Distributions of Quadratic Forms

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SUMMARY

Exact expressions for the distribution function of a random variable of the form

$$c_1\chi_m^2 + c_2\chi_n^2$$

are given where χ_m^2 and χ_n^2 are independent chi-square random variables with m and n degrees of freedom respectively. (The positive c_i are distinct). In particular, the exact asymptotic distribution function for the average Kendall tau statistic is written as a function of tables of Solomon (1960) and some found in Abramowitz and Stegun's Handbook of Mathematical Functions.

Section 1. Introduction

For independent chi-squared variables χ^2_m and χ^2_n, with m and n degrees of freedom, respectively, we consider the quadratic form

$$Q = c_1\chi^2_m + c_2\chi^2_n$$

where the positive c_i are distinct. (We distinguish between variables with the same degrees of freedom by writing ${}^{(1)}\chi^2_m$ and ${}^{(2)}\chi^2_m$.)

This paper gives exact finite expressions for the distribution of Q in terms of available functions such as the distribution function of chi-squared random variables, modified Bessel Functions, Dawson's integral (tabled in Abramowitz and Stegun (1964)) as well as the distribution of $c_1{}^{(1)}\chi^2_1 + c_2{}^{(2)}\chi^2_1$ (tabled in Solomon (1960)). These formulas are useful for checking the accuracy of approximations and tables of the distribution of Q and provide a simple alternative in their absence.

For large m and n, reasonable approximations to the distribution of Q are available. For the general quadratic form Williams (1984) compares algorithms for truncations of infinite series expansions of the distribution. (See Johnson and Kotz (1970).) Oman and Zacks (1981) give a mixture approximation and Davies (1980) provides an algorithm for an approximation. For small values of m and n, tables for the distribution of Q are given by Harter (1960), Johnson and Kotz (1967), Marsaglia (1960), Owen (1962), and Solomon (1960).

Distributions of the form Q arise in a number of applications. Solomon (1961) noted that probabilities of hitting targets frequently reduce to the distribution of quadratic forms of the type Q. Pillai and Young (1973) show that the trace of a 2-dimensional Wishart matrix is distributed as Q with m and n equal. The variable $Q^{\frac{1}{2}}$ arises in the engineering literature described as a weighted unbiased Rayleigh variate of dimension two. (See Miller (1975)). A very important application is the distribution of chi-squared goodness-of-fit tests with estimated parameters. Certain two-sample chi-squared tests described by Moore and Spruill (1975) have asymptotic distributions of the form Q. Alvo, Cabillo and Fiegen (1982) show this for the average Kendall tau statistic. The distribution of Q for small m

and n for the average Kendall tau statistic is provided as an example in Section 3.

The exact expressions for the distribution of Q may also be useful for approximations for more general quadratic forms, especially in the case where there are essentially two groups of coefficients nearly alike within groups, i.e. the distribution of Q is an approximation for the distribution of

$$Q' = \sum_{i=1}^{m+n} a_i{}^{(i)} \chi_1^2$$

where $a_1 \approx c_1, i = 1, \ldots, m$ and $a_i \approx c_2, i = m+1, \ldots, m+n$. The exact expressions for the distribution function of Q are given in the next section.

Section 2. Exact expressions for the distribution function of a linear combination of two chi-squared random variables

The results in this section give exact expressions for the distribution function of

$$Q = c_1 \chi_m^2 + c_2 \chi_n^2$$

where the positive c_i are distinct and χ_m^2 and χ_n^2 are independent chi-squared random variables with m and n degrees of freedom respectively. The first theorem handles the case where at least one of m and n are even. Corollary 2.5 gives an expression for the distribution of Q in terms of that of a quadratic form with fewer degrees of freedom. This corollary can be applied repeatedly to give the distribution function of

$$c_1 \chi_{2k+1}^2 + c_2 \chi_{2\ell+1}^2$$

in terms of modified Bessel functions I_0 and I_1 and the distribution function of

$$Q_1 = c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2$$

Tables of the distribution function of Q_1 are given by Solomon (1960) and tables of I_0 and I_1 are given in Abramowitz and Stegun (1964). In an example, a representation for the distribution function of $c_1{}^{(1)} \chi_3^2 + c_2{}^{(2)} \chi_3^2$ is given.

