THE CENTRAL LIMIT THEOREM FOR THE RIGHT EDGE OF SUPERCRITICAL ORIENTED PERCOLATION

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Running head: CLT FOR ORIENTED PERCOLATION

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ABSTRACT

This paper contains a proof of the conjectured central limit theorem for the growth of the right hand edge for supercritical oriented percolation. The technique of proof, finding points with regeneration type properties termed "break points", may also apply to other processes.

Abbreviated Title: Central Limit Theorem for Percolation.

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1. INTRODUCTION

In a recent special invited paper concerning oriented percolation, Durrett (1984), it was noted as an open problem that a central limit theorem held for the growth of the right edge of the process in the supercritical case. The purpose of this paper is to prove the central limit theorem for the right edge by showing the existence of points with regeneration-type properties, referred to as "break points". The results then follow immediately by invoking a very simple version of a central limit theorem. In a recent paper, Galves and Presutti (1987) obtain the central limit theorem for the contact process case. The technique presented here is an improvement in the sense that it is shorter, simpler, gives more information on the "structure" of edge behavior, and invokes a much simpler central limit theorem to obtain the result. The i.i.d. nature of edge growth should also make it possible to improve large deviation results for the process.

2. DEFINITIONS AND NOTATION

We develop notation here which is generally consistent with that of sections two and three of Durrett (1984). First of all, we say that (y,m) can be reached from (x,n), denoted $(x,n) \to (y,m)$, if there is an open path from (x,n) to (y,m); that is, there is a sequence $(x_0,n_0) = (x,n)$, $(x_1,n_1),\dots,(x_j,n_j) = (y,m)$ where $j = m-n$ such that the arc from (x_{k-1},n_{k-1}) to (x_k,n_k) is open for each $1 \le k \le (m-n)$. If we write $(x,n) \not\to (y,m)$, then no open path exists from (x,n) to (y,m). Let

$$C_{(x,n)} = \{(y,m)\colon\ (x,n) \to (y,m)\},$$

the set of all points we can reach from (x,n). Events of particular interest in what follows are

$$\Omega_{(x,n)} = \{|C_{(x,n)}| = \infty\},$$

i.e., the event that there is an infinite open path starting at (x,n), where $|A|$ denotes the number of points in A. As in section 3 of Durrett (1984), we let $\mathcal{L} = \{(m,n) \in \mathbb{Z}^2\colon\ m+n$ is even, $n \ge 0\}$.

Now define

$$\xi_n^{(y,m)} = \{x\colon\ \ (x,n+m) \in \mathcal{L} \text{ and } (y,m) \to (x,n+m)\}$$

and

$$r_n^{(y,m)} = \sup \xi_n^{(y,m)} \qquad (\sup \varnothing = -\infty).$$

For the sake of readability, we will delete the superscript and write $r_n^{(0,0)}$ as r_n noting that r_n has the same meaning here as in Durrett (1984). Complications to later arguments arise when $\xi_n^{(0,0)} = \varnothing$. To avoid them we define

$$\begin{aligned}
\xi'_0 &= \xi_0^{(0,0)} \\
\xi'_1 &= \xi_1^{(0,0)} \text{ if } \xi_n^{(0,0)} \neq \varnothing \\
&= \{1\} \text{ otherwise.} \\
\xi'_{n+1} &= \{x\colon \ (x, n+1) \in \mathcal{L} \text{ and} \\
&\qquad (y,n) \to (x, n+1) \text{ for} \\
&\qquad \text{some } y \in \xi'_n\} \text{ if} \\
&\qquad \text{this set is nonempty} \\
&= \{n+1\} \text{ otherwise.}
\end{aligned}$$

and

$$r'_n = \sup \xi'_n.$$

Note that

$$\begin{aligned}
\xi_n^{(0,0)} \subset \xi'_n &\subset \{-n, -n+1, \dots, n-1, n\}, \\
-n &\le r'_n \le n, \\
r_n &\le r'_n \le n,
\end{aligned}$$

and

$$\text{on } \{\xi_n^{(0,0)} \neq \varnothing\} \qquad r_n = r'_n.$$

DEFINITION 2.1. A point $(x,n) \in \mathcal{L}$ is a percolation point if and only if $I_{\Omega_{(x,n)}} = 1$. Thus any particular point is a percolation point if there is an infinite open path originating from that point.

