First we show that there exists $K' < \infty$ such that for every $y \in \mathcal{F}_\delta$, $p_0 \in \partial\Omega$, and $\varepsilon > 0$

$$Pr\{\tau_\varepsilon(y) \leq T_1|P_0 = p_0\} \leq K'\varepsilon. \tag{4.2}$$

Recall that if $y \in \mathcal{F}_\delta$ then dist $(y(t), \partial\Omega) \geq \delta$ and $|y'(t)| \leq v < 1$ for all t. In order for the event $\{\tau_\varepsilon(y) \leq T_1\}$ to occur with $\tau_\varepsilon(y) = t$ it must be the case that dist $(y(t), P_0) \in [t - \varepsilon, t + \varepsilon]$, because $X(\cdot)$ traverses the chord P_0P_1 at unit velocity. Say that $y(\cdot)$ is "in range" of $p_0 \in \partial\Omega$ at time $t > 0$ if dist $(y(t), p_0) \in [t - \varepsilon, t + \varepsilon]$. Since $|y'(t)| \leq v < 1$ for all t, the set of times t when y is in range of p_0 is contained in an interval of length $\leq 2\varepsilon(1 - v)^{-1}$. Thus $\{z \in \Omega\text{: dist } (z, y(t)) \leq \varepsilon \text{ at some } t \text{ when } y \text{ is in range of } p_0\}$ is contained in a disc D of radius $\varepsilon + 2\varepsilon v(1 - v)^{-1}$ whose center z_0 satisfies dist $(z_0, p_0) \geq \delta$. The set of $\theta \in (0, \pi)$ such that the ray emanating from p_0 at angle θ to $\partial\Omega$ intersects D is an interval whose length ℓ satisfies

$$\begin{aligned} \sin(\ell/2) =& \varepsilon\big(1 + 2v(1 - v)^{-1}\big)/ \text{ dist } (z_0, p_0) \\ \leq& \varepsilon\big(1 + 2v(1 - v)^{-1}\big)/\delta. \end{aligned}$$

Since $\tau_\varepsilon(y) \leq T_1$ occurs only if Θ_0 is in this set, since the probability density function of Θ_0 is bounded, and since $\sin x \sim x$ as $x \to 0$, this proves (4.2).

For $p \in \partial\Omega$ define

$$f(p) = Pr\{\tau_\varepsilon(y) \leq T_1 | P_0 = p\};$$

by (4.2), $f(p) \leq K'\varepsilon$ for all $p \in \partial\Omega$. By the Markov property of $\{P_n\}$,

$$\begin{aligned} Pr\{\tau_\varepsilon(y) \leq& C|P_0 = p_0\} \\ =& \sum_{n=1}^{\infty} Pr\left\{\sum_{i=1}^{n-1} T_i < \tau_\varepsilon(y) \leq \sum_{i=1}^{n} T_i \leq C|P_0 = p_0\right\} \\ \leq& \sum_{n=0}^{\infty} f(P_n) Pr\left\{\sum_{i=1}^{n} T_i \leq C|P_0 = p_0\right\} \\ \leq& K'\varepsilon \sum_{n=0}^{\infty} Pr\left\{\sum_{i=1}^{n} T_i \leq C|P_0 = p_0\right\} \end{aligned}$$

Therefore, to prove (4.1) it suffices to show that $E(N_C|P_0 = p)$ is uniformly bounded for

$p \in \partial\Omega$, where

$$N_C = \inf\{n : \sum_{i=1}^{n} T_i > C\}.$$

It is easily proved that there exist $t > 0$, $\alpha > 0$ such that $Pr\{T_i \geq t | P_{i-1} = p\} \geq \alpha$ for every $p \in \partial\Omega$ and $i = 1, 2, \ldots$, and it follows by routine arguments that $E(N_C | P_0 = p)$ is uniformly bounded in p.

$\square$

5. A Lower Bound on the Expected Number of Hits

Recall that T_i is the length of the line segment $P_{i-1}P_i$. The successive visits to $\partial\Omega$ by $X(t)$ occur at the points P_n at the times $\sum_{i=1}^{n} T_i$; the renewal measure U for the Markov renewal process consisting of these successive visits is

$$U(ds, dp|p_0) = \sum_{n=0}^{\infty} Pr\left\{\sum_{i=1}^{n} T_i \in ds; P_n \in dp | P_0 = p_0\right\} \tag{5.1}$$

By the Radon-Nikodym and Lebesgue decomposition theorems

$$U(ds, dp|p_0) = u(s, p|p_0) ds\nu(dp) + U^{\perp}(ds, dp|p_0),$$

where $u(s, p|p_0)$ is nonnegative and locally integrable relative to $ds\nu(dp)$, and $U^{\perp}(ds, dp|p_0)$ is singular relative to $ds\nu(dp)$.

