SELECTING THE BEST EXPONENTIAL POPULATION BASED ON TYPE-I CENSORED DATA: A BAYESIAN APPROACH *

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Technical Report#87-4

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February 1987
Revised October 1989
Revised May 1992

* This research was partially supported by the Office of Naval Research Contract N00014-89-K-0170 and NSF Grants DMS-8606964, DMS-8923071 at Purdue University. A part of this paper was presented in the 46th Session of the ISI in Tokyo, in September, 1987.

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Abstract

We investigate the problem of selecting the exponential population having the largest mean life among several exponential populations. A Bayes rule based on type-I censored data is derived. A monotone property of the Bayes selection rule is discussed. An early selection rule is also proposed. Finally, an example is presented to illustrate the implementation of these two selection rules.

AMS 1980 Subject Classification: 62F07; 62C10

KEY WORDS: Type-I censoring, early selection rule, Bayes rule, life-testing, best population, total time.

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* This research was partially supported by the Office of Naval Research Contract N00014-89-K-0170 and NSF Grants DMS–8606964, DMS–8923071 at Purdue University. A part of this paper was presented in the 46th Session of the ISI in Tokyo, in September, 1987.

1. Introduction

Let $\pi_1, \ldots, \pi_k$ denote $k(k \geq 2)$ independent exponential populations with density functions $h(x|\theta_i) = \frac{1}{\theta_i} e^{-\frac{x}{\theta_i}}, x > 0$, where the values of the scale parameters $\theta_i, 1 \leq i \leq k$, are positive but unknown. Let $\theta_{[1]} \leq \ldots \leq \theta_{[k]}$ denote the ordered values of the parameters $\theta_1, \ldots, \theta_k$. It is assumed that the exact pairing between the ordered and the unordered parameters is unknown. The population associated with the largest value $\theta_{[k]}$ is considered as the best population. The problem of selecting the exponential population having the largest scale parameter $\theta_{[k]}$ has been studied by Sobel (1956) using sequential approach. Gupta (1963) studied some selection rules for gamma populations via subset selection approach. His selection rules can be applied for the exponential populations case. Recently, Huang and Huang (1980), and Berger and Kim (1985) also studied this selection problem using either subset selection approach or indifference zone approach with type-II censored data. The purpose of this paper is to derive Bayes rules to select the best exponential population based on type-I censored data.

Type-I censored data arise in many situations such as industrial life-testing, clinical trials and biological experiments. To motivate this study, consider a life-testing experiment, where m items from each of the k independent exponential populations are independently put on test at the outset and are not replaced on failure. Due to the time restriction, the experiment terminates at a prespecified time T. The failure time of an item is observable if it fails before time T. If an item still functions at the close of the experiment, its failure time is not observable. The item then is said to be censored at time T. This type of time censoring is known as type-I censoring. Type-I censoring scheme has received much attention in the statistical literature. See Bartholomew (1963), Yang and Sirvanci (1977), Spurrier and Wei (1980), and Mann, Schafer and Han (1982), among others.

Let $X_{ij}, 1 \leq j \leq m$, denote the failure times of the m items taken from population π_i based on a life-test experiment. According to the time censoring scheme, we only observe $\min(X_{ij}, T)$. Let $C_{ij} = 1$ if $X_{ij} < T$ and $C_{ij} = 0$ otherwise. Then, $N_i = \sum_{j=1}^{m} C_{ij}$ is the number of uncensored observations of the m items up to time T. Let $Y_{i1} \leq Y_{i2} \leq \ldots \leq Y_{iN_i}$ denote the ordered values of the N_i observable failure times and let $Y_i = \sum_{j=1}^{N_i} Y_{ij} + (m - N_i)T$. That is, Y_i is the total life time of the m items upto the time T. Then, $(Y_{i1}, \ldots, Y_{iN_i}, N_i)$ has a joint probability density function of the form:

