A Connection Between the Expansion of Filtrations and Girsanov's Theorem

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Summary

We give sufficient conditions under which a continuous local martingale remains a continuous local martingale under a simultaneous initial expansion of the filtration of σ-fields and change to an equivalent probability law. In particular this gives a method for a Brownian motion to remain a Brownian motion under such a double transformation. The classical example of K. Itô is treated in detail.

Key words and phrases: Brownian motion, local martingale, semimartingale, initial expansion of filtrations, Girsanov's theorem.

The connections between the theory of the expansion of filtrations and Girsanov's theorem have long been intriguing. The relationship is problematic, however, as T. Jeulin and M. Yor originally pointed out in 1979 [5]. Nevertheless we show here that in some cases one is able to "undo the damage" done by an expansion of the filtration, with a change to an equivalent probability measure.

We assume given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $F = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, each $t \geq 0$, and $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. We let $G = (\mathcal{G}_t)_{t\geq 0}$ denote a filtration containing F. Typically G will arise by the

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initial expansion of F. For example, if Z is an $\mathcal{F}$-measurable random variable, one might define G by

$$\mathcal{G}_t = \bigcap_{s>t} \{\mathcal{F}_s \vee \sigma(Z)\}.$$

It then often happens that if M is an (F,P) continuous local martingale, M is a (G,P) semimartingale with the (unique) decomposition

$$\begin{aligned} M_t &= (M_t - \int_0^t L_s \, d\langle M,M\rangle_s) + \int_0^t L_s \, d\langle M,M\rangle)_s \\ &= \tilde{M}_t + \int_0^t L_s \, d\langle M,M\rangle_s \end{aligned} \tag{1}$$

for a certain G-predictable process $L = (L_t)_{t\geq 0}$. The now classical example of K. Itô [2] for a Brownian motion $B = (B_t)_{0\leq t\leq 1}$ is a case in point:

$$\text{here } \mathcal{G}_t = \bigcap_{s>t} \{\mathcal{F}_s \vee \sigma(B_1)\} \qquad \text{and}$$

$$\tilde{B}_t = B_t - \int_0^t \frac{B_1 - B_s}{1-s} \, ds, \qquad 0 \leq t < 1.$$

(This formula holds as well for Lévy processes; see [4]). Recently Jacod [3] and Yor [7] have given general sufficient conditions for a decomposition such as (1) to hold.

THEOREM. Let M be an (F,P) continuous local martingale, $M_0 = 0$, and let G be an expansion of the filtration F such that there exists a G-predictable process L making

$$\tilde{M}_t = M_t - \int_0^t L_s \, d\langle M,M\rangle_s$$

a (G,P) continuous local martingale. Suppose further $E\{\exp(\frac{1}{2}\int_0^{t_0} L_s^2 \, d\langle M,M\rangle_s)\} < \infty$ for some t_0. Then there exists a probability Q equivalent to P such that M is a (G,Q) continuous local martingale, $0 \leq t \leq t_0$.

Proof: Let t be less than t_0 (t_0 can equal ∞). Define

$$\hat{M}_t = \int_0^t -L_s \, d\tilde{M}_s.$$

Then $\hat{M}$ is a (G,P) continuous local martingale; the stochastic integral is well defined for $t \leq t_0$ as a consequence of Novikov's condition, since $\langle \tilde{M}, \tilde{M}\rangle$, computed under (G,P), is equal to $\langle M, M\rangle$, computed under (F, P); this is because the quadratic variation of a *continuous* local martingale M can be written

$$\lim_{n\to\infty} \sum_{t_i \varepsilon \pi_n[0,t]} (M_{t_{i+1}} - M_{t_i})^2 = \langle M, M\rangle_t$$

where $\lim_{n\to\infty}$ mesh $(\pi_n) = 0$ and convergence is in probability (that is, it is computed without involving the filtration). In this case $E(e^{\frac{1}{2}\langle \hat{M}, \hat{M}\rangle_{t_0}}) = E(e^{\frac{1}{2}\int_0^{t_0} L_s^2 \, d\langle M,M\rangle_s})$ and Novikov's condition is satisfied for $\hat{M}$.