LEMMA 4: For any finite interval $J \subset \mathbb{R}$ there exists $K < \infty$ such that

$$U\big((s + J) \times \partial\Omega | p_0\big) \leq K \tag{5.2}$$

for all $s \in \mathbb{R}$, $p_0 \in \partial\Omega$. Moreover for each arc $A \subset \partial\Omega$,

$$\lim_{s \to \infty} U\big((s + J) \times A | p_0\big) = |J| \nu(A) (\pi |\Omega|)^{-1} \tag{5.3}$$

and

$$\lim_{s\to\infty} U^{\perp}((s+J)\times\partial\Omega|p_0)=0 \tag{5.4}$$

uniformly for all $p_0\in\partial\Omega$. *Finally, for each* $\beta>0$ *there exists* $s_\beta<\infty$ *such that if* $s\geq s_\beta$ *then*

$$u(s,p|p_0)\geq(\pi|\Omega|)^{-1}(1-\beta) \tag{5.5}$$

for all $p,p_0\in\partial\Omega$.

The proof will be given in section 6. Except for (5.4) and (5.5), which will be needed for Lemmas 9–10, the result follows from standard results in Markov renewal theory.

To get a handle on the distribution of $\tau_\varepsilon(y)$ we shall obtain an asymptotic lower bound for the expected number of chords of $X(t)$ that pass within ε of the path $y(t)$ during a fixed period of time. Each chord of $X(t)$ is determined by a time $s\in\mathbb{R}$ (the time at which $X(t)$ begins traversing the chord), a point $p\in\partial\Omega$ (the initial point), and an angle $p\in\partial\Omega$ (the angle the chord makes with $\partial\Omega$ at p). Consider the Markov renewal process consisting of the triples (s,p,θ) specifying the successive chords of $X(t)$. By Lemma 4, the asymptotic form of the renewal measure is a scalar multiple of the measure μ on $\mathbb{R}\times\partial\Omega\times(0,\pi)$ defined by

$$\mu(ds,\ dp,\ d\theta)=\frac{1}{2}\sin\theta ds\ \nu(dp)d\theta.$$

For each $(s,p,\theta)\in\mathbb{R}\times\partial\Omega\times(0,\pi)$ and $t\geq s$ let $\gamma(t;s,p,\theta)$ be the point at distance $t-s$ from p on the ray originating at p at angle θ to $\partial\Omega$. Let $y(t)$ be a continuous path in

Ω, and for $\alpha,\ \varepsilon > 0$ define

$$F_{\varepsilon,\alpha}(y) = \{(s,p,\theta) \in \mathbb{R} \times \partial\Omega \times (0,\pi) : \text{ for some } t \in [0,\alpha)$$
$$\text{dist } \big(y(t), \gamma(t;s,p,\theta)\big) = \varepsilon \text{ and dist } \big(y(t'), \gamma(t';s,p,\theta)\big) > \varepsilon$$
$$\text{for all } t' \in [s,t)\}.$$

The key to Theorems 1, 2 is the following

LEMMA 5: For each $\delta > 0$, $\alpha > 0$, and $0 < \varepsilon < \delta$, if $y \in \mathcal{F}_\delta$ then

$$\mu\big(F_{\varepsilon,\alpha}(y)\big) \geq 2\pi\alpha\varepsilon - 4\pi\varepsilon^2. \tag{5.5}$$

Furthermore, for $q \in \Omega$ such that dist $(q, \partial\Omega) \geq \delta$,

$$\mu\big(F_{\varepsilon,\alpha}(y_q)\big) = 2\pi\alpha\varepsilon. \tag{5.6}$$

PROOF: Fix $p \in \partial\Omega$ and $\theta \in (0,\pi)$. Consider the ray emanating from p at angle θ to $\partial\Omega$: let φ be the angle between the horizontal and this ray. In the coordinate system (s,p,φ) the measure μ takes the form

$$\mu(ds, dp, d\varphi) = \frac{1}{2} r(p,\varphi) ds \nu_\varphi(dp) d\varphi \tag{5.7}$$

where $r(p,\varphi) = 1$ if the ray $\gamma^*(t;s,p,\varphi)$ points into Ω and $r(p,\varphi) = 0$ if $\gamma^*(t;s,p,\varphi)$ points out of Ω (cf. Lemma 1). For each $\varphi \in \mathbb{K}$ define a measure λ_φ on $\mathbb{R} \times \partial\Omega$ by

$$\lambda_\varphi(ds, dp) = r(p,\varphi) ds \nu_\varphi(dp).$$

Let $\varphi \in \mathbb{K}$, $s \in \mathbb{R}$, and $p \in \partial\Omega$; for $t \geq s$ define $\gamma^*(t;s,p,\varphi)$ to be the point at distance $t - s$ from p on the ray originating at p at angle φ to the horizontal. For $y \in \mathcal{F}_\delta$ and α,