$$
\begin{aligned}
f_i(y_{i1}, \ldots, y_{in}, n|\theta_i) &= \frac{m!}{(m-n)!}\theta_i^{-n} exp\{-\theta_i^{-1}[\sum_{j=1}^{n} y_{ij} + (m-n)T]\} \\
&= \frac{m!}{(m-n)!}\theta_i^{-n} exp\{-\theta_i^{-1} y_i\} \qquad (1.1)
\end{aligned}
$$

where $0 \leq n \leq m, 0 < y_{i1} \leq y_{i2} \leq \ldots \leq y_{in} < T, y_i = \sum_{j=1}^{n} y_{ij} + (m-n)T$ and $\sum_{j=1}^{n} \equiv 0$ if $n = 0$. Note that $(m-n)T \leq y_i \leq m\ T$. For convenience, we denote the expression at the right-hand-side of (1.1) by $f_i(y_i, n|\theta_i)$.

In this paper, we investigate the problem of selecting the best exponential population from among several exponential populations. A Bayes selection rule based on type-I censored data is derived in Section 2. A monotone property of the Bayes selection rule is discussed in Section 3. We also prove that the posterior density functions given the observed type-I censored data have the monotone likelihood ratio (MLR) property. Based on the MLR property, an early selection rule is proposed in Section 4. Finally, an example is presented to illustrate the implementation of the two selection rules.

2. A Bayes Selection Rule

Let $\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$ and let $\underset{\sim}{N} = (N_1, \ldots, N_k)$ where $(Y_i, N_i), 1 \leq i \leq k$, are defined in Section 1. Let $\mathcal{N}$ be the sample space generated by $\underset{\sim}{N}$ and conditional on $\underset{\sim}{N} = \underset{\sim}{n} = (n_1, \ldots, n_k)$, let $\mathcal{Y}_{\underset{\sim}{n}}$ be the sample space generated by $\underset{\sim}{Y}$. Thus, for $\underset{\sim}{y} = (y_1, \ldots, y_k) \epsilon \mathcal{Y}_{\underset{\sim}{n}}, (m - n_i)T \leq y_i \leq m\ T, 1 \leq i \leq k$.

Let $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ and let $\Omega = \{\underset{\sim}{\theta} | \theta_i > 0, 1 \leq i \leq k\}$ be the parameter space. Let $\mathcal{A} = \{1, \ldots, k\}$ be the action space. Action i corresponds to the selection of population π_i as the best population. For a given $\underset{\sim}{\theta} \epsilon \Omega$, and an action i, the associated loss function $L^*(\underset{\sim}{\theta}, i)$ is defined by

$$L^*(\underset{\sim}{\theta}, i) = L(\theta_{[k]} - \theta_i) \tag{2.1}$$

where $L(x)$ is a nonnegative, nondecreasing function of x, $x \geq 0$, such that $L(0) = 0$.

Let $g(\underset{\sim}{\theta}) = \prod_{i=1}^{k} g_i(\theta_i)$ be the prior density function over the parameter space Ω. It is assumed that $\int L(\theta_{[k]}) g(\underset{\sim}{\theta}) d\underset{\sim}{\theta} < \infty$.

A selection rule $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_k)$ is defined to be a measurable mapping from the sample space $(\mathcal{N}, (\mathcal{Y}_{\underset{\sim}{n}})_{\underset{\sim}{n} \epsilon \mathcal{N}})$ to $[0,1]^k$ such that $0 \leq \delta_i(\underset{\sim}{n}, \underset{\sim}{y}) \leq 1$ and $\sum_{i=1}^{k} \delta_i(\underset{\sim}{n}, \underset{\sim}{y}) = 1$ for all $\underset{\sim}{y} \epsilon \mathcal{Y}_{\underset{\sim}{n}}, \underset{\sim}{n} \epsilon \mathcal{N}$. The value of $\delta_i(\underset{\sim}{n}, \underset{\sim}{y})$ is the probability of selecting population π_i as the best population given the observation $(\underset{\sim}{n}, \underset{\sim}{y})$.