The following theorem gives the distribution of Q unless both m and n are odd.

Theorem 2.1

Let χ^2_m and χ^2_{2k} be independent chi-squared variables with m and $2k$ degrees of freedom, respectively. Then

$$P\left[\frac{\chi^2_{2k}}{a_0}+\frac{\chi^2_m}{a_1}>1\right]=P\left[\chi^2_m>a_1\right]+\sum_{j=0}^{k-1}\beta_j P\left[\chi^2_{2(k-j)}>a_0\right]\cdot\gamma_j$$

where

$$\beta_j=\left(\frac{a_0}{a_0-a_1}\right)^j\left(\frac{a_1}{|a_1-a_0|}\right)^{\frac{m}{2}}\frac{\Gamma\left(\frac{m}{2}+j\right)}{\Gamma\left(\frac{m}{2}\right)(j)!}.$$

If $a_1>a_0, \gamma_j$ is $P\left[\chi^2_{m+2j}<a_1-a_0\right]$. If $a_1<a_0$, and m is odd,

$$\gamma_j=e^{\frac{a_0-a_1}{2}}(-1)^{\frac{m-1}{2}}\left\{\frac{2\mathcal{D}\left(\sqrt{\frac{a_0-a_1}{2}}\right)}{\pi^{\frac{1}{2}}}-\sum_{t=0}^{j+\frac{(m-3)}{2}}\frac{\left(\frac{a_0-a_1}{2}\right)^{t+\frac{1}{2}}(-1)^t}{\Gamma(t+\frac{3}{2})}\right\}$$

where $\mathcal{D}(y)$ is Dawson's integral tabled in Abramowitz and Stegun (1964).

Remark: Note that the result in the theorem is completely general since we may write

$$P\left[c_1\chi^2_{2k}+c_2\chi^2_m>c\right]=\ P\left[\frac{\chi^2_{2k}}{a_0}+\frac{\chi^2_m}{a_1}>1\right]$$

where $a_0=cc_1^{-1}$ and $a_1=cc_2^{-1}$.

Proof:

$$\begin{aligned}(*)&=P\left[\frac{\chi^2_{2k}}{a_0}+\frac{\chi^2_m}{a_1}>1\right]-P\left[\chi^2_m>a_1\right]\\&=P\left[\chi^2_m<a_1 \text{ and } \chi^2_{2k}>\frac{a_1-\chi^2_m}{a_1a_0^{-1}}\right]\\&=E\left[I\left(\chi^2_m<a_1\right)P\left[\chi^2_{2k}>\frac{a_1-\chi^2_m}{a_1a_0^{-1}}|\chi^2_m\right]\right].\end{aligned}$$

Hence,

$$\begin{aligned}(*)&=\int_0^{a_1}\frac{u^{\frac{m}{2}-1}e^{\frac{-u}{2}}}{\Gamma(\frac{m}{2})2^{\frac{m}{2}}}\left\{\sum_{j=0}^{k-1}\frac{\left(\frac{a_1-u}{2a_1a_0^{-1}}\right)^j}{j!}e^{\left(-\frac{a_1-u}{2a_1a_0^{-1}}\right)}\right\}du\\&=\frac{e^{\frac{-a_0}{2}}}{\Gamma(\frac{m}{2})2^{\frac{m}{2}}}\sum_{j=0}^{k-1}\frac{\left(\frac{a_0}{2a_1}\right)^j}{j!}\int_0^{a_1}(a_1-u)^ju^{\frac{m}{2}-1}e^{\frac{-u}{2}(1-\frac{a_0}{a_1})}du.\end{aligned}$$

Equation 3.393, #1, p. 318, of Gradshteyn and Ryzhik implies that the integral above is $a_1^{\frac{m}{2}+j}\ \beta(\frac{m}{2},j+1)\,{}_1F_1\ (\frac{m}{2},j+1+\frac{m}{2};(a_0-a_1)/2)$ where ${}_1F_1$ is the confluent hypergeometric function.