$\varepsilon > 0$ define

$$F^{\varphi}_{\varepsilon,\alpha}(y) = \{(s,p,) \in \mathbb{R} \times \partial\Omega : \text{ for some } t \in [0,\alpha)$$
$$\text{dist } \big(y(t); \gamma^*(t; s, p, \varphi)\big) = \varepsilon \text{ and}$$
$$\text{dist } \big(y(t'); \gamma^*(t'; s, p, \varphi)\big) > \varepsilon$$
$$\text{for all } t' \in [s,t)\}.$$

By (5.7) and Fubini's Theorem,

$$\mu\big(F_{\varepsilon,\alpha}(y)\big) = \int_{\varphi\in\mathbb{K}} \frac{1}{2}\lambda_{\varphi}\big(F^{\varphi}_{\varepsilon,\alpha}(y)\big)\, d\varphi.$$

To prove (5.5) and (5.6) we shall show that for each $\varphi \in \mathbb{K}$ and $y \in \mathcal{F}_\delta$

$$\lambda_{\varphi}(F^{\varphi}_{\varepsilon,\alpha}) + \lambda_{-\varphi}(F^{-\varphi}_{\varepsilon,\alpha}) \geq 2(2\alpha\varepsilon - 4\varepsilon^2), \tag{5.8}$$

and that if $y = y_q$ for some $q \in \partial\Omega$ such that dist $(q, \partial\Omega) > \varepsilon$,

$$\lambda_{\varphi}(F^{\varphi}_{\varepsilon,\alpha}) = 2\alpha\varepsilon. \tag{5.9}$$

This suffices, since $\int_{\mathbb{K}} d\varphi = 2\pi$.

Recall that for each arc $A \subset \partial\Omega$, $\nu_\varphi(A)$ is the length of the orthogonal projection of A (counting multiplicities) onto a line at angle $\varphi + \pi/2$ to the horizontal. Imagine a (2-dimensional) incompressible fluid with density 1 flowing through $\mathbb{R}^2$ with velocity 1 in the direction φ: then $\nu_\varphi(A)$ is the amount of fluid flowing through A in a unit of time. Consequently, for $-\infty < \alpha_1 < \alpha_2 < \infty$, $\lambda_\varphi([\alpha_1, \alpha_2) \times A) = (\alpha_2 - \alpha_1)\nu_\varphi(A)$ is the amount of fluid that flows through during the time interval $[\alpha_1, \alpha_2)$.

Consider $y \in \mathcal{F}_\delta$. The set $F^{\varphi}_{\varepsilon,\alpha}(y)$ consists of those $(s,p) \in \mathbb{R} \times \partial\Omega$ such that the ray $\gamma^*(t; s, p, \varphi)$ comes within ε of $y(t)$ for the first time at some $t \in [0, \alpha)$. It follows that

$\lambda_\varphi\big(F^\varphi_{\varepsilon,\alpha}(y)\big)$ is the amount of fluid that comes within ε of $y(t)$ for the first time at some $t \in [0, \alpha)$, since the fluid is incompressible.

Suppose $y(t) \equiv q$ for some $q \in \Omega$. Then $\lambda_\varphi\big(F^\varphi_{\varepsilon,\alpha}(y)\big)$ is the amount of fluid that enters the circle $D_\varepsilon(q) = \{q' \in \mathbb{R}^2\colon \text{dist } (q', q) \le \varepsilon\}$ at some $t \in [0, \alpha)$. (Fluid can only enter $D_\varepsilon(q)$ once, because the direction of the flow is constant and $D_\varepsilon(q)$ is convex.) The amount of fluid entering $D_\varepsilon(q)$ during $[0, \alpha)$ equals the area occupied by that fluid at time 0, since the flow is incompressible and the density of the fluid is 1. It is easily seen that this area is $2\alpha\varepsilon$ (Fig. 3): this proves (5.9).