Let $R(\underset{\sim}{\delta}, g)$ denote the Bayes risk associated with the selection rule $\underset{\sim}{\delta}$. Then,

$$R(\underset{\sim}{\delta}, g) = \int_{\Omega} \sum_{\underset{\sim}{n} \epsilon \mathcal{N}} \int_{\mathcal{Y}_{\underset{\sim}{n}}} \sum_{i=1}^{k} L(\theta_{[k]} - \theta_i) \delta_i(\underset{\sim}{n}, \underset{\sim}{y}) f(\underset{\sim}{y}, \underset{\sim}{n} | \underset{\sim}{\theta}) g(\underset{\sim}{\theta}) d\underset{\sim}{y} d\underset{\sim}{\theta} \tag{2.2}$$

where $f(\underset{\sim}{y}, \underset{\sim}{n} | \underset{\sim}{\theta}) = \prod_{i=1}^{k} f_i(y_i, n_i | \theta_i)$. Now, let

$$f_i(y_i, n_i) = \int_0^{\infty} f_i(y_i, n_i | \theta_i) g_i(\theta_i) d\theta_i, f(\underset{\sim}{y}, \underset{\sim}{n}) = \prod_{i=1}^{k} f_i(y_i, n_i),$$

$$g_i(\theta_i|y_i,n_i) = \frac{f_i(y_i,n_i|\theta_i)g_i(\theta_i)}{f_i(y_i,n_i)} \text{ and } g(\underset{\sim}{\theta}|\underset{\sim}{y},\underset{\sim}{n}) = \prod_{i=1}^{k} g_i(\theta_i|y_i,n_i).$$

Using Fubini's theorem, it is easily seen that

$$R(\underset{\sim}{\delta},g) = \sum_{\underset{\sim}{n}\epsilon\mathcal{N}} \int_{\mathcal{Y}_{\underset{\sim}{n}}} \sum_{i=1}^{k} \delta_i(\underset{\sim}{n},\underset{\sim}{y}) \int_{\Omega} L(\theta_{[k]}-\theta_i) g(\underset{\sim}{\theta}|\underset{\sim}{y},\underset{\sim}{n}) d\underset{\sim}{\theta} f(\underset{\sim}{y},\underset{\sim}{n}) d\underset{\sim}{y}. \tag{2.3}$$

For each $(\underset{\sim}{n},\underset{\sim}{y})$, let

$$\Delta_i(\underset{\sim}{n},\underset{\sim}{y}) = \int_{\Omega} L(\theta_{[k]}-\theta_i) g(\underset{\sim}{\theta}|\underset{\sim}{y},\underset{\sim}{n}) d\underset{\sim}{\theta}, \quad i=1,\ldots,k, \tag{2.4}$$

and let

$$A(\underset{\sim}{n},\underset{\sim}{y}) = \{i|\Delta_i(\underset{\sim}{n},\underset{\sim}{y}) = \min_{1\le j\le k} \Delta_j(\underset{\sim}{n},\underset{\sim}{y})\}. \tag{2.5}$$

Then, a uniformly randomized Bayes rule is $\underset{\sim}{\delta}_G = (\delta_{G1},\ldots,\delta_{Gk})$, where

$$\delta_{Gi}(\underset{\sim}{n},\underset{\sim}{y}) = \begin{cases} |A(\underset{\sim}{n},\underset{\sim}{y})|^{-1} & \text{if } i \,\epsilon\, A(\underset{\sim}{n},\underset{\sim}{y}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

3. A Monotonicity Property of $\underset{\sim}{\delta}_G$

In this section, we claim that the Bayes selection rule $\underset{\sim}{\delta}_G$ has the following monotone property.