For $a_0 < a_1$, a theorem of Bock, Judge and Yancey (1984) implies that for odd m,

$$\Gamma(\frac{m}{2})\,[(a_1-a_0)/2]^{\frac{1}{2}}\,{}_1F_1(\frac{m}{2},j+1+\frac{m}{2};(a_0-a_1)/2)$$

$$=\frac{(-1)^{(m-1)/2}}{(\Gamma(\frac{m}{2}+j+1))^{-1}}\sum_{\ell=(m+1)/2}^{j+(m+1)/2}\frac{\Gamma(\ell-\frac{1}{2})(\frac{2}{a_0-a_1})^{\ell-1}P\left[\chi^2_{2\ell-1}<a_1-a_0\right]}{(\ell-\frac{(m+1)}{2})!(j+\frac{(m+1)}{2}-\ell)!}.$$

Applying these to the integral we have we may write (*) as

$$\frac{e^{-\frac{a_0}{2}}a_1^{\frac{m}{2}}}{\Gamma(\frac{m}{2})(-2)^{\frac{m-1}{2}}(a_1-a_0)^{\frac{1}{2}}}\sum_{j=0}^{k-1}(\frac{a_0}{2})^j$$

$$\sum_{\ell=\frac{(m+1)}{2}}^{j+\frac{(m+1)}{2}}\frac{\Gamma(\ell-\frac{1}{2})(\frac{2}{a_0-a_1})^{\ell-1}P\left[\chi^2_{2\ell-1}<a_1-a_0\right]}{(\ell-\frac{(m+1)}{2})!(j+\frac{(m+1)}{2}-\ell)!}.$$

Interchanging the orders of summation and setting $i=\ell-\frac{(m+1)}{2}$ above gives (*) as

$$\frac{a_1^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}e^{\frac{-a_0}{2}}\sum_{i=0}^{k-1}\frac{\Gamma(i+\frac{m}{2})(\frac{2}{a_0-a_1})^{i+\frac{(m-1)}{2}}}{i!(-2)^{\frac{m-1}{2}}(a_1-a_0)^{\frac{1}{2}}}\sum_{j=n}^{k-1}\frac{(\frac{a_0}{2})^j}{(j-i)!}P\left[\chi^2_{2i+m}<a_1-a_0\right].$$

Because

$$e^{\frac{-a_0}{2}}\sum_{j=i}^{k-1}\frac{(\frac{a_0}{2})^j}{(j-i)!}=(\frac{a_0}{2})^iP\left[\chi^2_{2(k-i)}>a_0\right],$$

we can substitute this in the last expression for (*) and the theorem is shown for $a_0 < a_1$ and odd m. A corresponding evaluation of ${}_1F_1(\frac{m}{2},j+1+\frac{m}{2};(a_0-a_1)/2)$ for even m gives the same result here and the definitions of γ_j and β_j complete the proof of the result for $a_1 > a_0$. If $a_1 < a_0$, then a theorem of Bock, Judge and Yancey (1984) implies that for odd m,

$$\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+j+1)}{}_1F_1(\frac{m}{2},j+1+\frac{m}{2};(a_0-a_1)/2)$$

$$= e^{(a_0-a_1)/2}(-1)^{\frac{m-1}{2}}\sum_{s=0}^{j}\frac{(\frac{a_0-a_1}{2})^{s-j-(m-1)/2}\Gamma(j+\frac{m}{2}-s)}{s!(j-s)!}$$

$$\left\{\frac{2\mathcal{D}(\sqrt{\frac{a_0-a_1}{2}})}{\sqrt{\frac{\pi(a_0-a_1)}{2}}}-\sum_{t=0}^{j-s+\frac{(m-3)}{2}}\frac{(\frac{a_1-a_0}{2})^t}{\Gamma(t+\frac{3}{2})}\right\}.$$