Figure 3 here

Consider now the general case $y \in \mathcal{F}_\delta$. To calculate $\lambda_\varphi\big(F^\varphi_{\varepsilon,\alpha}(y)\big)$ we must calculate the amount of fluid passing within ε of $y(t)$ for the first time at some $t \in [0, \alpha)$. Think of $y(t)$ as the locus of a point moving through the fluid; imagine that this point is the center of a rigid disc of radius ε that moves with the point. Consider the reference frame R_φ in which the fluid is at rest. The amount of fluid that enters the disc for the first time during $[0, \alpha)$ is the area covered for the first time by the disc during $[0, \alpha)$ in the reference frame R_φ. Recall that in the original reference frame R the velocity of the disc is $|y'(t)| \le v < 1$ and the velocity of the fluid is 1 (in the direction φ). Consequently, if $v_\varphi(t)$ is the φ-component of the disc's velocity vector in the reference frame R_φ at time t, then

$$v_\varphi(t) \ge 1 - v > 0.$$

It follows that the area covered for the first time by the disc (in the reference frame R_φ)

during $[0, \alpha)$ is at least

$$2\varepsilon \int_0^\alpha v_\varphi(t)\,dt$$

(Fig. 4). This, then, is a lower bound for $\lambda_\varphi(F^\varphi_{\varepsilon,\alpha}(y))$. To prove (5.8) we merely note that

$$\nu_\varphi(t) + \nu_{-\varphi}(t) \equiv 2,$$

because the relative velocity of R_φ with respect to $R_{-\varphi}$ is 2. □

(Figure 4 here)

6. The Renewal Measure

In this section we prove Lemma 4. The primary difficulty is the lower bound (5.5); for this it is necessary to establish first that the renewal measure has a nontrivial absolutely continuous component.

LEMMA 6: There exists a nonnegative, continuous function $f_2(s, p|p_0)$, where $s \in [0, \infty)$ and $p, p_0 \in \partial\Omega$, such that for every $p_0 \in \Omega$

$$f_2(s, p|p_0) > 0 \tag{6.1}$$

for some $s > 0$, $p \in \partial\Omega$, and such that for all $s > 0$ and $p, p_0 \in \partial\Omega$,

$$Pr\{T_1 + T_2 \in ds;\ P_2 \in dp|P_0 = p_0\} \geq f_2(s, p|p_0)ds\nu(dp). \tag{6.2}$$

PROOF: It suffices to prove that for each $p_* \in \partial\Omega$ there exists a nonnegative continuous function $f_{p_*}(s, p|p_0)$ satisfying (6.2) and such that $f_{p_*}(s, p|p_*) > 0$ for some $s \in \mathbb{R}$, $p \in \partial\Omega$.

For if this is true, there exists a neighborhood $N(p_*)$ of p_* in $\partial\Omega$ such that $f_{p_*}(s,p|p_0) > 0$ for all $p_0 \in N(p_*)$; by the compactness of $\partial\Omega$ there exist $p_*^{(1)}, p_*^{(2)}, \ldots, p_*^{(n)}$ such that $\partial\Omega = \cup_{i=1}^n N(p_*^{(i)})$, and hence $f_2(s,p|p_0) = \max_{1\le i\le n} f_{p_*^{(i)}}$ has properties (6.1) and (6.2).

Fix $p_*, p \in \partial\Omega$; consider the function $g : \partial\Omega \to \mathbb{R}$ defined by $g(p') = |p_*p'| + |p'p|$, where $|p_*p'|, |p'p|$ are the lengths of the line segments $p_*p', p'p$. Clearly, g is C^∞, since $\partial\Omega$ is smooth; moreover, g is not constant, because if $p' \neq p_*, p' \neq p$ then $g(p') > g(p)$, by the convexity of Ω. Therefore, by Sard's theorem ([6], sec. 1.7) there exists $p \in \partial\Omega$ such that $p' \neq p_*$, $p' \neq p$, and such that the differential of g at p' is nonsingular.

Consider now the map $G_{p_*} : \partial\Omega\times\partial\Omega \to \mathbb{R}\times\partial\Omega$ defined by $G_{p_*}(p',p) = (|p_*p'|+|p'\ |, p)$. By the argument of the preceding paragraph, for each $p \in \partial\Omega$ there exists $p' \in \partial\Omega$, $p' \neq p_*, p$, such that the differential of G_{p_*} at (p',p) is nonsingular. Since $G_{p_*}(p',p)$ is C^∞ in (p_*,p',p), the Inverse Function Theorem implies that there exist a neighborhood $\mathcal{N}$ of p_* and a neighborhood $\mathcal{M}$ of (p',p) such that for each $p_0 \in \mathcal{N}$ the differential of G_{p_*} is nonsingular in $\mathcal{M}$ and G_{p_0} is a diffeomorphism of $\mathcal{M}$ onto a neighborhood of $G_{p_*}(p',p)$.