Thus we call the rightmost point of ξ'_n, (r'_n, n) a percolation point if it is located at a percolation point. So in particular, if $\xi_n^{(0,0)} \neq \varnothing$ and (r_n, n) is located at a percolation point, then we would call (r_n, n) a percolation point.

3. BREAK POINTS

We wish to simplify the study of the behavior of the right edge of the process by breaking it into independent pieces. To this end define a sequence of random variables $\{T_i\}$ by

$$T_1 = \inf\{n \geq 1\colon\ (r'_n, n) \text{ is a percolation point}\} \quad (\inf \emptyset = \infty),$$

$$T_2 = \inf\{n \geq T_1 + 1\colon\ (r'_n, n) \text{ is a percolation point}\}$$

$$\vdots$$

$$T_{m+1} = \inf\{n \geq T_m + 1\colon\ (r'_n, n) \text{ is a percolation point}\}.$$

Note that the event $\{T_i = n\}$ for $n \geq i$ is not measurable with respect to the σ-algebra of observable events up to and including time n, and so cannot be a stopping time with respect to this σ-algebra. This is because the event $\{T_i = n\}$ depends in part on future behavior. It will be shown, however, that members of the sequence $\{T_i\}$ have stopping time-like properties. Define

$$\tau_1 = T_1$$

$$\tau_2 = T_2 - T_1$$

$$\vdots$$

$$\tau_{n+1} = T_{n+1} - T_n,$$

where τ_{i+1} is set equal to zero if T_{i+1} and/or T_i is infinite. Also define

$$X_1 = r'_{T_1}$$

$$X_2 = r'_{T_2} - r'_{T_1}$$

$$\vdots$$

$$X_{n+1} = r'_{T_{n+1}} - r'_{T_n},$$

where X_{i+1} is set equal to zero if $r'_{T_{i+1}}$ and/or r'_{T_i} are undefined. We refer to the collection of points $\{(r'_{T_i}, T_i)\}$ as *break points*, since, as the following theorem shows, they break the behavior of the right hand edge into i.i.d. pieces when the origin is a percolation point.

THEOREM: Conditioned on the event $\Omega_{(0,0)}$ the random vectors $\{(X_i, \tau_i)\}$ are independent and identically distributed with all moments.

PROOF: It will first be shown that τ_1 has all moments unconditionally. This will imply that τ_1 has all moments conditioned on any set of positive probability. In particular, this will imply that τ_1 has all moments conditioned on the event that the origin is a percolation point. An immediate consequence is that X_1 has all moments conditioned on the event that the origin is a percolation point. It will then be shown that, conditioned on the event that the origin is a percolation point, the random vectors are independent and identically distributed.

In order to prove the existence of all moments for τ_1, we will first write τ_1 as a random sum of random variables which have bounded tails. To this end, define

$$Y(x,n) = \inf\{m: \ \xi_m^{(x,n)} = \varnothing\} \quad (\inf \varnothing = \infty).$$

In section 12 of Durrett (1984), it was shown that if $p > p_c$ that there exist constants c and γ such that

$$P(n < Y(0,0) < \infty) \leq ce^{-\gamma n}.$$

If we let $p_0 = P(Y(0,0) < \infty)$ and $q_0 = 1 - p_0$, then

$$P(n < Y(0,0) | Y(0,0) < \infty) \leq (c/p_0)e^{-\gamma n}.$$

Since the array $\{Y(x,n): \ (x,n) \in \mathcal{L}\}$ is identically distributed, though not independent,

$$P(n < Y(x,n) | Y(x,n) < \infty) \leq (c/p_0)e^{-\gamma n}.$$

In order to write τ_1 as a random sum of random variables, we define the following events and random variables recursively. In the first step, let

$$A_1 = \{(r_1', 1) \text{ is a percolation point}\}$$

and

$$Y_1 = Y(r_1', 1).$$

Then let

$$A_2 = A_1^c \cap \{(r_{Y_1+1}', Y_1 + 1) \text{ is a percolation point}\}$$

and

$$Y_2 = Y(r'_{Y_1+1}, Y_1 + 1)$$
$$= \infty \text{ if } Y_1 = \infty.$$

Then

$$A_3 = A_1^c \cap A_2^c \cap \{(r'_{Y_1+Y_2+1}, Y_1 + Y_2 + 1) \text{ is a percolation point}\}$$

and

$$Y_3 = Y(r'_{Y_1+Y_2+1}, Y_1 + Y_2 + 1),$$
$$= \infty \text{ if } Y_1 + Y_2 = \infty$$

so in general

$$A_{n+1} = A_1^c \cap \ldots \cap A_n^c \cap \{(r'_{Y_1+\ldots+Y_n+1}, Y_1 + \ldots + Y_n + 1) \text{ is a percolation point}\}$$

and

$$Y_{n+1} = Y(r'_{Y_1+\ldots+Y_n+1}, Y_1 + \ldots + Y_n + 1)$$
$$= \infty \text{ if } Y_1 + \ldots + Y_n = \infty$$