Theorem 3.1. For each $i = 1,\ldots,k$, $\delta_{Gi}(\underset{\sim}{n},\underset{\sim}{y})$ is nondecreasing in y_i and also in $n_j, j \neq i$ when all other variables are kept fixed, and nonincreasing in n_i and also in $y_j, j \neq i$ when all other variables are kept fixed.

To prove this theorem, we need the following two lemmas.

Lemma 3.1. Let $0 \le n_i^* \le n_i \le m, \quad 0 < y_i \le y_i^* \le m\ T$. Consider the likelihood ratio $r_i(\theta|y_i, n_i, y_i^*, n_i^*)$ defined by

$$r_i(\theta|y_i, n_i, y_i^*, n_i^*) = \begin{cases} \frac{g_i(\theta|y_i^*, n_i^*)}{g_i(\theta|y_i, n_i)} & \text{if } g_i(\theta|y_i, n_i) \neq 0 \\ 0 & \text{if both } g_i(\theta|y_i, n_i) = 0 \text{ and } g_i(\theta|y_i^*, n_i^*) = 0. \end{cases} \tag{3.1}$$

Then,

a) As $n_i = n_i^*$ and $y_i < y_i^*$, then $r_i(\theta|y_i, n_i, y_i^*, n_i^*)$ is nondecreasing in θ.

b) As $y_i = y_i^*$ and $n_i < n_i^*$, then $r_i(\theta|y_i, n_i, y_i^*, n_i^*)$ is nonincreasing in θ.

Proof: Note that as $g_i(\theta|y_i, n_i) \neq 0$, after simplification, we have

$$\begin{aligned} r_i(\theta|y_i, n_i, y_i^*, n_i^*) &= g_i(\theta|y_i^*, n_i^*)/g_i(\theta|y_i, n_i) \\ &= c(y_i, n_i, y_i^*, n_i^*)\theta^{n_i - n_i^*} \exp\{\theta^{-1}(y_i - y_i^*)\} \end{aligned} \tag{3.2}$$

where

$$c(y_i, n_i, y_i^*, n_i^*) = \int_0^\infty \theta^{-n_i} \exp\{-\theta^{-1} y_i\} g_i(\theta) d\theta \Big/ \int_0^\infty \theta^{-n_i^*} \exp\{-\theta^{-1} y_i^*\} g_i(\theta) d\theta > 0.$$

Thus, the proof of this lemma is completed by following a straightforward argument.

Lemma 3.2. Let $\Delta_i(\underset{\sim}{n}, \underset{\sim}{y})$ be that defined in (2.4), for each $i = 1, \ldots, k$. Then, $\Delta_i(\underset{\sim}{n}, \underset{\sim}{y})$ is nonincreasing in y_i and also in $n_j,\ j \neq i$, when all the other variables are kept fixed, and nondecreasing in n_i and also in $y_j,\ j \neq i$, when all the other variables are kept fixed.

Proof: We prove that $\Delta_i(\underset{\sim}{n}, \underset{\sim}{y})$ is nonincreasing in y_i when all the other variables are kept fixed only. The others can be proved in a similar way.

Let $\underset{\sim}{\theta}^i = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_k), \Omega^i = \{\underset{\sim}{\theta}^i | \theta_j > 0 \text{ for } j = 1, \ldots, k, \ j \neq i\}, \underset{\sim}{y} = (y_1, \ldots, y_k)$ and $\underset{\sim}{y}^* = (y_1^*, \ldots, y_k^*)$ where $y_j^* = y_j$ for $j \neq i$ and $y_i^* > y_i$. Then,

$$\Delta_i(\underset{\sim}{n}, \underset{\sim}{y}) = \int_{\Omega^i} [\int_{\theta_i=0}^{\infty} L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i, n_i) d\theta_i] \prod_{\substack{j=1 \\ j \neq i}} g_j(\theta_j | y_j, n_j) d\underset{\sim}{\theta}^i.$$