Thus

$$(*) = \frac{e^{\frac{-a_0}{2}}}{\Gamma(\frac{m}{2})}(\frac{a_1}{2})^{\frac{m}{2}}\sum_{j=0}^{k-1}(\frac{a_0}{2})^j(-1)^{\frac{m-1}{2}}e^{\frac{(a_0-a_1)}{2}}$$

$$\sum_{s=0}^{j}\frac{(\frac{a_0-a_1}{2})^{s-j-(\frac{m-1}{2})}\Gamma(j+\frac{m}{2}-s)}{s!(j-s)!}$$

$$\left\{\frac{2\mathcal{D}(\sqrt{\frac{a_0-a_1}{2}})}{\sqrt{\frac{\pi(a_0-a_1)}{2}}}-\sum_{t=0}^{j-s+\frac{(m-3)}{2}}\frac{(\frac{a_1-a_0}{2})^t}{\Gamma(t+\frac{3}{2})}\right\}.$$

Setting $i=j-s$ and interchanging the order of summation for i and j gives

$$(*) = \frac{e^{\frac{-a_1}{2}}}{\Gamma(\frac{m}{2})}(\frac{a_1}{2})^{\frac{m}{2}}\sum_{i=0}^{k-1}\frac{(\frac{a_0-a_1}{2})}{i!}^{-i-\frac{(m-1)}{2}}\Gamma(i+\frac{m}{2})(-1)^{\frac{m-1}{2}}$$

$$\cdot\left\{\frac{2\mathcal{D}(\sqrt{\frac{a_0-a_1}{2}})}{\sqrt{\frac{\pi(a_0-a_1)}{2}}}-\sum_{t=0}^{i+(\frac{m-3}{2})}\frac{(\frac{a_1-a_0}{2})^t}{\Gamma(t+\frac{3}{2})}\right\}\sum_{j=i}^{k-1}\frac{(\frac{a_0}{2})^j}{(j-i)!}.$$

The result of the theorem follows because

$$\sum_{j=i}^{k-1}\frac{e^{\frac{-a_0}{2}}(\frac{a_0}{2})^{j-i}}{(j-i)!} = P\left[\chi^2_{2(k-i)} > a_0\right].$$

The corollary to the next theorem gives a representation for the distribution of Q in terms of its density.

Theorem 2.2. Let W be a continuous nonnegative random variable and assume χ^2_n has a central chi-squared (n) distribution independent of W. Let $f_{Q_n}(x)$ be the density of

$Q_n = W + c_0\chi_n^2$ where $c_0 > 0$. For $c > 0$, and $n > 2$,

$$P\left[W + c_0\chi_n^2 > c\right] = (2c_0) f_{Q_n}(c) + P\left[W + c_0\chi_{n-2}^2 > c\right].$$

If $n = 2$,

$$P\left[W + c_0\chi_2^2 > c\right] = (2c_0) f_{Q_2}(c) + P\left[W > c\right].$$

Corollary 2.3. For the quadratic form

$$Q = c_1\chi_m^2 + c_2\chi_n^2,$$

we have

$$P\left[Q > c\right] = 2c_2 f_Q(c) + P\left[c_1\chi_m^2 + c_2\chi_{n-2}^2 > c\right]$$

where $f_Q(x)$ is the density of Q and $\chi_0^2 \equiv 0$.

Proof of Theorem 2.2. Let $Q_n = W + c_0\chi_n^2$. Let $f_{Q_n}(x)$ be the density of Q_n. Then

$$f_{Q_n}(c) = \frac{d}{dc}\left[P\left[W + c_0\chi_n^2 < c\right]\right].$$

We may write

$$P\left[W + c_0\chi_n^2 < c\right] = \int_0^{c/c_0} \frac{u^{n/2-1}e^{-u/2}}{\Gamma(\frac{n}{2})2^{n/2}} \left[\int_0^{c-c_0 u} dF_W\right] du$$

$$= \int_0^{c} \frac{(\frac{c-t}{c_0})^{n/2-1}}{c_0} \frac{e^{-1/2(c-t)/c_0}}{\Gamma(\frac{n}{2})2^{n/2}} \left[\int_0^{t} dF_W\right] dt$$

where $t = c - c_0 u$ is the change of variable.