Finally define N by

$$N = \inf\{i: \ I_{A_i} = 1\}.$$

The following series of lemmas will show that

$$\tau_1 = 1 + \sum_{i=1}^{N-1} Y_i.$$

LEMMA 1: If (x, n) is not a percolation point and $(x, n) \to (y, m)$, then (y, m) is not a percolation point.

PROOF: If $(y, m) \to (z, \ell)$, then $(x, n) \to (z, \ell)$ also. Therefore all points connected to (y, m) by an open arc are also connected to (x, n) by an open arc, which implies that if (y, m) were a percolation point, (x, n) would be also.

LEMMA 2: If (r'_n, n) is not a percolation point, then $(r'_{n+m}, n + m)$ is not a percolation point for $m = 0, 1, \ldots, Y(r'_n, n) - 1$.

PROOF: The result is immediate if $m = 0$ (i.e. $Y(r'_n, n) = 1$), so assume $Y(r'_n, n) > 1$. It will be shown that for $m = 1, 2, \ldots, Y(r'_n, n) - 1$, that $(r'_n, n) \to (r'_{n+m}, n + m)$. The result will then follow from lemma 1. By definition of $Y(r'_n, n)$, $\xi_m^{(r'_n,n)} \neq \emptyset$ for $m = 1, 2, \ldots,$ $Y(r'_n, n) - 1$. This being the case, it will suffice to show that $(r'_n, n) \to (r'_{n+m}, n+m)$, for $m = 1, 2, \ldots, Y(r'_n, n) - 1$. The result then follows from lemma 1. Given that $\xi_m^{(r'_n,n)} \neq \emptyset$ for $m = 1, 2, \ldots, Y(r'_n, n) - 1$, it follows that $\xi'_{n+m} = \{x\colon\ (y,n) \to (x, n+m)$ for some $y \in \xi'_n\}$. Now suppose that $(x, n) \to (r'_{n+m}, n+m)$ for some $x \in \xi'_n$. Since $\xi_m^{(r'_n,n)} \neq \emptyset$, $\exists\, y \in \xi'_{n+m}$ such that $(r'_n, n) \to (y, n+m)$. Since $y \leq r'_{n+m}$, these paths must intersect, which implies $(r'_n, n) \to (r'_{n+m}, n+m)$ (see figure 2 of Durrett (1984)). □

LEMMA 3: Given that $Y_1, Y_2, \ldots, Y_K$ are defined and finite, $P(Y_{K+1} \leq m | Y_1, \ldots, Y_K) = P(Y(0,0) \leq m)$.

PROOF: Let $n_1, \ldots, n_K$ be integers such that $1 \leq n_i < \infty$ for $i = 1, 2, \ldots, K$, and let $M = \sum_{i=1}^{K} n_i$. The event $\{Y_1 = n_1, Y_2 = n_2, \ldots, Y_K = n_K\}$ is determined by open or closed bonds between lattice points in the set $\{(x,n)\colon\ (x,n) \in \mathcal{L}$ and $n = 0, 1, \ldots, \sum_{i=1}^{K} n_i\}$. The event $\{Y_{K+1} \leq m\}$ is determined by open or closed bonds between lattice points in the set $\{(x,n)\colon\ (x,n) \in \mathcal{L}$ and $n = \sum_{i=1}^{K} n_i, \sum_{i=1}^{K} n_i + 1, \ldots, \sum_{i=1}^{K} n_i + m\}$. The bonds between lattice points in one set are open or closed independently of the behavior of bonds between lattice points in the other set. Noting that

$$P(Y_{K+1} \leq m | Y_1 = n_1, \ldots, Y_K = n_K) = P(Y(r'_M, M) \leq m | Y_1 = n_1, \ldots, Y_K = n_K),$$

by the previous remark

$$P(Y(r'_M, M) \leq m | Y_1 = n_1, \ldots, Y_K = n_K) = P(Y(r'_M, M) \leq m).$$

Since $\{Y(x,n)\}$ are identically distributed,

$$P(Y(r'_N, N) \leq m) = P(Y(0,0) \leq m)$$

□

LEMMA 4: $P(N = K) = q_0 p_0^{K-1}$, $K = 1, 2, \ldots$.