Since for each fixed $\underset{\sim}{\theta}^i$, $L(\theta_{[k]} - \theta_i)$ is nonincreasing in θ_i and $g_i(\theta_i | y_i, n_i)$ has monotone likelihood ratio in y_i and θ_i (see Lemma 3.1.a), for $y_i^* > y_i$, we have

$$\int_{\theta_i=0}^{\infty} L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i, n_i) d\theta_i \geq \int_{\theta_i=0}^{\infty} L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i^*, n_i) d\theta_i,$$

and hence $\Delta_i(\underset{\sim}{n}, \underset{\sim}{y}) \geq \Delta_i(\underset{\sim}{n}, \underset{\sim}{y}^*)$.

Now, we see that Theorem 3.1 is a direct result of Lemma 3.2, (2.5) and (2.6).

4. An Early Selection Rule

In this section, we consider the following linear loss function: $L^*(\underset{\sim}{\theta}, i) = \theta_{[k]} - \theta_i$, the difference between the parameters of the best and the selected populations. Thus, the set $A(\underset{\sim}{n}, \underset{\sim}{y})$ given in (2.5) turns out to be:

$$A(\underset{\sim}{n}, \underset{\sim}{y}) = \{i | E[\theta | y_i, n_i] = \max_{1 \leq j \leq k} E[\theta | y_j, n_j]\}, \tag{4.1}$$

where $E[\theta | y_i, n_i] = \int \theta g_i(\theta | y_i, n_i) d\theta$, the posterior mean of θ_i given $(Y_i, N_i) = (y_i, n_i)$. By Lemma 3.2, we have the following result.

Lemma 4.1. For each $i = 1, \ldots, k$, $E[\theta | y_i, n_i]$ is increasing in y_i and decreasing in n_i.

We will use this monotonicity property of $E[\theta|y_i, n_i]$ to derive a modified selection rule. This modified selection rule is designed to make a selection earlier than the termination time T of the life-testing experiment.

At time t, $0 < t < T$, let $N_i(t)$ denote the number of failures from population π_i up to time t. That is, $N_i(t) =$ number of $\{X_{ij} | 1 \leq j \leq m,\ X_{ij} < t\}$. Also, we let $Y_{i1} \leq \ldots \leq Y_{iN_i(t)}$ denote the $N_i(t)$ failure times up to the time t. At time t, exclude population π_i as a nonbest population if there exists some population π_h such that either

$$N_h(t) < m \text{ and } \int \theta g_h(\theta | y_h(t), m) d\theta \geq \int \theta g_i(\theta | y_i(t,T),\ N_i(t)) d\theta \tag{4.2.a}$$

or

$$N_h(t) = m \text{ and } \int \theta g_h(\theta | y_h(t), m) d\theta > \int \theta g_i(\theta | y_i(t,T), N_i(t)) d\theta \tag{4.2.b}$$

where

$$\begin{cases} y_h(t) = \sum_{j=1}^{N_h(t)} y_{hj} + (m - N_h(t))t, \\ y_i(t,T) = \sum_{j=1}^{N_i(t)} y_{ij} + (m - N_i(t))T. \end{cases} \tag{4.3}$$

We also let $S(t)$ denote the set of indices of the contending populations at time t. That is,

$$\begin{aligned} S(t) = \{i | N_h(t) < (=) m \text{ and } & \int \theta g_i(\theta | y_i(t,T), N_i(t)) d\theta \\ & > (\geq) \int \theta g_h(\theta | y_h(t), m) d\theta, h \neq i\}. \end{aligned} \tag{4.4}$$

The life-testing experiment terminates as soon as there is a time t, $0 < t < T$, such that $|S(t)| = 1$ and in this situation, we select the population with the index in the set $S(t)$ as the best population. Otherwise, the experiment goes on until the time T. At the time T, let

$$S(T) = \{i| \int \theta g_i(\theta|y_i, N_i)d\theta = \max_{j\epsilon S(T^-)} \int \theta g_j(\theta|y_j, N_j)d\theta\}, \tag{4.5}$$

where $S(T^-)$ denotes the set of the indices of those populations having not been eliminated before the time T. Then, a uniformly randomized selection is made over the set $S(T)$.