Let $h_1 : \partial\Omega \to [0,1]$ be a smooth function with support contained in $\mathcal{N}$, such that $h_1(p_*) > 0$; let $h_2 : \partial\Omega \times \partial\Omega \to [0,1]$ be a smooth function with support contained in $\mathcal{M}$, such that $h_2(p',p) > 0$. Consider the measure

$$\xi(ds, dp_2|p_0) = E\big(h_1(P_0)h_2(P_1,P_2); T_1 + T_2 \in ds; P_2 \in dp_2|P_0 = p_0\big).$$

Obviously, $\xi(ds, dp_2|p_0) \le Pr\{T_1 + T_2 \in ds;\ P_2 \in dp_2|P_0 = p_0)$. Since the transition kernel $k(p_0, p_1)$ of the Markov chain P_2 is smooth and strictly positive except for $p_1 = p_0$, the transformation theorem for multiple integrals implies that

$$\xi(ds, dp_2|p_0) = f_{p_*}(s, p_2|p_0)ds\nu(dp),$$

where $f_{p_*}(s,p_2|p_0)$ is jointly continuous in s,p_2,p_0 and $f_{p_*}(s,p_2|p_*) > 0$ for $(s,p_2) = G_{p_*}(p',p)$. $\square$

COROLLARY 1: There exist a constant $\rho < 1$ and nonnegative continuous functions $f_{2k}(s,p|p_0)$ such that

$$Pr\{\sum_{i=1}^{2k} T_i \in ds;\ P_{2k} \in dp|P_0 = p_0\} \geq f_{2k}(s,p|p_0)ds\ \nu(dp) \tag{6.3}$$

for all $s \in [0,\infty)$, p, $p_0 \in \partial\Omega$, and $k = 1,2,\ldots$, and such that

$$\int\int_{[0,\infty)\times\partial\Omega} f_{2k}(s,p|p_0)ds\nu(dp) \geq 1-\rho^k \tag{6.4}$$

for all $p_0 \in \partial\Omega$ and $k = 1,2,\ldots$.

PROOF: By induction on k. Let

$$1-\rho = \min_{p_0\in\partial\Omega}\int\int_{[0,\infty)\times\partial\Omega} f_2(s,p|p_0)ds\ \nu(dp);$$

by Lemma 6, $1-\rho > 0$, so (6.3) and (6.4) are valid for $k = 1$. Let

$$F_{2k}(ds,dp|p_0) = Pr\{\sum_{i=1}^{2k} T_i \in ds;\ P_{2k} \in dp|P_0 = p_0\}$$

and

$$\tilde{F}_{2k}(ds,dp|p_0) = F_{2k}(ds,dp|p_0) - f_{2k}(s,p|p_0)ds\ \nu(dp).$$

Then

$$\begin{aligned} F_{2k+2}(ds,dp|p_0) \geq &\Big\{\int\int_{(s',p')\in[0,\infty)\times\partial\Omega} f_2(s-s',p|p')F_{2k}(ds',dp'|p_0)\}ds\ \nu(dp) \\ &+\int\int_{(s',p')\in[0,\infty)\times\partial\Omega} \tilde{F}_2(ds-s',dp|p')f_{2k}(s',p'|p_0)ds'\ \nu(dp')\Big\} \end{aligned}$$

By the induction hypothesis, the total mass of the measure on the r. h. s. is $\geq 1-\rho^{k+1}$. The continuity of $f_2(s,p|p_0)$, the transition kernel $k(p,p')$, and the map $(p,p') \to |pp'|$, together with the compactness of $\partial\Omega$, imply that

$$\int\int_{(s',p')\in[0,\infty)\times\partial\Omega} f_2(s-s',p|p')F_{2k}(ds',dp'|p_0)$$

is a continuous function of s, p, p_0. Thus, to complete the proof it suffices to show that the measure

$$\int\int_{(s',p')\in[0,\infty)\times\partial\Omega} \tilde{F}_2(ds-s',dp|p')f_{2k}(s',p'|p_0)ds'\ \nu(dp')$$

has a density w. r. t. $ds\nu(dp)$ that is jointly continuous in s,p,p',p_0. For this it suffices to show that

$$\int\int_{s',p')\in[0,\infty)\times\partial\Omega} F_2(ds-s',dp|p')f_{2k}(s',p'|p_0)ds'\nu(dp') \tag{6.5}$$

has a continuous density, since $f_2(s,p|p')$ is continuous. But there exist probability measures $H_2(ds|p,p')$ such that

$$F_2(ds,dp|p') = H_2(ds|p,p')k_2(p',p)\nu(dp),$$

where $k_2(p',p)$ is the 2-step transition kernel for P_n ($k_2(p',p)$ is continuous). Clearly,

$$\int_{s'\in\mathbb{R}} H_2(ds-s'|p,p')f_{2k}(s',p'|p_0)ds'$$

has a density w. r. t. $ds\nu(dp)$ that is jointly continuous in s,p,p_0. It follows that the measure (6.5) has a continuous density. □