PROOF: $P(N = K) = P(A_1^c \cap \ldots \cap A_{K-1}^c \cap A_K)$ by definition of N. By lemma 3,

$$P(A_1^c \cap \ldots \cap A_{K-1}^c \cap A_K) = P(A_1^c \cap \ldots \cap A_{K-1}^c) \cdot P(A_K),$$

where $P(A_K) = q_0$. Lemma 3 also implies that

$$P(A_1^c \cap \ldots \cap A_{K-1}^c) = P(A_1^c \cap \ldots \cap A_{K-2}^c) \cdot P(A_{K-1}^c),$$

where $P(A_{K-1}^c) = p_0$. Repeating the argument,

$$P(A_1^c \cap \ldots \cap A_{K-1}^c) = p_0^{K-1},$$

yielding

$$P(A_1^c \cap \ldots \cap A_{K-1}^c \cap A_K) = p_0^{K-1} q_0$$

and, hence the result. □

LEMMA 5: $1 + \sum_{i=1}^{N-1} Y_i$ is finite with probability one.

PROOF: By lemma 4, $(N - 1)$ is finite with probability one. By definition of N, given that $\{N = n\}$, $Y_1, \ldots Y_{n-1}$ are finite and Y_n infinite with probability one. □

LEMMA 6: $\tau_1 = 1 + \sum_{i=1}^{N-1} Y_i$ with probability one.

PROOF: First note that $Y_N = \infty$, by definition of N, implying that $(r'_{1+\sum_{i=1}^{N-1} Y_i}, 1+\sum_{i=1}^{N-1} Y_i)$ is a percolation point. This implies that $\tau_1 \leq 1 + \sum_{i=1}^{N-1} Y_i$ almost surely. Now it remains to be shown that $1 + \sum_{i=1}^{N-1} Y_i$ is the first time, after time zero, that the rightmost point is a percolation point. Note that at times $1, 1+Y_1, \ldots, 1+\sum_{i=1}^{N-2} Y_i$, the rightmost point cannot be a percolation point, because the next Y_i in the sequence is finite. For time points in between elements of the sequence 1, $1 + Y_1, \ldots, 1 + \sum_{i=1}^{N-1} Y_i$, the rightmost point cannot be located at a percolation point, since lemma 2 would then imply that one of $(r'_1, 1)$, $(r'_{1+Y_1}, 1 + Y_1), \ldots, (r'_{1+\sum_{i=1}^{N-2} Y_i}, 1 + \sum_{i=1}^{N-2} Y_i)$ is a percolation point. □

In view of the preceding lemmas, it should be intuitively obvious that τ_1 is stochastically bounded by a geometric sum of an i.i.d. sequence of random variables with exponentially bounded tails, which are independent of the geometric random variable which we sum to. It is, however, necessary to show this formally since the sequence $\{Y_i\}$ is "not quite" independent of N, since it is used to define N.

LEMMA 7:

$$P(Y_K = m_K, \ldots, Y_1 = m_1 | N - 1 = K) = \prod_{i=1}^{K} P(Y(0,0) = m_i | Y(0,0) < \infty).$$

PROOF:

$$P(Y_K = m_K, \ldots, Y_1 = m_1 | N - 1 = K) = \frac{P(\{Y_K = m_K, \ldots, Y_1 = m_1\} \cap \{N - 1 = K\})}{P(N - 1 = K)}.$$

From lemma 4 and by definition of N,

$$\frac{P(\{Y_K = m_K, \ldots, Y_1 = m_1\} \cap \{N - 1 = K\})}{P(N - 1 = K)} = \frac{P(Y_k = m_K, \ldots, Y_1 = m_1, Y_{K+1} = \infty)}{q_0 p_0^K}.$$

Factoring the numerator of the latter term we obtain

$$P(Y_K = m_K, \ldots Y_1 = m_1, Y_{K+1} = \infty) =$$
$$P(Y_{K+1} = \infty | Y_K = m_K, \ldots, Y_1 = m_1) \cdot P(Y_K = m_K, \ldots, Y_1 = m_1).$$