From the above description, we see that this modified selection rule can make selection earlier than the termination time T. We denote this modified early selection rule by $\underset{\sim}{\delta}_G^*$ and let δ_{Gi}^* be the probability of selecting population π_i as the best population by applying the selection rule $\underset{\sim}{\delta}_G^*$. Note that the probability δ_{Gi}^*, $1 \leq i \leq k$, are functions of the data observed during the time interval $(0, T]$.

In the following, we will show

Theorem 4.1. Under the loss function $L^*(\underset{\sim}{\theta}, i) = \theta_{[k]} - \theta_i$, $\delta_{Gi}^* = \delta_{Gi}(\underset{\sim}{n}, \underset{\sim}{y})$ for all $1 \leq i \leq k$, $\underset{\sim}{y}\epsilon\mathcal{Y}_n$ and $\underset{\sim}{n}\epsilon\mathcal{N}$, where $\delta_{Gi}(\underset{\sim}{n}, \underset{\sim}{y})$ is defined by (4.1) and (2.6).

Note that $\delta_{Gi}(\underset{\sim}{n}, \underset{\sim}{y})$ is the probability of selecting population π_i as the best population based on the type-I censored data $(\underset{\sim}{n}, \underset{\sim}{y})$ obtained at the end of the time T.

Let $B = \{0 < t \leq T||S(t)| = 1\}$ and let

$$t_1 = \begin{cases} \inf B & \text{if } B \neq \phi, \\ T & \text{if } B = \phi, \end{cases} \tag{4.6}$$

where ϕ denotes an empty set. Note that if $B \neq \phi$, then $B = [t_1, T]$.

By a uniformly randomized selection over the set $S(T)$ when $t_1 = T$, Theorem 4.1 is equivalent to the following.

Theorem 4.2. $S(t_1) = A(\underset{\sim}{n}, \underset{\sim}{y})$ for all $(\underset{\sim}{n}, \underset{\sim}{y})$ where $A(\underset{\sim}{n}, \underset{\sim}{y})$ is defined in (4.1).

Proof: Case 1. As $t_1 < T$, then $|S(t_1)| = 1$. Without loss of generality, we let π_k be the population with index in the set $S(t_1)$. Since $A(\underset{\sim}{n}, \underset{\sim}{y})$ contains at least one element, it suffices to show that $i \notin A(\underset{\sim}{n}, \underset{\sim}{y})$ for all $i \neq k$. Since $i \notin S(t_1)$, it means that population π_i is eliminated at some time, say t_0, not later than t_1, by some population, say π_h. That is, at time t_0 either

$$N_h(t_0) < m \text{ and } \int \theta g_h(\theta|y_h(t_0), m)d\theta \geq \int \theta g_i(\theta|y_i(t_0, T), N_i(t_0))d\theta \qquad (4.7.a)$$

or

$$N_h(t_0) = m \text{ and } \int \theta g_h(\theta|y_h(t_0), m)d\theta > \int \theta g_i(\theta|y_i(t_0, T), N_i(t_0))d\theta. \qquad (4.7.b)$$

Now, note that $N_i(t)$ is an nondecreasing function of $t\epsilon(0, T]$ and $N_i(t) \leq m$. Also, by (4.3), $y_h(t)$ is nondecreasing in t and $y_i(t, T)$ is nonincreasing in t. In fact, we have

$$N_h = N_h(T) \leq m, \quad N_i(t) \leq N_i(T) = N_i, \; y_i(t_0, T) \geq y_i(T, T) \equiv y_i$$

$$y_h \equiv y_h(T) > (=) y_h(t_0) \text{ if } N_h(t_0) < (=) m.$$

Thus, when $N_h(t_0) = m$, then $N_h \equiv N_h(T) = m$. Then by Lemma 4.1, and (4.7.b),

$$\begin{aligned} \int \theta g_h(\theta|y_h, N_h)d\theta &= \int \theta g_h(\theta|y_h(t_0), m)d\theta \\ &> \int \theta g_i(\theta|y_i(t_0, T), N_i(t_0))d\theta \\ &\geq \int \theta g_i(\theta|y_i, N_i)d\theta. \end{aligned} \qquad (4.8)$$