Differentiating this last expression and using I for the indicator function implies

$$f_{Q_n}(c) = (2c_0)^{-1}\{I(n \geq 3) \int_0^c \frac{(\frac{c-t}{c_0})^{\frac{n}{2}-2}e^{-(c-t)/2c_0}}{c_0\Gamma(\frac{n-2}{2})2^{n/2-1}}[\int_0^t dF_W]dt$$

$$+ I(n = 2)[\int_0^c dF_W] + (-1)\int_0^c \frac{(\frac{c-t}{c_0})^{n/2-1}e^{-(c-t)/2c_0}}{c_0\Gamma(\frac{n}{2})2^{n/2}}[\int_0^t dF_W]dt\}$$

$$= (2c_0)^{-1}\{I(n \geq 3)P[W + c_0\chi_{n-2}^2 < c] + I(n = 2)P[W < c] - P[W + c_0\chi_n^2 < c]\}.$$

The following theorem gives the density of Q in terms of a confluent hypergeometric function.

Theorem 2.4. Let m and n be positive integers and let c_1, c_2 be positive. Then the density of

$$Q = c_1\chi_m^2 + c_2\chi_n^2$$

is

$$f_Q(y) = \frac{y^{(m+n)/2-1}e^{-y/2c_1}}{\Gamma(\frac{m+n}{2})(2c_1)^{m/2}(2c_2)^{n/2}}\,{}_1F_1(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1} - c_2^{-1})\frac{y}{2})$$

for $y \geq 0$ where χ_m^2 and χ_n^2 are independent chi-squared random variables and ${}_1F_1$ is the confluent hypergeometric function.

Proof. Let W_1 and W_2 be independent random variables such that W_1/c_1 has a chi-squared (m) distribution and W_2/c_2 has a chi-squared (n) distribution. Then the density of W_1 is

$$h_1(x) = \frac{x^{m/2-1}e^{-x/2c_1}}{\Gamma(\frac{m}{2})(2c_1)^{m/2}}$$

for $x \geq 0$.

The density of W_2 is

$$h_2(x) = \frac{x^{n/2-1}e^{-x/2c_2}}{\Gamma(\frac{n}{2})(2c_2)^{n/2}}$$

for $x \geq 0$.

Then the density of $Q = W_1 + W_2$ is

$$f_Q(y) = \int_0^y h_1(y-x)h_2(x)dx$$

$$= \frac{e^{-y/2c_1}\left[\int_0^y (y-x)^{m/2-1}x^{n/2-1}e^{-x/2(c_2^{-1}-c_1^{-1})}\right]}{(2c_1)^{m/2}(2c_2)^{n/2}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}.$$

The integral in parentheses can be written as

$$\frac{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})y^{(m+n)/2-1}}{\Gamma(\frac{m+n}{2})}\,{}_1F_1(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1} - c_2^{-1})\frac{y}{2})$$

Thus, for $y \geq 0$,

$$f_Q(y) = \frac{e^{-y/2c_1} y^{(m+n)/2-1} {}_1F_1(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1} - c_2^{-1})\frac{y}{2})}{\Gamma(\frac{m+n}{2})(2c_1)^{m/2}(2c_2)^{n/2}}. \|$$

The following is a direct result of Corollary 2.3 and Theorem 2.4.