Applying lemma 3,

$$P(Y_{K+1} = \infty | Y_K = m_K, \ldots, Y_1 = m_1) = q_0.$$

By factoring and repeatedly applying Lemma 3,

$$P(Y_K = m_K, \ldots, Y_1 = m_1) = \prod_{i=1}^{K} P(Y(0,0) = m_i),$$

so that

$$P(Y_K = m_K, \ldots, Y_1 = m_1, Y_{K+1} = \infty) = q_0 \prod_{i=1}^{K} P(Y(0,0) = m_i)$$

Finally noting that

$$P(Y(0,0)=m_i|Y(0,0)<\infty)=P(Y(0,0)=m_i)/p_0,$$

we have the result. $\square$

To complete the proof that τ_1 has all moments, let

$$g_{N-1}(s)=Es^{N-1}$$

and

$$g(s)=E(s^{Y(0,0)}|Y(0,0)<\infty)$$

denote the generating functions of $N-1$ and $Y(0,0)$, conditional on the event $\{Y_{(0,0)}<\infty\}$ respectively. From lemma 7, the generating function of $\sum_{i=1}^{N-1} Y_i$ is $g_{N-1}(g(s))$. Because $\tau_1=1+\sum_{i=1}^{N-1} Y_i$, τ_1 will have all moments if $\sum_{i=1}^{N-1} Y_i$ does, and $\sum_{i=1}^{N-1} Y_i$ has all moments if its moment generating function extends to a neighborhood of $s=1$. But this is true because $g_{N-1}(s)$ and $g(s)$ each extend beyond $s=1$.

In order to show the i.i.d. nature of $\{(X_i,\tau_i)\}$, a few more lemmas are in order.

LEMMA 8: If the origin is a percolation point, then $(0,0)\to(r_n,n)$ for $n=1,2,\ldots$.

PROOF: On $\Omega_{(0,0)}$, $\xi_n\neq\varnothing$, $n=1,2,\ldots$. Given this fact, note that $\xi_1'=\xi_1$. Given that $\xi_n'=\xi_n$ and $\xi_{n+1}\neq\varnothing$, $\xi_{n+1}'=\xi_{n+1}$, so that $\xi_n'=\xi_n$ on $\Omega_{(0,0)}$ for all n. $\square$

LEMMA 9: If $(0,0)\to(r_n,n)$ and (r_n,n) is a percolation point, then the origin is a percolation point and $(r_n,n)\to(r_{n+m},n+m)$ for $m=1,2,\ldots$.

PROOF: The first statement follows from Lemma 1. To see that $(r_n,n)\to(r_{n+m},n+m)$, note that $(x,n)\to(r_{n+m},n+m)$ for some $x\in\xi_{n+m}^{(0,0)}$. Since $(r_n,n)\to(y,n+m)$ for some $y\in\xi_{n+m}^{(0,0)}$ (because (r_n,n) is a percolation point), these arcs must cross so that $(r_n,n)\to(r_{n+m},n+m)$. (See figure 2 of Durrett (1984)). $\square$

Before proving the next lemma, define $S_m^{(x,n)}$ for $(x,n)\in\mathcal{L}$ and $m=1,2,\ldots$, by

$$S_1^{(x,n)}\equiv 1,$$

and for $m = 2, 3, \ldots,$

$$\begin{aligned} S_m^{(x,n)} &= m \text{ if } (r_i^{(x,n)}, i+n) \not\rightarrow (r_m^{(x,n)}, n+m) \quad i = 1, 2, \ldots, m-1 \\ &\qquad \text{and } (x,n) \rightarrow (r_m^{(x,n)},\ n+m) \\ &= m+1 \text{ otherwise.} \end{aligned}$$

LEMMA 10: $\{\{(X_1, \tau_1) = (x,m)\} \cap \Omega_{(0,0)}\} = \{\{S_m^{(0,0)} = m\} \cap \{r_m^{(0,0)} = x\} \cap \Omega_{(x,m)}\}$

PROOF: $(\supset)$. If (x,m) is a percolation point $(\Omega_{(x,m)})$ and $r_m^{(0,0)} = x$, then by lemma 9, the origin is a percolation point $(\Omega_{(0,0)})$. To see that (x,m) is the first time the rightmost point is a percolation point, note that if any previous rightmost point were a percolation point, say (r_n, n) $1 \leq n < m$, then $(r_n, n) \rightarrow (r_m, m) = (x,m)$ by lemma 9. This contradicts $\{S_m = m\}$ which states $(r_n, n) \not\rightarrow (r_m, m)$, $n = 1, 2, \ldots m-1$.