LEMMA 7: Let $g : \mathbb{R}\times\partial\Omega \to \mathbb{R}$ be any continuous function with compact support. For $(t,p_0) \in \mathbb{R}\times\partial\Omega$ define

$$G(t,p_0) = \int\int_{\mathbb{R}\times\partial\Omega} g(t-s,p)U(ds,dp|p_0);$$

then $G(t, p_0)$ is uniformly continuous on $\mathbb{R} \times \partial\Omega$, and

$$\lim_{t \to \infty} G(t, p_0) = (\pi|\Omega|)^{-1} \int\int_{\mathbb{R}\times\partial\Omega} g(s,p)ds\, \nu(dp). \tag{6.6}$$

uniformly for $p_0 \in \partial\Omega$.

PROOF: First we show that for any $\alpha > 0$ there exists $C_\alpha < \infty$ such that

$$U([s, s+\alpha] \times \partial\Omega | p_0) \leq C_\alpha \tag{6.7}$$

for all $(s, p_0) \in \mathbb{R} \times \partial\Omega$. It suffices to consider $s = 0$ (condition on the first visit to $[s, s+\alpha]$ by $\sum_{i=1}^n T_i$ and use the Markov property). Now

$$\begin{aligned} &U([0,\alpha] \times \partial\Omega|p_0) \\ &= \sum_{n=0}^{\infty} Pr\{\sum_{i=1}^{n} T_i \leq \alpha | P_0 = p_0\} \\ &\leq 1 + \sum_{n=0}^{\infty} \int_{\partial\Omega} Pr\{\sum_{i=1}^{n} T_i \leq \alpha | P_0 = p_1\} k(p_0, p_1)\, \nu(dp_1) \\ \leq &\left(\max_{\partial\Omega\times\partial\Omega} k(p_0, p_1) \right) \int_{\partial\Omega} U([0,\alpha] \times \partial\Omega | p_0)\, \nu(dp_0). \end{aligned}$$

Here $k(p_0, p_1)$ is the transition kernel for P_n; since it is continuous on $\partial\Omega \times \partial\Omega$, it is bounded. That

$$\begin{aligned} &\int_{\partial\Omega} U([0,\alpha] \times \partial\Omega|p_0)\, \nu(dp_0) \\ &= \sum_{n=0}^{\infty} Pr\{T_1 + \ldots T_n \leq \alpha\} \\ &< \infty \end{aligned}$$

follows because there exist $t, \eta > 0$ such that $Pr\{T_{i+1} \geq t | P_i\} \geq \eta$.

To prove the uniform continuity of $G(t, p_0)$ we observe that

$$\begin{aligned} &G(t,p_0) = g(t,p_0) + \\ &\int\int\int_{\partial\Omega\times\mathbb{R}\times\partial\Omega} g(t - s - |p_0p_1|, p) U(ds, dp|p_1) k(p_0,p_1) \nu(dp_1). \end{aligned}$$

Since $k(p_0, p_1)$ is uniformly continuous in p_0 and $g(t,p)$ is uniformly continuous in (t,p) and has compact support, it follows from (6.7) that $G(t,p_0)$ is uniformly continuous in (t,p_0).

For $p_0 \in \partial\Omega$, (6.6) is a consequence of ([8], Th. 4). (Conditions II.1, II.2 of [8] follow from Proposition 1 and Lemma 2; Condition II.3 follows from Lemma 6). The uniformity in p_0 holds because $G(t,p_0)$ is uniformly continuous. $\square$.