Corollary 2.5. Let c_0, c_1 and c be positive and assume that chi-squared variables are independent in the following expressions. Then if $m > 2$,

$$P[c_0\chi_m^2 + c_1\chi_n^2 > c] = P\left[c_0\chi_{m-2}^2 + c_1\chi_n^2 > c\right] +$$

$$\frac{c_0(\frac{c}{2})^{m+n/2-1} e^{-c/2c_0}}{\Gamma(\frac{m+n}{2}) c_0^{m/2} c_1^{n/2}} {}_1F_1(\frac{n}{2}, \frac{m+n}{2}; \frac{c}{2}(c_0^{-1} - c_1^{-1})).$$

For $m = 2$,

$$P[c_0\chi_2^2 + c_1\chi_n^2 > c] = P\left[\chi_n^2 > \frac{c}{c_1}\right] +$$

$$\frac{(\frac{c}{2c_1})^{n/2}}{\Gamma(\frac{n}{2}+1)} e^{-c/2c_0} {}_1F_1(\frac{n}{2}, \frac{n}{2}+1; \frac{c}{2}(c_0^{-1} - c_1^{-1})).$$

Remark: Repeated applications of Corollary 2.5 enable one to evaluate the distribution of Q when m and n are odd since

$${}_1F_1(\frac{m}{2}, m, y) = \Gamma(\frac{m+1}{2}) e^{\frac{y}{2}} (\frac{y}{4})^{\frac{(m-1)}{2}} \cdot I_{(\frac{m-1}{2})}(\frac{y}{2}),$$

where $I_{(\frac{m-1}{2})}$ is the modified Bessel function. (See Equation 13.6.3 of Abramowitz and Stegun (1964).)

Examples:

(a) For $c_2 < c_1$ and $y = \frac{c}{2}(c_2^{-1} - c_1^{-1})$, we have

$$P[c_1\chi_1^2 + c_2\chi_2^2 > c] = P\left[\chi_1^2 > \frac{c}{c_1}\right]$$
$$+ e^{\frac{-c}{2c_1}} (\frac{2c}{\pi c_1 y})^{\frac{1}{2}} D(y^{\frac{1}{2}})$$

and

$$P[c_1\chi_1^2 + c_2\chi_4^2 > c] = P[\chi_1^2 > \frac{c}{c_1}]+$$

$$e^{\frac{-c}{2c_1}}(\frac{2c}{\pi c_1 y})^{\frac{1}{2}}\left[\mathcal{D}(y^{\frac{1}{2}})(1+\frac{c}{4c_2}\left[1+y^{-1}\right]) - \frac{c}{4c_2 y^{\frac{1}{2}}}\right].$$

(b) For $d_i = c/4c_i, i = 1, 2,$

$$P[c_1\chi_3^2 + c_2\chi_1^2 > c] = P[c_1{}^{(1)}\chi_1^2 + c_2{}^{(2)}\chi_1^2 > c]+$$

$$\sqrt{4d_1d_2}\; e^{-(d_1+d_2)}\left\{I_0(d_2-d_1) + I_1(d_2-d_1)\right\}.$$

and

$$P[c_1{}^{(1)}\chi_3^2 + c_2{}^{(2)}\chi_3^2 > c] = P[c_1{}^{(1)}\chi_1^2 + c_2{}^{(2)}\chi_1^2 > c]+$$

$$e^{-(d_1+d_2)}(4d_1d_2)^{\frac{1}{2}}\left\{I_0(d_1-d_2) + \frac{(d_1+d_2)}{(d_1-d_2)}I_1(d_1-d_2)\right\}.$$

For instance with $c_1 = .25, c_2 = .75$ and $c = 1.8$, we get after substitution,

$$P[.25\chi_3^2 + .75\chi_1^2 > 1.8] = .292.$$

Furthermore with $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}$ and $c = 8$, we get after substitution

$$P\left[\frac{1}{3}{}^{(1)}\chi_3^2 + \frac{2}{3}{}^{(2)}\chi_3^2 > 8\right] = .018318.$$

Section 3. Example: the average Kendall tau statistic

For the rankings of r objects by n judges, the average Kendall tau statistic, $\bar{\tau}_n$, is the average of Kendall's rank correlation between each of the $\binom{n}{2}$ pairs of judges. The null hypothesis is that the r rankings of the judges are picked at random from a uniform distribution on the $r!$ possible rankings. As $n \to \infty$, the null distribution of

$$(n\bar{\tau}_n + 1) \cdot \frac{3r(r-1)}{2}$$

is that of

$$Q = (r+1)\chi^2_{r-1} + \chi^2_{r^*},$$

where r^* is $\binom{r-1}{2}$ the binomial coefficient. (See Alvo, Cabilio and Fiegin (1982) for this result and discussion.) The results in this section are derived from the results of Section 2 using algebra.