$(\subset)$ If the origin is a percolation point $(\Omega_{(0,0)})$ then $(0,0) \rightarrow (r_m, m)$. If $(X_1, \tau_1) = (x,m)$, then $(r_m, m) = (x,m)$ giving $\Omega_{(x,m)}$ and $\{r_m^{(0,0)} = x\}$. Since (x,m) is the first percolation point, $(r_n, n) \not\rightarrow (r_m, m)$ for $n = 1, 2, \ldots, m-1$, by lemma 1 (contrapositive).□

Before proceeding to the final section of the proof, note that by lemma 10

$$P((X_1, \tau_1) = (x,m) | \Omega_{(0,0)}) = \frac{P(\{S_m^{(0,0)} = m\} \cap \{r_m^{(0,0)} = x\} \cap \Omega_{(x,m)})}{P(\Omega_{(0,0)})}$$

Note also that the events $\{S_m^{(0,0)} = m\}$ and $\{r_m^{(0,0)} = x\}$ are determined by bonds between points of $\mathcal{L}' = \mathcal{L} \cap \mathbb{Z}^1 \times \{0, 1, \ldots, m\}$, whereas $\Omega_{(x,m)}$ is determined by bonds between points of $\mathcal{L}'' = \mathcal{L} \cap \mathbb{Z}^1 \times \{m, m+1, \ldots\}$ and so

$$P(\{S_m^{(0,0)} = m\} \cap \{r_m^{(0,0)} = x\} \cap \Omega_{(x,m)}) = P(\{S_m^{(0,0)} = m\} \cap \{r_m^{(0,0)} = x\}) \cdot P(\Omega_{(x,m)}).$$

Since $P(\Omega_{(x,m)}) = P(\Omega_{(0,0)})$,

$$P((X_1, \tau_1) = (x,m) | \Omega_{(0,0)}) = P(\{S_m^{(0,0)} = m\} \cap \{r_m^{(0,0)} = x\}).$$

To finish the proof, let

$$\ell_j = \sum_{i=1}^{j} x_i \qquad (\ell_0 \equiv 0)$$

and

$$t_j = \sum_{i=1}^{j} m_i \qquad (t_0 \equiv 0).$$

By lemma 10,

$$\left\{ \bigcap_{i=1}^{K} \{(X_i, \tau_i) = (x_i, m_i)\} \right\} \cap \Omega_{(0,0)}$$
$$= \{S_{m_1}^{(0,0)} = m_1\} \cap \{r_{m_1}^{(0,0)} = x_1\} \cap \Omega_{(x_1,m_1)} \bigcap_{i=2}^{K} \{(X_i, \tau_i) = (x_i, m_i)\}.$$

Relabeling the origin as (x_1, m_1) in lemma 10 and noting that (τ_2) is the first time after time m_1 that the rightmost point is a percolation point, we have

$$\{(X_2, \tau_2) = (x_2, m_2)\} \cap \Omega_{(x_1,m_1)}$$
$$= \{S_{m_2}^{(x_1,m_1)} = m_2\} \cap \{r_{m_2}^{(x_1,m_1)} = x_2\} \cap \Omega_{(\ell_2,t_2)}.$$

Repeating the argument,

$$\left\{ \bigcap_{i=1}^{K} \{(X_i, \tau_i) = (x_i, m_i)\} \right\} \cap \Omega_{(0,0)}$$
$$= \left\{ \bigcap_{i=1}^{K} \{S_{m_i}^{(\ell_{i-1},t_{i-1})} = m_i\} \cap \{r_{m_i}^{(\ell_{i-1},t_{i-1})} = x_i\} \right\} \cap \Omega_{(\ell_K,t_K)}.$$