PROOF of Lemma 4: We have already proved (5.2) (cf. (6.7)). Moreover, (5.3) follows from (6.6) by a routine approximation argument. Now observe that by (6.3), for every $k = 1,2,\ldots$

$$U(ds,dp|p_0) \geq \left\{ \int \int_{\mathbb{R}\times\partial\Omega} f_{2k}(s-s',p|p')U(ds',dp'|p_0) \right\} ds\nu(dp);$$

thus (6.6) and (6.4) imply (5.5). Finally, (5.4) is a consequence of (5.5) and (5.3). $\square$

7. Proof of Theorems 1 and 2

We shall use the results of Sections 4, 5 to prove the following

LEMMA 8: For each $\beta > 0$, $\delta > 0$ there exist $\alpha > 0$, $\varepsilon_* > 0$ such that for every $y \varepsilon \mathcal{F}_\delta$, $0 < \varepsilon \leq \varepsilon_*$, and $n = 0,1,2,\ldots$

$$Pr\{\tau_\varepsilon(y) > n\alpha\} \leq \left(1 - 2\alpha\varepsilon|\Omega|^{-1}(1-\beta)\right)^n, \tag{7.1}$$

and for every $q \in \Omega$ satisfying dist $(q,\partial\Omega) \geq \delta$

$$|Pr\{\tau_\varepsilon(y_q) \leq (n+1)\alpha | \tau_\varepsilon(y_q) > n\alpha\} - 2\alpha\varepsilon|\Omega|^{-1}| \leq \beta\alpha\varepsilon. \tag{7.2}$$

Theorems 1–2 follow easily from Lemma 8. For $y \in \mathcal{F}_\delta$ and $\varepsilon < \varepsilon_*$, (7.1) implies

$$\begin{aligned} E\tau_\varepsilon(y) &= \int_0^\infty Pr\{\tau_\varepsilon(y) > t\}dt \\ &\leq \sum_{n=0}^\infty \alpha Pr\{\tau_\varepsilon(y) > n\alpha\} \\ &\leq |\Omega|\{2\varepsilon(1-\beta)\}^{-1}. \end{aligned}$$

As $\beta > 0$ may be arbitrarily small, it follows that

$$\limsup_{\varepsilon \to 0} \sup_{y \varepsilon \mathcal{F}_\delta} 2\varepsilon|\Omega|^{-1} E\tau_\varepsilon(y) \leq 1. \tag{7.3}$$

The inequality (7.2) implies that as $\varepsilon \to 0$

$$Pr\{2\varepsilon|\Omega|^{-1}\tau_\varepsilon(y_q) > t\} \to e^{-t} \tag{7.4}$$

for all $t > 0$, uniformly for $q \in \Omega$ satisfying dist $(q, \partial\Omega) \geq \delta$. Now (7.3) and (7.4) together imply that as $\varepsilon \to 0$

$$2\varepsilon|\Omega|^{-1} E\tau_\varepsilon(y_q) \to 1 \tag{7.5}$$

provided that dist $(q, \partial\Omega) \geq \delta$. Theorems 1–2 are immediate consequences of (7.3), (7.4), and (7.5).

Before proving Lemma 8 we shall record some consequences of Lemmas 4–5.

LEMMA 9: For any $\beta > 0$ there exists $\alpha > 0$ so large that for every $y \in \mathcal{F}_\delta$, $p_0 \in \partial\Omega$, and $\varepsilon > 0$

$$\begin{aligned} &\int\int\int_{F_{\alpha,\varepsilon}(y)} \frac{1}{2}\sin\theta U(ds, dp|p_0)d\theta \\ &\geq \left(2\alpha\varepsilon(1-\beta) - 4\varepsilon^2\right)|\Omega|^{-1}. \end{aligned} \tag{7.6}$$

PROOF: By Lemma 4 there is an $\alpha' > 0$ so large that for every $s \geq \alpha' - \text{diam}\Omega$ and all $p, p_0 \in \partial\Omega$,

$$U(ds, dp|p_0) \geq (1-\beta)^{1/2}/(\pi|\Omega|)ds\nu(dp). \tag{7.7}$$

Let $\alpha > \alpha'$; define $F_{\varepsilon,\alpha',\alpha}(y) = F_{\varepsilon,\alpha}(y) \cap \{(s,p,\theta) : s \geq \alpha' - \text{diam}\Omega\}$. By Lemma 5 and the translation invariance of μ in s (cf. (5.7)),

$$\mu\big(F_{\varepsilon,\alpha',\alpha}(y)\big) \geq 2\pi(\alpha - \alpha')\varepsilon - 4\pi\varepsilon^2. \tag{7.8}$$

It follows from (7.7) and (7.8) that if $(\alpha - \alpha') \geq \alpha(1-\beta)$ then (7.6) holds. □

LEMMA 10: For any $\beta > 0$ there exist $\alpha > 0$, $\varepsilon_* > 0$ such that for every $p_0 \in \partial\Omega$, $q \in \partial\Omega$ satisfying dist $(q, \partial\Omega) \geq \delta$, and $0 < \varepsilon < \varepsilon_*$

$$\left| \int\int\int_{F_{\alpha,\varepsilon}(y_q)} \frac{1}{2}\sin\theta U(ds, dp|p_0)d\theta - 2\alpha\varepsilon|\Omega|^{-1} \right| < \beta\alpha\varepsilon. \tag{7.7}$$