For $r = 3$,

$$P\left[4\chi^2_2 + \chi^2_1 > t\right] = P\left[\chi^2_1 > t\right] + \frac{2}{\sqrt{3}} e^{\frac{-t}{8}} P\left[\chi^2_1 < \frac{3t}{4}\right].$$

For $r = 4$,

$$P\left[5\,{}^{(1)}\chi^2_3 + {}^{(2)}\chi^2_3 > t\right] = P\left[5\,{}^{(1)}\chi^2_1 + {}^{(2)}\chi^2_1 > t\right]$$
$$+ e^{-.3t} \frac{t}{2\sqrt{5}} \left\{ I_0\left(\frac{t}{5}\right) + 1.5\ I_1\ \left(\frac{t}{5}\right) \right\}$$

where I_0 and I_1 are modified. Bessel functions tabled in Abramowitz and Stegun (1964), Tables of $P\left[c_1{}^{(1)}\chi^2_1 + c_2{}^{(2)}\chi^2_1 > c\right]$ are given in Solomon (1960). If tables of non-central chi-squared distribution functions are available, we may use the exact expression that follows where A and B are non-centrally parameters:

$$P[5\,{}^{(1)}\chi^2_1 + {}^{(2)}\chi^2_1 > t]$$
$$= P[\chi^2_{2,A} < B] - P[\chi^2_{2,B} < A]$$

where

$$A = \frac{t}{72}(3 - 5^{\frac{1}{2}})$$
$$B = \frac{t}{72}(3 + 5^{\frac{1}{2}}).$$

Now for $r = 5$,

$$P\left[6\chi^2_4 + \chi^2_6 > t\right]$$
$$= P\left[\chi^2_6 > t\right] + P\left[\chi^2_8 < \frac{5t}{6}\right] \left(\frac{-3}{5}\right)\left(\frac{6}{5}\right)^3 P\left[\chi^2_2 > \frac{t}{6}\right]$$
$$+ P\left[\chi^2_6 < \frac{5t}{6}\right] \left(\frac{6}{5}\right)^3 P\left[\chi^2_4 > \frac{t}{6}\right]$$
$$= P\left[\chi^2_6 > t\right] + .1\left(\frac{t}{2}\right)^3 e^{\frac{-t}{2}} +$$
$$+ e^{\frac{-t}{12}} (.6912 + .144t) P\left[\chi^2_6 < \frac{5t}{6}\right].$$

The asymptotic distribution of $\bar{\tau}$ is summarized in the table below for small values of r:

r = number of items ranked	Q = asymptotic distribution of $(n\bar{\tau}_n + 1)\frac{3r(r-1)}{2}$ as $n \to \infty$	$P[Q > t]$
3	$4\chi_2^2 + \chi_1^2$	$P[\chi_1^2 > t] + \frac{2}{\sqrt{3}} e^{-\frac{t}{8}} P\left[\chi_1^2 < \frac{3t}{4}\right]$
4	$5\ {}^{(1)}\chi_3^2 + {}^{(2)}\chi_3^2$	$P[5\ {}^{(1)}\chi_1^2 + {}^{(2)}\chi_1^2 > t] + e^{-.3t} \frac{t}{2\sqrt{5}} \{I_0(.2t)$ $+1.5 I_1(.2t)\}$
5	$6\chi_4^2 + \chi_6^2$	$P[\chi_6^2 > t] + .1(\frac{t}{2})^3 e^{\frac{-t}{2}} +$ $e^{\frac{-t}{12}}(.6912 + .144t) P[\chi_6^2 < \frac{5t}{6}]$

Section 4. Acknowledgements

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