When, $N_h(t_0) < m$, then $y_h \equiv y_h(T) > y_h(t_0)$ and $N_h = N_h(T) \leq m$. Therefore, by Lemma 4.1 and (4.7.a),

$$\begin{aligned} \int \theta g_h(\theta|y_h, N_h)d\theta &> \int \theta g_h(\theta|y_h(t_0), m)d\theta \\ &\geq \int \theta g_i(\theta|y_i(t_0, T), N_i(t_0))d\theta \\ &\geq \int \theta g_i(\theta|y_i, N_i)d\theta. \end{aligned} \tag{4.9}$$

In either situations, we see that $i \notin A(\underset{\sim}{n}, \underset{\sim}{y})$.

Case 2. As $t_1 = T$, we need to prove that

(a) $i \notin S(T) \Rightarrow i \notin A(\underset{\sim}{n}, \underset{\sim}{y})$ and

(b) $i \epsilon S(T) \Rightarrow i \epsilon A(\underset{\sim}{n}, \underset{\sim}{y})$.

We prove part (a) first. Suppose $i \notin S(T)$. Then, π_i is eliminated at a time $t_0 \leq T$ by some other π_h.

If $t_0 < T$, this reduces to the situation discussed in Case 1.

If $t_0 = T$, then by (4.5), $\int \theta g_h(\theta|y_h, N_h)d\theta > \int \theta g_i(\theta|y_i, N_i)d\theta$. Therefore, by the definition of $A(\underset{\sim}{n}, \underset{\sim}{y})$, $i \notin A(\underset{\sim}{n}, \underset{\sim}{y})$.

Note that the statement in part (a) is equivalent to that

$$A(\underset{\sim}{n}, \underset{\sim}{y}) \subset S(T). \tag{4.10}$$

Now, part (b) is a direct consequence of (4.5) and (4.10). Therefore, we complete the proof of this theorem.

5. An Illustrative Example

We use the insulating fluid example (taken from Table 4.1, page 462 of Nelson (1982)) to illustrate the way to implement the selection rules $\underset{\sim}{\delta}_G$ and $\underset{\sim}{\delta}^*_G$. There are six groups of insulating fluid. The purpose is to identify which group of insulating fluid has the largest life-time when subjected to high voltage stress. Ten items from each group are put on a life-test experiment which is subjected to high voltage stress. The record of the times to breakdown in minutes is shown in Table 1. The result of Nelson (1982) indicates that the data in each group follows an exponential distribution.

Table 1: Times to Insulating Fluid Breakdown

Group	1	2	3	4	5	6
	1.89	1.30	1.99	1.17	8.11	2.12
	4.03	2.75	0.64	3.87	3.17	3.97
	1.54	0.00	2.15	2.80	5.55	1.56
	0.31	2.17	1.08	0.70	0.80	1.34
	0.66	0.66	2.57	3.82	0.20	1.49
	1.70	0.55	0.93	0.02	1.13	8.71
	2.17	0.18	4.75	0.50	6.63	2.10
	1.82	10.60*	0.82	3.72	1.08	7.21
	9.99*	1.63	2.06	0.06	2.44	3.83
	2.24	0.71	0.49	3.57	0.78	5.13

Suppose that time censoring scheme is adopted before the life-testing and the censoring time T is set to be 9 minutes. Therefore, in Table 1, the two failure times 9.99 and 10.60 marked with '*' should be censored data according to this censoring scheme. Then, we have