PROOF: It follows from (4.2) that

$$\left| \int\int\int_{F_{\alpha,\varepsilon}(y_q)} \frac{1}{2}\sin\theta\,\big(U(ds, dp|p_0) - (\pi|\Omega|)^{-1}ds\nu(dp)\big)\,d\theta \right|$$
$$\leq K'\varepsilon \int_{p\in\partial\Omega}\int_{s\in[-\text{diam}\,\Omega,\alpha]} \big|U(ds, dp|p_0) - (\pi|\Omega|)^{-1}ds\nu(dp)\big|.$$

Lemma 4 implies that the double integral is $o(\alpha)$ as $\alpha \to \infty$. The result now follows from the observation that

$$\int\int_{F_{\alpha,\varepsilon}(y_q)}\int (\frac{1}{2}\sin\theta)(\pi|\Omega|^{-1}ds\nu(dp)d\theta = \mu\big(F_{\alpha,\varepsilon}(y_q)\big)/(\pi|\Omega|)$$

and Lemma 5. □

PROOF of Lemma 8: Let $N_{\alpha,\varepsilon}(y)$ be the number of chords of $X(t)$ that pass within ε of $y(t)$ before time t; i.e.,

$$N_{\alpha,\varepsilon}(y) = \#\{n \geq 0: \text{ for some } t \leq \alpha \text{ satisfying } \sum_{i=1}^{n} T_i \leq t < \sum_{i=1}^{n+1} T_i, \text{ dist } (X(t), y(t)) \leq \varepsilon\}.$$

Lemma 3 provides an upper bound for $Pr\{N_{\alpha,\varepsilon}(y) \geq 1 | P_0 = p_0\}$. Together with the Markov property of $\{P_n\}_{n\geq 0}$ this bound implies that for each $\alpha > 0$ there exists $K_\alpha < \infty$ such that

$$Pr\{N_{\alpha,\varepsilon}(y) \geq n | P_0 = p_0\} \leq K_\alpha \varepsilon^n \tag{7.8}$$

for every $n \geq 0$, $p_0 \in \partial\Omega$, and $y \in \mathcal{F}_\delta$.

Consider now $E(N_{\alpha,\varepsilon}(y)|P_0 = p_0)$. This may be written in term of the renewal measure (5.1) as

$$E(N_{\alpha,\varepsilon}(y)|P_0 = p_0) = \int\int\int_{F_{\alpha,\varepsilon}(y)} \frac{1}{2} \sin\theta U(ds, dp|p_0) d\theta.$$

Lemma 9 implies that for each $\beta > 0$ there exists $\alpha > 0$ so large that for every $p_0 \in \partial\Omega$ and $y \in \mathcal{F}_\delta$

$$E(N_{\alpha,\varepsilon}(y)|P_0 = p_0) \geq \left(2\alpha\varepsilon(1-\beta) - 4\varepsilon^2\right)|\Omega|^{-1}. \tag{7.9}$$

Together, (7.8) and (7.9) imply that for each $\beta > 0$ there exist $\alpha > 0$, $\varepsilon_* > 0$ such that if $0 < \varepsilon < \varepsilon_*$, $p_0 \in \partial\Omega$, and $y \in \mathcal{F}_\delta$ then

$$\begin{aligned} &Pr\{N_{\alpha,\varepsilon}(y) \geq 1 | P_0 = p_0\} \\ =&Pr\{\tau_\varepsilon(y) \leq \alpha | P_0 = p_0\} \\ \geq&2\alpha\varepsilon|\Omega|^{-1}(1-\beta). \end{aligned}$$

The statement (7.1) (with a different α) follows by a routine argument using the Markov property of P_n and the fact that $T_i \leq$ diam Ω.

A similar argument using Lemma 10 shows that for each $\beta > 0$ there exist $\alpha > 0$, $\varepsilon_* > 0$ such that if $0 < \varepsilon < \varepsilon_*$, $p_0 \in \partial\Omega$, and $q \in \Omega$ satisfies dist $(q, \partial\Omega) \geq \delta$ then

$$\left|Pr\{\tau_\varepsilon(y_q) \leq \alpha | P_0 = p_0\} - 2\alpha\varepsilon|\Omega|^{-1}\right| \leq \beta\alpha\varepsilon.$$

The relation (7.2) follows by a routine argument. □

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