The event $\{\{S_{m_i}^{(\ell_{i-1} t_{i-1})} = m_i\} \cap \{r_{m_i}^{(\ell_{i-1},t_{i-1})} = x\}\}$ depends only on the behavior of bonds between points in $\mathcal{L} \times \{t_{i-1}, t_{i-1}+1, \ldots, t_i\}$, implying that

$$P(\bigcap_{i=1}^{K} \{\{S_{m_i}^{(\ell_{i-1},t_{i-1})} = m_i\} \cap \{r_{m_i}^{(\ell_{i-1},t_{i-1})} = x_i\}\} \cap \Omega_{(\ell_K,t_K)})$$
$$= \prod_{i=1}^{K} P(\{S_{m_i}^{(\ell_{i-1},t_{i-1})}\} \cap \{r_{m_i}^{(\ell_{i-1},t_{i-1})} = x_i\}) \cdot P(\Omega_{(\ell_K,t_K)})$$
$$= \prod_{i=1}^{K} P(\{S_{m_i}^{(0,0)} = m_i\} \cap \{r_{m_i}^{(0,0)} = x_i\}) \cdot P(\Omega_{(0,0)}).$$

This finally gives

$$P(\{(X_1, \tau_1) = (x_1, m_1)\} \cap \ldots \cap \{(X_K, \tau_K) = (x_K, m_K)\} | \Omega_{(0,0)})$$
$$= \prod_{i=1}^{K} P(\{S_{m_i}^{(0,0)} = m_i\} \cap \{r_{m_i}^{(0,0)} = x_i\}).$$

□

COROLLARY 1: On the set of non-extinction, $(r_n - \alpha n)/\sqrt{n} \xrightarrow{\mathcal{L}} N(0, \sigma^2)$.

PROOF: Let $N_n = \sup\{m: \sum_{i=1}^{m} \tau_i \leq n\}$, so that r_{N_n+1} is the location of the right hand edge at the first "regeneration point" after time n. Observe that r_{N_n+1} and r_n are offspring of r_{N_n} so that

$$|r_{N_n+1} - r_{N_n}| \leq \sum_{i=1}^{N_n+1} \tau_i - \sum_{i=1}^{N_n} \tau_i = \tau_{N_n+1}$$

and

$$|r_n - r_{N_n}| \leq n - \sum_{i=1}^{N_n} \tau_i \leq \tau_{N_n+1},$$

implying

$$|r_{N_n+1} - r_n| \leq 2\tau_{N_n+1}.$$

It is clear from renewal theory, eg. Feller (1966), that τ_{N_n+1} converges in law to a proper random variable, implying that

$$(r_n - r_{N_n+1})/\sqrt{n} \xrightarrow{P} 0.$$

So it is only necessary to show

$$(r_{N_n+1} - \alpha n)/\sqrt{n} \xrightarrow{\mathcal{L}} N(0, \sigma^2).$$

The proof is not technically difficult, but a proof may be found in Siegmund (1975) which concerns the time until ruin in collective risk. In this paper it is assumed (keeping our notation), that the sequence of random vectors $\{(X_i, \tau_i)\}$ are independent and identically distributed with finite means and variances, and that $\tau_i \geq 0$. Then on page 158, lemma 2 states (again using our notation):

Lemma 2. Assume $EX_1 \geq 0$ and let Φ denote the standard normal distribution function. Then

$$\lim_{n\to\infty} P\{(r_{N_n+1} - (EX_1/E\tau_1) \cdot n)/n^{1/2} \leq x\}$$
$$= \Phi((E\tau_1)^{3/2} x/(E(X_i \cdot E\tau_1 - \tau_1 EX_1)^2)^{1/2}).$$

To complete the proof of the corollary, it only remains to show that

$$\sigma^2 = E(X_1 \cdot E\tau_1 - \tau_1 \cdot EX_1)^2 > 0.$$

This will occur if X_1 and τ_1 are not constant multiples of each other. However, keeping in mind that

$$P(\{(X_1, \tau_1) = (x_1, m_1)\}) = P(\{S_{m_1}^{(0,0)} = m_1\} \cap \{r_{m_1}^{(0,0)} = x_1\},$$

it is easy to see that

$$P(\{S_{m_1}^{(0,0)} = m_1\} \cap \{r_{m_1}^{(0,0)} = x_1\}) > 0$$

if m_1 is a positive integer greater than 1 and $x_1 = -m_1, -m_1 + 2, \ldots, m_1 - 2$. □

4. DISCUSSION

The study of any interacting particle system, even one defined on $\mathbb{Z}^1$, is complicated by the complex stochastic conditioning involved. In the case studied here, proving the existence of break points greatly simplifies the study of the process. Break points do not occur at stopping times, since we condition on a future event. However, what occurs is that we stop at a stopping time and ask a question about the (independent) future. This is why break points occur at times which have stopping time like properties.

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