$y_1 = 25.36,\quad y_2 = 18.95,\quad y_3 = 17.48,\quad y_4 = 20.23,\quad y_5 = 29.89,\quad y_6 = 37.46,$
$n_1 = 9,\quad n_2 = 9,\quad n_3 = 10,\quad n_4 = 10,\quad n_5 = 10,\quad n_6 = 10.$

We also assume that the six scale random parameters $\theta_1, \ldots, \theta_6$ are iid with a common prior density function $g_i(\theta) = \theta^{-3} e^{-\frac{1}{\theta}}$. Therefore, $E(\theta_i|y_i, n_i) = \int_0^\infty \theta g_i(\theta|y_i, n_i) d\theta = (y_i +$

$1)/(n_i+1)$, $1 \le i \le k$. Hence, $E(\theta_1|y_1,n_1) = 2.636$, $E(\theta_2|y_2,n_2) = 1.995$, $E(\theta_3|y_3,n_3) = 1.68$, $E(\theta_4|y_4,n_4) = 1.93$, $E(\theta_5|y_5,n_5) = 2.808, E(\theta_0|y_6,n_6) = 3.496$. According to the selection rule $\underset{\sim}{\delta}_G$, we select Group 6 as the best group.

However, if the modified selection rule $\underset{\sim}{\delta}_G^*$ is applied, for the same data set in Table 1, the selection can be made before the termination time T. According to the selection rule $\underset{\sim}{\delta}_G^*$, with the same prior distribution given above, at time t, $0 < t \le T = 9$, remove π_i from further consideration if there exists some h such that either

a) $N_h(t) < 10$ and

$$(\sum_{j=1}^{N_h(t)} y_{hj} + (10 - N_i(t))t + 1)/11 \ge (\sum_{j=1}^{N_i(t)} y_{ij} + (10 - N_i(t))T + 1)(N_i(t)+1)^{-1}, \text{ or}$$

b) $N_h(t) = 10$ and

$$(\sum_{j=1}^{10} y_{hj} + 1)/11 > (\sum_{j=1}^{N_i(t)} y_{ij} + (m - N_i(t))T + 1)(N_i(t)+1)^{-1}.$$

Table 2 indicates the times (in minutes) at each of which some group is removed from the set of contending groups; and the life-testing experiment can be ended at time $t_1 = 6.63$. We then select Group 6 as the best group.

Table 2: Times to Reduce the Size of the Set of Contending Groups

Group	1	2	3	4	5	6
time	4.03	2.75	3.079	3.87	6.63	

Note that the modified experiment and the procedure lead to early selection and a saving of time $T - t_1 = 2.37$(minutes). Also, in Table 2, Group 3 is excluded as a non-best group at time $t = 3.079$ which is not a failure time for any item in Group 3. While for Group 1, 2, 4 and 5, the time at which the associated group is excluded as a non-best group is also a failure time of some item in that group.

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BEFORE COMPLETING FORM

1. REPORT NUMBER Technical Report #87-4	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SELECTING THE BEST EXPONENTIAL POPULATION BASED ON TYPE-I CENSORED DATA: A BAYESIAN APPROACH		5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER Technical Report #87-4
7. AUTHOR(s) Shanti S. Gupta and TaChen Liang		8. CONTRACT OR GRANT NUMBER(s) N00014-88-K-0170 and NSF Grant DMS-8606964
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Statistics West Lafayette, IN 47907		10. PROGRAM ELEMENT, PROJECT, TASK, AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Washington, DC		12. REPORT DATE February 1987/Revised 9/89 13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release, distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Type-I censoring, early selection rule, Bayes rule, life-testing, best population, total time.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We investigate the problem of selecting the exponential population having the largest mean life among several exponential populations. A Bayes rule based on type-I censored data is derived. A monotone property of the Bayes selection rule is discussed. An early selection rule is also proposed. Finally, an example is presented to illustrate the implementation of these two selection rules.		

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