Next define $N = (N_t)_{0\le t\le t_0}$ to be the solution of

$$N_t = 1 + \int_0^t N_s \, d\hat{M}_s,$$

the stochastic exponential of $\hat{M}$. By Novikov's theorem (see [6]) we have $E(N_t) = E(N_{t_0}) = 1$, $0 \le t \le t_0$. We define Q by

$$dQ = N_{t_0} \, dP.$$

Then Q is equivalent to P (that is, Q has the same null sets as does P). Under Q the process $\tilde{M}$ is still a $(\mathcal{G}, Q)$-semimartingale, and it has a decomposition

$$\tilde{M}_t = (\tilde{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \tilde{M}\rangle_s) + \int_0^t \frac{1}{N_s} \, d\langle N, \tilde{M}\rangle_s$$

by the Meyer-Girsanov theorem (cf [1, p. 238]). In particular $\tilde{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \tilde{M}\rangle_s$ is a (G, Q) continuous local martingale. However

$$\begin{aligned}
\int_0^t \frac{1}{N_s} \, d\langle N, \tilde{M}\rangle_s &= \int_0^t \frac{1}{N_s} \, N_s d\langle \hat{M}, \tilde{M}\rangle_s \\
&= \int_0^t \frac{N_s}{N_s} (-L_s) d\langle \tilde{M}, \tilde{M}\rangle_s \\
&= -\int_0^t L_s \, d\langle M, M\rangle_s.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\tilde{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \tilde{M}\rangle_s \\
&= (M_t - \int_0^t L_s \, d\langle M, M\rangle_s) - (-\int_0^t L_s \, d\langle M, M\rangle_s) \\
&= M_t
\end{aligned}$$

is a (G,Q) continuous local martingale, and therefore so is M. $\square$

For an example we can apply the theorem to Itô's classical example mentioned earlier. Fix $t_0 < \frac{1}{2}$ and let $\gamma > 0$ be such that $E\{e^{\gamma(B_1^*)^2}\} < \infty$, where B is a standard Brownian motion and $B_1^* = \sup_{0\le s\le 1} |B_s|$. Any $\gamma < \frac{1}{2}$ can be taken. Let G be the filtration $\mathcal{G}_t = \bigcap_{s>t} \{\mathcal{F}_s \vee \sigma(B_1)\}$. Then $\tilde{B}_t = B_t - \int_0^t \frac{B_1 - B_s}{1-s}\, ds = B_t - \int_0^t L_s\, ds$ is a (G,P) martingale. Moreover

$$E\{\exp(\frac{1}{2}\int_0^{t_0} L_s^2\, d\langle B,B\rangle_s)\} = E\{\exp(\frac{1}{2}\int_0^{t_0} \left(\frac{B_1-B_s}{1-s}\right)^2 ds)\}$$
$$\le E\{\exp\left[\left(\frac{t_0}{2(1-t_0)}\right) \sup_{0<s\le t_0} (B_1 - B_s)^2\right]\}.$$

Since $\sup_{0<s\le t_0} (B_1 - B_s)^2$ has the same distribution as $(B_{t_0}^*)^2$, the above is less than

$$E\{\exp(\gamma(B_1^*)^2)\} < \infty,$$

by our choice of γ. Therefore the hypotheses of the theorem are satisfied and we conclude there exist a law Q equivalent to P such that B is a (G,Q) continuous local martingale. However $\langle B,B\rangle_t = t$ under (G,Q) as well as under (F,P), hence by Lévy's theorem B is also a (G,Q) Brownian motion, for $0 \le t \le t_0 < \frac{1}{2}$.

One can apply the theorem as well to diffusions. For example let Z be an $\mathcal{F}$-measurable random variable and let B be an $(\mathcal{F}_t)_{t\ge 0}$ Brownian motion. Let $\mathcal{G}_t = \bigcap_{s>t} \{\mathcal{F}_s \vee \sigma(Z)\}$ and suppose Z is such that $\tilde{B}_t = B_t - \int_0^t L_s\, ds$ is a $(\mathcal{G}_t, P)$-local martingale. Suppose also $E\{\exp(\frac{1}{2}\int_0^t L_s^2\, ds)\} < \infty$, some $\alpha > 0$. Let X be a solution of

$$X_t = Z + \int_0^t f(X_s)\, dB_s + \int_0^t g(X_s) ds.$$

Since X has an anticipating initial condition it cannot be an (F,P)-diffusion. However by the Theorem B is a (G,Q) Brownian motion and $Z \varepsilon \mathcal{G}_0$. Thus X is a (G,Q) diffusion. This allows for the consideration of a theory of expansion of filtrations for diffusions.

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